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Pressure jump and radial stationary solutions of the degenerate Cahn–Hilliard equation

Saut de pression et solutions stationnaires radiales de l'équation dégénérée de Cahn–Hilliard

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Abstract. The Cahn–Hilliard equation with degenerate mobility is used in several areas including the modeling of living tissues, following the theory of mixtures. We are interested in quantifying the pressure jump at the interface between phases in the case of incompressible flows. To do so, we depart from the spherically symmetric dynamical compressible model and include an external force. We prove existence of stationary states as limits of the parabolic problems. Then we prove the incompressible limit and characterize compactly supported stationary solutions. This allows us to compute the pressure jump in the small dispersion regime and in particular the force dependent curvature effect.

Résumé. L'équation de Cahn–Hilliard avec mobilité dégénérée est utilisée dans différents domaines, en particulier la description de tissus vivants suivant la théorie des mélanges. Nous visons à quantifier le saut de pression à l'interface entre phases dans le cas de flots incompressibles. Pour cela, nous considérons des solutions à symmétrie radiale du problème compressible. Nous démontrons l'existence d'états stationnaires comme limite du problème d'évolution. Nous prouvons ensuite la limite incompressible et caratérisons les solutions à support compact. Ceci nous permet de calculer le saut de pression dans le régime de faible dispersion et en particulier d'obtenir la dépendance en la courbure suivant la force appliquée.

Keywords. Degenerate Cahn–Hilliard equation, Asymptotic Analysis, Incompressible limit, Hele–Shaw equations, Surface tension, Pressure jump.

Mots-clés. Equation de Cahn–Hilliard dégénérée, Analyse asymptotique, Limite incompressible, Equation de Hele–Shaw, Saut de pression.

Mathematical subject classification (2010). 35B40, 35B45, 35G20, 35Q92.

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1. Introduction

The degenerate Cahn–Hillard equation is now commonly used in tumor growth modeling and takes into account surface tensions at the interface between different types of cells, leading to a jump of pressure. In order to compute this jump, we propose to set the problem in a spherically symmetric domain with a boundary determined by the radius R_b, and to include an external force. Therefore we consider, in two dimensions for simplicity, the equation

$$\frac{\partial(rn)}{\partial t} - \frac{\partial}{\partial r}\left(rn\frac{\partial(\mu+V)}{\partial r}\right) = 0, \quad \text{in} \quad (0,+\infty)\times I_{R_b}, \tag{1}$$

$$\mu = n^\gamma - \frac{\delta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right), \quad \text{in} \quad (0,+\infty)\times I_{R_b}, \tag{2}$$

where $I_{R_b} = (0,R_b)$ is the line segment of length R_b. Equations (1)-(2) are equipped with Neumann boundary conditions

$$\left.\frac{\partial n}{\partial r}\right|_{r=0} = \left.\frac{\partial n}{\partial r}\right|_{r=R_b} = \left.n\frac{\partial(\mu+V)}{\partial r}\right|_{r=0} = \left.n\frac{\partial(\mu+V)}{\partial r}\right|_{r=R_b} = 0, \tag{3}$$

and with an initial condition satisfying

$$n_0 \in H^1\left(I_{R_b}\right), \qquad n_0 \geq 0. \tag{4}$$

We only consider nonnegative solutions and thus the term n^γ is well defined and the (normalized by a factor $\frac{\pi}{2}$) total mass is

$$m := \int_0^{R_b} r\,n_0(r)\,\mathrm{d}r = \int_0^{R_b} r\,n(t,r)\,\mathrm{d}r. \tag{5}$$

Finally, the confining potential $V(r)$ is of class C^1.

1.1. *Main results*

Our first result concerns the existence of solutions of (1)-(2), their regularity, and asymptotic behaviour.

Theorem 1 (Existence of solutions and long term asymptotic). *There exists a global weak solution of* (1)-(4) *in the sense of Definition 4 and it satisfies estimates as in Remark 5. Moreover, up to a subsequence,* $\{r\,n(t+k,r)\}_k$ *converges locally in time uniformly in space to a stationary solution* $r\,n_\infty(r)\geq 0$ *where* $n_\infty \in C^1(\overline{I_{R_b}})$ *satisfies* $m = \int_0^{R_b} r\,n_\infty(r)\,\mathrm{d}r$ *and*

$$r\,n_\infty\frac{\partial(\mu_\infty+V)}{\partial r} = 0, \qquad \mu_\infty = n_\infty^\gamma - \frac{\delta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_\infty}{\partial r}\right) \qquad n_\infty'(0) = n_\infty'(R_b) = 0. \tag{6}$$

Our second result characterizes possible stationary states and shows we can distinguish an interval where $n_\infty = 0$ and another where $\mu_\infty + V$ is constant as expected from the first equation in (6). From now on, we consider the confining potential $V(r) = r^2$ for simplicity. The proof may be adapted to any increasing potential.

Theorem 2 (Characterization of the stationary states). *Let* $n_\infty \in C^1([0,R_b])$, $n_\infty \geq 0$, *be a solution of* (6) *as built in Theorem 1.*

(**A**) *Then,* n_∞ *is nonincreasing and it satisfies* $0 \leq n_\infty(R_b) < \frac{2m}{R_b^2}$.

(**B**) *Assume* $n_\infty(R_b) = 0$ *and let* $R > 0$ *be the smallest argument such that* $n_\infty(R) = 0$ *and thus* $n_\infty > 0$ *in* $[0,R)$. *Then, there is* $\lambda_\infty \in (0,R^2)$ *such that*

$$\begin{cases} n_\infty^\gamma - \frac{\delta}{r}n_\infty' - \delta n_\infty'' = R^2 - r^2 - \lambda_\infty & \text{in } (0,R), \\ n_\infty(R) = n_\infty'(R) = 0. \end{cases} \tag{7}$$

and, given $R > 0$, *there is at most one couple* (n,λ) *solving* (7).

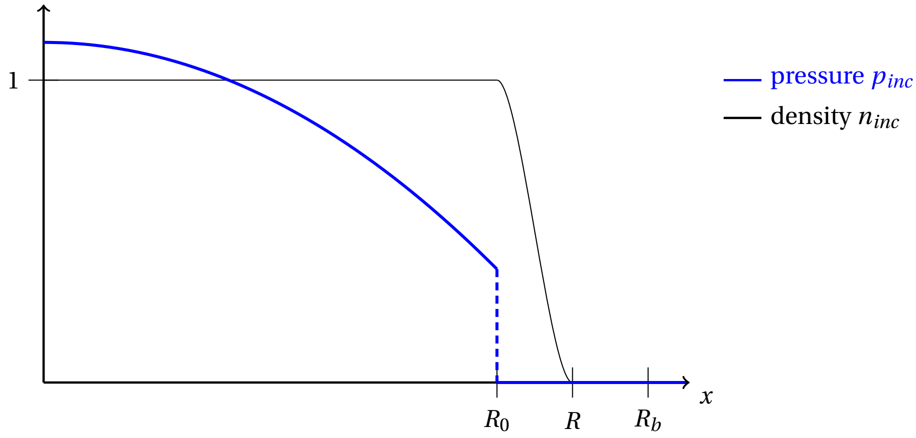


Figure 1. Plot of the limiting profile n_{inc}, as $\gamma_k \to \infty$, for the potential $V(r) = r^2$. We can observe that the pressure has a discontinuity at R_0 with $(0, R_0) = \{n_{inc} = 1\} = \{p_{inc} > 0\}$, while the density remains C^1.

(**C**) *Fix $\delta \in (0,1)$. There exists $R(m)$, independent of γ, such that when $R_b > R(m)$, then $n_\infty(R_b) = 0$.*

Next, we focus on the incompressible limit of the solutions of (7), that is when $\gamma_k \to \infty$. We denote by n_k the steady state associated with γ_k and assume that R_b is large enough so that $n_k(R_b) = 0$.

Theorem 3 (Incompressible limit of the stationary states). *Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be any sequence such that $\gamma_k \to \infty$. Let $\{n_k\}_{k\in\mathbb{N}}$ be a sequence of stationary states with the same mass m and with radius R_k, being the smallest argument such that $n_k(r) = 0$.*

Then, $n_k \to n_{inc}$ in $C^1([0,R_b])$ and $R_k \to R$, where n_{inc} and R are uniquely defined in Proposition 11. Moreover, the sequence of pressures $\{p_k := n_k^{\gamma_k}\}_{k\in\mathbb{N}}$ converges weakly to some pressure p_{inc} such that $p_{inc}(n_{inc} - 1) = 0$ and p_{inc} has a jump at $\partial\{n_{inc} = 1\}$

$$[\![p_{inc}]\!] \approx \sqrt[3]{6}\, R^{2/3}\, \delta^{1/3}, \qquad \text{as } \delta \to 0.$$

The profile n_{inc} obtained for the incompressible limit of stationary states is depicted in Figure 1. The density is equal to 1 on a certain interval $(0, R_0)$ where the pressure is positive. Then, the pressure vanishes and the density decreases to 0 on a small interval (R_0, R). At the boundary point R_0 the pressure undergoes a jump, which depends on the surface tension coefficient δ and on the shape of the confinement potential V. More precisely, for a general potential $V(r)$, this jump is determined by

$$[\![p_{inc}]\!] \approx \frac{\sqrt[3]{12}}{2}\, \delta^{1/3}\, (V'(R))^{2/3} \quad \text{as } \delta \to 0, \tag{8}$$

where R is the smallest value where $n(R) = 0$ and we have the estimate $R^2 - R_0^2 \approx \frac{2\sqrt[3]{12}\delta^{1/3} R}{\sqrt[3]{V'(R)}}$. We point out that the limiting profile (including parameters R_0 and R) is uniquely determined in terms of mass m, δ and V cf. Proposition 11.

In the above statements, the main novelty concerns the incompressible limit $\gamma \to \infty$ for the stationary states. A previous work in this direction [1] made use of viscosity relaxation, which provided additional estimates implying compactness. In our case, assuming the radial symmetry of the problem, we are able to characterize the incompressible limit of the sequence of compactly supported stationary solutions. While our setting is restrictive, it allows performing many computations explicitly. In particular, we find how the pressure jump depends on V and δ, cf. (8).

