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Blow-up time of solutions to a class of pseudo-parabolic equations

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Abstract. In this paper, we study the Dirichlet problem for a semilinear pseudo-parabolic equation. By using the energy estimates and ordinary differential inequalities, we studied the upper and lower bounds of blow-up time of the solutions. The results of this paper extend and complete the results on this model.

Keywords. Pseudo-parabolic equation, Blow-up, Blow-up time, Potential well method, Global existence.

Mathematical subject classification (2010). 35K70, 35A01.

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1. Introduction

The pseudo-parabolic equation

$$u_t - a\Delta u_t - \Delta u = f(u) \quad (1)$$

with $a > 0$ can be used to describe many interesting physical and biological phenomena [1–6], for example, the non-stationary process in semiconductors in the presence of sources, where $a\Delta u_t - u_t$ stands for the free electron density rate, Δu stands for the linear dissipation of free charge current and the nonlinear term stands for the source of free electron current (see [3, 5]). Showalter and Ting [7] investigated the initial boundary value problem of (1) with $f(u) = 0$, and the global existence, uniqueness and regularity of solutions were studied. When $f(u)$ is a polynomial, i.e., $f(u) = u^{p-1}$ or $f(u) = |u|^{p-2}u$, the initial and boundary value problem of (1) was studied in [8–13], and the existence, asymptotic behavior of the global solutions and global nonexistence of solutions were studied. When $f(u)$ is a logarithmic function, i.e., $f(u) = u \ln |u|$, the initial boundary value problem of (1) was studied in [14], where global existence, infinite-time blow-up of solutions, and behavior of vacuum isolation of solutions were studied. When $f(u)$ is a nonlocal function, i.e., $f(u) = |u|^{p-2}u - \frac{1}{|\Omega|} \int_{\Omega} |u|^{p-2}u dx$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain, problem (1) with homogeneous Neumann boundary and initial value was studied [15],

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and the existence, uniqueness and asymptotic behavior of the global solution and the blow-up phenomena of solution were studied. For the study of global existence and finite time blow-up of solutions for pseudo-parabolic equation with potential terms, we refer the readers to [16, 17].

In this paper, we consider the following initial boundary-value problem (IBVP) for a class of pseudo-parabolic equation proposed by Zhu et al. in [18]:

$$\begin{cases} u_t - \Delta u_t - \Delta u + u = |u|^{p-2}u, & x \in \Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \\ u(x, t) = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{2}$$

where $p \in (2, 2^*)$ and either $\Omega = \mathbb{R}^N$ ($N \geq 3$) or $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $u_0 \in H_0^1(\Omega)$. Here

$$2^* = \frac{2N}{N-2}.$$

When $\Omega = \mathbb{R}^N$, the boundary condition that $u(x, t) = 0$ for $x \in \partial\Omega$ and $t > 0$ is ineffective.

Problem (2) was studied by Zhu et al. in [18] via the potential well method (see, for example, [19–21]), where the existence, uniqueness, polynomial decay of the global solutions, and blow-up of solutions were studied. Zhou [22] improve the results of [18] by showing the global solutions decay exponentially. *The main purpose of this paper is to complete the blow-up result given in [18] by estimating the upper and lower bounds of the blow-up time.* To introduce the previous results and give the main results of the present paper, let's firstly recall some notations used in [18]. Throughout this paper, we use $\|\cdot\|_p$ as the usual L^p -norm, (\cdot, \cdot) and $\|\cdot\|$ as the inner product and the associated norm on $H_0^1(\Omega)$ respectively, that is

$$(u, v) = \int_{\Omega} (\nabla u \cdot \nabla v + uv) dx \tag{3}$$

and

$$\|u\| = \sqrt{(u, u)}. \tag{4}$$

By [18, Theorem 1.3], when $p \in (2, 2^*)$ and $u_0 \in H_0^1(\Omega)$, IBVP (2) admits a (weak) solution

$$u \in C^1([0, T_{\max}), H_0^1(\Omega)), \tag{5}$$

which satisfies $u(0) = u_0 \in H_0^1(\Omega)$ and

$$(u'(t), v) + (u(t), v) = \int_{\Omega} |u(t)|^{p-2}u(t)v dx, \quad \forall v \in H_0^1(\Omega) \tag{6}$$

where T_{\max} is the maximum existence time.

