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The scientific legacy of Roland Glowinski / *L'héritage scientifique de Roland Glowinski*

Glowinski and numerical control problems

Glowinski et le contrôle numérique

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Abstract. This paper is devoted to recall several contributions to the numerical control of PDE's that have origin in Glowinski's work. I will consider null controllability problems for linear and nonlinear heat equations and some free-boundary systems. We will also deal with some bi-objective optimal control problems. Additionally, some new methods and results will be announced.

Résumé. Dans cet article, on rappelle quelques contributions sur contrôle numérique des EDPs issues du travail de Roland Glowinski. On considérera des problèmes de contrôlabilité nulle pour des équations de la chaleur linéaires et non linéaires et aussi pour des systèmes à frontière libre. Nous regarderons aussi quelques problèmes de contrôle optimal bi-objectif. En outre, quelques méthodes et résultats nouveaux seront annoncés.

Keywords. Controllability of linear and nonlinear PDE's, control of free-boundary problems, bi-objective optimal control problems, Nash equilibria, numerical methods.

Mots-clés. Contrôlabilité des EDPs linéaires et non linéaires, contrôle de problèmes de frontières libre, Problèmes de contrôle bi-objectif, Equilibria de Nash, Méthodes numériques.

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1. Introduction

The aims of this paper are to review some contributions by Ronald Glowinski to the numerical solution of control problems for PDE's and, also, to mention some recent advances in the field.

The paper is organized as follows.

Section 2 deals with null controllability problems for linear and nonlinear heat-like equations and systems. First, the original formulation by Glowinski and Lions and the related numerical difficulties and repairs will be recalled. Then, I will recall some results, derived in collaboration with A. Münch and others, where we have been able to use the so called Fursikov–Imanuvilov formulation for numerical purposes. I will also present some new methods, based on Fortin–Glowinski's ideas. In the final part of this section, we will deal with a null controllability problem

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for the free-boundary two-phase Stephan problem, together with the results provided by an appropriate iterative algorithm.

In Section 3, some bi-objective optimal control problems for the stationary Navier–Stokes equations will be recalled. More precisely, the search and computation of a Nash equilibrium will be described and, again, this will be illustrated with some numerical results. This research has been performed in collaboration with I. Martín-Gayte.

I will end this introduction with some personal regards and comments. First, a reference to my work in the old times, carried out under the direction of Glowinski. Then, other works inspired in his contributions and main interests.

I am indebted to Glowinski for having been first my PhD Thesis advisor and then a friend. In the beginning, he proposed me to work in the analysis and numerical resolution of the following two problems:

(1) Vortex rings for ideal fluids.

This is a problem previously considered and analyzed by H. Berestycki whom I had the pleasure to collaborate with. The goal is to find $u = u(\mathbf{x})$ and $K > 0$ such that

$$\begin{cases} -\Delta u = F(u - Wx_1 - K), & \mathbf{x} \in \Omega \\ u = 0, & \mathbf{x} \in \Gamma_0 \\ \dfrac{\partial u}{\partial n} = 0, & \mathbf{x} \in \Gamma_1 \\ \displaystyle\int_\Omega |\nabla u|^2 \, d\mathbf{x} = \eta, \end{cases} \tag{1}$$

where $\Omega = (0, L_1) \times (-L_2, L_2)$, $W, \eta > 0$ and $F \in C^0(\mathbb{R})$ is a non-decreasing function satisfying

$$F(s) = 0 \;\; \forall\, s \le 0 \quad \text{and} \quad F(s) \le Cs \;\; \forall\, s > 0.$$

The solution furnishes the stream function of a plane vortex ring $z := u - Wx_1 - K$ and the amount of flux K that flows between the axis $\{x_1 = 0\}$ and the ring (W and η are respectively the speed and the kinetic energy of the ring).

Several theoretical and numerical results for (1) can be found in [1].

(2) Two-dimensional semiconductor process modelling.

In this subject, I collaborated with E. Caquot and A. Marrocco. In this case, we want to find u and v with

$$\begin{cases} u_t - \nabla \cdot \big(D_{11}(u,v) \nabla u + D_{12}(u,v) \nabla v \big) = 0 & \text{in } Q \\ v_t - \nabla \cdot \big(D_{21}(u,v) \nabla u + D_{22}(u,v) \nabla v \big) = 0 & \text{in } Q \\ + \dots \end{cases}$$

where Ω is again a rectangle, $T > 0$ is given and the dots must contain boundary and initial conditions.

The unknowns u and v must be viewed as concentrations of impurities (for instance As and B). In the most simple situation, one has

$$D_{11} = a_{11} \left(1 + \frac{u}{\sqrt{(u-v)^2 + 4}} \right), \quad D_{12} = -a_{12} \frac{u}{\sqrt{(u-v)^2 + 4}}$$

for some positive constants a_{ij} and similar expresions hold for D_{22} and D_{21}.

Some results on the existence, numerical approximation and realistic simulation can be found in [2–4].

Then, he suggested me to work on the numerical analysis of Navier–Stokes equations. Thanks to this, I was involved during several years in the convergence analysis of time and space approximation schemes, parallelization methods, related stability issues and then optimal design analysis, controllability, etc.

Specifically, his advances in the use of operator-splitting techniques for the solution of nonlinear PDE's as well as other more recent contributions are briefly reviewed in [5].

Consequently, it is for me absolutely in order to thank him deeply for his crucial and very positive influence in my career.

2. Numerical null controllability

A substantial part of Glowinski's work concerns numerical control. In particular, he worked on the computation of null controls for the heat and wave equations.

2.1. *Linear heat equations*

In the case of the linear heat equation, together with J.-L. Lions, he started from the following formulation, where $\varepsilon > 0$ is a small constant:

$$\begin{cases} \text{Minimize } J(v) := \dfrac{1}{2} \displaystyle\iint_{\omega\times(0,T)} |v|^2 \, d\mathbf{x} \, dt \\ \text{Subject to } v \in L^2(\omega \times (0,T)),\ (3),\ \|y|_{t=T}\| \le \varepsilon \end{cases} \tag{2}$$

where the control v and the state y are related through the system

$$\begin{cases} y_t - \Delta y = v 1_\omega & \text{in } Q := \Omega \times (0,T) \\ y = 0 & \text{on } \Sigma := \partial\Omega \times (0,T) \\ y|_{t=0} = y^0 & \text{in } \Omega \end{cases} \tag{3}$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded connected open set ($N \ge 1$ is an integer), $\omega \subset \Omega$ is a (small) nonempty open subset, $T > 0$ and $y^0 \in L^2(\Omega)$.

From convex analysis, it is known that (2)–(3) is equivalent to the unconstrained (dual) extremal problem

$$\begin{cases} \text{Minimize } K_\varepsilon(\varphi_T) := \dfrac{1}{2} \displaystyle\iint_{\omega\times(0,T)} |\varphi|^2 \, d\mathbf{x} \, dt + \varepsilon \|\varphi_T\|_{L^2} - \int_\Omega \varphi(\mathbf{x},0) y^0(\mathbf{x}) \, d\mathbf{x} \\ \text{Subject to } \varphi_T \in L^2(\Omega),\ (5), \end{cases} \tag{4}$$

where (5) is the so called adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in } Q \\ \varphi = 0 & \text{on } \Sigma \\ \varphi|_{t=T} = \varphi_T & \text{in } \Omega. \end{cases} \tag{5}$$

In fact, the following identity is satisfied by the solution $\widehat{v}$ to (2)–(3) and the adjoint state $\widehat{\varphi}$ associated to the solution $\widehat{\varphi}_T$ to (4)–(5):

$$\widehat{v} = \widehat{\varphi}|_{\omega\times(0,T)} \,.$$

Unfortunately, this dual problem is numerically ill-posed for small ε. More precisely, in his first experiments, Glowinski found that the solution behaves as in Figure 1, exhibiting unpleasant oscillations as t approaches the final time T.

This is explained by the fact that, for $\varepsilon = 0$, (4)–(5) has no solution in $L^2(\Omega)$ in general. Indeed, there is no reason to have the coercivity of K_0 in this space. In particular, it becomes clear that, without modifications, (4)–(5) with $\varepsilon = 0$ is not a good formulation of the null control problem.

