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The scientific legacy of Roland Glowinski / *L'héritage scientifique de Roland Glowinski*

A review of implicit algebraic constitutive relations for describing the response of nonlinear fluids

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Abstract. We review the class of implicit algebraic constitutive relations for fluids which includes in its ambit those whose material properties depend on the invariants of the stress, the symmetric part of the velocity gradient, as well as their mixed invariants. Such constitutive relations can describe the response of complex fluids whose material properties depend on the mechanical pressure, shear rate, etc. The class of models under consideration can describe the non-monotone relationship between the shear stress and the shear rate observed in experiments on colloids, as well as other novel response characteristics of non-Newtonian fluids. Constitutive relations for power-law fluids, generalized Stokesian fluids and Piezo-viscous fluids are special sub-classes of the class of fluids considered herein.

Keywords. implicit constitutive relations, colloids, stress power-law fluids, non-monotone stress-shear rate relations, monotone graphs.

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1. Introduction

The general class of Stokesian fluids[1] is described by providing an explicit expression for the stress as a function of the density and the symmetric part of the velocity gradient. The classical Navier–Stokes constitutive relation is a special sub-class of Stokesian fluids where the stress depends arbitrarily on the density and is linear in the symmetric part of the velocity gradient. The Euler fluid (see [1,2] which is a special sub-class of the Navier–Stokes fluid (see [3–7]) is characterized by the stress being given in terms of the density. The classical Navier–Stokes fluids, or for that matter the general Stokesian fluids, are incapable of describing the response characteristics of a very large class of fluids. Fluids that cannot be characterized by the Navier–Stokes fluids are referred

[1] By a Stokesian fluid we mean a fluid whose constitutive relation takes the form $\mathbf{T} = \mathbf{f}(\rho, \mathbf{D})$, where $\mathbf{T}$ is the Cauchy stress, $\mathbf{D}$ is the symmetric part of the velocity gradient and ρ is the density.

to as non-Newtonian[2] fluids and such fluids can be classified under three categories[3], fluids of the differential type, fluids of the rate type and fluids of the integral type (see [11–14]). Rate type constitutive theories are markedly different from Stokesian fluids in that the constitutive relation is an implicit relationship between the stress, its time derivatives, and kinematical measures such as the velocity gradients, and their time derivatives. Such constitutive relations have been in place for a long time. In the celebrated constitutive relation due to [15], which seems to be the first constitutive relation to describe viscoelastic fluids, while the stress and the time rate of the stress appear in the constitutive relation, it is not an implicit relation as the symmetric part of the velocity gradient can be expressed explicitly as a function of the stress and the time rate of stress. On the other hand, the constitutive relations proposed by [16] and [17] which relate the stress and its time rate and the symmetric part of the velocity gradient and its time derivative are implicit constitutive relations. Most of the implicit constitutive relations used in non-Newtonian mechanics to describe viscoelastic fluids are of the rate type, that is the relation is between the stress and its time rate, the symmetric part of the velocity gradient and its time rates. By an algebraic implicit constitutive relation, we mean an implicit relationship between just the stress and the symmetric part of the velocity gradient, with their rates not appearing in the relationship[4]. The general Stokesian constitutive relation is a special sub-class of these algebraic implicit constitutive relations wherein the stress is an explicit function of the symmetric part of the velocity gradient, and the classical power-law constitutive relation and the Navier–Stokes fluids are also in our terminology algebraic constitutive relations as they are in fact sub-classes of the Stokesian fluids.

There is a very important philosophical distinction underpinning implicit constitutive relations with respect to explicit constitutive relations like the Navier–Stokes constitutive relation as it is usually expressed, and this has to do with causality. Since, forces or stresses are the cause, and the deformation the effect, it is more meaningful to express the effect (kinematics) in terms of the cause (stress), rather than the cause (stress) in terms of the effect (kinematics). That is, in the case of the Navier–Stokes fluid, it would be philosophically more sensible to express the velocity gradient in terms of the stress rather than the stress in terms of the velocity gradient[5]. A much more general approach to the development of constitutive relations is to just assume that various relevant quantities are related by implicit relations[6] (see [18, 19] for a discussion of the role of causality in the development of constitutive relations).

[2]A fluid described by the Navier–Stokes constitutive relation is often times referred to as a Newtonian fluid, but this is misattributing credit to Newton for the works of Navier, Poisson, Stokes and St. Venant. The balance of linear momentum for a fluid described by the Navier–Stokes constitutive relation, is referred to as the Navier–Stokes equation. At the time of Newton, there was no notion of partial derivatives or balance laws for continua, and as pointed out by [8] it was James Bernoulli that first advocated the balance law for a continuum. Newton did not have a clear understanding of the constitutive relation that is now referred to as the Navier–Stokes fluid. In fact, [8] makes the statement "*Newton's theories of fluids are largely false*". As [9] points out Newton advanced two theories for describing the behavior of fluids, the first "*a schematic theory of fluids, which he considered to be formed of an aggregate of elastic particles, which repelled each other, were arranged at equal distances from each other, and were free*", and "*In the second theory the particles of the fluid are contiguous*". The two theories led to results to the resistance due to the translation of a solid cylinder in the fluid to differ by a factor of four.

[3][10] for a recent classification of incompressible fluids.

[4]Algebraic constitutive relations can be viewed as implicit differential type constitutive relations of order zero or rate type constitutive relations of order zero (that is no derivative appears in the constitutive relation).

[5]Doing so makes it apparent why assumptions like the "Stokes assumption", which has been clearly shown to be incorrect in numerous experiments but which researchers yet persist in appealing to, is inapt (see [18]). Also, in doing so, one can incorporate the constraint of incompressibility in a much more natural manner.

[6]In situations wherein one has numerous fields: electric field, magnetic field, thermal field, stress, strain, density, moisture, etc., we might only be in a position to conclude that all these fields are related and not know how any one of them depends on the others.

The celebrated gas law due to [20] that states that the product of the pressure and volume is a constant, that due to Charles that states that when pressure is held constant, the volume is directly proportional to the temperature ([21] attributes this famous result to Charles though Charles himself did not publish it), and the law due to [21] that states that when volume is kept fixed the pressure is directly proportional to the temperature (this law is attributed to Amontons) and the ideal gas law, are all essentially statements concerning an implicit relationship between pressure, volume and temperature of the gas. Based on which quantity is viewed as the effect, and which is viewed as the cause, the appropriate laws due to Boyle, Charles and Gay-Lussac are arrived at.