Open question

In this paper, we prove that the stationary states are compactly supported or at least zero on the boundary if the domain is large enough. It is logical to ask whether the solutions of the parabolic equation are compactly supported for a large domain and a strong confining potential. This question is still open. However, a work in this direction [2] has proved that in dimension 1, one could expect the solutions of the Cahn–Hilliard equation without confining potential to propagate with finite speed. By adding this potential, we can expect to have a better result, and compactly supported solutions, with time-independent support.

Contents of the paper

The above theorems are proved in the following sections. Section 2, is devoted to the convergence to stationary states stated in Theorem 1. In Section 3 we prove Theorem 2 and in Section 4 we give the proof of our main new result, namely Theorem 3. Numerical simulations of the model with a source term and no confining potential are presented in Section 5.

Notations

For a function $n(x,t)$ we associate a function in radial coordinates that is still denoted by $n(r,t)$. For $1 \le p, s \le +\infty$ or $s = -1$ and Ω a domain, $L^p(\Omega), H^s(\Omega)$ denote the usual Lebesgue and Sobolev spaces. When $s = -1$, $H^{-1}(\Omega)$ is the topological dual of $H_0^1(\Omega)$. Here $H^s(\Omega) = W^{s,2}(\Omega)$ in the usual notation. We also consider the Bochner spaces $L^p(0,T;H^s(\Omega))$ associated with the norm

$$\|f\|_{L^p(0,T;H^s(\Omega))} = \left(\int_0^T \|f\|_{H^s(\Omega)}^p\right)^{1/p}.$$

The partial derivative with respect to the radial variable is written as $\partial_r u(r) = \frac{\partial u}{\partial r}(r) = u'(r)$. Finally, C denotes a generic constant which appears in inequalities and whose value can change from one line to another. This constant can depend on various parameters unless specified otherwise.

1.2. *Literature review and biological relevancy of the system*

Tissue growth models and Hele–Shaw limits.

Development of tissue growth models is presently a major line of research in mathematical biology. Nowadays, number of models are available [3–5] with the common feature that they use the tissue internal pressure as the main driver of both the cell movement and proliferation. The simplest example of a mechanical model of living tissue is the *compressible* equation

$$\partial_t n = \operatorname{div}\big(n\nabla p\big) + nG(p), \quad p = P_\gamma(n) := n^\gamma, \tag{9}$$

in which $p(t,x) = P(n(t,x))$, with P a law of state, is the pressure and n the density of cell number. Here, the cell velocity is given via Darcy's law which captures the effect of cells moving away from regions of high compression. Dependence on growth function pressure has also been used to model the sensitivity of tissue proliferation to compression (contact inhibition, [6]).

An important problem is to understand the so-called incompressible limit (i.e. $\gamma \to \infty$) of this model. Perthame et al. [7] have shown that in this limit, solutions of (9) converge to a limit solution (n_∞, p_∞) of a Hele–Shaw-type free boundary limit problem for which the speed of the free boundary is given by the normal component of ∇p_∞, see also other approaches in [8, 9]. In this limit, the solution of (9) is organized into 2 regions: $\Omega(t)$ in which the pressure is positive (corresponding to the tissue) and outside of this zone where $p = 0$. Furthermore, the free boundary problem is supplemented by a complementary equation that indicates that the pressure satisfies

$$-\Delta p_\infty = G(p_\infty), \quad \text{in} \quad \Omega(t), \quad \text{or similarly} \quad p_\infty\left(\Delta p_\infty + G\left(p_\infty\right)\right) = 0 \quad \text{a.e. in } \Omega. \tag{10}$$

In this model, the pressure stays continuous in space, with jumps in time, and is equal to 0 at the interface. This is because only repulsive forces were taken into account. Hence, the crucial role of the cell-cell adhesion and thus the pressure jump at the surface of the tissue is not retrieved at the limit. Additionally, as pointed out by Lowengrub et al. [5], the velocity of the free surface should depend on its geometry and more precisely on the local curvature denoted by κ.

This motivated considering variants of the general model (9), where other physical effects of mechanical models of tissue growth are introduced. One of them is the addition of the effect of viscosity in the model, which has been made to represent the friction between cells [10, 11] through the use of Stokes' or Brinkman's law. Moreover, as pointed out by Perthame and Vauchelet [12], Brinkman's law leads to a simpler version of the model and, therefore, is a preferential choice for its mathematical analysis. Adding viscosity through the use of Brinkman's law leads to the model

$$\begin{cases} \partial_t n = \operatorname{div}\left(n\nabla\mu\right) + nG(p), & \text{in} \quad (0, +\infty)\times\Omega, \\ -\sigma\Delta\mu + \mu = p, & \text{in} \quad (0, +\infty)\times\Omega. \end{cases} \tag{11}$$

The incompressible limit of this system also yields the complementary relation (see [12])

$$p_\infty\left(p_\infty - \mu_\infty - \sigma G\left(p_\infty\right)\right) = 0, \quad \text{a.e. in } \Omega.$$

In the incompressible limit, notable changes compared to the system with Darcy's law are found. First, the previous complementary relation is different compared to Equation (10), and the pressure p_∞ in the limit is discontinuous, i.e. there is a jump of the pressure located at the surface of $\Omega(t)$. However, the pressure jump is related to the potential μ and not to the local curvature of the free boundary $\partial\Omega(t)$. The authors already indicated that a possible explanation for this is that the previous model does not include the effect of surface tension.

Surface tension and pressure jump

Surface tension is a concept associated with the internal cohesive forces between the molecules of a fluid: hydrogen bonds, van der Waals forces, metallic bonds, etc. Inside the fluid, molecules are attracted equally in all directions leading to a net force of zero; however molecules on the surface experience an attractive force that tends to pull them to the interior of the fluid: this is the origin of the surface energy. This energy is equivalent to the work or energy required to remove the surface layer of molecules in a unit area. The value of the surface tension will vary greatly depending on the nature of the forces exerted between the atoms or molecules. In the case of solid tumor cells in a tissue, it reflects the cell-cell adhesion tendency between the cells and depends on the parameter δ and the geometry of the tumor.

In the previous definition, the surface tension is associated with a single body that has an interface with the vacuum. When one considers two bodies, the surface energy of each body is modified by the presence of the other and we speak of interfacial tension. The latter depends on the surface tension of each of the two compounds, as well as the interaction energy between the two compounds. In the system considered above, it is then possible to imagine that the vacuum in which the tumor grows is in fact another body that has an internal pressure of the form $V(r)$ which increases with respect to r so that the tumor is stopped at some point and we can consider the stationary states.

For such a tumor to be in equilibrium, it is necessary that the interior is overpressured relative to the exterior by an amount. This amount is called the pressure jump and is computed explicitly in our case.

Surface tension effects can be introduced in the Hele–Shaw model as follows (see e.g. [13])

$$\begin{cases} -\Delta\mu = 0 & \text{in} \quad \Omega \setminus \partial\Omega(t), \\ \mu = \sigma\kappa & \text{on} \quad \partial\Omega(t). \end{cases} \tag{12}$$

where σ is a positive constant, called a surface tension and κ is a mean curvature of $\partial\Omega(t)$. This correct Hele–Shaw limit has been formally obtained as the sharp-interface asymptotic model of the Cahn–Hilliard equation [14]; see also [15] for a convergence result in a weak varifold formulation. This suggests that the Cahn–Hillard equation is an appropriate model to capture surface tension effects.

The Cahn–Hilliard equation

Cahn–Hillard type models for tissue growth have been developed based on the theory of mixtures in mechanics, see [16–18] and the references therein. Nowadays, they are widely used, in particular for tumor growth, and analysed, [19–25]. Originally introduced in the context of materials sciences [26], they are currently applied in numerous fields, including complex fluids, polymer science, and mathematical biology. For the overview of mathematical theory, we refer to [27].

Usually, in mechanical models, the Cahn–Hillard equation takes the form

$$\partial_t\varphi = \operatorname{div}\big(b(\varphi)\nabla\left(\psi'(\varphi) - \delta\Delta\varphi\right)\big) \Longleftrightarrow \begin{cases} \partial_t\varphi & = \operatorname{div}\big(b(\varphi)\nabla\mu\big), \\ \mu & = -\delta\Delta\varphi + \psi'(\varphi), \end{cases} \tag{13}$$

where φ represents the relative density of cells $\varphi = n_1/(n_1+n_2)$, b is the mobility, ψ is the potential while μ is the quantity of chemical potential, which is a quantity related to the effective pressure. From the point of view of mathematical biology, the most relevant case is $b(\varphi) = \varphi(1-\varphi)$, which is referred to as degenerate mobility.

In our context, (1) models the motion of a population of cells constituting a biological tissue in the form of a continuity equation. It takes into account pressure, the surface tension occurring at the surface of the tissue and its viscosity. More precisely, the equation for μ (i.e. equation (2)) includes the effects of both the pressure, through the term n^γ with $\gamma > 1$ that controls the stiffness of the pressure law, and surface tension by $-\delta\Delta n$, where $\sqrt{\delta}$ is the width of the interface in which partial mixing of the two components n_1, n_2 occurs.

A similar Cahn–Hilliard problem, without radial symmetry assumption, has previously been considered in [1], but including a relaxation (viscosity) term and a proliferation source term in place of the confinement potential. In the incompressible limit, the authors obtain a jump in pressure at the interface at all times for the relaxed system. The aim here is to justify a rigorous limit without viscosity relaxation and mostly to compute the pressure jump by analyzing the stationary states of a system with confining potential.