Since the 80's, several remedies have been introduced to arrange the situation by Carthel, Glowinski J.-L. Lions, Boyer, Trélat, Le Rousseau, Zuazua, …Among others, we can mention the following techniques:

- Penalization, by adding to $J(y)$ terms of the form $\frac{k}{2}\|y|_{t=T}\|^2_{L^2}$ and/or $\frac{a}{2}\iint_Q |y|^2\,d\mathbf{x}\,dt$, see [6, 7].
- Regularization, relying on the Tychonov method and others, see [6].
- Good choice of approximation parameters in time and space, see [8].
- High order time-approximation, conjugate gradient pre-conditioning and other numerical adjustments, see [9], etc.

These methods are, in general terms, satisfactory. In particular, in [8] and [9], a careful numerical work has been done to make them efficient and accurate.

However, they provide in practice "only" approximate controllability. Indeed, it becomes difficult to get very small values of $\|y|_{t=T}\|_{L^2}$ with reasonable sizes of the penalization and regularization parameters. For instance, it was shown in [10] that, if we penalize with a term of the form $\frac{k}{2}\|y|_{t=T}\|^2_{L^2}$, in order to get a control for which $\|y|_{t=T}\|_{L^2} = \varepsilon$, we need a constant $k \sim e^{1/\varepsilon}$.

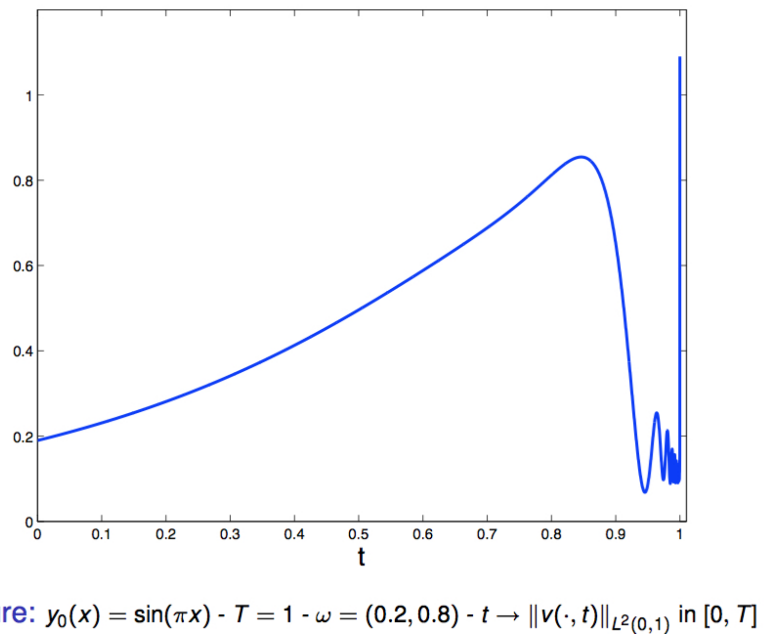

Figure 1. The adjoint state φ for small ε. Courtesy of A. Münch.

In [11], we proposed a numerical alternative, based on Carleman estimates and, more precisely, on the so called Fursikov–Imanuvilov formulation of the null controllability problem, see [12].

The main ideas are the following. Let us introduce appropriate weights $\rho, \rho_0 \sim e^{C(\mathbf{x})/(T-t)}$ and let us consider the extremal problem

$$\begin{cases} \text{Minimize } \displaystyle\iint_Q \left(\rho^2|y|^2 + 1_\omega \rho_0^2 |v|^2\right)\,d\mathbf{x}\,dt \\ \text{Subject to } v \in L^2(\omega \times (0,T)),\ y \in L^2(Q), (3) \end{cases} \tag{6}$$

It is clear that any solution (y,v) to this problem automatically satisfies $y|_{t=T}=0$ (and also, if v is regular enough, $v|_{t=T}=0$).

The existence and uniqueness of a solution to (6) can be established in different ways, see [12].

Let us set $Ly := y_t - \Delta y$ and $L^*p := -p_t - \Delta p$. From the Euler-Lagrange principle, it is not difficult to prove that the unique solution to (6) satisfies

$$y = \rho^{-2} L^* p, \quad v = -\rho_0^{-2} p|_{\omega\times(0,T)},$$

where p solves the 2nd order in time / 4th order in space linear problem

$$\begin{cases} L\big(\rho^{-2}L^*p\big) + \rho_0^{-2}1_\omega p = 0 & \text{in } Q \\ p = 0, \ \ \rho^{-2}L^*p = 0 & \text{on } \Sigma \\ \rho^{-2}L^*p\big|_{t=0} = y^0, \ \ \rho^{-2}L^*p\big|_{t=T} = 0 & \text{in } \Omega. \end{cases}$$

The rigorous (weak) formulation of this problem is the following:

$$\begin{cases} \displaystyle\iint_Q \big(\rho^{-2}L^*p\,L^*\varphi + \rho_0^{-2}1_\omega p\,\varphi\big)\,d\mathbf{x}\,dt = \int_\Omega y^0(\mathbf{x})\,\varphi(\mathbf{x},0)\,d\mathbf{x} \\ \forall\,\varphi\in P, \ p\in P, \end{cases}$$

where P is an appropriate Hilbert space. More precisely, P must be the completion of the space

$$P_0 := \Big\{\varphi\in C^2(\overline{Q}) : \varphi = 0 \quad \text{on} \quad \Sigma\Big\}$$

for the norm

$$\varphi\mapsto\left(\iint_Q\Big(\rho^{-2}\left|L^*\varphi\right|^2 + \rho_0^{-2}1_\omega|\varphi|^2\Big)\right)^{1/2}.$$

This can be written in the form

$$m(p,\varphi) = \langle \ell,\varphi\rangle \quad \forall\,\varphi\in P; \ \ p\in P \tag{7}$$

for a bilinear form $m(\cdot,\cdot)$ and a linear form ℓ on P.

The good choice of the weights ρ and ρ_0 makes it possible to employ global Carleman estimates for the functions in P_0. As a consequence,

$$P := \left\{\varphi : \iint_Q\Big(\rho^{-2}\left|L^*\varphi\right|^2 + \rho_0^{-2}1_\omega|\varphi|^2\Big) < +\infty, \ \ \varphi = 0 \ \text{on} \ \Sigma\right\},$$

any $\varphi\in P$ satisfies (among other things) $\varphi\in C^0([0,T/2];L^2(\Omega))$ and the $L^2(\Omega)$-valued linear mapping $\varphi\mapsto\varphi(\cdot,0)$ is continuous on P. Hence, *Lax–Milgram Theorem* can be applied in this context. This confirms the existence and uniqueness of a solution to (7) and also opens the possibility to introduce finite dimensional approximations.

Thus, a first way to get approximations to (7) relies on introducing a finite dimensional subspace P_k of P and simply replace (7) by

$$m\big(p_k,\varphi_k\big) = \big\langle \ell,\varphi_k\big\rangle \quad \forall\,\varphi_k\in P_k; \ \ p_k\in P_k. \tag{8}$$

As indicated in [11], this is not difficult and numerically efficient in the 1D case.

Thus, assume that $\Omega=(0,1)$, let N_x and N_t be two (large) positive integers and set $\Delta x = 1/N_x$, $\Delta t = T/N_t$ and $k=(\Delta x,\Delta t)$. A very natural $\mathbb{Q}_3-\mathbb{Q}_1$ technique is the following:

(1) First, let $\mathcal{Q}_k$ be the square mesh formed by the rectangles of size $\Delta x\cdot\Delta t$ with vertices $(n\Delta t, m\Delta x)$. For any $K\in\mathcal{Q}_k$, we denote by $\mathbb{P}_{3,1}(K)$ the space of polynomial functions on K of degree ≤ 3 in x and degree ≤ 1 in t.