In order to describe stress relaxation that is observed in many real fluids, Maxwell, and later Burgers, Oldroyd and others developed rate type constitutive relations wherein the stress and its rate and the symmetric part of the velocity gradient and its rates are related. Fluids that exhibit stress relaxation can also be described by integral type constitutive relations. Fluids that exhibit stress relaxation are referred to as viscoelastic fluids. However, there are many fluids that do not exhibit viscoelasticity, but they cannot be described by the general class of Stokesian fluids. It transpires that several such fluids can be described by the class of algebraic constitutive relations. Examples of such fluids are those whose viscosity is pressure dependent (referred to as piezo-viscous fluids), and also colloids and slurries wherein one has non-monotonic relationship between the shear stress and shear rate in simple shear flows. The class of fluids that are describable by such algebraic constitutive relations are in fact much larger than the class wherein the stress is explicitly defined in terms of the kinematics, as the material moduli characterizing such fluids depend on the invariants of stress, invariants of the appropriate kinematical quantities and also mixed invariants of the stress and appropriate kinematical quantities. Such constitutive relations can also be the embarking point for the development of constitutive relations to describe turbulence wherein the invariants of the fluctuations of stress, shear rate and the mixed invariants might play a role in the response of the fluid.

The organization of the paper is as follows. After introducing the kinematics in the next section, incompressible fluids defined through implicit constitutive relations are discussed in Section 3. Implicit algebraic constitutive relations and stress-power law fluids are discussed in Sections 4 and 5, respectively, and Section 6 is devoted to a discussion of a class of implicit constitutive relations that have been shown to describe the response of colloidal solutions. Fluids with pressure dependent viscosity are considered in Section 7 and fluids with viscosity that is dependent on both pressure and shear rate are treated in Section 8. In Section 9 we provide a brief discussion of constitutive equations specified in terms of monotone graphs. The final Section 10 is dedicated to a discussion of some interesting new applications for implicit constitutive relations.

2. Preliminaries

Let $\mathfrak{B}$ denote a body. By a placer κ, we mean a one-to-one mapping

$$\kappa : \mathfrak{B} \to \mathfrak{E} \tag{1}$$

where $\mathfrak{E}$ denotes an Euclidean space of dimension three. $\kappa(\mathfrak{B})$ is referred to as a configuration of the body in the three-dimensional Euclidean space. By the motion of a body, one refers to a one-parameter family of placers κ_t, $t \in \mathbb{R}$, $\mathbb{R}$ being the set of real numbers and the parameter being time. The members $\mathscr{P} \in \mathfrak{B}$ are called particles of the abstract body. For $\mathscr{P} \in \mathfrak{B}$, let $\mathbf{X} := \kappa_{\mathfrak{R}}(\mathscr{P})$, where $\kappa_{\mathfrak{R}}$ is one of the one parameter family of placers that we shall identify as a reference placement of the body, with $\kappa_{\mathfrak{R}}(\mathfrak{B})$ being identified as a reference configuration. Let $\mathbf{x} := \kappa_t(\mathscr{P})$, where κ_t is the placement at time t, which we shall identify with the current placement of the body. We shall refer to $\kappa_t(\mathfrak{B})$ as the configuration of the body at time t or as the current configuration of the body.

With the one parameter family of placers $\{\kappa_{\mathfrak{R}}, \cdots, \kappa_t, \cdots\}$, and hence the corresponding one parameter family of configurations $\{\kappa_{\mathfrak{R}}(\mathfrak{B}), \cdots, \kappa_t(\mathfrak{B}), \cdots\}$, we can identify a one to one mapping $\chi_{\kappa_{\mathfrak{R}}}$ which is defined through

$$\chi_{\kappa_{\mathfrak{R}}} : \kappa_{\mathfrak{R}}(\mathfrak{B}) \times \mathbb{R} \to \mathfrak{E} \tag{2}$$

which takes the reference configuration to the configuration that it occupies at time t, such that

$$\mathbf{x} = \chi_{\kappa_{\mathfrak{R}}}(\mathbf{X}, t). \tag{3}$$

Let

$$\xi = \chi_{\kappa_{\mathfrak{R}}}(\mathbf{X}, \tau) \tag{4}$$

denote the position occupied by the point $\mathbf{X}$ at time τ. Since the mapping $\chi_{\kappa_{\mathfrak{R}}}$ is one to one, we can express ξ as

$$\xi = \chi_{\kappa_{\mathfrak{R}}}(\mathbf{X}, \tau) = \chi_{\kappa_{\mathfrak{R}}}\left(\chi^{-1}_{\kappa_{\mathfrak{R}}}(\mathbf{x}, t), \tau\right) = \chi_{\kappa_t}(\mathbf{x}, \tau), \tag{5}$$

and χ_{κ_t} is referred to as the "Relative Motion" (see [22]).

The velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are defined through

$$\mathbf{v} = \frac{\partial \chi_{\kappa_{\mathfrak{R}}}}{\partial t}, \quad \mathbf{a} = \frac{\partial^2 \chi_{\kappa_{\mathfrak{R}}}}{\partial t^2} \tag{6}$$

and the velocity gradient $\mathbf{L}$, and its symmetric part $\mathbf{D}$ and its skew part through

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \quad \mathbf{D} = \frac{1}{2}\left[\mathbf{L} + \mathbf{L}^T\right], \quad \mathbf{W} = \frac{1}{2}\left[\mathbf{L} - \mathbf{L}^T\right]. \tag{7}$$

A general fluid of the differential type of order n is defined through[7] (see [11])

$$\mathbf{T} = \mathbf{f}\left(\rho, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n\right), \tag{8}$$

where ρ is the density and

$$\mathbf{A}_1 = \left[\mathbf{L} + \mathbf{L}^T\right] = 2\mathbf{D} \tag{9}$$

and $\mathbf{A}_n$ is given by the recursive relation

$$\mathbf{A}_n = \frac{d}{dt}\mathbf{A}_{n-1} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^T\mathbf{A}_{n-1} \tag{10}$$

The tensors $\mathbf{A}_n$, $n = 1, 2, \ldots$ are referred to as Rivlin–Ericksen tensors (see [24]).