2. Existence, regularity, and long term behavior

The existence of solutions and their regularity is standard for the Cahn–Hilliard equation, see [28, 29]. Thus, we admit here the first part of Theorem 1 and refer to an extended version of the present paper in [30] for details. Weak solutions are defined as follows:

Definition 4 (Weak solutions). *We say that $n(t,r)$ is a global weak solution of the equation* (1)-(2) *provided that*

- *n is nonnegative,*
- *$r\,n$ is continuous in $[0,\infty)\times\overline{I_{R_b}}$, $\sqrt{r}\,n\in L^\infty((0,\infty)\times I_{R_b})$ and $r\,\partial_t n\in L^2((0,\infty);H^{-1}(I_{R_b}))$,*
- *$\sqrt{r\,n}\partial_r\mu\in L^2((0,\infty)\times\overline{I_{R_b}}\setminus\{r\,n=0\})$ and μ is defined in* (2)*,*
- *for every test function $\varphi\in L^2((0,\infty);H^1(I_{R_b}))\cap C_c^1([0,\infty)\times I_{R_b})$ the two relations hold*

$$\int_0^T r\left\langle\partial_t n,\varphi\right\rangle_{H^{-1},H^1}\mathrm{d}t+\int_0^T\int_0^{R_b}\mathbb{1}_{rn>0}\,r\,n\,\partial_r(\mu+V)\partial_r\varphi\,\mathrm{d}r\,\mathrm{d}t=0,$$
$$\int_0^T r\left\langle\partial_t n,\varphi\right\rangle_{H^{-1},H^1}\mathrm{d}t=-\int_0^T\int_0^{R_b}r\,n\partial_t\varphi\,\mathrm{d}r\,\mathrm{d}t-\int_0^{R_b}\varphi(0,r)n_0(r)\,\mathrm{d}r,$$

- *$n'(t,R_b)=0$ for a.e. $t\in(0,T)$.*

Remark 5 (Energy, entropy properties of weak solutions). In fact, we can construct solutions satisfying additionally mass, energy, and entropy relations as follows: for a.e. $\tau\in[0,T]$

$$\int_0^{R_b}r\,n(\tau,r)\,\mathrm{d}r=\int_0^{R_b}r\,n_0(r)\,\mathrm{d}r, \tag{14}$$

$$\mathscr{E}[n(\tau,\cdot)]+\int_0^\tau\int_0^{R_b}\mathbb{1}_{rn>0}\,r\,n\left|\partial_r(\mu+V)\right|^2\mathrm{d}r\,\mathrm{d}t\le\mathscr{E}[n_0], \tag{15}$$

$$\Phi[n(\tau,\cdot)]+\int_0^\tau\int_0^{R_b}\left(\gamma r\,n^{\gamma-1}|\partial_r n|^2+\delta r\,|\partial_{rr}n|^2+\delta\frac{|\partial_r n|^2}{r}+r\,\partial_r n\,\partial_r V\right)\mathrm{d}r\,\mathrm{d}t\le\Phi[n_0], \tag{16}$$

where energy and entropy are defined as follows:

$$\mathscr{E}[n]=\int_0^{R_b}r\left(\frac{n^{\gamma+1}}{\gamma+1}+\frac{\delta}{2}|\partial_r n|^2+n\,V\right)\mathrm{d}r,\qquad\Phi[n]=\int_0^{R_b}r\,\phi(n)\,\mathrm{d}r,$$

and $\phi(n)=n(\log(n)-1)+1$. Equations (14)-(16) provide the basic a priori estimates. Moreover, we construct Hölder continuous solutions; there is a constant C, such that for all $r,r_1,r_2\in[0,R_b]$, $t,t_1,t_2\in[0,\infty)$

$$|r_1\,n(t,r_1)-r_2\,n(t,r_2)|\le C\,|r_1-r_2|^{1/2}, \tag{17}$$

$$|r\,(n(t_2,r)-n(t_1,r))|\le C\,|t_2-t_1|^{1/8}. \tag{18}$$

2.1. *Proof of Theorem 1 (Long term asymptotics)*

With global solutions at hand, we can study the long term behaviour. For that purpose, we fix k,T, $k\ge T$ and define $n_k(t,x)=n(t+k,x)$, $\mu_k(t,x)=\mu(t+k,x)$. Consider the solution n in the interval $(-T+k,T+k)$, it satisfies

$$\int_{-T+k}^{T+k}r\left\langle\partial_t n,\varphi\right\rangle_{H^{-1},H^1}\mathrm{d}t+\int_{-T+k}^{T+k}\int_0^{R_b}\mathbb{1}_{rn>0}\,r\,n\,\partial_r(\mu+V)\partial_r\varphi\,\mathrm{d}r\,\mathrm{d}t=0,$$

and a change of variables yields

$$\int_{-T}^{T}r\left\langle\partial_t n_k,\varphi\right\rangle_{H^{-1},H^1}\mathrm{d}t+\int_{-T}^{T}\int_0^{R_b}\mathbb{1}_{rn_k>0}\,r\,n_k\,\partial_r(\mu_k+V)\partial_r\varphi\,\mathrm{d}r\,\mathrm{d}t=0. \tag{19}$$

We also recall the Neumann boundary condition $n_k'(t,R_b)=0$ and the conservation of mass $\int_0^{R_b}r\,n_k\,\mathrm{d}r=\int_0^{R_b}r\,n_0\,\mathrm{d}r$. We want to pass to the limit $k\to\infty$ in this equation and prove the

Proposition 6. *Let (n,μ) be a weak solution of* (1)-(2)*. Then, we can extract a subsequence, still denoted by the index k, of (n_k,μ_k) such that $\sqrt{r}n_k\to\sqrt{r}n_\infty$ strongly in $L^\infty((-T,T)\times I_{R_b})$ and $\sqrt{r\,n_k}\partial_r(\mu_k+V)\rightharpoonup\sqrt{r\,n}\partial_r(\mu_\infty+V)$ weakly in $L^2((-T,T)\times I_{R_b}\setminus\{r\,n=0\})$. We have $n_\infty\in C^1(\mathbb{R}\times\overline{B}_{R_b})$ and the relations*

$$r\,n_\infty\partial_r\left(\mu_\infty+V\right)=0,\qquad\mu_\infty=n_\infty^\gamma-\frac{\delta}{r}\partial_r\left(r\partial_r n_\infty\right), \tag{20}$$

with the Neumann boundary conditions

$$\frac{\partial n_\infty}{\partial r}\Big|_{r=0} = \frac{\partial n_\infty}{\partial r}\Big|_{r=R_b} = 0.$$

The mass $\int_0^{R_b} r\, n_\infty(t)\,\mathrm{d}r$ *is constant and equal to the initial mass* $\int_0^{R_b} r\, n_0\,\mathrm{d}r$.

This proposition implies the assertions of Theorem 1.

Before proving this proposition, we first state a useful lemma that we admit, and refer the reader to [30] for the details.

Lemma 7. *When* $n(t,r) : [0,\infty)\times[0,R_b] \to \mathbb{R}$ *satisfies* $\int_0^{R_b} r\, n(t,r)\,\mathrm{d}r = m$ *and* $\sqrt{r}\,\partial_r n \in L^\infty(0,T;L^2(0,R_b))$ *then*

$$\left|\sqrt{r}\, n(t,r)\right| \le C\left(R_b, m, \left\|\sqrt{r}\,\partial_r n\right\|_{L^\infty(0,T;L^2(0,R_b))}\right).$$

With this lemma we can prove the proposition.

Proof of Proposition 6.

Step 1. Bounds coming from the energy. We claim that the following uniform estimates (with respect to k) are true:

$$\{\sqrt{r}\,\partial_r n_k\} \quad \text{in} \quad L^\infty\left((-T,T);L^2\left(I_{R_b}\right)\right), \tag{B1}$$

$$\{\sqrt{r}\, n_k\} \quad \text{in} \quad L^\infty\left((-T,T)\times I_{R_b}\right), \tag{B2}$$

$$\{r\,\partial_t n_k\} \quad \text{in} \quad L^2\left((-T,T);H^{-1}\left(I_{R_b}\right)\right), \tag{B3}$$

$$|r_2 n_k(t_2,r_2) - r_1 n_k(t_1,r_1)| \le C\left(|t_2-t_1|^{1/8} + |r_2-r_1|^{1/2}\right), \tag{B4}$$

$$L_k(T) := \int_{-T}^{T}\int_0^{R_b} \mathbb{1}_{r n_k>0}\, r\, n_k \left|\partial_r(\mu_k+V)\right|^2 \xrightarrow[k\to+\infty]{} 0. \tag{B5}$$

The energy decay estimate (15) and assumption $\mathscr{E}[n_0]<\infty$ imply that $\mathscr{E}[n_k(t)]$ remains bounded with respect to k for all $k>T$. Therefore, (B1) follows directly from (15) and then (B2) follows from Remark 7. Estimate (B3) is a consequence of the dissipation of the energy and (B4) follows from (17)-(18). Finally, to see (B5), we note that

$$\int_0^\infty\int_0^{R_b} \mathbb{1}_{rn>0}\, r\, n \left|\partial_r(\mu+V)\right|^2 \mathrm{d}r\,\mathrm{d}t \le \mathscr{E}(n_0),$$

so by change of variables we obtain

$$L_k(T) \le \int_{k-T}^{\infty}\int_0^{R_b} \mathbb{1}_{rn>0}\, r\, n \left|\partial_r(\mu+V)\right|^2 \mathrm{d}r\,\mathrm{d}t \xrightarrow[k\to+\infty]{} 0.$$

Step 2. Bounds coming from the entropy. We prove now uniform estimates

$$\{\sqrt{r}\,\partial_{rr} n_k\} \quad \text{in} \quad L^2\left((-T,T);L^2\left(I_{R_b}\right)\right), \tag{C1}$$

$$\left\{\frac{\partial_r n_k}{\sqrt{r}}\right\} \quad \text{in} \quad L^2\left((-T,T);L^2\left(I_{R_b}\right)\right). \tag{C2}$$

To this end, we integrate the entropy relation (16) between $k-T$ and $k+T$ and perform a change of variables to obtain

$$\int_{-T}^{T}\int_0^{R_b}\left(\gamma r n_k^{\gamma-1}|\partial_r n_k|^2 + \delta r|\partial_{rr} n_k|^2 + \delta\frac{|\partial_r n_k|^2}{r}\right)\mathrm{d}r\,\mathrm{d}t$$
$$\le \Phi[n_k(-T,\cdot)] - \Phi[n_k(T,\cdot)] + \int_{-T}^{T}\int_0^{R_b} r\,\partial_r n_k\,\partial_r V\,\mathrm{d}r\,\mathrm{d}t.$$