(2) Then, we take

$$P_k = \Big\{p_k\in C^{1,0}_{x,t}(\overline{Q}) : p_k\Big|_K\in\mathbb{P}_{3,1}(K) \ \ \forall\,K\in\mathcal{Q}_k, \ \ p_k = 0 \text{ on } \Sigma\Big\}.$$

We have denoted by $C^{1,0}_{x,t}(\overline{Q})$ the space of continuous functions on $\overline{Q}$ whose spatial derivatives exist and are continuous in $\overline{Q}$.

(3) Finally, we replace (7) by (8).

This approach was introduced in [11]. A complete analysis of the convergence of the resulting approximations was given there.

Nevertheless, it is not so simple to apply a similar technique in higher dimensions since, in practice, we need spaces of continuous functions in $\overline{Q}$ that must have continuous spatial derivatives in that set.

In view of this inconvenience, a second method has been introduced in [11, 13].

Specifically, a mixed formulation of (7) was given, with new unknowns $z = \rho^{-1}L^* p$, $q = \rho_0^{-1} p$ and a multiplier λ associated to the "constraint" $z = \rho^{-1}L^*(\rho_0 q)$. Then, with the help of a small $\varepsilon > 0$, we *regularized* the problem with a L^2 term involving the multiplier. Finally, after integration by parts, we found the system

$$\begin{cases} \displaystyle\iint_Q (z\zeta + 1_\omega q\psi)\,d\mathbf{x}\,dt + \beta((\zeta,\psi),\lambda) = \int_\Omega y^0(\mathbf{x})\,\rho_0(\mathbf{x},0)\psi(\mathbf{x},0)\,d\mathbf{x} \quad \forall\,(\zeta,\psi)\in X \\ \displaystyle -\beta((z,\psi),\mu) + \varepsilon\iint_Q \lambda\mu\,d\mathbf{x}\,dt = 0 \quad \forall\,\mu\in M, \end{cases} \tag{9}$$

where we have used the notation

$$\beta((\zeta,\psi),\mu) := \iint_Q \left(\mu\left(\zeta + \rho^{-1}(\rho_0\psi)_t\right) - \nabla(\rho^{-1}\mu)\cdot\nabla(\rho_0\psi)\right)\,d\mathbf{x}\,dt,$$

the solution is searched in $X \times M$,

$$X = \left\{(\zeta,\psi)\in L^2(Q)^2 : \rho^{-1}(\rho_0\psi)_t, \nabla(\rho_0\psi)\in L^2(Q)\right\}$$

and

$$M = \left\{\mu\in L^2(Q) : \nabla(\rho^{-1}\mu)\in L^2(Q)\right\}.$$

Now, it is found that standard easy-to-construct spaces of finite elements can be used to approximate the problem. For instance, with the help of a triangulation $\mathcal{T}_h$ of Q, we can consider the finite dimensional spaces

$$X_h = \left\{(\zeta_h,\psi_h)\in C^0(\overline{Q})^2 : (\zeta_h,\psi_h)|_K \in \mathbb{P}_1(K)\times\mathbb{P}_2(K) \quad \forall\,K\in\mathcal{T}_h,\ \ \psi_h|_{t=T} = 0\right\} \tag{10}$$

and

$$M_h = \left\{\mu_h\in C^0(\overline{Q})^2 : \mu_h|_K\in\mathbb{P}_1(K)\ \ \forall\,K\in\mathcal{T}_h\right\} \tag{11}$$

and introduce the approximated-regularized mixed problem

$$\begin{cases} \displaystyle\iint_Q \left(z_h\zeta_h + 1_\omega q_h\psi_h\right)\,d\mathbf{x}\,dt + \beta\left((\zeta_h,\psi_h),\lambda_h\right) \\ \displaystyle\qquad = \int_\Omega y^0(\mathbf{x})\,\rho_0(\mathbf{x},0)\psi_h(\mathbf{x},0)\,d\mathbf{x} \quad \forall\,(\zeta_h,\psi_h)\in X_h \\ \displaystyle -\beta\left((z_h,\psi_h),\mu_h\right) + \varepsilon\iint_Q \lambda_h\mu_h\,d\mathbf{x}\,dt = 0 \quad \forall\,\mu_h\in M_h, \end{cases} \tag{12}$$

where the solution is of course searched in $X_h \times M_h$.

More details can be found in [13].

Let us present the results of some experiments.

The first of them corresponds to the null control of the 1D heat equation

$$\begin{cases} y_t - \alpha y_{xx} = v 1_\omega, & (x,t) \in (0,1)\times(0,T) \\ y(0,t) = y(1,t) = 0, & t \in (0,T) \\ y(x,0) = y^0(x), & x \in (0,1), \end{cases} \tag{13}$$

where, as before, $v = v(x,t)$. It is well known that (13) is null-controllable for all $\alpha > 0$ and any ω and T.

We will assume here that $\alpha = 0.1$, $\omega = (a,b)$ with $0 < a < b < 1$, $T = 0.5$ and $y_0(x) \equiv \sin(\pi x)$. The solution to (8) on a regular mesh $\mathcal{Q}_k$ leads to the results depicted in Figure 2.

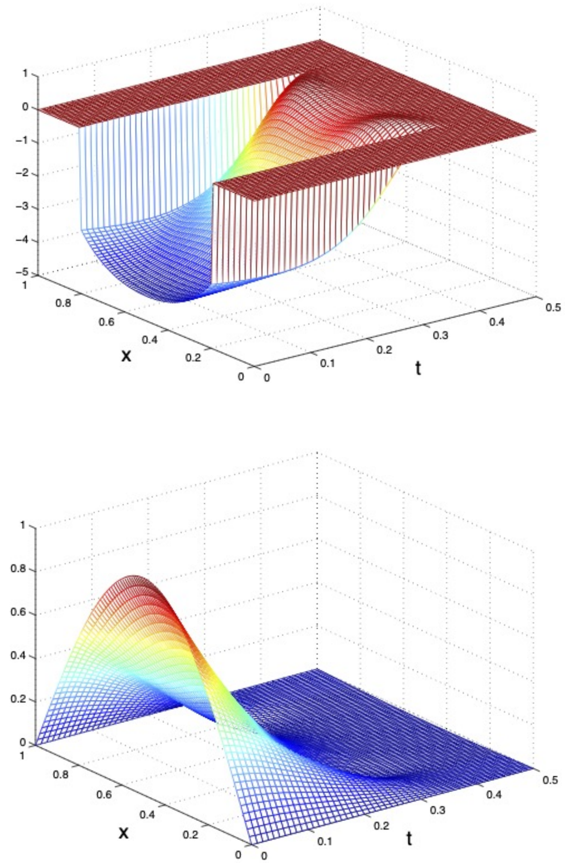

Figure 2. A null controllability experiment for (13). The control (left) and the state (right).

Our second experiment concerns the null controllability of the following 2D linear parabolic PDE:

$$\begin{cases} y_t - \Delta y + G(\mathbf{x})y = v1_\omega & \text{in } \Omega \times (0,T) \\ y = 0 & \text{on } \partial\Omega \times (0,T) \\ y|_{t=0} = y^0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{14}$$

where $\Omega = (0,1) \times (0,1)$.

It is also known that (14) is null-controllable for all $G \in L^\infty(\Omega)$, any open set $\omega \subset \Omega$ and any $T > 0$.

Now, we have taken $G(\mathbf{x}) \equiv -10$, $\omega = (0.3, 0.7) \times (0.3, 0.7)$, $T = 1$ and $y_0(\mathbf{x}) \equiv 10^3 \sin(\pi x_1) \sin(\pi x_2)$. We have solved (12), with the spaces X_h and M_h respectively given by (10) and (11) corresponding to a space-time mesh $\mathcal{T}_h$ with 2496 vertices, 12210 elements and 18055 nodes. Note that the number of variables is $3 \times 18055 = 54165$. The control and the state computed from the solution to (12) have been depicted in Figure 3.