Differential type fluids described by (8) are incapable of exhibiting stress relaxation and instantaneous elastic response. While most researchers in the field refer to them as viscoelastic fluids, they are not viscoelastic fluids as we shall identify the ability to stress relax and possess the capability for instantaneous elastic response as characteristics of viscoelastic fluids. Fluids of the differential type are elasticoviscous fluids. They are essentially viscous fluids with a smidgen of elastic behavior.

If one starts with the assumption that we have a fluid of the differential type of complexity 1, that is a Stokesian fluid

$$\mathbf{T} = \mathbf{f}\left(\rho, \mathbf{A}_1\right), \tag{11}$$

and require that the stress is linear in $\mathbf{A}_1$, then we would obtain the constitutive expression

$$\mathbf{T} = -p_{th}(\rho)\mathbf{I} + \lambda(\rho)(\mathrm{tr}\mathbf{D})\mathbf{I} + 2\mu(\rho)\mathbf{D}, \tag{12}$$

the compressible Navier–Stokes fluid. In the above expression, p_{th} is the thermodynamic pressure and $\lambda(\rho)$ and $\mu(\rho)$ are the bulk and shear viscosities.

[7] One can define more general class of fluids referred to as fluids of the differential type of Complexity (m,n) but this would require us to introduce a thermodynamic framework in which to cast the constitutive relations. We shall not do so here, the interested reader can find an extended discussion in [23].

3. Incompressible fluids defined through implicit constitutive relations

We shall primarily be interested in discussing the response of incompressible fluids. The stress in an incompressible fluid of complexity n has the representation

$$\mathbf{T} = -p\mathbf{I} + \mathbf{f}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n), \tag{13}$$

where p denotes the indeterminate part of the stress due to the constraint of incompressibility. Since an incompressible fluid can undergo only isochoric motions, it follows that

$$\mathrm{div}\mathbf{v} = \mathrm{tr}\mathbf{D} = 0. \tag{14}$$

A special subclass of these fluids that has attracted a lot of attention are fluids of grade n (see [11]). As these fluid models are used to describe the response of dilute polymeric liquids which are idealized as being incompressible, the constitutive relations are those that model incompressible fluids. An important sub-class of fluids of the differential type are Stokesian fluids wherein the stress is expressed in the form

$$\mathbf{T} = -p\mathbf{I} + \mathbf{f}(\mathbf{A}_1), \tag{15}$$

where $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility. The constitutive relations due to [25] and [26] are special sub-classes of the constitutive relation (15).

The classical incompressible Navier–Stokes fluid is given by the constitutive relation

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}, \tag{16}$$

where μ, the viscosity is a constant. However, Stokes was well aware that viscosity was not constant but dependent on the pressure as these remarks of his make evident ([7]): "*. . . If we suppose μ to be independent of the pressure. . .*" and "*Let us now consider in what cases it is allowable to suppose μ to be independent of the pressure*".

[27][8] had proposed an exponential dependence of the viscosity on pressure as shown below:

$$\mathbf{T} = -p\mathbf{I} + 2\mu(p)\mathbf{D}, \tag{17}$$

where

$$\mu(p) = \mu_0 \exp(\alpha p), \alpha > 0. \tag{18}$$

Recent experiments by [30] and others shows that the viscosity increases to much greater values than previously thought. For Napthalmic mineral oil, a change of density of 16% (computed using a correlation developed by [31]) can be accompanied by a change in viscosity of approximately 5×10^8%, and thus in comparison to the change in viscosity due to pressure, the changes in density can be neglected (see [19] for details regarding the same). Fluids whose viscosity are pressure dependent and approximated as being incompressible are described by constitutive relations of the form:

$$\begin{aligned} \mathbf{T} &= -p\mathbf{I} + \mathbf{S}, \\ \mathbf{S} &= 2\mu(p)\mathbf{D}, \end{aligned} \tag{19}$$

where $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility, and $\mathbf{S}$ is the deviatoric part of the stress. It follows from (14), on taking the trace of (19), that

$$p = -\frac{1}{3}\mathrm{tr}\mathbf{T}. \tag{20}$$

This immediately implies that (17) is no more an explicit expression for the Cauchy stress $\mathbf{T}$ in terms of $\mathbf{D}$, but it can however be re-written to express the symmetric part of the velocity gradient

[8] The terminology "pressure" is one of the most abused terminologies in fluid mechanics as it is used to define a variety of quantities that are quite distinct (see [28, 29]). Here, by pressure we mean the mean normal stress.

in terms of the stress. We notice that the Cauchy stress and the symmetric part of the velocity gradient are related through

$$\mathbf{T} - \frac{1}{3}(\mathrm{tr}\mathbf{T})\mathbf{I} = 2\mu\left(-\frac{1}{3}\mathrm{tr}\mathbf{T}\right)\mathbf{D}. \tag{21}$$

Suppose the viscosity also depends on the second invariant of the symmetric part of the velocity gradient in the above constitutive relation, then the viscosity will be $\mu(\mathrm{tr}\mathbf{T}, |\mathbf{D}|^2)$ and will lead, in our terminology to an implicit algebraic constitutive relation. In general, neither the stress nor the symmetric part of the velocity gradient can be expressed as an explicit function of the other.

The constitutive relation (12) is an explicit expression for the stress, it expresses the stress as a function of the density and the symmetric part of the velocity gradient, and the constitutive relation (13) is also an explicit expression for the deviatoric stress in terms of the symmetric part of the velocity gradient. We shall be interested in the class of implicit constitutive relations for incompressible fluids wherein the Cauchy stress and the symmetric part of the velocity gradient are related through:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},$$
$$\mathbf{f}(\mathbf{S}, \mathbf{D}) = \mathbf{0}.$$

Such constitutive relations have several interesting applications in the field of elastohydrodynamics, and in modeling colloids, slurries, etc. We shall discuss these applications in what follows.

4. Implicit Algebraic Constitutive Relations

As mentioned earlier, implicit constitutive relations wherein one has an implicit relationship between the stress and its frame indifferent time derivatives and the symmetric part of the velocity gradient and its various frame-indifferent time derivatives, have been in use for a long time. However, constitutive relations wherein no derivative of the stress or the symmetric part of the velocity gradient appear but where just the stress and symmetric part of the velocity gradient are related implicitly were not considered despite the fact that such constitutive relations would be the natural ones to use to describe materials with pressure dependent viscosity, amongst many other applications.