As in [18], the energy functional $J(u)$ can be defined through

$$J(u) := \frac{p-2}{2p} \|u\|^2 + \frac{1}{p} I(u), \quad \forall u \in H_0^1(\Omega), \tag{7}$$

where

$$I(u) := \|u\|^2 - \|u\|_p^p, \quad \forall u \in H_0^1(\Omega). \tag{8}$$

By (5), $J(u(t))$ and $I(u(t))$ are well-defined and are continuous differentiable with respect to t .

Let

$$\mathcal{N} := \{u \in H_0^1(\Omega) \setminus \{0\} : I(u) = 0\} \tag{9}$$

denote the Nehari manifold. We also define two sets \mathcal{N}_{\pm} related to \mathcal{N} as follows:

$$\begin{aligned} \mathcal{N}_+ &:= \{u \in H_0^1(\Omega) : I(u) > 0\}, \\ \mathcal{N}_- &:= \{u \in H_0^1(\Omega) : I(u) < 0\}. \end{aligned} \tag{10}$$

The main result on global existence and blow-up of [18] is the following theorem (see [18, Theorem 1.4]):

Theorem 1. *Let $u = u(t)$ be a solution of IBVP (2), whose maximal existence time is T_{\max} . Then $T_{\max} < \infty$ if and only if there exists a $t_0 \in [0, T_{\max})$ such that $u(t_0) \in \mathcal{N}_-$. Moreover, if $T_{\max} < \infty$, then $\lim_{t \uparrow T_{\max}} \|u(t)\| = \infty$.*

When blow-up occurs, the blow-up time T_{\max} cannot usually be computed exactly. It is therefore of great importance in practice to determine the upper and lower bounds for T_{\max} . So we complete Theorem 1 by showing the upper and lower bounds of T_{\max} :

Theorem 2. *Let $u = u(t)$ be a solution of IBVP (2) with initial value $u_0 \in \mathcal{N}_-$, i.e., $I(u_0) = \|u_0\|^2 - \|u_0\|_p^p < 0$. Then*

$$\frac{1}{2} \int_{\|u_0\|^2}^{\infty} \frac{d\xi}{\kappa^p \xi^{p-2} - \xi} \leq T_{\max} \leq \frac{4(p-1)\|u_0\|^2}{(p-2)^2 (\|u_0\|_p^p - \|u_0\|^2)}, \tag{11}$$

where κ is the optimal constant of the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, i.e.,

$$\kappa = \sup_{\substack{u \in H_0^1(\Omega) \\ \|u\|=1}} \|u\|_p. \tag{12}$$

Remark 3.

- (1) Since $p \in (2, 2^*)$, it follows $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$;
- (2) We show that

$$\frac{1}{2} \int_{\|u_0\|^2}^{\infty} \frac{d\xi}{\kappa^p \xi^{p-2} - \xi} < \frac{4(p-1)\|u_0\|^2}{(p-2)^2 (\|u_0\|_p^p - \|u_0\|^2)}. \tag{13}$$

So, (11) makes sense. By using $I(u_0) < 0$, it follows from (8) and (12) that

$$\|u_0\|^2 < \|u_0\|_p^p \leq \kappa^p \|u_0\|^p,$$

which implies

$$\|u_0\| > \kappa^{\frac{-p}{p-2}}. \tag{14}$$

Let $f(\xi) = \sigma \kappa^p \xi^{p-2} - \xi$ for $\xi \geq 0$, where

$$\sigma = \kappa^{-p} \|u_0\|^{2-p}.$$

Since $p > 2$, by (14),

$$\sigma < \kappa^{-p} \left(\kappa^{\frac{-p}{p-2}} \right)^{2-p} = 1.$$

Then, for $\xi \in [\|u_0\|^2, \infty)$,

$$\begin{aligned} f'(\xi) &= \frac{p}{2} \sigma \kappa^p \xi^{p-3} - 1 = \frac{p}{2} \sigma \|u_0\|^{2-p} \xi^{p-3} - 1 \\ &\geq \frac{p}{2} \sigma \|u_0\|^{2-p} (\|u_0\|^2)^{p-3} - 1 \\ &= \frac{p}{2} \sigma - 1 > 0. \end{aligned} \tag{15}$$