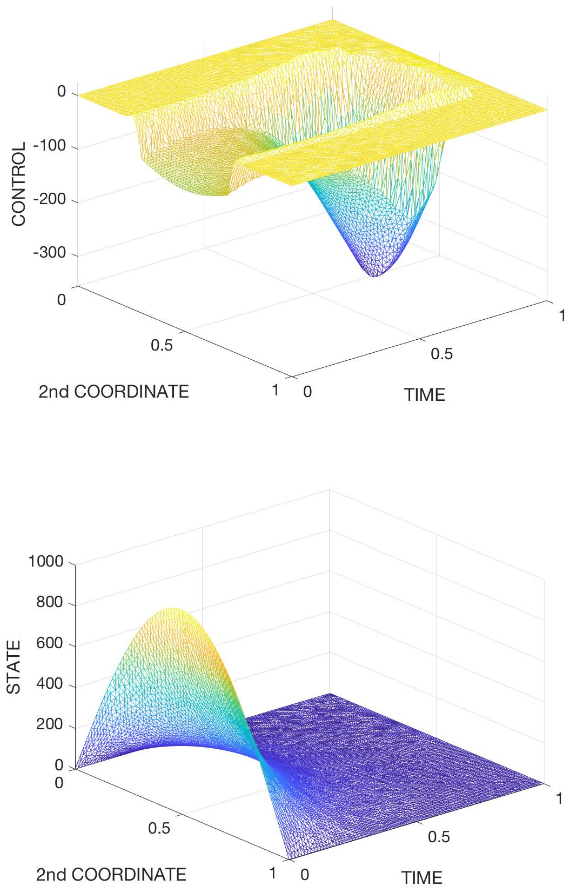

Figure 3. A null controllability experiment for (14). Sections at $x_1 = 0.5$ of the control (left) and the state (right).

At this point, let us describe a new method for the resolution of the null controllability problem for (3). It is similar (but different) to the previous one and relies on some old ideas used by Glowinski for the solution of constrained extremal problems; see [14].

The point is to consider again (6), introduce a Lagrangian and an *augmented Lagrangian* reformulation and then search for an associated saddle-point through an appropriate iterative method. More precisely, we can rewrite (6) as follows:

$$\begin{cases} \text{Minimize } F(v) + G(y) \\ \text{Subject to } v \in L^2(\omega \times (0,T)),\ Mv + \overline{y} = y, \end{cases} \tag{15}$$

where we have set

$$F(v) := \iint_{\omega \times (0,T)} \rho_0^2 |v|^2 \, d\mathbf{x}\, dt \quad \text{and} \quad G(y) := \iint_Q \rho^2 |y|^2 \, d\mathbf{x}\, dt,$$

the linear mapping $M : L^2(\omega \times (0,T)) \mapsto L^2(Q)$ associates to each control v the solution to (3) with y^0 replaced by zero and $\overline{y}$ is the solution to (3) corresponding to $v \equiv 0$.

It is not difficult to see that (15) possesses the following two equivalent Lagrangian formulations:

$$\min_{v,y}\max_{p}\, clL(v,y;p) \quad \text{and} \quad \min_{v,y}\max_{p}\, clL_R(v,y;p)$$

where we have used the notations

$$clL(v,y;p) := F(v) + G(y) + (p, Mv + \overline{y} - y)_{L^2(Q)}$$

and

$$clL_R(v,y;p) := F(v) + G(y) + (p, Mv + \overline{y} - y)_{L^2(Q)} + \frac{R}{2}\left\| Mv + \overline{y} - y \right\|^2_{L^2(Q)}$$

and the minima (resp. the maxima) are searched in $L^2(\omega \times (0,T)) \times L^2(Q)$ (resp. in $L^2(Q)$).

Also, it can be proved that a solution (i.e. a saddle-point for clL and clL_R) can be found by solving the dual problems

$$\max_{p}\min_{v,y}\, clL(v,y;p) \quad \text{and} \quad \max_{p}\min_{v,y}\, clL_R(v,y;p). \tag{16}$$

Thus, we can apply (for instance) *Uzawa algorithm* to the first of these problems. Taking into account the definitions of F and G, this leads to the following iterates:

ALG 1 (Uzawa):

(1) Choose p^1 and fix $r > 0$.

(2) For given $n \ge 1$ and p^n, find v^{n+1}, y^{n+1} and p^{n+1} as follows:

$$v^{n+1} = -\rho_0^{-2} M^* p^n, \quad y^{n+1} = \rho^{-2} p^n, \quad p^{n+1} = p^n + r\left(Mv^{n+1} + \overline{y} - y^{n+1}\right).$$

In order to accelerate the convergence of the p^n, it is convenient to consider the augmented Lagrangian $\mathscr{L}_R$ instead of $\mathscr{L}$. Now, the second dual problem in (16) can be solved by the *Arrow-Hurwicz* algorithm:

ALG 2 (Arrow–Hurwicz):

(1) Choose p^1 and fix $R > 0$, $r > 0$ and $s > 0$.

(2) For given $n \ge 1$ and p^n, find d^n, v^{n+1}, y^{n+1} and p^{n+1} as follows:

$$d^n = Mv^n + \overline{y} - y^n,$$
$$v^{n+1} = v^n - s\left(v^n + \rho_0^{-2} M^*\left(p^n + Rd^n\right)\right), \quad y^{n+1} = y^n - s\left(y^n - \rho^{-2}\left(p^n + Rd^n\right)\right),$$
$$p^{n+1} = p^n + r\left(Mv^{n+1} + \overline{y} - y^{n+1}\right).$$

More details and a convergence analysis of the sequences produced for **ALG 1** and **ALG 2** will appear in a forthcoming paper.

Let us end this section with the results of a new experiment.

This time, we consider the null control of the modified 1D heat equation

$$\begin{cases} y_t - \alpha y_{xx} + G(x)y = v1_\omega, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0, & t \in (0,T) \\ y(x,0) = y^0(x), & x \in (0,1), \end{cases} \tag{17}$$

where $G(x) \equiv -5$ and again $\alpha = 0.1$, $\omega = (a,b)$ with $0 < a < b < 1$, $T = 0.5$ and $y_0(x) \equiv \sin(\pi x)$.

We have applied **ALG 2** to a space and time approximation to (3). In this case, we have used the `pdepe` function from MatLab to compute the solutions to the state equations (17) (that is, the Mv^{n+1}) and the similar adjoint systems (i.e. the $M^* p^n$). Recall that `pdepe` uses *second order* finite difference approximation in space and a variable-step, *fifth-order* time discretization scheme based on numerical differentiation, see [15].

In the limit, the v^n and y^n have given the functions depicted in Figure 4.

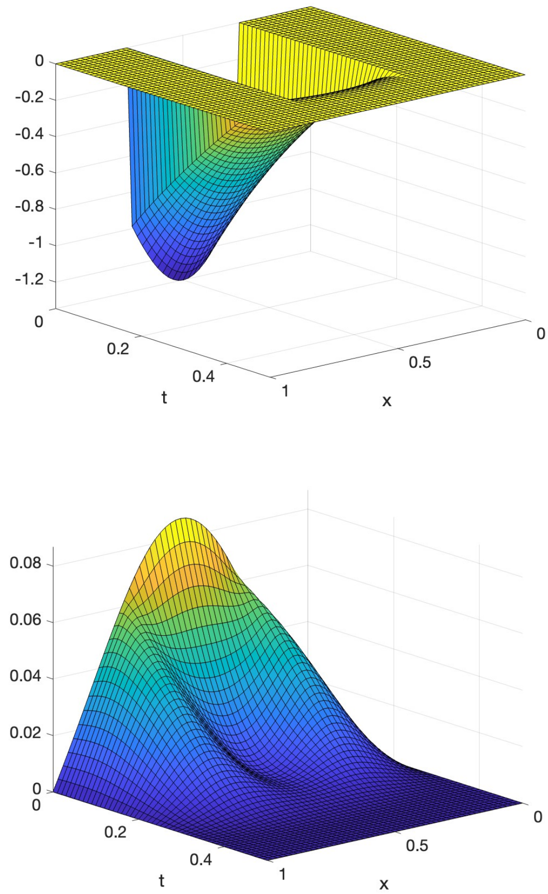

Figure 4. A null controllability experiment for (17). Results given by **ALG 2** on a regular rectangular mesh in the (x,t) plane with 50×50 points. The control (left) and the state (right).