We are interested in studying the class of implicit algebraic constitutive relations for incompressible fluids defined through (22). The assumption that the fluid is isotropic leads to (see [32])

$$\alpha_0\mathbf{I} + \alpha_1\mathbf{S} + \alpha_2\mathbf{D} + \alpha_3\mathbf{S}^2 + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{DS} + \mathbf{SD}) + \alpha_6\left(\mathbf{S}^2\mathbf{D} + \mathbf{DS}^2\right) + \alpha_7\left(\mathbf{SD}^2 + \mathbf{D}^2\mathbf{S}\right) + \alpha_8\left(\mathbf{S}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{S}^2\right) = \mathbf{0}, \tag{22}$$

where the material functions α_i, $i = 0, \ldots, 8$ depend on the invariants

$$\mathrm{tr}\mathbf{S}, \mathrm{tr}\mathbf{S}^2, \mathrm{tr}\mathbf{D}^2, \mathrm{tr}\mathbf{S}^3, \mathrm{tr}\mathbf{D}^3, \mathrm{tr}(\mathbf{SD}), \mathrm{tr}\left(\mathbf{S}^2\mathbf{D}\right), \mathrm{tr}\left(\mathbf{SD}^2\right), \mathrm{tr}\left(\mathbf{S}^2\mathbf{D}^2\right). \tag{23}$$

Before discussing fluids belonging to the class of incompressible fluids defined by (22) and (23), we shall discuss a class of compressible fluids that has been recently introduced by [33], namely implicit relations that are a generalization of the classical Euler and Korteweg fluids (see [1, 2, 34]). First, let us consider the implicit constitutive relation

$$\mathbf{f}(\rho, \mathbf{T}) = \mathbf{0}, \tag{24}$$

where ρ is the density. We notice that the above is a generalization of the classical Euler fluid. It can be shown that many constitutive relations that belong to the class (24) can describe the simple static solution exhibited by the Euler fluid that is at rest in the presence of gravity. One is confronted with the question whether one of these constitutive relations might prove useful in describing specific flows of perfectly elastic fluids which the classical Euler fluid is incapable of doing so.

Yet another generalization of a compressible fluid is the implicit constitutive relation

$$\mathbf{f}\left(\rho, \frac{\partial \rho}{\partial x}\mathbf{T}\right) = \mathbf{0}, \tag{25}$$

which is a a generalization of the constitutive relation due to [34]. It follows from representation theorems that

$$\alpha_1 \mathbf{I} + \alpha_2 \mathbf{T} + \alpha_3 \left(\frac{\partial \rho}{\partial \mathbf{x}} \otimes \frac{\partial \rho}{\partial \mathbf{x}}\right) + \alpha_4 \mathbf{T}^2 + \alpha_5 \left(\frac{\partial \rho}{\partial \mathbf{x}} \otimes \mathbf{T}\frac{\partial \rho}{\partial \mathbf{x}} + \mathbf{T}\frac{\partial \rho}{\partial \mathbf{x}} \otimes \frac{\partial \rho}{\partial \mathbf{x}}\right) + \alpha_6 \left(\frac{\partial \rho}{\partial \mathbf{x}} \otimes \mathbf{T}^2\frac{\partial \rho}{\partial \mathbf{x}} + \mathbf{T}^2\frac{\partial \rho}{\partial \mathbf{x}} \otimes \frac{\partial \rho}{\partial \mathbf{x}}\right) = \mathbf{0}, \tag{26}$$

where the material moduli $\alpha_i, i = 1, \cdots, 6$ depend on the density ρ and the following invariants:

$$\mathrm{tr}\mathbf{T}, \mathrm{tr}\mathbf{T}^2, \mathrm{tr}\mathbf{T}^3, \mathrm{tr}\left(\frac{\partial \rho}{\partial \mathbf{x}} \otimes \frac{\partial \rho}{\partial \mathbf{x}}\right), \mathrm{tr}\left(\frac{\partial \rho}{\partial \mathbf{x}} \otimes \mathbf{T}\frac{\partial \rho}{\partial \mathbf{x}}\right), \mathrm{tr}\left(\frac{\partial \rho}{\partial \mathbf{x}} \otimes \mathbf{T}^2\frac{\partial \rho}{\partial \mathbf{x}}\right). \tag{27}$$

We shall not discuss these implicit relations further and refer the reader to [33] for the same.

A special sub-class of the constitutive relations described by (22) is the constitutive class wherein the symmetric part of the velocity gradient is expressed explicitly in terms of the stress in the following manner

$$\mathbf{D} = \mathbf{g}(\mathbf{S}). \tag{28}$$

The above constitutive expression is not an implicit relation between the stress and the symmetric part of the velocity gradient, it is an explicit expression for the symmetric part of the velocity gradient. If the expression (28) is invertible, then we recover the classical incompressible Stokesian fluid. However, it is possible that (28) is not invertible and such constitutive expressions are new and were only considered after the introduction of implicit equations by [18] as special subclasses of the implicit constitutive equations (22). Even when (28) is invertible, expressing the Stokesian fluid as (28) provides us a new perspective to such constitutive relations.

We notice that the classical incompressible Navier–Stokes constitutive relation can be expressed as

$$\mathbf{D} = \frac{1}{2\mu}\mathbf{S}. \tag{29}$$

A comment concerning the compressible Navier–Stokes constitutive relation is warranted at this juncture. Instead of starting from stress being a function of the density and the symmetric part of the velocity gradient if one were to assume that

$$\mathbf{D} = \mathbf{f}(\rho, \mathbf{T}), \tag{30}$$

then, one arrives at the constitutive relation for the compressible Navier–Stokes fluid in the form

$$\mathbf{D} = \beta_1(\rho)\mathbf{I} + \beta_2(\rho)(\mathrm{tr}\mathbf{T})\mathbf{I} + \beta_3(\rho)\mathbf{T}, \tag{31}$$

where $\beta_1(\rho)$, $\beta_2(\rho)$, and $\beta_3(\rho)$ are material functions. The important consequence of the representation (31) is that one would forthwith arrive at the conclusion that (see [35] for details)

$$3\beta_2(\rho) + \beta_3(\rho) \neq 0, \tag{32}$$

which would immediately imply that the Stokes assumption is untenable, as substantiated by numerous experiments (see [36–39]). Stokes himself had serious doubts about the validity of his assumption. Unfortunately, an erroneous "proof" for the assumption offered by Maxwell gave credibility to the incorrect assumption (see [40]).