We need to bound the right-hand side. Concerning the entropy term, we recall the inequality $\log n \le n-1$ valid for $n>0$ so that, by bound (B2),

$$\Phi(n_k(T,\cdot)) = \int_0^{R_b} r\left(n_k(T,r)\left(\log n_k(T,r)-1\right)+\right)\mathrm{d}r$$
$$\le \int_0^{R_b} r\left((n_k(T,r))^2 + n_k(T,r)\right)\mathrm{d}r \le C\left\|\sqrt{r}\,n_k\right\|_\infty \le C.$$

The same estimate is satisfied by $\Phi(n_k(T))$. Concerning $\int_{-T}^{T}\int_0^{R_b} r\,\partial_r n_k\,\partial_r V\,\mathrm{d}r\,\mathrm{d}t$, we estimate it using (B1) and uniform bound on $\partial_r V$. Therefore,

$$\int_{-T}^{T}\int_0^{R_b}\left(\gamma r n_k^{\gamma-1}|\partial_r n_k|^2 + \delta r\,|\partial_{rr} n_k|^2 + \delta\frac{|\partial_r n_k|^2}{r}\right)\mathrm{d}r\,\mathrm{d}t \le C(T,\mathscr{E}(n_0)).$$

Step 3. Convergence in equation (19)**.** Using standard compactness argument we obtain in the limit $k\to\infty$ (up to a subsequence)

$$\int_{-T}^{T} r\left\langle \partial_t n_\infty, \varphi\right\rangle_{H^{-1},H^1}\mathrm{d}t + \int_{-T}^{T}\int_0^{R_b} \mathbb{1}_{r n_\infty>0}\, r\, n_\infty\,\partial_r\left(\mu_\infty+V\right)\partial_r\varphi\,\mathrm{d}r\,\mathrm{d}t = 0.$$

We can show even better, namely that $\partial_t n_\infty = 0$. Indeed, from the Cauchy–Schwarz inequality we obtain that for every test function χ compactly supported in $(-T,T)\times(0,R_b)$,

$$\left|\int_{-T}^{T}\int_0^{R_b} r\partial_t n_k\chi\right| = \left|\int_{-T}^{T}\int_0^{R_b}\mathbb{1}_{r n_k>0}\, r\, n_k\partial_r\left(\mu_k+V\right)\partial_r\chi\right|$$
$$\le C(T,R_b)\left\|\partial_r\chi\right\|_{L^\infty}\int_{-T}^{T}\int_0^{R_b}\mathbb{1}_{r n_k>0}\, r\, n_k\left|\partial_r(\mu_k+V)\right|^2 \xrightarrow[k\to\infty]{} 0.$$

where we used (B2) and (B5). This means that in the limit, n_∞ does not depend on the time variable t. Then, in the limit, we obtain that, for every test function χ,

$$\int_{-T}^{T}\int_0^{R_b}\mathbb{1}_{r n_\infty>0}\, r\, n_\infty\partial_r(\mu_\infty+V)\partial_r\chi = 0.$$

Step 4. n_∞' is uniformly continuous and n_∞ satisfies Neumann boundary condition $n_\infty'(0)=0$. We recall that n_∞ does not depend on time. Moreover, the estimate (C1) implies that n_∞' is continuous on $(0,R_b]$. Furthermore, from the estimates (C1)-(C2), we obtain the absolute continuity in space of the derivative of n_∞. Indeed, for every $r_1,r_2\in(0,R_b)$ we obtain

$$(\partial_r n_\infty(r_2))^2 - (\partial_r n_\infty(r_1))^2 = 2\int_{r_1}^{r_2}\partial_r n_\infty(r)\,\partial_{rr} n_\infty(r)\,\mathrm{d}r$$
$$= 2\int_{r_1}^{r_2}\frac{\partial_r n_\infty(r)}{\sqrt{r}}\sqrt{r}\,\partial_{rr} n_\infty(r)\,\mathrm{d}r$$
$$\le 2\left(\int_{r_1}^{r_2}\frac{|\partial_r n_\infty(r)|^2}{r}\,\mathrm{d}r\right)^{1/2}\left(\int_{r_1}^{r_2} r\,|\partial_{rr} n_\infty(r)|^2\,\mathrm{d}r\right)^{1/2}.$$

Thanks to Sobolev embeddings, this implies that $\partial_r n_\infty$ is bounded and n_∞ is continuous

$$n_\infty(r_2) - n_\infty(r_1) = \int_{r_1}^{r_2}\partial_r n_\infty(r)\,\mathrm{d}r.$$

Next, we discover that $(0,R_b]\ni r\mapsto(\partial_r n_\infty(r))^2$ is uniformly continuous, so that by Lemma 8 below, $n_\infty'(r)$ is uniformly continuous on $(0,R_b]$. Therefore, there is the unique extension of $r\mapsto n_\infty'(r)$ to $[0,R_b]$, which is uniformly continuous. Furthermore, in view of

$$\int_0^{R_b}\frac{|\partial_r n_\infty|^2}{r}\,\mathrm{d}r \le C,$$

this extension satisfies $n_\infty'(0)=0$.

It remains to prove that n_∞ is differentiable (in the classical sense) at $r = 0$ and $n'_\infty(0) = 0$. To this end, we write

$$\left|\frac{n_\infty(r) - n_\infty(0)}{r}\right| \le \frac{1}{r}\int_0^r |\partial_r n_\infty(u)|\,\mathrm{d}u \le \sup_{u\in(0,r]} |\partial_r n_\infty(u)| \to 0$$

as $r \to 0$ by uniform continuity which, again, implies that $n'_\infty(0)$ exists and $n'_\infty(0) = 0$.

Step 5. Neumann boundary condition $n'_\infty(R_b) = 0$. For a fixed $k \in \mathbb{N}$, there is a set of times $\mathcal{N}_k \subset (0,T)$ of full measure such that, when $t \in \mathcal{N}_k$, we have $n'_k(t,R_b) = 0$ and $n_k(t,\cdot) \in H^2(R_0,R_b)$. Let $\mathcal{N} = \cap_{k\in\mathbb{N}}\mathcal{N}_k$, which is again the set of full measure. For $t \in \mathcal{N}$ and $\phi \in C^1[a,b] \cap H^2(a,b)$ we have (via approximation),

$$\int_{R_0}^{R_b} n'_k(t,r)\,\phi'(r) + n''_k(t,r)\,\phi(r)\,\mathrm{d}r = 0.$$

We multiply by a smooth test function $\eta(t)$ and pass to the weak limit $k \to \infty$ to deduce

$$\int_0^T \eta(t)\,\mathrm{d}t \int_{R_0}^{R_b} \left(n'_\infty(r)\,\phi'(r) + n''_\infty(r)\,\phi(r)\right)\mathrm{d}r = 0.$$

As $n_\infty \in H^2(R_0,R_b)$ we deduce $n'_\infty(R_b) = 0$. □

Lemma 8. *Let $f : (a,b) \to \mathbb{R}$ be a continuous function such that f^2 is uniformly continuous. Then $|f|$ and f are also uniformly continuous.*

3. Properties of the stationary states

The stationary solution built previously has compact support for R_b large enough. This is the main content of Theorem 2 which we prove here. We still use, to simplify notations, the potential $V(r) = r^2$.

3.1. *Proof of Theorem 2* (**A**)

We recall that, from Theorem 6, $n_\infty \ge 0$ is C^1, $n'_\infty(R_b) = 0$, $n'_\infty(0) = 0$.

Proof of Theorem 2 (A). To prove that n_∞ is non-increasing, the main idea is to show that it cannot have a local maximum except at the point $r = 0$.

To do so, by contradiction, we assume there is local maximum at $R_2 \in (0,R_b]$. This implies that $n'_\infty(R_2) = 0$, $n''_\infty(R_2) \le 0$. Also by C^1 regularity, in a neighborhood of R_2 the equation hold

$$n^\gamma_\infty(r) - \frac{\delta}{r}n'_\infty(r) - \delta n''_\infty(r) = C - r^2,$$

for some constant C. This equation implies that the local maximum is strict.

Also, still by C^1 regularity, in this neighborhood of R_2 there is a point $0 < R_1 < R_2$ such that $0 < n_\infty(R_1) < n_\infty(R_2)$ and $n'_\infty(R_1) > 0$. Evaluating the equation at the points R_1 and R_2, and eliminating the constant C, we obtain

$$\delta n''_\infty(R_1) = R_1^2 - R_2^2 + n^\gamma_\infty(R_1) - n^\gamma_\infty(R_2) - \frac{\delta}{R_1}n'_\infty(R_1) + \delta n''_\infty(R_2) < 0.$$

Therefore n_∞ is strictly concave at R_1. Consequently, R_1 can be continued to smaller values, $n_\infty(R_1)$ staying concave increasing (and thus n'_∞ larger and larger as R_1 decreases) until either $R_1 = 0$ or $n_\infty(R_1) = 0$. In both cases we get a contradiction with the condition $n'_\infty(R_1) = 0$ which holds at 0 and at values where $n_\infty(R_1) = 0$.

Consequently, the only possible local maximum is at 0 and n_∞ is non-increasing.

The upper bound on $n_\infty(R_b)$ is just to say that $n_\infty(r) \ge n_\infty(R_b)$ on the full interval. □

3.2. *Proof of Theorem 2* (**B**)

We now consider a stationary state such that $n_\infty(R_b) = 0$. Theorem 2 (**A**) asserts that there is $R \in [0, R_b]$ such that $n_\infty(r) = 0$ on $[R, R_b]$ and n_∞ is positive on $[0, R)$. Hence, on $[0, R]$, the relation (20) shows that there exists a constant, that we write $R^2 - \lambda$, such that n_∞ solves

$$\begin{cases} n^\gamma(r) - \frac{\delta}{r} n'(r) - \delta n''(r) = R^2 - r^2 - \lambda, \qquad 0 \le r \le R, \\ \\ n(R) = 0. \end{cases} \tag{21}$$

Because it is C^1, the stationary solution also satisfies $n'_\infty(R) = 0$ (and this is also true for $R = R_b$ as stated in Theorem 1.

We prove that there exists only one value λ such that the solution of Equation (21) also satisfies the condition $n'(R) = 0$.

Firstly, we exclude some values of λ. Here, we use the notation n^γ for $\max(0, n)^\gamma$.