2.2. *Semilinear and nonlinear parabolic equations*

Techniques of this kind, used in combination with fixed-point, conjugate gradient, Newton and quasi-Newton algorithms, have been applied satisfactorily in the context of the numerical null controllability of semi-linear and nonlinear parabolic PDE's; see [13, 16, 17].

For instance, let us consider the following system for a semilinear heat equation:

$$\begin{cases} y_t - \Delta y + f(y) = v 1_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{18}$$

where $f \in C^1(\mathbb{R})$ and $f(0) = 0$.

We have a global null controllability result when f is at most weakly superlinear:

Theorem 1 ([18]). *Assume that*

$$\limsup_{|s|\to\infty} \frac{f(s)}{s\log^{3/2}(1+|s|)} = 0. \tag{19}$$

Then (18) *is null-controllable and approximately controllable for any ω at any $T > 0$.*

The proof relies on a fixed-point reformulation of the null controllability problem and appropriate sharp estimates of the cost of controllability. More precisely, let $y^0 \in L^\infty(\Omega)$ be given and let us introduce the function $g \in C^0(\mathbb{R})$, with

$$g(s) = \begin{cases} \dfrac{f(s)}{s} & \text{if } s \neq 0 \\ f'(0) & \text{otherwise.} \end{cases}$$

Thanks to (19) and the usual parabolic regularity results, we can construct a multi-valued mapping $\Lambda : L^\infty(Q) \rightrightarrows L^\infty(Q)$ with the following properties:

- Λ associates to each $z \in L^\infty(Q)$ a family of solutions y_z to the linearized systems

$$\begin{cases} y_t - \Delta y + g(z)y = v_z 1_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(0) = y^0 & \text{in } \Omega, \end{cases}$$

  with the controls $v_z \in L^\infty(\omega \times (0,T))$ such that

$$y_z(T) = 0 \text{ in } \Omega.$$

- Λ is weakly upper semi-continuous and there exists a convex compact set $K \subset L^\infty(Q)$ such that $\Lambda(z) \subset K$ for all $z \in L^\infty(Q)$.

Consequently, Kakutani's Fixed-Point Theorem can be applied to Λ and this implies the existence of a control in $L^\infty(\omega \times (0,T)$ such that the associated solution to (18) vanishes at $t = T$.

In view of this argument, we get for the computation of a null control a first algorithm:

ALG 3 (fixed-point):

(1) Choose y^1.

(2) For given $n \geq 1$ and y^n, find v^{n+1} and y^{n+1} by solving the null controllability problem (6) with (3) replaced by

$$\begin{cases} y_t - \Delta y + g(y^n)y = v & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(0) = y^0 & \text{in } \Omega. \end{cases}$$

Let us now present a different way to linearize and compute a null control.

It is possible to introduce two Hilbert spaces Y and W, respectively formed by state-control pairs (y, v) and right hand side-initial data pairs (f, y^0), such that the mapping $F : Y \mapsto W$ with

$$F(y, v) = (y_t - \Delta y + f(y) - v1_\omega, y(0)) \quad \forall\, (y, v) \in Y \tag{20}$$

is well-defined and C^1 and possesses a continuous inverse.

In particular, for any $(0, y^0) \in W$, the equation

$$F(y, v) = (0, y^0), \;\; (y, v) \in Y \tag{21}$$

is solvable and the solution furnishes a null control and the associated state.

Thus, it makes sense to apply (for instance) Newton's method to (21). The corresponding iterates are as follows:

ALG 4 (Newton):

(1) Choose y^1.

(2) For given $n \geq 1$ and y^n, find v^{n+1} and y^{n+1} by solving the null controllability problem (6) with (3) replaced by

$$\begin{cases} y_t - \Delta y + f'\left(y^n\right) y = v - R^n & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(0) = y^0 & \text{in } \Omega, \end{cases}$$

where $R^n = f(y^n) - f'(y^n)y^n$.

Note that both **ALG 3** and **ALG 4** rely on the solution of a sequence of null control problems for linear parabolic PDE's.

Remark 2. A different proof of Theorem 1 that relies on the convergence of a contracting algorithm has been presented in the recent paper [19], together with illustrating numerical experiments. □

Let us present the results of a numerical experiment with $N = 1$. The data are the following: $\Omega = (0,1)$, $\omega = (0.2, 0.8)$, $T = 0.5$, $f(y) \equiv -5y\log^{1.4}(1+|y|)$, $y^0(x) \equiv 40\sin(\pi x)$. Note that the chosen exponent is below but close to the critical value $3/2$.

We have applied **ALG 3** and **ALG 4** with identical results. In order to approximate the linear systems, we have used the mixed formulation (12), with X_h and M_h respectively given by (10) and (11).

An adaptative mesh technique has been used. More precisely, the following has been done:

- First, we have taken the initial mesh $\mathcal{T}_h^1$ in Figure 5 and we have computed a first solution and a first associated control-state pair (v_*^1, y_*^1).
- Then, for any given $\ell \geq 1$, $\mathcal{T}_h^\ell$ and (v_*^n, y_*^n), we have used y_*^n to adapt the mesh and get a new $\mathcal{T}_h^{\ell+1}$ and a new control-state pair $(v_*^{\ell+1}, y_*^{\ell+1})$.

  The adaptation process is described in [20].

The iterates were stopped when the convergence test

$$\frac{\|y_*^{\ell+1} - y_*^\ell\|_{L^2}}{\|y_*^\ell\|_{L^2}} \leq 10^{-7}$$

was satisfied.

The final adapted mesh is shown in Figure 6. The computed control and state are displayed in Figure 7.

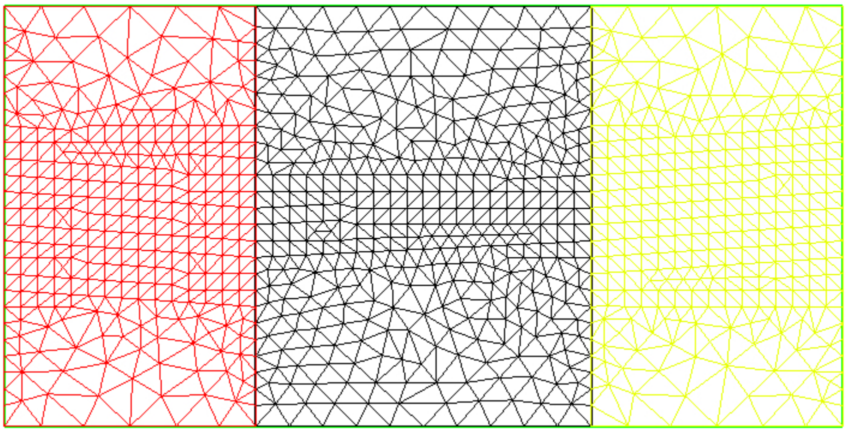

Figure 5. The initial mesh — Nodes: 1765, Elements: 3512

More recently, we have been able to extend the previous theoretical and numerical arguments and methods to the framework of free-boundary systems. Without too many details, let us recall a theoretical result and let us exhibit the computed control and state corresponding to a particular case.