It is worth observing that expressing the symmetric part of the velocity gradient in terms of the Cauchy stress allows one to express the constraint of incompressibility in a natural manner without having to introduce a Lagrange multiplier. This is also true with regard to the general implicit theory for a variety of constraints. For example, the incompressible Navier–Stokes fluid expressed as

$$\mathbf{D} = \frac{1}{3}\left\{\mathbf{T} - \frac{1}{3}(\mathrm{tr}\mathbf{T})\mathbf{I}\right\}, \tag{33}$$

automatically satisfies the constraint that the fluid can undergo only isochoric motion. We have not introduced a Lagrangian multiplier "p" which in fact happens to be the mean normal stress. We now have to solve the balance of mass, the balance of linear momentum and the constitutive relation, ten partial differential equations for the density, the components of the velocity and the components of the stress, that is ten unknown scalars (since the stress is symmetric), simultaneously. In using the expression (33), in conjunction with the balance of mass and the balance of linear momentum it might seem like we have to solve a larger system of equations for more unknowns than is usual when one substitutes the constitutive relation for the stress into the balance of linear momentum to obtain an equation for the velocity and the pressure, and the equation due to the constraint of incompressibility, namely that the divergence of the velocity field is zero. However, the standard approach increases the order of the balance of linear momentum to a second order partial differential equation for the velocity field, while solving for the balance of mass, the balance of linear momentum and the constitutive (33) is a system of first order equations. This immediately has implications regarding the smoothness of solutions as well as the boundary conditions that are required.

5. Stress Power-Law Fluids

Many fluids shear-thin and shear-thicken, but do not exhibit significant stress-relaxation or normal stress differences in simple shear flows, and such fluids are usually described quite adequately by the classical power-law fluid wherein the Cauchy stress is expressed as a power-law function of the symmetric part of the velocity gradient (constitutive relations due to [41–44] and others). A counterpart to such power-law fluids are stress power-law fluids wherein the symmetric part of the velocity gradient is a power-law function of the stress (see [45]). We can only invert the stress power-law constitutive relation to obtain the classical power-law relation for a certain range of power-law values. A modification of the stress power-law model for which there is no classical equivalent in that the stress can not be expressed as a function of the symmetric part of the velocity gradient was introduced by [46], which we will discuss later, can be used to describe the response of colloidal solutions.

Recently [47] introduced a power-law constitutive relation that mimics the response of the viscoplastic fluids in that the viscosity (resistance to flow) blows up, as the shear rate tends to zero, while the shear stress is yet finite. The Cauchy stress in their fluid takes the form

$$\mathbf{T} = -p\mathbf{I} + \left\{\frac{\alpha_1}{\|\mathbf{D}\|} - \frac{\alpha_1 \exp\left(\alpha_2\|\mathbf{D}\|\right)}{\|\mathbf{D}\|}\right\}\|\mathbf{D}\|, \tag{34}$$

where α_1 and α_2 are positive constants, and $\|\mathbf{D}\| = (\mathrm{tr}\mathbf{D}^2)^{1/2}$. They determined the constants α_1 and α_2 for various non-Newtonian fluids such as Kaolin-water, meat extract, paint, etc., by corroborating against available experimental data. The ability of the constitutive relation to mimic viscoplastic fluids stems from the viscosity blowing up at zero shear rate. Recently, some boundary value problems have been studied within the context of (34) (see [48, 49]).

The constitutive relation for the stress power-law fluid takes the form

$$\mathbf{D} = \alpha\left\{\left[1+\beta\left\|\mathrm{tr}\mathbf{S}^2\right\|\right]^n\right\}\mathbf{S};\quad \mathbf{T} = -p\mathbf{I}+\mathbf{S}, \tag{35}$$

where $-\infty < n < \infty$, $\alpha > 0$, $\beta > 0$.

The fluid described by the constitutive relation (35) is an incompressible fluid as $\mathrm{tr}\mathbf{D} = \mathbf{0}$. The relationship between the shear stress and shear rate for a classical power-law fluid and a stress power-law fluid for a simple shear flow is portrayed in Figures 1 and 2. We notice that the response depicted in Figure 2, when $n < \frac{-1}{2}$ is not possible within the context of the classical power-law fluid. [45] have studied a fluid described by (35). We also notice that when $n = \frac{-1}{2}$, the fluid exhibits a limiting shear-rate, that is the fluid can never be made to flow with a higher shear-rate, however high the applied shear stress.

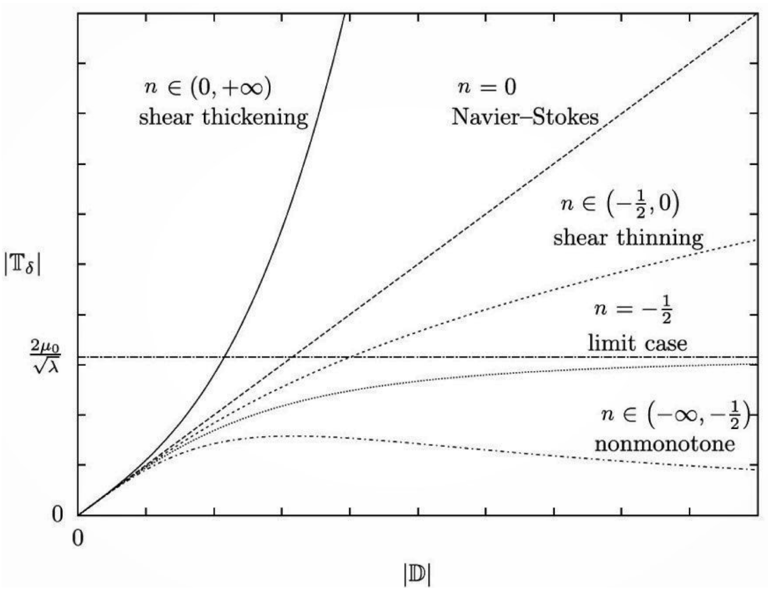

Figure 1. Qualitative behavior of the classical power-law model (from [45, Figure 2 (a)]). $\mathbf{T}_\delta$ denotes the deviatoric part of the stress and $\mathbf{D}$ is the symmetric part of the velocity gradient.