Lemma 9. *Being given $\lambda \in \mathbb{R}$, let n be the solution of Equation* (21). *We have*

- *when $\lambda \ge R^2$,* $\quad n(r) \le 0 \quad \forall\, r \in [0, R]$ *and* $\; n'(R) > 0$,
- *when $\lambda \le 0$,* $\quad n(r) \ge 0 \quad \forall\, r \in [0, R]$ *and* $\; n'(R) < 0$.

Proof. For $\lambda \ge R^2$, $n \le 0$ is a consequence of the maximum principle and it follows immediately that $n'(R) \ge 0$. If we had $n'(R) = 0$, the equation gives $n''(R) = \frac{\lambda}{\delta} > 0$ which is in contradiction with the fact that n is nonpositive in a small left neighborhood of R.
For $\lambda \le 0$, $n \ge 0$ is a consequence of the maximum principle and it follows that $n'(R) \le 0$. To exclude the possibility that $n'(R) = 0$, we suppose by contradiction that $n'(R) = 0$. Since we also have $n(R) = 0$, we find that $n''(R) = \frac{\lambda}{\delta}$. As before, for $\lambda < 0$, we find contradiction. For $\lambda = 0$, we have $n''(R) = 0$. Differentiating the equation, we find

$$\gamma\, n^{\gamma-1}(r)\, n'(r) - \delta n^{(3)}(r) - \delta \frac{n''(r)}{r} + \delta \frac{n'(r)}{r^2} = -2r,$$

and thus $n^{(3)}(R) = 2R/\delta > 0$. As $n(R) = n'(R) = n''(R) = 0$, it follows that in a small neighbourhood of R, n has to be negative raising a contradiction. The lemma is proved. □

Secondly, from Lemma 9, we may conclude that there is at least one value $\lambda \in (0, R^2)$ such that the Neumann condition is satisfied. This value is unique.

Lemma 10. *There exists only one $\lambda \in (0, R^2)$ such that the solution of* (21) *satisfies $n'(R) = 0$.*

Proof. Suppose there are two solutions n_1, n_2 of (21) with $0 < \lambda_1 < \lambda_2 < R^2$ such that $n_i(R) = n'_i(R) = 0$ for $i = 1, 2$. From (21), we find $n''_i(R) = \frac{\lambda_i}{\delta}$. Therefore $0 < n''_1(R) < n''_2(R)$ and we conclude by a Taylor expansion that n'_2 is smaller than n'_1 in a small left neighborhood of R which contradicts that n decreases with λ. This proves Lemma 10. □

Proof of Theorem 2 (B). Clearly, n_∞ is a solution to the problem (21) with some λ. By Lemma 9, we know that $\lambda \in (0, R^2)$ and then Lemma 10 yields the unique value of λ.

For the second assertion, if there are two solutions (n_1, λ_1), (n_2, λ_2) of (7), Lemma 10 applies and we obtain that $\lambda_1 = \lambda_2$. The conclusion follows from uniqueness of solutions of the elliptic PDE (21). □

3.3. *Proof of Theorem 2* (**C**)

Proof of Theorem 2 (C). Consider a solution n_∞ of (6) with $a := n_\infty(R_b) > 0$. From Theorem 2 (**A**), we know that n_∞ is C^1 and $n'_\infty \le 0$ so that $n_\infty > 0$. Therefore, from the equation for n_∞, $r\, n_\infty(r)$ is C^2, and (6) boils down to

$$\begin{cases} n_\infty^\gamma - \frac{\delta}{r} n' - \delta n'' = R_b^2 - r^2 - \lambda \quad \text{in } (0, R_b), \\ n_\infty(R_b) = a > 0,\ n'(R_b) = 0, \\ m = \int_0^{R_b} r\, n_\infty(r)\, \mathrm{d}r, \end{cases} \tag{22}$$

where λ is some constant. Our goal is to prove that if R_b is sufficiently large with respect to m, there is no such solution of (22). Therefore we now assume that $R_b > 2$.

A useful formula in the sequel is, because of radial symmetry and after integration between 0 and r,

$$\int_0^r \bar{r}\, n_\infty^\gamma(\bar{r})\, \mathrm{d}\bar{r} - \delta r \partial_r n_\infty = \left(R_b{}^2 - \lambda\right) \frac{r^2}{2} - \frac{r^4}{4}. \tag{23}$$

Another useful general observation is that we may assume

$$n_\infty(R_b)^\gamma \le R_b{}^2.$$

Otherwise, by Theorem 2 (**A**), we have $m \ge \frac{R_b{}^{2+2/\gamma}}{2}$ which proves the result.

Firstly, we provide lower and upper bounds on admissible values of the constant λ

$$-R_b{}^2 \le -n_\infty(R_b)^\gamma \le \lambda \le \frac{R_b{}^2}{2}. \tag{24}$$

The first inequality is the above restriction on $n_\infty(R_b)^\gamma$. The second inequality is valid because $n''_\infty(R_b) \ge 0$ since n_∞ is decreasing and $n'_\infty(R_b) = 0$. The third inequality is just (23) at $r = R_b$.

Secondly, we provide a control of $n_\infty(0)$. To do so, using (23), $\partial_r n_\infty \le 0$ and the above upper bound on λ, we estimate $|\partial_r n_\infty|$ from above as

$$\delta\, |\partial_r n_\infty| \le R_b{}^2 r.$$

This gives

$$n_\infty(r) \ge n_\infty(0) - \frac{R_b{}^2}{2\delta} r^2$$

and, with $\alpha > 0$ such that $\alpha^2 = \frac{\delta}{2R_b{}^2} \le 1$,

$$m \ge \int_0^{\alpha R_b} r\, n_\infty(r)\, \mathrm{d}r \ge \frac{\alpha^2 R_b{}^2}{2} \left(n_\infty(0) - \frac{R_b{}^2}{2\delta} \frac{\alpha^2 R_b{}^2}{2} \right) = \frac{\alpha^2 R_b{}^2}{2} \left(n_\infty(0) - \frac{R_b{}^2}{8} \right).$$

As a conclusion of this step, we may assume

$$n_\infty(0) \le \frac{R_b{}^2}{4},$$

otherwise $m \ge \frac{\alpha^2 R_b{}^2}{2} \frac{R_b{}^2}{8} = \delta \frac{R_b{}^2}{32}$ and the result is proved again.

Thirdly, we prove that with this control from above of $n_\infty(0)$, the derivative $|\partial_r n_\infty|$ is large, thus again there is a control on the mass since n_∞ is decreasing. To do so, we use again (23) and the third inequality in (24). This gives

$$\delta r |\partial_r n_\infty| \ge -n_\infty(0)^\gamma \frac{r^2}{2} + \frac{R_b{}^2}{2} \frac{r^2}{2} - \frac{r^4}{4} \ge \frac{r^2}{2} \left(-\frac{R_b{}^2}{4} + \frac{R_b{}^2}{2} - \frac{r^2}{2} \right) = \frac{r^2}{4} \left(\frac{R_b{}^2}{2} - r^2 \right)$$

where we have used the smallness assumption on $n_\infty(0)$ and $\gamma \ge 1$. On the range $r \in (0, \frac{R_b}{2})$, we control

$$\delta |\partial_r n_\infty| \ge \frac{r}{4} \frac{R_b{}^2}{4}, \quad \text{thus} \quad n_\infty(r) \ge \frac{R_b{}^2}{32\delta} \left(\frac{R_b{}^2}{4} - r^2 \right),$$

and thus

$$m \geq \frac{{R_b}^2}{32\delta}\int_0^{R_b/2}\left(\frac{{R_b}^2}{4}-r^2\right)\mathrm{d}r \geq \frac{{R_b}^5}{4\cdot 128\,\delta}.$$

Again we have the desired control and Theorem 2 (**C**) is proved. □

4. Proof of Theorem 3

To study the incompressible limit of stationary states of the Cahn–Hilliard equation, the difficulty comes from the singularity of the pressure. However it is possible to fully characterize them, and calculate the pressure jump at the tumor boundary. We begin with establishing the existence and uniqueness for the solution n_{inc} of the limiting equation. Then, we show that all limits of n_k's are determined by this profile n_{inc}.

4.1. *Preliminary steps*

If a sequence $\gamma_k \to \infty$ of stationary states n_k converges to n_{inc} and the sequence of pressures $p_k = n_k^{\gamma_k}$ converges to p_{inc}. Then we expect that $p_{inc}(n_{inc}-1)=0$. Therefore, there should be a 'tumor zone' where $n_{inc}=1$ and the pressure vanishes outside. This leads us to study the following problem in the zone (R_0,R) where $p_{inc}=0$:

$$\begin{cases} -\frac{\delta}{r}u_c' - \delta u_c'' = R^2 - r^2 - \lambda_c & \text{in } (R_0,R),\\ u_c(R) = u_c'(R) = 0, & u_c(R_0)=1,\, u_c'(R_0)=0,\\ \int_0^R r\,n_{inc}(r)\,\mathrm{d}r = m, \end{cases} \tag{25}$$

where n_{inc} is the extension of u_c by 1 on $[0,R_0]$.

In a later subsection, we prove the convergence of the stationary states n_k to this limiting profile.

Notice that System (25) has three free parameters (R, R_0, λ_c) and three constraints (2 additional boundary conditions and mass m). The following proposition gives the existence of a solution.