So,

$$f(\xi) \geq f(\|u_0\|^2) = 0, \quad \xi \in [\|u_0\|^2, \infty).$$

Then, note $\sigma \in (0, 1)$, we can estimate the integral to the left of (13) as

$$\begin{aligned} \frac{1}{2} \int_{\|u_0\|^2}^{\infty} \frac{d\xi}{\kappa^p \xi^{p/2} - \xi} &= \frac{1}{2} \int_{\|u_0\|^2}^{\infty} \frac{d\xi}{(1-\sigma)\kappa^p \xi^{p/2} + f(\xi)} \\ &\leq \frac{1}{2(1-\sigma)\kappa^p} \int_{\|u_0\|^2}^{\infty} \frac{d\xi}{\xi^{p/2}} = \frac{1}{(1-\sigma)\kappa^p(p-2)} \|u_0\|^{2-p} \\ &= \frac{1}{(1-\kappa^{-p}\|u_0\|^{2-p})\kappa^p(p-2)} \|u_0\|^{2-p} \\ &= \frac{1}{(p-2)(\kappa^p\|u_0\|^{p-2} - 1)}. \end{aligned} \tag{16}$$

By (12), we can estimate the right of (13) as

$$\begin{aligned} \frac{4(p-1)\|u_0\|^2}{(p-2)^2(\|u_0\|_p^p - \|u_0\|^2)} &\geq \frac{4(p-1)\|u_0\|^2}{(p-2)^2(\kappa^p\|u_0\|^p - \|u_0\|^2)} \\ &= \frac{4(p-1)}{(p-2)^2(\kappa^p\|u_0\|^{p-2} - 1)}. \end{aligned} \tag{17}$$

Since $4(p-1)/(p-2) > 4$, (13) follows from (16) and (17).

(3) It obvious that

$$\int_{\|u_0\|^2}^{\infty} \frac{d\xi}{\kappa^p \xi^{p/2} - \xi} \geq \int_{\|u_0\|^2}^{\infty} \frac{d\xi}{\kappa^p \xi^{p/2}} = \frac{2}{\kappa^p(p-2)} \|u_0\|^{2-p}.$$

Then it follows from (11) that

$$\frac{1}{\kappa^p(p-2)} \|u_0\|^{2-p} \leq T_{\max} \leq \frac{4(p-1)\|u_0\|^2}{(p-2)^2(\|u_0\|_p^p - \|u_0\|^2)}.$$

2. Proof of Theorem 2

In this section, we give the proof of Theorem 2 by using the following lemma:

Lemma 4 ([23]). *Suppose that $0 < T \leq +\infty$ and suppose a nonnegative function $F(t) \in C^2[0, T]$ satisfies*

$$F''(t)F(t) - (1 + \gamma)(F'(t))^2 \geq 0$$

for some constant $\gamma > 0$. If $F(0) > 0, F'(0) > 0$, then

$$T \leq \frac{F(0)}{\gamma F'(0)} < \infty$$

and $F(t) \rightarrow +\infty$ as $t \uparrow T$.

Proof of Theorem 2. Let $u = u(t)$ be a solution of problem (2) with initial value $u_0 \in \mathcal{N}_-$. By Theorem 1, the maximal existence $T_{\max} < \infty$. So we only need to prove the inequality (11).

(1) Upper bound estimate. In the part, we will give the upper bound estimate, i.e., show that

$$T_{\max} \leq \frac{4(p-1)\|u_0\|^2}{(p-2)^2(\|u_0\|_p^p - \|u_0\|^2)}, \tag{18}$$

For $t \in [0, T_{\max})$, let

$$\xi(t) := \left(\int_0^t \|u(s)\|^2 ds \right)^{\frac{1}{2}},$$

$$\eta(t) := \left(\int_0^t \|u'(s)\|^2 ds \right)^{\frac{1}{2}}.$$

The following facts can be found in [18, Lemma 3.2 and the proof of Theorem 5.1]:

- (R₁) $J(u(t)) + \eta^2(t) = J(u_0)$, $t \in [0, T_{\max})$;
- (R₂) $\frac{d}{dt} \|u(t)\|^2 = -2I(u(t))$, $t \in [0, T_{\max})$;
- (R₃) $u(t) \in \mathcal{N}_-$, i.e., $I(u(t)) < 0$, $t \in [0, T_{\max})$.