In simple shear flow, the constitutive relation for a fluid described by (35) reduces to

$$\kappa = \alpha\left\{\left[1+\beta\tau^2\right]^n\right\}\tau, \tag{36}$$

where κ is the shear rate and τ is the shear stress. $\alpha\{[1+\beta\tau^2]^n\}$ is the inverse of the viscosity of the fluid and is usually referred to as the "fluidity" of the fluid. We notice that in simple shear flow, the generalized fluidity α_g is given by

$$\alpha_g(\tau) = \alpha\left[1+\beta\tau^2\right]^n \tag{37}$$

where $\alpha_g(\tau)$ is the generalized fluidity. When $n > 0$, the generalized fluidity at zero shear stress is α, and $\kappa(0) = 0$. The generalized fluidity tends to infinity as the shear stress tends to infinity as does κ. We notice that if $n < 0$, then

$$\alpha_g(0) = \alpha \quad \text{and} \quad \alpha_g(\tau) \to 0 \,\text{as}\, \tau \to \infty. \tag{38}$$

Thus, in the limit of zero shear stress we have the generalized fluidity tends to a finite value. However, while $\alpha_g(\tau) \to 0 \,\text{as}\, \tau \to \infty$ when $n < 0$, the shear rate can tend to a finite value for other

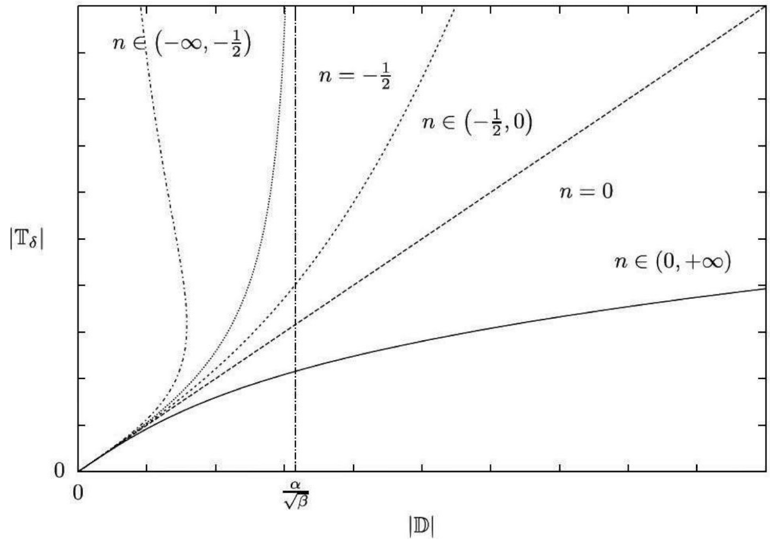

Figure 2. Qualitative behavior of a typical power-law fluid.

values of n. For instance, if $n = \frac{-1}{2}$, $\kappa(0) = 0$, and we have a limiting strain rate $\frac{1}{(\beta)^{1/2}}$ as $\tau \to \infty$. If $n < \frac{-1}{2}$, $\kappa = \alpha\{[1+\beta\tau^2]^n\}\tau$ is such that

$$\kappa(0) = 0 \quad \text{and} \quad \left(\alpha\left\{\left[1+\beta\tau^2\right]^n\right\}\tau\right) \to 0 \, \text{as} \, \tau \to \infty. \tag{39}$$

Such a response is not possible in a classical power-law fluid. A modification of this response allows us to develop a constitutive relation to describe the response of colloids. Before turning to this discussion, we should briefly discuss another stress power-law model that exhibits completely different behavior though it looks very similar in form to (35). Consider the constitutive relation

$$\mathbf{D} = \alpha\left\{1 + \left[\beta \left\|\mathrm{tr}\mathbf{S}^2\right\|\right]^n\right\}\mathbf{S}; \quad \mathbf{T} = -p\mathbf{I} + \mathbf{S}. \tag{40}$$

The slight modification to the constitutive relation leads to a markedly different response. In simple shear flow, we find that

$$\alpha_g f(\tau) = \alpha\left\{1 + \left[\beta\tau^2\right]^n\right\}, \tag{41}$$

where $\alpha_g(\tau)$ is the generalized fluidity. We notice that if $n < 0$, then $\alpha_f(0) = \infty$, and $\alpha_f(\tau) \to \alpha \, \text{as} \, \tau \to \infty$

Next, since

$$\kappa = \alpha\left\{1 + \left[\beta\tau^2\right]^n\right\}\tau, \tag{42}$$

we conclude that when $n < -\frac{1}{2}$, $\kappa(0) = \infty$, and $\kappa(\tau) \to \infty$ as $\tau \to \infty$. If $0 < n < -\frac{1}{2}$, $\kappa(0) = 0$, and $\kappa(\tau) \to \infty$ as $\tau \to \infty$. Notice that when $n > 0$, the generalized fluidity at zero shear stress is a finite value α while the fluidity $\alpha_f(\tau) \to \infty \, \text{as} \, \tau \to \infty$. Also, $\kappa(0) = 0$, and $\kappa(\tau) \to \infty$ as $\tau \to \infty$.

[45] studied several simple flows within the context of the constitutive relation (35). They studied Plane Couette, Plane Poiseuille, Hagen–Poiseuille and cylindrical Couette flow, and the solutions that they find are markedly different from the classical solution for these problems. Later [50]) studied the flow of such fluids between two plates rotating with the same angular speed about non-coincident axis. They also found solutions that are very different from that for a Navier-Stokes fluid with pronounced boundary layers developing in the case of a stress power-law fluid while no such boundary layer manifests itself in the case of the Navier–Stokes fluid. [51] have examined the flow of a stress power-law fluid down an inclined plane, while [52]

analysed a class of unsteady flows. Recently, [53] studied stress power-law fluids undergoing squeeze flow. Since all these investigations indicate that the solutions with regard to stress power-law fluids can be distinctly different from that for the Navier–Stokes and classical power-law fluids, for a range of power-law parameter values, it would be worthwhile solving other boundary value problems as they might explain phenomena that cannot be explained by the classical Navier–Stokes or the usual class of power-law fluids.