Proposition 11 (Unique limiting profile). *Let $m > 72\,\delta^{1/2}$. There exist uniquely determined $R>0$, $\lambda_c \in (0,R^2)$ and $R_0 \in (0,R)$ such that Equation* (25) *has a solution. Furthermore,*

$$R_0 = \sqrt{R^2 - 2\lambda_c} \qquad \textit{and} \qquad \lambda_c \approx \sqrt[3]{6}\,R^{2/3}\,\delta^{1/3} \qquad \textit{for small } \delta > 0. \tag{26}$$

We postpone the proof of this proposition to the next subsection. Its proof uses an explicit solution obtained by the following problem. Find a couple (λ_u, u) such that

$$\begin{cases} -\frac{\delta}{r}u' - \delta u'' = R^2 - r^2 - \lambda_u & \text{in } (0,R)\\ u(R) = u'(R) = 0. \end{cases} \tag{27}$$

Proposition 12 (Lower bound profile). *Let $\lambda_u \in [0,R^2]$, then the solution u of* (27) *satisfies*

(A) *the explicit formula for u*

$$u(r) = \frac{R^2}{4\delta}(R^2-2\lambda_u)\ln\left(\frac{r}{R}\right) + \frac{(r^2-R^2)^2}{16\delta} + \frac{R^2-2\lambda_u}{8\delta}(R^2-r^2),$$

$$u'(r) = \frac{(R^2-r^2)(R^2-r^2-2\lambda_u)}{4\delta\,r},$$

(B) *the function $u(r)$ is decreasing and positive for r such that $0 < R^2 - r^2 < 2\lambda_u$,*

(C) *for a solution n_∞ of* (7) *as in Theorem 2, if $\lambda_\infty \geq \lambda_u$ then $n_\infty(r) \geq u(r)$ for $r \in (0,R]$.*

Proof of Proposition 12. To prove (A), we first compute $u'(r)$ and $u''(r)$:

$$u'(r) = \frac{R^2}{4\delta}\left(R^2 - 2\lambda_u\right)\frac{1}{r} + \frac{\left(r^2 - R^2\right) r}{4\delta} - \frac{R^2 - 2\lambda_u}{4\delta} r = \frac{\left(R^2 - r^2\right)\left(R^2 - r^2 - 2\lambda_u\right)}{4\delta\, r},$$
$$u''(r) = \frac{-4r\left(R^2 - r^2\right) + 4\lambda_u r}{4\delta\, r} - \frac{\left(R^2 - r^2\right)\left(R^2 - r^2 - 2\lambda_u\right)}{4\delta\, r^2} = -\frac{1}{\delta}\left(R^2 - r^2 - \lambda_u\right) - \frac{u'(r)}{r}.$$

Therefore, we obtain the desired equation (27).

The statement (B) is an immediate consequence of the formula for $u'(r)$.

Finally, we prove (C). We introduce $h(r) = n_\infty(r) - u(r)$ and we have to prove that $h(r) \geq 0$. From the equations we get

$$n^\gamma - \frac{\delta}{r} h' - \delta h'' = \lambda_u - \lambda \text{ in } (0, R].$$

So, thanks to our assumptions and letting $g'(r) = r h'(r)$, we have

$$h''(r) + \frac{h'(r)}{r} \geq 0, \qquad g''(r) \geq 0.$$

Integrating this from r to R and using the boundary conditions, we obtain

$$g'(r) \leq 0 \implies r\, h'(r) \leq 0 \implies h'(r) \leq 0.$$

Integrating this once again and using boundary conditions, we discover $h(r) \geq 0$ as desired. □

4.2. *Proof of Proposition 11*

The explicit solution built in Proposition 12 allows us to characterize the parameters λ_c and R_0. Indeed, we are looking for λ_c and R_0 such that

$$u_c'(R_0) = \frac{\left(R^2 - R_0^2\right)\left(R^2 - R_0^2 - 2\lambda_c\right)}{4\delta\, r} = 0, \tag{28}$$

$$u_c(R_0) = \frac{R^2}{4\delta}\left(R^2 - 2\lambda_c\right)\ln\left(\frac{R_0}{R}\right) + \frac{\left(R_0^2 - R^2\right)^2}{16\delta} + \frac{R^2 - 2\lambda_c}{8\delta}\left(R^2 - R_0^2\right) = 1. \tag{29}$$

Lemma 13 (Solving for R_0 and λ_c). *Let $R > 0$. Then* (28)-(29) *has a unique solution if and only if* $16\delta < R^4$. *Moreover, the solution is given by*

$$R_0 = \sqrt{R^2 - 2\lambda_c}, \qquad \lambda_c = \frac{R^2 x_c}{2}, \tag{30}$$

where $x_c \in (0,1)$ is the unique solution of

$$(1 - x_c)\ln(1 - x_c) + \frac{1}{2}x_c^2 + (1 - x_c)\, x_c = \frac{8\delta}{R^4}. \tag{31}$$

Proof. We split the reasoning into several steps.

Step 1. Equation for R_0. Because $R_0 = R$ cannot fit (29), from (28) we immediately deduce the formmula for R_0 in (30).

Step 2. Equation for λ_c. We plug the formula for R_0 into (29) to deduce

$$\frac{R^2}{4\delta}\left(R^2 - 2\lambda_c\right)\ln\left(\frac{\sqrt{R^2 - 2\lambda_c}}{R}\right) + \frac{4\lambda_c^2}{16\delta} + \frac{R^2 - 2\lambda_c}{8\delta} 2\lambda_c = 1.$$

Using properties of logarithm and simple algebra, we have

$$\frac{R^2}{8\delta}\left(R^2 - 2\lambda_c\right)\ln\left(1 - \frac{2\lambda_c}{R^2}\right) + \frac{\lambda_c^2}{4\delta} + \frac{R^2 - 2\lambda_c}{4\delta}\lambda_c = 1.$$

Introducing the auxiliary variable $x_c = \frac{2\lambda_c}{R^2}$ and after multiplication by $\frac{8\delta}{R^4}$, this equation is equivalent to Equation (31).

Step 3. Existence and uniqueness of x_c and λ_c. We prove that if $16\delta < R^4$, equation (31) has a unique solution. To this end, we define

$$f(x) := (1-x)\ln(1-x) - \frac{1}{2}x^2 + x, \qquad f(0) = 0, \quad f(1) = \frac{1}{2}. \tag{32}$$

Then, we compute

$$f'(x) = -\ln(1-x) - x, \qquad f''(x) = \frac{1}{1-x} - 1. \tag{33}$$

Since $f'(0) = 0$ and $f''(x) > 0$ for $x \in (0,1)$, it follows that $f'(x) > 0$ so that $f(x)$ is increasing. It follows that f is one-to-one from $(0,1)$ into $(0, \frac{1}{2})$. Therefore, when $16\delta < R^4$, there exists a unique $x_c \in (0,1)$ such that $f(x_c) = \frac{8\delta}{R^4}$. □

Lemma 14 (Estimates for x_c). *Let x_c be a solution to* (31) *and* $16\delta < R^4$. *Then, we have*

$$x_c \approx 2\sqrt[3]{6}\,\delta^{1/3}R^{-4/3}, \qquad \lambda_c \approx \sqrt[3]{6}\,\delta^{1/3}R^{2/3} \qquad (\text{as } \delta \to 0). \tag{34}$$

More precisely, we have

$$x_c \le 2\sqrt[3]{6}\,\delta^{1/3}R^{-4/3}. \tag{35}$$

Moreover, if $6^4\,8\,\delta < R^4$, *we have*

$$x_c \ge 2\sqrt[3]{5}\,\delta^{1/3}R^{-4/3}. \tag{36}$$

Proof. For δ small, Equation (31) shows that x_c is small. More precisely, using (32), (33), we obtain $f(0) = f'(0) = f''(0) = 0$ and $f^{(3)}(0) = 1$ since $f^{(3)}(x) = \frac{1}{(1-x)^2}$. Hence, by the Taylor expansion, for small x, $f(x) \approx \frac{x^3}{6}$. Plugging this approximation into (31), we obtain Estimate (34).

Next, we observe that

$$f^{(k)}(x) = \frac{(k-2)!}{(1-x)^{k-1}}, \qquad f^{(k)}(0) = (k-2)!.$$

In particular, the Taylor expansion around $x = 0$ gives

$$f(x) = \sum_{k\ge 3} \frac{(k-2)!}{k!}x^k = \sum_{k\ge 3} \frac{1}{k(k-1)}x^k.$$

Therefore, $f(x)$ is controlled by

$$\frac{x^3}{6} \le f(x) \le \frac{x^3}{6}\sum_{k\ge 0} x^k = \frac{x^3}{6(1-x)}. \tag{37}$$

The control (35) follows from the lower bound.

Finally, using this, we can find δ such that $2\sqrt[3]{6}\,\delta^{1/3}R^{-4/3} \le \frac{1}{6}$, namely $6^4 8\delta \le R^4$. Then, we have $x_c \le \frac{1}{6}$ so that $1 - x_c \ge \frac{5}{6}$ and then the estimate (37) gives us

$$\frac{8\delta}{R^4} = f(x_c) \le \frac{x_c^3}{6(1-x_c)} \le \frac{x_c^3}{5}$$

so that

$$\frac{8\delta}{R^4} \le \frac{x_c^3}{5} \iff \frac{40\delta}{R^4} \le x_c^3 \iff 2\sqrt[3]{5}\,\delta^{1/3}R^{-4/3} \le x_c. \qquad \square$$

Lemma 15. *Let u_c and n_{inc} be a as in Equation* (25), *then the total mass of n_{inc} satisfies*

$$\mathcal{M}(n_{inc}) := \int_0^R r\, n_{inc}(r)\,\mathrm{d}r = \frac{R^6\, x_c^3(R)}{96\delta}.$$

Moreover, the map $R \mapsto \frac{R^6\, x_c^3(R)}{96\delta}$ is increasing if $R^4 x_c^3(R) > 32\delta$.