By (R₁)-(R₃), it is easy to see

- (R₄) $\|u(t)\|^2$ is strictly increasing with respect to t ;
- (R₅) the function $\phi(t) := (p-2)\|u(t)\|^2 - 2p(J(u(t)))$ satisfies

$$\begin{aligned} \phi(t) &\geq (p-2)\|u_0\|^2 - 2p(J(u_0) - \eta^2(t)) \\ &= \underbrace{2(\|u_0\|_p^p - \|u_0\|^2)}_{\text{by (7) and (8)}} + 2p\eta^2(t), \quad t \in [0, T_{\max}). \end{aligned}$$

Consider the following functional

$$F(t) := \xi^2(t) + (T_{\max} - t)\|u_0\|^2 + \beta(t + \alpha)^2, \quad t \in [0, T_{\max}), \tag{19}$$

where α and β are two positive constants to be determined later. Then by (R₄), we have

$$\begin{aligned} F'(t) &= \|u(t)\|^2 - \|u_0\|^2 + 2\beta(t + \alpha) \\ &\geq 2\beta(t + \alpha) > 0, \quad t \in [0, T_{\max}), \end{aligned} \tag{20}$$

which implies

$$F(t) \geq F(0) = T_{\max}\|u_0\|^2 + \beta\alpha^2 > 0, \quad t \in [0, T_{\max}) \tag{21}$$

and (by (R₂), (7) and (R₅))

$$\begin{aligned} F''(t) &= -2I(u(t)) + 2\beta \\ &= \phi(t) + 2\beta \\ &\geq 2(\|u_0\|_p^p - \|u_0\|^2) + 2p\eta^2(t) + 2\beta, \quad t \in [0, T_{\max}). \end{aligned} \tag{22}$$

By (3), the Cauchy–Schwarz inequality and Hölder’s inequality, we have

$$\begin{aligned} \frac{1}{2} \int_0^t \frac{d}{ds} \|u(s)\|^2 ds &= \int_0^t (u(s), u'(s)) ds \\ &\leq \int_0^t \|u(s)\| \|u'(s)\| ds \\ &\leq \xi(t)\eta(t), \quad t \in [0, T_{\max}), \end{aligned}$$

which, together with (19), implies

$$\begin{aligned}
 (F(t) - (T_{\max} - t)\|u_0\|_2^2)(\eta^2(t) + \beta) &= (\xi^2(t) + \beta(t + \alpha)^2)(\eta^2(t) + \beta) \\
 &= \xi^2(t)\eta^2(t) + \beta\xi^2(t) + \beta(t + \alpha)^2\eta^2(t) + \beta^2(t + \alpha)^2 \\
 &\geq \xi^2(t)\eta^2(t) + 2\xi(t)\eta(t)\beta(t + \alpha) + \beta^2(t + \alpha)^2 \\
 &\geq (\xi(t)\eta(t) + \beta(t + \alpha))^2 \\
 &\geq \left(\frac{1}{2} \int_0^t \frac{d}{ds} \|u(s)\|^2 ds + \beta(t + \alpha)\right)^2, \quad t \in [0, T_{\max}].
 \end{aligned}$$

Then it follows from (20) and the above inequality that

$$\begin{aligned}
 (F'(t))^2 &= 4 \left(\frac{1}{2} \int_0^t \frac{d}{ds} \|u(s)\|^2 ds + \beta(t + \alpha)\right)^2 \\
 &\leq 4F(t)(\eta^2(t) + \beta), \quad t \in [0, T_{\max}].
 \end{aligned} \tag{23}$$