Thus, let $L > 0$, $T > 0$, $\ell^0 \in (0, L)$, $y^0 \in C^1([0, \ell^0])$ and $z^0 \in C^1([\ell^0, L])$ be given, with $y^0(0) = y^0(\ell^0) = z^0(\ell^0) = z^0(L) = 0$ and let $\omega_l \subset\subset (0, \ell^0)$ and $\omega_r \subset\subset (\ell^0, L)$ be (small) non-empty open sets. Consider the following null controllability problem for the two-phase Stefan problem: Find $v_l \in L^2(\omega_l \times (0,T))$, $v_r \in L^2(\omega_r \times (0,T))$, ℓ, y and z with

$$\begin{cases} y_t - d_l y_{xx} = v_l 1_{\omega_l}, \; x \in (0, \ell(t)), \; t \in (0,T) \\ z_t - d_r z_{xx} = v_r 1_{\omega_r}, \; x \in (\ell(t), L), \; t \in (0,T) \\ y|_{x=0} = y|_{x=\ell(t)} = z|_{x=\ell(t)} = z|_{x=L} = 0, \; t \in (0,T) \\ \ell(0) = \ell^0 \\ y|_{t=0} = y^0(x), \; x \in \left(0, \ell^0\right) \\ z|_{t=0} = z^0(x), \; x \in \left(\ell^0, L\right) \\ (d_l y_x - d_r z_x)|_{x=\ell(t)} = -k\ell'(t), \;\; t \in (0,T) \end{cases} \tag{22}$$

and

$$\begin{cases} y(x,T) = 0, \;\; x \in (0, \ell(T)) \\ z(x,T) = 0, \;\; Zx \in (\ell(T), L) \\ \ell(T) = \ell_T. \end{cases} \tag{23}$$

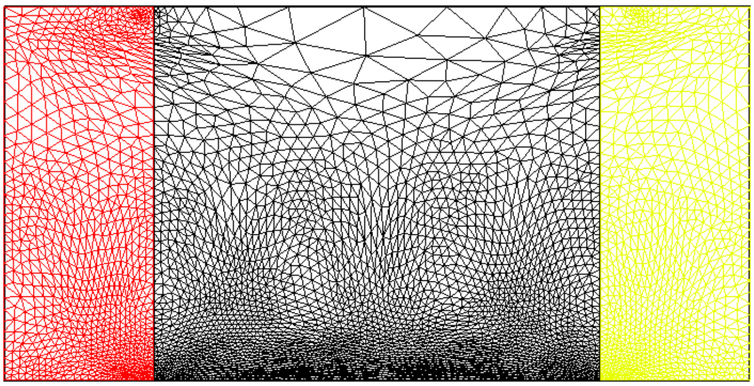

Figure 6. The final mesh — Nodes: 4175, Elements: 8128

The following local null controllability result holds:

Theorem 3 ([21]). *There exists $\varepsilon > 0$ such that, if the data y_0, z_0, ℓ_0 and ℓ_T satisfy*

$$\|y_0\|_{H_0^1} + \|z_0\|_{H_0^1} + |\ell_0 - \ell_T| \le \varepsilon,$$

there exist v_l, v_r, ℓ, y and z solving (22) *such that* (23) *holds.*

Let us take now $L = 10$, $T = 10$, $d_l = 2$, $d_r = 2.15$, $k = 0.16$, $\ell^0 = \ell^T = 5$, $y^0(x) \equiv 3\sin(\pi x/5)$, $z^0(x) \equiv -2\sin(\pi(x-5)/10)$, $\omega_l = (0,3)$ and $\omega_r = (12,15)$.

The corresponding problem (22)–(23) has been solved numerically.

To this purpose, we have first introduced an equivalent reformulation of the fixed-point kind $\ell = \Xi(\ell)$, where the computation of $\Xi(\ell)$ involves the resolution of null controllability problems

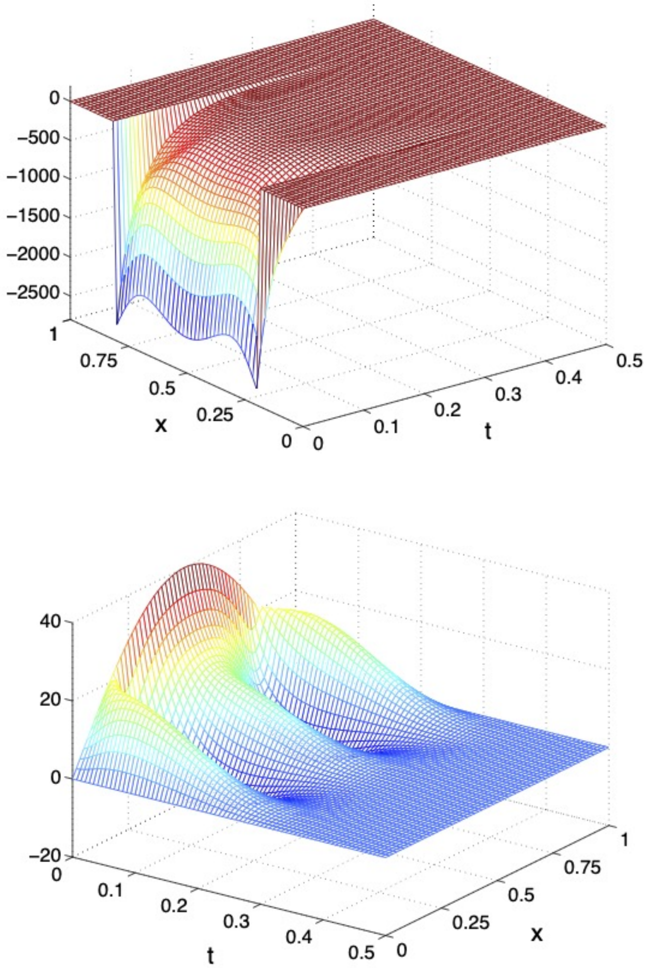

Figure 7. A null controllability experiment for (18). The computed control (left) and state (right).

on the left and on the right of ℓ constrained to satisfy an integral identity. Then, the null control problems have been approximated as in (12) for some appropriate X_h and M_h. More details wil be given in [22].

The final mesh and the computed controls and state are shown in Figures 8–10.

3. Bi-objective optimal control

Glowinski also contributed to the solution of several bi-objective optimal control problems; see [23, 24].

In order to motivate this research and understand the situation, consider the functions $J_i = J_i(v_1, v_2)$ depicted in Figure 11.

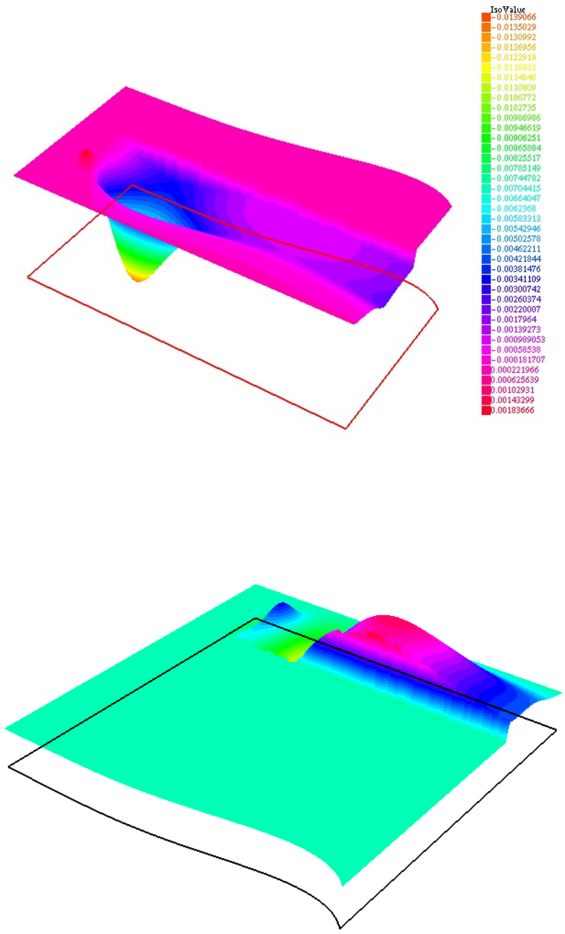

Figure 8. A null controllability experiment for (22). The computed controls v_l (left) and v_r (right).

It is clear that it is impossible to find a common minimizer. Consequently, if we have interest in getting simulatneously small values of J_1 and J_2, we have to look for an *equilibrium.* A way to proceed is to search for a couple $(\hat{v}_1, \hat{v}_2)$ satisfying

$$J_1(\hat{v}_1, \hat{v}_2) \le J_1(v_1, \hat{v}_2) \quad \forall\, v_1 \quad \text{and} \quad J_2(\hat{v}_1, \hat{v}_2) \le J_2(\hat{v}_1, v_2) \quad \forall\, v_2. \tag{24}$$

This is called a (noncooperative) *Nash equilibrium;* see again Figure 11.

In [23] and [24], Glowinski, Périaux and Ramos did the following:

- They proved the existence and performed the numerical computation of Nash equilibria for bi-objective optimal control problems for linear PDE's.
- They got similar results for systems governed by the viscous Burgers' PDE. In this case, they worked with pointwise controls.