Proof. Because n_{inc} is a C^1 function, integrating by parts, we find

$$\mathcal{M}(n_{inc}) = \int_0^R r\, n_{inc}(r)\,\mathrm{d}r = -\frac{1}{2}\int_0^R r^2 n_{inc}'(r)\,\mathrm{d}r = -\frac{1}{2}\int_{R_0}^R r^2 u_c'(r)\,\mathrm{d}r.$$

Inserting the formula for $u_c'(r)$ stated in Proposition 12, we deduce that

$$\mathcal{M}(n_{inc}) = -\frac{1}{8\delta}\int_{R_0}^R r\left(R^2-r^2\right)\left(R^2-r^2-2\lambda_c\right)\mathrm{d}r.$$

With the notations $\lambda_c := R^2 x_c/2$ and $R_0 = R\sqrt{1-x_c}$, we obtain

$$\mathcal{M}(n_{inc}) = -\frac{1}{8\delta}\int_{R\sqrt{1-x_c}}^R r\left(R^2-r^2\right)\left(R^2-r^2-R^2x_c\right)\mathrm{d}r.$$

We change variables $\tau = R^2 - r^2$ to get the desired formula

$$\mathcal{M}(n_{inc}) = -\frac{1}{16\delta}\int_0^{R^2x_c} \tau\left(\tau - R^2x_c\right)\mathrm{d}\tau = \frac{R^6x_c^3}{96\delta}.$$

For the second assertion, it is sufficient to prove that the map $R \mapsto R^6x_c^3(R)$ is strictly increasing. Note that $x_c(R)$ is given implicitly via equation (31). Differentiating it with respect to R, we discover that

$$\frac{\mathrm{d}x_c}{\mathrm{d}R}(x_c + \log(1-x_c)) = \frac{32\delta}{R^5} \implies \frac{\mathrm{d}x_c}{\mathrm{d}R} = \frac{32\delta}{R^5\,(x_c+\log(1-x_c))}.$$

Then, we study the derivative of $R^6\,x_c^3(R)$,

$$\frac{\mathrm{d}\left(R^6x_c^3(R)\right)}{\mathrm{d}R} = 6R^5x_c^3 + 3R^6x_c^2\frac{\mathrm{d}x_c}{\mathrm{d}R} = 6R^5x_c^3 + \frac{96\delta R x_c^2}{(x_c+\log(1-x_c))}.$$

Using a simple Taylor estimate, we have $\frac{1}{x+\log(1-x)} \ge \frac{-2}{x^2}$ and we conclude that $R^6x_c^3(R)$ is increasing since

$$\frac{\mathrm{d}\left(R^6x_c^3(R)\right)}{\mathrm{d}R} \ge 6R^5x_c^3 - 192\delta R = 6R\left(R^4x_c^3 - 32\delta\right). \qquad \square$$

Proof of Proposition 11. First, we notice that if (R_0, R, λ) satisfy conditions of the Proposition 11, then R_0, λ_c are given by (30) (Lemma 13) and $\frac{R^6x_c^3}{96\delta} = m$ (Lemma 15). Then, by the control $6^2\,2\delta^{1/2} \le m$ as well as an upper bound on x_c, cf. (35), we deduce

$$6^2\,2\delta^{1/2} \le m = \frac{R^6\,x_c^3}{96\delta} \le \frac{R^2}{2} \implies 6^4 4^2\delta \le R^4.$$

This means that we can apply the lower bound (36) to deduce

$$R^4x_c^3 \ge 40\delta^{1/3}.$$

It follows that the necessary condition for existence of (R_0, R, λ) is $6^44^2\delta \le R^4$ which implies $R^4x_c^3 \ge 40\delta^{1/3}$. Therefore, by Lemma 15, the map $R \mapsto R^6x_c^3(R)$ is invertible and we can find uniquely determined R such that

$$m = \frac{R^6x_c^3(R)}{96\delta}.$$

With such a value R (because $16\delta < R^4$), we can find unique R_0 and λ_c solving (28)-(29) so that the formula for the mass is satisfied and the conclusion follows. $\square$

4.3. *Proof of Theorem 3*

The solutions of Theorem 2 satisfy, with $\lambda_k \in (0, R^2)$, $R_k > 0$,

$$\begin{cases} n_k^{\gamma_k} - \delta n_k'' - \frac{\delta}{r} n_k' = R_k^2 - r^2 - \lambda_k \quad \text{in } (0, R_k), \quad n_k = 0 \quad \text{in } (R_k, R_b), \\ n_k(R_k) = n_k'(R_k) = 0, \qquad\qquad n_k'(0) = 0. \end{cases}$$

Thanks to the maximum principle, the sequence $\{n_k^{\gamma_k}\}_k$ is bounded in $L^\infty(I_{R_b})$. Moreover, multiplying this equation by n_k'' and integrating by parts, we obtain

$$\int_0^{R_k} \left(\delta \left(n_k''\right)^2 + \gamma n_k^{\gamma_k - 1} \left(n_k'\right)^2 + \delta \frac{\left(n_k'\right)^2}{2r^2} \right) \mathrm{d}r = \int_0^{R_k} n_k'' \left(R_k^2 - r^2 - \lambda_k\right) \mathrm{d}r.$$

Since $0 \le R_k, \lambda_k \le R_b$, the right-hand side is bounded by $\frac{\delta}{2} \int_0^{R_k} (n_k'')^2 + C(\delta, R_b)$. Thus, $\{n_k''\}_k$ is uniformly bounded in $L^2(0, R_b)$. Therefore, up to a subsequence, as $k \to \infty$

$$n_k^{\gamma_k} \rightharpoonup p_{inc} \ge 0 \quad \text{weakly}^* \text{ in } L^\infty(I_{R_b}), \qquad n_k \to n_{inc} \le 1 \quad \text{in } C^1(\overline{I_{R_b}}).$$

We also have the algebraic relation $p_{inc}(n_{inc} - 1) = 0$. The inequality $p_{inc}(n_{inc} - 1) \le 0$ is straightforward using $p_{inc} \ge 0$ and $n_{inc} \le 1$. It remains to show that $p_{inc}(n_{inc} - 1) \ge 0$. For $\nu > 0$, there exists γ_0 such that for $\gamma_k \ge \gamma_0$

$$n_k^{\gamma_k + 1} \ge n_k^{\gamma_k} - \nu$$

because the function $x \mapsto x^\gamma (x - 1)$ is nonpositive on $[0, 1]$ and attains its minimum $-(\frac{\gamma}{\gamma+1})^\gamma \frac{1}{\gamma+1}$ $\to 0$ as $\gamma \to \infty$. Then, from the strong convergence of n_k and the weak convergence of $n_k^{\gamma_k}$ we know that $n_k^{\gamma_k} n_k$ converges weakly to $p_{inc}\, n_{inc}$. Passing to the limit, we obtain

$$p_{inc}\, n_{inc} \ge p_{inc} - \nu,$$

for every $\nu > 0$. Letting $\nu \to 0$ yields the result.

Since $\{\lambda_k\}_k$ and $\{R_k\}_k$ are also bounded subsequences, we can extract converging subsequences to λ and R respectively. Thanks to the C^1 convergence we know that n satisfies the boundary condition $n_{inc}(R) = n_{inc}'(R) = 0$ and n_{inc} is radially decreasing as the uniform limit of radially decreasing functions. Finally, we can pass to the limit in the equation of mass conservation and obtain $\int_0^R r\, n_{inc}\, \mathrm{d}r = m$. To sum up, in the limit we obtain a C^1, nonincreasing function n_{inc} satisfying

$$\begin{cases} p_{inc} - \frac{\delta}{r} n_{inc}' - \delta n_{inc}'' = R^2 - r^2 - \lambda \quad \text{in } (0, R), \\ n_{inc}(R) = n_{inc}'(R) = n_{inc}'(0) = 0, \\ \int_0^R r\, n_{inc}(r)\, \mathrm{d}r = m, \\ p_{inc}(n_{inc} - 1) = 0. \end{cases}$$

The limiting ODE is satisfied on $(0, R)$ because the ODE for n_k is satisfied on $(0, \inf_{l \ge k} R_k)$. Passing to the limit, we obtain the ODE on $(0, \lim_{k\to\infty} \inf_{l \ge k} R_k) = (0, R)$ because $R = \lim_{k\to\infty} R_k$.

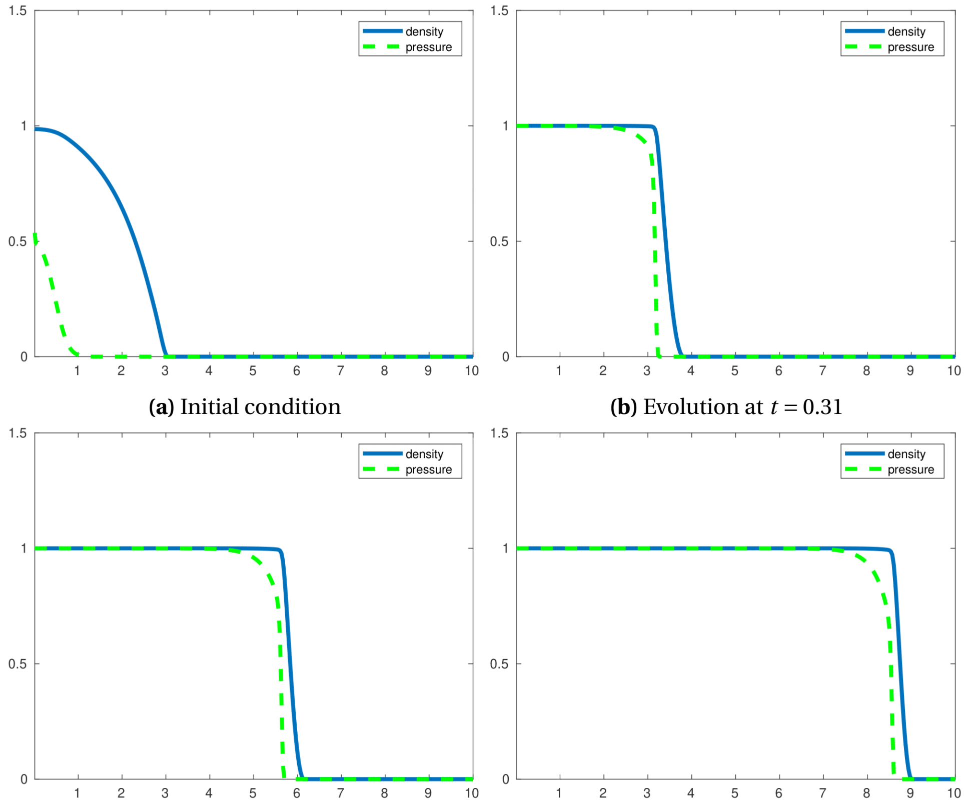
We claim that n_{inc} reaches the value 1. By contradiction, if $n_{inc} < 1$ on $[0, R]$, then $p_{inc} = 0$ so that n_{inc} is a C^1 solution to the following ODE on $[0, R]$:

$$-\frac{\delta}{r} n_{inc}' - \delta n_{inc}'' = R^2 - r^2 - \lambda, \qquad n_{inc}(R) = n_{inc}'(R) = n_{inc}'(0) = 0.$$

By Proposition 12 (A) such a solution does not exist.