6. A constitutive Relation to describe the response of colloidal solutions

[46] modified the constitutive relation (35) in the following manner:

$$\mathbf{D} = \alpha\{\left[1 + \beta \| \mathrm{tr}\mathbf{S}^2 \| \right]^n + \gamma\}\mathbf{S}; \quad \mathbf{T} = -p\mathbf{I} + \mathbf{S}. \tag{43}$$

In simple shear flow, the shear-rate versus shear-stress changes sign twice depending on the values of the material parameters (see Figure 3). [46] felt such constitutive relations could be used to describe the flow characteristics in materials that are modeled as fluids, but which have an internal structure or constituents (such as blood or other biological fluids, slurries, colloids) that can change their characteristics due to the shear rate. For instance, in blood at different shear rates one can have rouleaux forming and breaking whereby the response changes. Similarly, in colloids which are essentially molecules of one substance being dispersed through a second substance, one can have coalescence and breakup of the molecules of the first substance. Their hypothesis seems to have been justified as the constitutive relation (43) has been shown to describe the behavior of colloidal (see [54–59] for experimental results on colloids). [60] have shown that the constitutive relation (43) fits the complex experimental data of [56] for simple shear flow of colloids exceptionally well (see Figures 4 and 5). None of the popular models available in the non-Newtonian fluid mechanics literature come close to fitting such data as the derivative of the stress-shear rate curve changes sign more than once. Recently, [61] studied mathematical issues concerning the equations governing the flow of a fluid described by (43). They established the existence of weak solutions and on making additional assumptions were able to establish uniqueness of those solutions. Much remains to be done with regard to establishing both rigorous mathematical results as well as the study of interesting initial-boundary value problems whose results can be corroborated against available experimental results.

[62] studied the flow of a fluid described by (43) in between two rotating cylinders, the classic cylindrical Couette flow problem, and between parallel plates with a blockage. Since one value of the shear-rate can be engendered by several values of the shear-stress, this presents difficulties with regard to determining numerical solutions to the governing equations. They found interesting flow patterns that are very distinct from those for the classical Navier–Stokes fluid. Recently, [63] considered the flow of a fluid described by (43) flowing past a porous plate and were able to find an exact solution to the problem and they also studied the stability of such solutions. They found that suction stabilizes the flow. Since the constitutive relation (43) is not only capable of describing flows of colloids observed by [56], but is also on sound footing and could possibly describe other phenomena associated with colloids as well as slurries, the model and variants of the same warrant scrutiny.

7. Fluids with pressure dependent viscosity

As we have discussed in Section 3, there has been a great deal of work concerning the flows of fluids with pressure dependent viscosity[9], wherein the fluid is approximated as an incompress-

[9] We shall restrict our discussion to incompressible fluids that have a pressure dependent viscosity.

ible fluid in virtue of the fact that the changes in viscosity due to the increase in pressure is several orders of magnitude greater than the changes of the density of the liquid due to the increase in pressure. The constitutive relations for such fluids are given through (17). While the viscosity suggested by [27], namely (18) is a popular choice, other forms of pressure dependence of viscosity have been used in the literature. Another popular assumption is that the viscosity depends linearly on the pressure.

To date, there is no global existence theory for both steady and unsteady flows of fluids whose viscosity depends purely on the pressure. Previous studies by [64, 65] addressed either existence of solutions for small data for short time or under the conditions namely $\frac{\nu(p)}{p} \to 0$ as $p \to \infty$, but this condition is clearly contradicted by experiments. A detailed discussion of many of fluids whose viscosity depends on pressure can be found in [66].

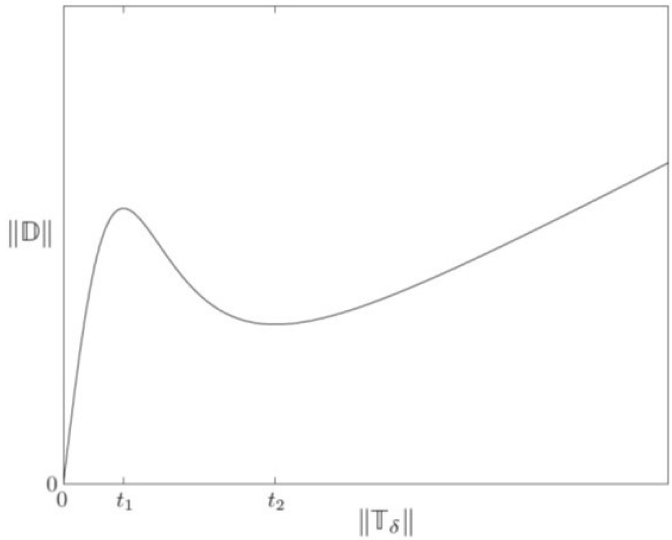

Figure 3. Qualitative response of the constitutive relation (43). t_1 and t_2 are values of the norm of the deviatoric stress where the tangent to the response function is zero.

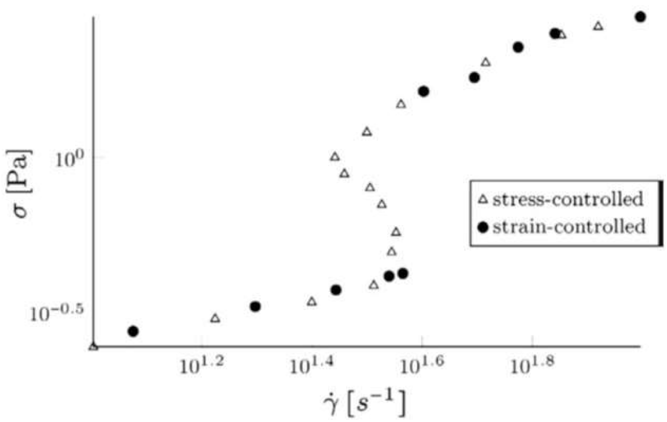

Figure 4. Shear stress versus shear rate in the experiments of [56].

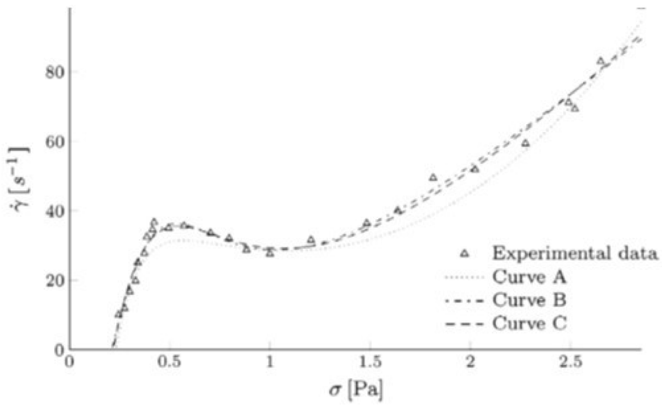

Figure 5. Fit of experimental data of [56] by [60]. The qualitative nature of the curve C belongs to precisely the same class of models introduced by [46].