Other contributions to bi-objective optimal control are due to J.-L. Lions, Díaz and others; see [25] and the references therein.

In some relatively recent papers, we have considered problems of this kind for the stationary Navier–Stokes equations with satisfactory theoretical and numerical results.

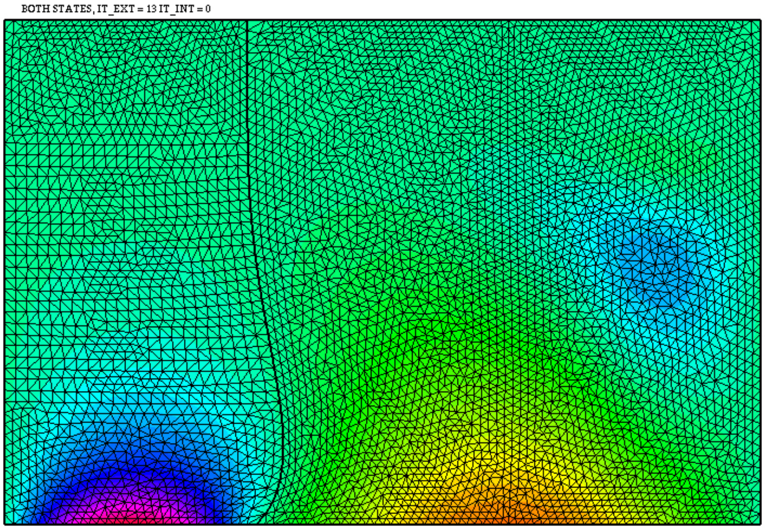

Figure 9. A null controllability experiment for (22). The final mesh and the computed state.

More precisely, let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set and let the function $\mathbf{u}_\Gamma \in H^1(\Omega)^N$ with $\nabla \cdot \mathbf{u}_\Gamma = 0$ and the non-empty open sets $\omega_i \subset\subset \Omega$ $(i = 1,2)$ be given. For any $(\mathbf{f}_1, \mathbf{f}_2)$ with $\mathbf{f}_i \in L^2(\omega_i)^N$, let $(\mathbf{u}, p)$ be a solution to the system

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}_1 1_{\omega_1} + \mathbf{f}_2 1_{\omega_2} & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{u}_\Gamma & \text{on } \partial\Omega \end{cases} \tag{25}$$

and let us set

$$J_i(\mathbf{f}_1, \mathbf{f}_2; \mathbf{u}) := \frac{a}{2}\int_{O_i} |\mathbf{u} - \mathbf{u}_{i,d}|^2 \, d\mathbf{x} + \frac{\mu}{2}\int_{\omega_i} |\mathbf{f}_i|^2 \, d\mathbf{x} \quad \text{for } i = 1,2 \tag{26}$$

where the $O_i \subset \Omega$ are non-empty open sets, the $\mathbf{u}_{i,d} \in L^2(O_i)^N$ and $a, \mu > 0$.

By definition, it will be said that $(\widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_2, \widehat{\mathbf{u}})$ is a Nash-equilibrium for (25)–(26) if one has

$$\begin{cases} J_1\left(\widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_2; \widehat{\mathbf{u}}\right) \le J_1\left(\mathbf{f}_1, \widehat{\mathbf{f}}_2; \mathbf{u}\right) & \forall\, \mathbf{f}_1 \in L^2(\omega_1)^N, \quad \forall \text{ associated } \mathbf{u} \\ J_1\left(\widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_2; \widehat{\mathbf{u}}\right) \le J_1\left(\widehat{\mathbf{f}}_1, \mathbf{f}_2; \mathbf{u}\right) & \forall\, \mathbf{f}_2 \in L^2(\omega_2)^N, \quad \forall \text{ associated } \mathbf{u}. \end{cases} \tag{27}$$

The existence of a Nash equilibrium is delicate. In fact, the following holds:

Theorem 4 ([26]). *Let Ω, the ω_i, O_i, $\mathbf{u}_{i,d}$, a and μ be as before. Then:*

(1) *If $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{u})$ is a Nash equilibrium, it is also a Nash quasi-equilibrium, i.e. there exist couples $(\mathbf{w}_i, q_i)$ $(i = 1,2)$ such that*

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}_1 1_{\omega_1} + \mathbf{f}_2 1_{\omega_2}, \ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ -\nu\Delta\mathbf{w}_i - (\mathbf{u}\cdot\nabla)\mathbf{w}_i + (\nabla\mathbf{u})^T\mathbf{w}_i + \nabla q_i = (\mathbf{u} - \mathbf{u}_{i,d})1_{O_i}, \ \nabla\cdot\mathbf{w}_i = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{u}_\Gamma, \ \ \mathbf{w}_i = 0 & \text{on } \partial\Omega \\ \mathbf{f}_i = -\dfrac{a}{\mu}\mathbf{w}_i\big|_{\omega_i}. \end{cases} \tag{28}$$

(2) *The coupled system* (28) *is always solvable.*

(3) *If ν is sufficiently large (depending on Ω and a/μ), all Nash quasi-equilibria are in fact Nash equilibria. Consequently, for large ν, Nash equilibria exist.*

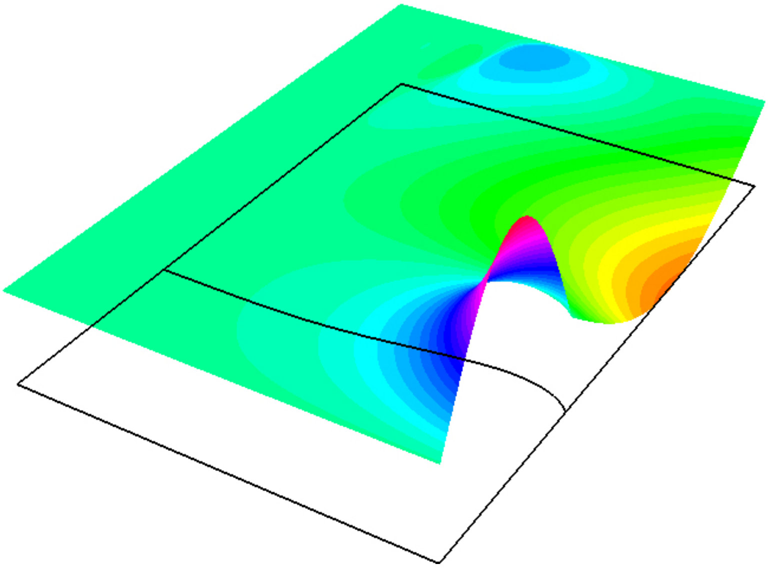

Figure 10. A null controllability experiment for (22). The computed state.

The proof is given in [26].

In order to prove part 1, it suffices to observe that, if $(\mathbf{f}_1,\mathbf{f}_2,\mathbf{u})$ is a Nash equilibrium, the "partial derivative" of J_i with respect to $\mathbf{f}_i$ at $(\mathbf{f}_1,\mathbf{f}_2,\mathbf{u})$ must vanish. Arguing as in the case of a "classical" optimal control problem, we easily deduce the optimality system (28).

The proof of part 3 relies on the fact that, if $(\mathbf{f}_1,\mathbf{f}_2,\mathbf{u})$ is a Nash quasi-equilibrium and ν is large enough, the functions $\mathbf{f}_1 \mapsto J_1(\mathbf{f}_1,\widehat{\mathbf{f}}_2,\mathbf{u})$ and $\mathbf{f}_2 \mapsto J_2(\widehat{\mathbf{f}}_1,\mathbf{f}_2,\mathbf{u})$ are well-defined, C^1 and convex, whence their critical points are in fact global minimizers.