By monotonicity and the fact that n_{inc} reaches value 1 we deduce that there are two zones. In the zone $\{p_{inc} > 0\}$ we have $n_{inc} = 1$, and thus $p_{inc} = R^2 - r^2 - \lambda$. Then, when $n_{inc} < 1$ (n_{inc} is decreasing), let us say at $r = R_0$ we have $p = 0$. The pressure jump is equal to $[\![p_{inc}]\!] = R^2 - R_0^2 - \lambda$. Finally, the convergence of the whole sequence follows from uniqueness of the limiting profile as stated in Proposition 11.

5. Conclusion and numerical simulations



(a) Initial condition **(b)** Evolution at $t = 0.31$

(c) Evolution at $t = 1.14$ **(d)** Evolution at $t = 2.11$

Motivated by the pressure jump imposed in free boundary problems of tissue growth, [4, 7–9], we included surface tension in such compressible models. We established that radially symmetric stationary solutions of the Cahn–Hilliard system with a confining potential $V(r)$ exist and are decreasing. In the incompressible limit, they present a jump of pressure at the boundary of the saturation set $\{n = 1\}$. We computed explicitly this pressure jump which is proportional to $\delta^{1/3} V'(R)^{2/3}$. There is a vacuum zone $\{n = 0\}$ that induces a degeneracy which is the main difficulty when establishing the a priori estimates.

It is an open question to prove a similar result, for a propagating wave, when the system is driven by a source term rather than a confining potential, as in [1] for instance. However, we provide numerical simulations in radial coordinates. More precisely, we focus on the system

$$\begin{aligned} &\frac{\partial (rn)}{\partial t} - \frac{\partial}{\partial r}\left(rn\frac{\partial \mu}{\partial r}\right) = nG(p), \quad \text{in} \quad (0,+\infty)\times I_{R_b}, \\ &\mu = p - \frac{\delta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right), \qquad p = n^{\gamma}. \end{aligned} \tag{38}$$

When $\gamma \to \infty$, we expect to find the incompressible limit

$$\begin{cases} -\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = G(p), \qquad p(n-1) = 0, \quad \text{in } \{n = 1\}, \\ [\![p]\!] = -\delta \left[\!\!\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right)\right]\!\!\right] \quad \text{on } \partial\{n = 1\}. \end{cases}$$

These equations are obtained formally after setting $n = 1$ in (38) and using the relation $p(n-1) = 0$. The main open question is to link the value of the pressure jump to the other parameters of the model, i.e. the source term G, the parameter δ and boundary's curvature. In radial settings the curvature is $\frac{1}{R(t)}$ where $R(t)$ is the radius of the tumor. We present below some numerical simulations for the evolution of the density and the pressure of the tumor. If the pressure jump seems to be decreasing as the tumor grows, it is not numerically clear how to determine the pressure jump.

Numerical settings

For the source term, we take $G(p) = 10(1-p)$. We use an explicit scheme, with time step $dt = 1\text{e}{-7}$, final time $t = 2.11$ and the interval is $I_{R_b} = [0, 10]$ with 300 points. The initial condition is a truncated arctangent. To remove the degeneracy $r = 0$ in the numerical scheme, we consider $r + \varepsilon$ instead of r for some small $\varepsilon > 0$.
The pressure p reaches the value 1, as the density, because we choose the homeostatic pressure $p_h = 1$ in the source term $G(p) = 10(p_h - p)$. The homeostatic pressure is interpreted as the lowest level of pressure that prevents cell multiplication due to contact-inhibition.

Conflicts of interest

The authors have no conflict of interest to declare.

References

[1] C. Elbar, B. Perthame, A. Poulain, "Degenerate Cahn-Hilliard and incompressible limit of a Keller-Segel model", *Commun. Math. Sci.* **20** (2022), no. 7, p. 1901-1926.

[2] B. Chen, C. Liu, "Finite speed of propagation for the cahn-hilliard equation with degenerate mobility", *Appl. Anal.* **100** (2021), no. 8, p. 1693-1726.

[3] H. Byrne, L. Preziosi, "Modelling solid tumour growth using the theory of mixtures", *Math. Med. Biol.* **20** (2003), no. 4, p. 341-366.

[4] A. Friedman, "Mathematical analysis and challenges arising from models of tumor growth", *Math. Models Methods Appl. Sci.* **17** (2007), no. Suppl., p. 1751-1772.

[5] J. S. Lowengrub, H. B. Frieboes, F. Jin, Y.-L. Chuang, X. Li, P. Macklin, S. M. Wise, V. Cristini, "Nonlinear modelling of cancer: bridging the gap between cells and tumours", *Nonlinearity* **23** (2010), no. 1, p. R1-R9.

[6] H. Byrne, D. Drasdo, "Individual-based and continuum models of growing cell populations: a comparison", *J. Math. Biol.* **58** (2009), no. 4, p. 657-687.

[7] B. Perthame, F. Quirós, J. L. Vázquez, "The Hele-Shaw asymptotics for mechanical models of tumor growth", *Arch. Ration. Mech. Anal.* **212** (2014), no. 1, p. 93-127.

[8] I. Kim, O. Turanova, "Uniform convergence for the incompressible limit of a tumor growth model", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **35** (2018), no. 5, p. 1321-1354.

[9] J.-G. Liu, X. Xu, "Existence and incompressible limit of a tissue growth model with autophagy", *SIAM J. Math. Anal.* **53** (2021), no. 5, p. 5215-5242.

[10] M. Basan, T. Risler, J. Joanny, X. Sastre-Garau, J. Prost, "Homeostatic competition drives tumor growth and metastasis nucleation", *HFSP Journal* **3** (2009), no. 4, p. 265-272.

[11] T. Bittig, O. Wartlick, A. Kicheva, M. González-Gaitán, F. Jülicher, "Dynamics of anisotropic tissue growth", *New J. Phys.* (2008), no. 10, article no. 063001.

[12] B. Perthame, N. Vauchelet, "Incompressible limit of a mechanical model of tumour growth with viscosity", *Philos. Trans. R. Soc. Lond., Ser. A* **373** (2015), no. 2050, article no. 20140283 (16 pages).

[13] J. Escher, G. Simonett, "Classical solutions for Hele-Shaw models with surface tension", *Adv. Differ. Equ.* **2** (1997), no. 4, p. 619-642.

[14] N. D. Alikakos, P. W. Bates, X. Chen, "Convergence of the Cahn-Hilliard equation to the Hele-Shaw model", *Arch. Ration. Mech. Anal.* **128** (1994), no. 2, p. 165-205.

[15] X. Chen, "Global asymptotic limit of solutions of the Cahn-Hilliard equation", *J. Differ. Geom.* **44** (1996), no. 2, p. 262-311.

[16] H. B. Frieboes, F. Jin, Y.-L. Chuang, S. M. Wise, J. S. Lowengrub, V. Cristini, “Three-dimensional multispecies nonlinear tumor growth—II: Tumor invasion and angiogenesis”, *J. Theor. Biol.* **264** (2010), no. 4, p. 1254-1278.

[17] C. Chatelain, T. Balois, P. Ciarletta, M. Ben Amar, “Emergence of microstructural patterns in skin cancer: a phase separation analysis in a binary mixture”, *New J. Phys.* **13** (2011), article no. 115013 (22 pages).

[18] G. Sciumè, “Mechanistic modeling of vascular tumor growth: an extension of biot’s theory to hierarchical bi-compartment porous medium systems”, *Acta Mech.* **232** (2021), no. 4, p. 1445-1478.

[19] H. Garcke, K. F. Lam, E. Sitka, V. Styles, “A Cahn-Hilliard-Darcy model for tumour growth with chemotaxis and active transport”, *Math. Models Methods Appl. Sci.* **26** (2016), no. 6, p. 1095-1148.

[20] A. Agosti, P. F. Antonietti, P. Ciarletta, M. Grasselli, M. Verani, “A Cahn-Hilliard-type equation with application to tumor growth dynamics”, *Math. Methods Appl. Sci.* **40** (2017), no. 18, p. 7598-7626.

[21] H. Garcke, K. F. Lam, R. Nürnberg, E. Sitka, “A multiphase Cahn-Hilliard-Darcy model for tumour growth with necrosis”, *Math. Models Methods Appl. Sci.* **28** (2018), no. 3, p. 525-577.

[22] S. Frigeri, K. F. Lam, E. Rocca, G. Schimperna, “On a multi-species Cahn-Hilliard-Darcy tumor growth model with singular potentials”, *Commun. Math. Sci.* **16** (2018), no. 3, p. 821-856.

[23] M. Ebenbeck, H. Garcke, “On a Cahn-Hilliard-Brinkman model for tumor growth and its singular limits”, *SIAM J. Math. Anal.* **51** (2019), no. 3, p. 1868-1912.

[24] M. Ebenbeck, H. Garcke, R. Nürnberg, “Cahn-Hilliard-Brinkman systems for tumour growth”, *Discrete Contin. Dyn. Syst.* **14** (2021), no. 11, p. 3989-4033.

[25] B. Perthame, A. Poulain, “Relaxation of the Cahn-Hilliard equation with singular single-well potential and degenerate mobility”, *Eur. J. Appl. Math.* **32** (2021), no. 1, p. 89-112.

[26] J. W. Cahn, J. E. Hilliard, “Free energy of a nonuniform system. I. interfacial free energy”, *J. Chem. Phys.* **28** (1958), no. 2, p. 258-267.

[27] A. Miranville, *The Cahn-Hilliard equation. Recent advances and applications*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 95, Society for Industrial and Applied Mathematics, 2019.

[28] J. Yin, C. Liu, “Radial symmetric solutions of the Cahn-Hilliard equation with degenerate mobility”, *Electron, J. Qual. Theory Differ. Equ.* **2001** (2001), no. 2, article no. 2 (14 pages).

[29] J. Yin, “On the Cahn-Hilliard equation with nonlinear principal part”, *J. Partial Differ. Equations* **7** (1994), no. 1, p. 77-96.

[30] C. Elbar, B. Perthame, J. Skrzeczkowski, “Pressure jump and radial stationary solutions of the degenerate Cahn-Hilliard equation.Extended version”, https://hal.science/hal-03696545v3, 2022.