[67] studied the flow of such fluids in pipes and they found that the velocity profiles can be markedly different from those for the classical Navier–Stokes fluid[10]. This study was followed by that due to [70] for the flow between two plates rotating with the same angular velocity about distinct axes, in the presence of gravity. Due to the variation of the pressure due to gravity, and the dependence of viscosity on pressure, they show that very pronounced boundary layers can develop. [71] studied the flow of such a fluid due to the rolling of a rigid cylinder on an elastic cylinder, a boundary value problem that has relevance to the flows in elastohydrodynamics. [72–78] and several others have studied specific boundary value problems which have relevance to several problems in elastohydrodynamic lubrication.

There is a very important distinction between the classical Navier–Stokes fluid whose viscosity is a constant and a fluid whose viscosity depends on the pressure, that requires careful discussion. The structure of the flow of a classical Navier–Stokes fluid in a pipe depends only on the pressure difference between the inlet and outlet (the pressure gradient) and not on the absolute value of the pressure. On the other hand, in the flow due to a fluid whose viscosity depends on the pressure, the value of the viscosity depends on the value of the pressure and it could be different by orders of magnitude. While the pressure field can be determined by fixing the value at one point, it is better to fix it based on a mean value (see the discussion in [79]).

8. Fluids with pressure and shear dependent viscosity

The viscosity of many fluids not only depends on the pressure, it can also depend on the shear-rate. It turns out that if the dependence of the viscosity on the shear rate has a certain mathematical structure, then one can establish interesting global existence of weak solutions to the equations governing the flows of such fluids. [67, 80, 81] established existence of global weak solution for unsteady flows, but their assumptions for the dependence of the viscosity on the pressure is contradicted by experiments. [82] obtained numerical solutions for a fluid with pressure and shear dependent viscosity.

[83] proved the existence of weak solutions for steady flows, and [84, 85] extended these results for flows in bounded domains when the fluid obeys Navier slip on the boundary. Establishing global existence of solutions for viscosities that have a dependence on the pressure and viscosity

[10]As pointed out by [68], one of the many solutions found by [69] is physically unacceptable as the pressure field corresponding to the solution is not continuous (see the discussion in [69] concerning the same).

which is qualitatively consistent with experimental results is an interesting open problem that deserves study, especially in view of its relevance to important technological problems such as elastohydrodynamics. We recall that in fluids with pressure and shear rate dependent viscosity, the Cauchy stress is given through:

$$\mathbf{T} = -p\mathbf{I} + 2\mu\left(p, |\mathbf{D}|^2\right)\mathbf{D}, \tag{44}$$

with tr$\mathbf{D} = \mathbf{0}$. Thus, we have

$$\mathbf{T} = -\frac{1}{3}(\mathrm{tr}\mathbf{T})\mathbf{I} + 2\mu\left(\frac{1}{3}\mathrm{tr}\mathbf{T}, |\mathbf{D}|^2\right)\mathbf{D}. \tag{45}$$

[86] studied the flow of a fluid whose viscosity depends on the pressure and shear rate in an infinite pipe and were able to establish explicit exact solutions. Special flows between parallel plates for such fluids have been studied by [82] in pipes, flows past slots and projections, and other technologically relevant flow domains. They found that the fluid exhibits distinctly different flow characteristics from the classical Navier–Stokes fluid. [79] also studied the flow of the fluid between two flat plates using a kinematic viscosity of the form:

$$\nu = \nu(p, \mathbf{D}) = \alpha p |\mathbf{D}|^{r-2}, \tag{46}$$

where α and r are constants. They were able to show that the equations governing the flows of such fluids exhibit multiplicity of solutions that are not possible in classical Navier–Stokes fluids or classical Power-Law fluids. For example, when $r = \frac{3}{2}$, three solutions are possible of which one is that which occurs in a Navier–Stokes fluid. [79] also establish an explicit exact solution for the flow of a fluid between plates, when the viscosity depends exponentially on the pressure, the flow taking place in the presence of gravity.

9. Implicit Constitutive Relations Characterized by Maximal Graphs

Careful mathematical studies concerning fluids described by implicit constitutive relations characterized by maximal monotone (possibly multivalued) graphs have been carried out by [87–89].

Implicit constitutive relations have also been used to describe generalizations of the response of dilute polymeric liquids that are obtained using a kinetic theory approach. Such generalizations that have been considered also lead to maximal monotone graphs. [90] have established the existence of global weak solutions for the flows of such fluids (see [90]).

10. A possible application for fluids defined through implicit constitutive relations

The most important open problem in fluid mechanics is the problem of turbulence. One phenomenological approach to turbulence within the context of continuum mechanics is the approach wherein one assumes a mean velocity and mean pressure and fluctuations for the velocity and pressure from the mean while writing down the Navier–Stokes equation. This introduces a closure problem wherein one needs to provide a constitutive relation for the product of the density and the tensor product of the fluctuating velocity, referred to in the literature as “Reynolds stress”, with closure being accomplished by assuming a constitutive relation for the fluctuations. Boussinesq was the first to address this closure problem and introduced the concept of eddy viscosity ([91]). Various other closure models have been proposed such as the $\kappa-\epsilon$, $\kappa-\omega$, and other models. As it is the stresses that cause changes in the velocity, the fluctuations in the stress ought to cause fluctuations in the velocities. Thus, it does not seem appropriate to just consider the fluctuations in the velocity and not consider the fluctuations in its cause, the stress. It is also peculiar that fluctuations are considered for the pressure, just the mean normal stress, but not the individual components of the stress, including the shear stress components. It would be more appropriate to consider implicit constitutive relations wherein one assumes a mean value for the stress as well as the velocity and admit fluctuations to both. Starting with algebraic implicit constitutive relations that include the mean values for the stress and velocity and allowing for fluctuations, one will arrive at an equation involving the fluctuations in terms of the mean quantities. Once closure relations are provided, we will have a constitutive relation that can incorporate the invariants of both the stresses and the symmetric part of the velocity gradient, as well as mixed invariants involving the stress and the symmetric part of the velocity gradient. Such a constitutive relation will be much more general than the usual closure models. However, just starting with an algebraic constitutive relation will not allow for the final system of constitutive relations to describe relaxation effects that are observed in turbulence. For such effects to be incorporated, one would have to start with implicit constitutive relations wherein one allows for the time derivative of the stress to also be involved, that is one would need to use a rate-type implicit model. Given the importance of the problem of turbulence, it might well be worth the effort to see if the approach outlined above might provide some useful insights.

Conflicts of interest

The author has no conflict of interest to declare.

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