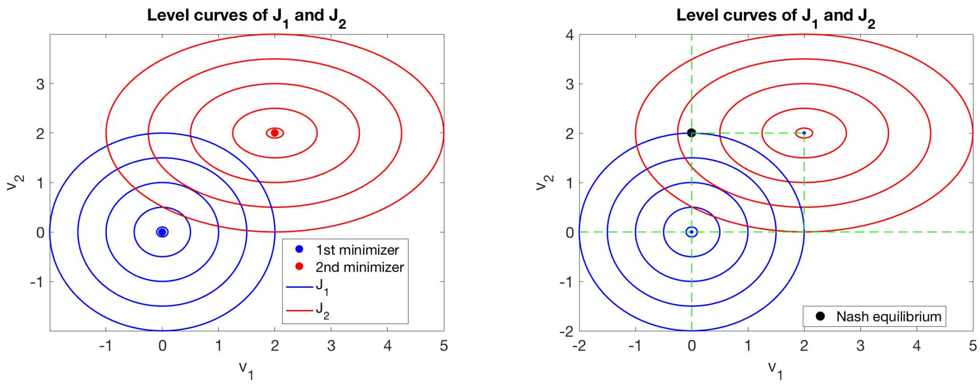

Figure 11. Illustration of a bi-objective extremal problem (left). The Nash equilibrium (right).

For the computation of a Nash equilibrium, we have solved the optimality system (28) with a Newton-like algorithm.

More precisely, we have rewritten the optimality system in the form

$$\Phi\left(\mathbf{u}, p, \mathbf{w}_1, q_1, \mathbf{w}_2, q_2\right) = 0$$

for some well-defined and C^1 function Φ. Then, by introducing the notation $\mathbf{a} := (\mathbf{u}, p, \mathbf{w}_1, q_1, \mathbf{w}_2, q_2)$ and starting from some $\mathbf{a}^1$, we have set for any $k \geq 1$

$$\mathbf{a}^{k+1} = \mathbf{a}^k - \mathbf{b}^k,$$

where $\mathbf{b}^k$ is the solution to the linear system

$$(DF^k)\mathbf{b}^k = F^k \tag{29}$$

and F^k and DF^k are respectively given by F and DF at $\mathbf{a}^k$.

In practice, it is not difficult to solve numerically (29) for instance by introducing a standard mixed weak formulation and reducing to finite dimension with the help of $\mathbb{P}_2 - \mathbb{P}_1$ finite elements.

A particular 2D experiment corresponds to the domain and the mesh indicated in Figure 12.

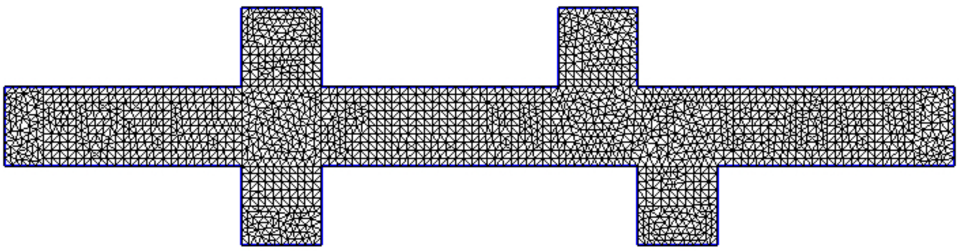

Figure 12. Computation of a Nash equilibrium for (25)–(26). The domain and the mesh; Ω is composed of the main pipe, two first rectangles (ω_1 and ω_2), a second upper rectangle $\mathcal{O}_1$ and a second lower rectangle $\mathcal{O}_2$. Nodes: 1541, Triangles: 2774.

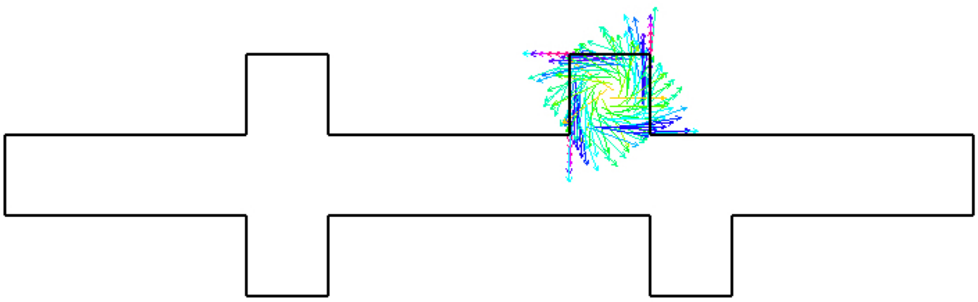

Figure 13. Computation of a Nash equilibrium for (25)–(26). The function $\mathbf{u}_{1,d}$; $\mathbf{u}_{2,d} = 0$.

The uncontrolled velocity field corresponding to (25) for $Re = 1200$ is depicted in Figure 14.

Let Γ_0 be the left vertical segment of $\partial\Omega$, with endpoints $(\bar{x}_1, \bar{x}_{2,1})$ and $(\bar{x}_1, \bar{x}_{2,2})$. We have taken $\mathbf{u}_\Gamma$ satisfying $\mathbf{u}_\Gamma = (U(x_2 - \bar{x}_{2,1})(\bar{x}_{2,2} - x_2), 0)$ on Γ_0 and $\mathbf{u}_\Gamma = 0$ on $\partial\Omega \setminus \Gamma_0$, where $U > 0$. On the other hand, the "desired" velocity fields $\mathbf{u}_{i,d}$ are exhibited in Figure 13.

Finally, the velocity field associated to a computed Nash equilibrium for $a = 1.99$ and $\mu = 0.01$ is shown in Figure 15. Note that the state associated to the computed controls seems to furnish good approximations of $\mathbf{u}_{1,d}$ and $\mathbf{u}_{2,d}$ respectively in O_1 and O_2.

Recently, other multiple-goal problems for the Navier–Stokes equations have been considered; see [27].

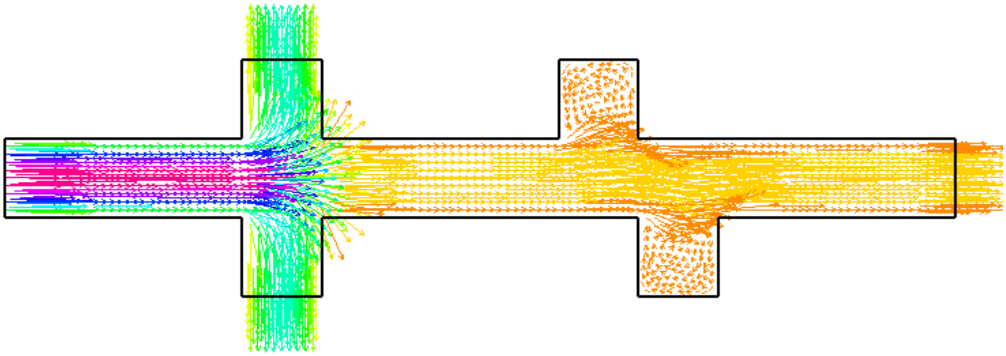

Figure 14. Computation of a Nash equilibrium for (25)–(26). The "free" parabolic profile for $Re = 1200$.

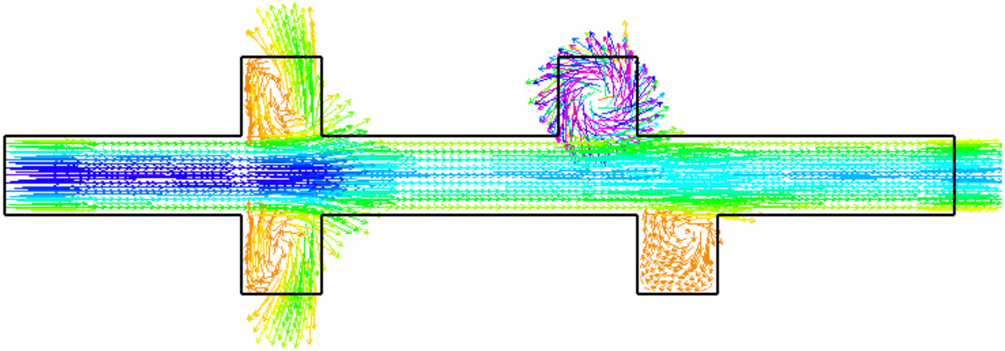

Figure 15. Test 2 - The final computed velocity field (Newton) for $Re = 1200$, $a = 1.99$, $\mu = 0.01$.

Conflicts of interest

The author has no conflict of interest to declare.

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