In view of (21), (22) and (23), we have

$$\begin{aligned}
 F(t)F''(t) - \frac{p}{2}(F'(t))^2 &\geq F(t)(2(\|u_0\|_p^p - \|u_0\|^2) + 2p\eta^2(t) + 2\beta - 2p(\eta^2(t) + \beta)) \\
 &= F(t)(2(\|u_0\|_p^p - \|u_0\|^2) - 2(p - 1)\beta), \quad t \in [0, T_{\max}],
 \end{aligned}$$

which is nonnegative if we take β small enough such that

$$0 < \beta \leq \frac{\|u_0\|_p^p - \|u_0\|^2}{p - 1}. \tag{24}$$

Then it follows from Lemma 4 that

$$\begin{aligned}
 T_{\max} &\leq \frac{F(0)}{(\frac{p}{2} - 1)F'(0)} \\
 &= \frac{1}{p - 2} \left(\alpha + \frac{\|u_0\|^2}{\beta\alpha} T_{\max}\right).
 \end{aligned} \tag{25}$$

By taking α large enough such that

$$\alpha > \frac{\|u_0\|^2}{(p - 2)\beta}, \tag{26}$$

we get from (25) that

$$T_{\max} \leq \frac{\beta\alpha^2}{(p - 2)\beta\alpha - \|u_0\|^2}. \tag{27}$$

The above analysis shows that ($\rho := \beta\alpha$)

$$T_{\max} \leq \inf_{(\rho, \alpha) \in \Phi} f(\rho, \alpha), \tag{28}$$

where

$$\begin{aligned}
 \Phi &:= \left\{(\rho, \alpha) : \rho > \frac{\|u_0\|^2}{p - 2}, \alpha \geq \frac{(p - 1)\rho}{\|u_0\|_p^p - \|u_0\|^2}\right\}, \\
 f(\rho, \alpha) &:= \frac{\rho\alpha}{(p - 2)\rho - \|u_0\|^2}.
 \end{aligned}$$

Since $f(\rho, \cdot)$ is increasing, we get

$$\begin{aligned} T_{\max} &\leq \inf_{\rho > \frac{\|u_0\|^2}{p-2}} f\left(\rho, \frac{(p-1)\rho}{\|u_0\|_p^p - \|u_0\|^2}\right) \\ &= \inf_{\rho > \frac{\|u_0\|^2}{p-2}} \frac{(p-1)\rho^2}{((p-2)\rho - \|u_0\|^2)(\|u_0\|_p^p - \|u_0\|^2)} \\ &= \frac{(p-1)\rho^2}{((p-2)\rho - \|u_0\|^2)(\|u_0\|_p^p - \|u_0\|^2)} \Big|_{\rho = \frac{2\|u_0\|^2}{p-2}} \\ &= \frac{4(p-1)\|u_0\|^2}{(p-2)^2(\|u_0\|_p^p - \|u_0\|^2)}. \end{aligned}$$

So, (18) is true.

(2) **Lower bound estimate.** In the part, we will give the low bound estimate, i.e., show that

$$T_{\max} \geq \frac{1}{2} \int_{\|u_0\|^2}^{\infty} \frac{d\xi}{\kappa^p \xi^{p/2} - \xi}. \tag{29}$$

By (R_2) , (8), and (12), we have

$$\begin{aligned} \frac{d}{dt} \|u(t)\|^2 &= -2I(u(t)) \\ &= -2\|u(t)\|^2 + 2\|u(t)\|_p^p \\ &\leq -2\|u(t)\|^2 + 2\kappa^p \|u(t)\|^p, \quad t \in [0, T_{\max}). \end{aligned}$$

Since $I(u(t)) < 0$ (see (R_3)), we get $-2\|u(t)\|^2 + 2\kappa^p \|u(t)\|^p \geq -2I(u(t)) > 0$, then the above inequality can be rewritten as

$$\frac{\frac{d}{dt} \|u(t)\|^2}{-2\|u(t)\|^2 + 2\kappa^p \|u(t)\|^p} \leq 1, \quad t \in [0, T_{\max}). \tag{30}$$

Since u blow up at T_{\max} , by Theorem 1,

$$\lim_{t \uparrow T_{\max}} \|u(t)\|^2 = \infty,$$

integrating (30) from 0 to T_{\max} , we get (29). □

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