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Physical Science in Microgravity within the Thematic Group Fundamental and Applied Microgravity / *Sciences physiques en microgravité au sein du GDR Micropesanteur Fondamentale et Appliquée*

Bending effects distorting axisymmetric capillary bridges. Generalized Young–Laplace equation and associated capillary forces

Distorsion des ponts capillaires axisymétriques due aux effets de flexion. Équation de Young-Laplace généralisée et forces capillaires associée

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Abstract. This study proposes a theoretical contribution to the problem of the various distortions affecting axisymmetric capillary bridges, due to gravity or to bending effects linked to the Gaussian curvature. We deduce a clear hierarchization of effects between various reference configurations and put in a prominent position an exact first integral for the Young–Laplace equations, classical or generalized. These relationships are taken advantage of to obtain the theoretical expression of the varying inter-particle force, quantified effects of flexural strength. Finally, we establish a generalization of the classical "gorge method" to calculate accurately the capillary force of a profile subjected to distorsion due to bending when the gravity effects are negligible or not taken into account.

Résumé. Cette étude propose une contribution théorique au problème des distorsions affectant les ponts capillaires axisymétriques, dues à la gravité ou aux effets de flexion liés à la courbure gaussienne. Nous en déduisons une hiérarchisation claire de ces effets pour différentes configurations de référence et nous mettons en évidence une intégrale première exacte pour les équations de Young–Laplace, classiques ou généralisées. Ces relations sont mises à profit pour obtenir une expression théorique de la force capillaire, tenant compte des effets de flexion, qui n'est plus constante. Enfin, nous établissons une généralisation de la "gorge method" classique pour calculer avec précision la force capillaire d'un doublet capillaire soumis à une distorsion due aux effets de flexion lorsque les effets de la gravité sont négligeables ou non pris en compte.

Keywords. Distortion of capillary bridges, Mean and Gaussian curvatures impact, Generalized Young–Laplace equation, Bending effects.

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Mots-clés. Distorsion des ponts capillaires, Courbure moyenne et courbure gaussienne, Équation de Young–Laplace généralisée, Effets de flexion.

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1. Introduction

It is well known that the constant mean curvature surfaces, highly studied, are obtained by minimizing the only surface tension energy at fixed volume, the constant corresponding to the Lagrange multiplier [1–6]. Implicitly, this means that the Gaussian curvature (or the total curvature) is not taken into account and that therefore, the bending energy is disregarded or a priori considered as having negligible effects compared to the effects of surface tension[1] [8–14].

Admittedly, the spherical or distorted water drops, freely evolving in the air, exactly agree with this simplifying assumption. However, this result is not valid when the drop, or any capillary bridge, is subjected to contact boundary conditions. In that case, the bending energy can be directly linked to an integral on the boundaries using Gauss–Bonnet theorem and must be taken into account. Let us quote that in the modeling of fluid membranes, an energy of the same type containing the Gaussian curvature is introduced (see for example [15]).

This study proposes a theoretical contribution to the problem of the various distortions affecting axisymmetric capillary bridges, due to gravity or to bending effects linked to the Gaussian curvature, in order to establish a structured and practical framework for experimentation and numerical approach [16]. We deduce a clear hierarchization of effects between various reference configurations. In Sections 3 and 4, we put in a prominent position an exact first integral for the Young–Laplace equations, classical or generalized. These relationships, which are actually total energy conservation laws, are taken advantage of to obtain the theoretical expression of the varying inter-particle force, quantified effects of flexural strength. When considering the only bending effects, the method allows to easily obtain a parameterization of the profile by generalizing together a Delaunay formula related to constant mean curvature surfaces [17], and the resolution method of the Young–Laplace equation as an inverse problem developed in [18–23].

Moreover, we establish an original generalization of the classical "gorge method", based on energy conservation principle, to calculate accurately the capillary force of a profile subjected to distorsion due to bending when the gravity effect is negligible or not taken into account.

2. The generalized Young–Laplace equation with gravity

In this work, we assume that the shape of the capillary surfaces remains axisymmetric in the deformations.

The strong distorsions of capillary bridges for which the bending effects, bending the interface[2], may be modeled by an additional curvature-related term: the introduction of C_K, a multiplier coefficient of the Gaussian curvature K, at the dimension of a force and standing for the bending stress. Under appropriate boundary conditions, the shape of a capillary interface between two fluids is then described by the so-called generalized Young-Laplace equation, involving both mean and Gaussian curvatures. Structurally analogous to the Gullstrand equation of

[1] The principal curvatures are intrinsically the two eigenvalues of the shape operator, the Gauss curvature being its determinant and the mean curvature is its trace [7].

[2] Bending the interface, i.e. changing its curvature, in a first approach.

geometrical optics[3], the resulting equation, at the upward vertical measurement x linked to the value Δp_0 at $x = 0$, comes in the following form [24–26]:

$$\gamma\left(\frac{1}{\rho_c} + \frac{1}{N}\right) + C_K \frac{1}{\rho_c\, N} = \Delta p_0 - \Delta\rho\, gx, \tag{1}$$

where the force C_K divided by the area $\rho_c N$ stands for the bending stress, ρ_c and N for the principal radii of curvature (evaluated algebraically, positively when the curvature is turned into the interior of the capillary bridge) and the pressure deficiency is Δp_0 at $x = 0$.

Thenceforth, a major difficulty is to estimate the influence of C_K on two determinant data: the modified contact angles and the spontaneous curvature $\frac{\Delta p_0}{\gamma}$ at $x = 0$ after distortion.

It is also reported [27] that in electro-capillarity, at the nanoscale, the presence of electric fields leads to an extra stress term to be added in the Young–Laplace equation.

The length $\frac{|C_K|}{\gamma}$ that occurs in the generalized Young–Laplace equation allows in a certain way to assess the relative importance of bending effects (a typical scale of tension versus bending). In particular, for this purpose, the smallness or not of the dimensionless number $\frac{|C_K|}{2\gamma Y^*}$ appears significant, Y^* being the gorge radius of the distorted bridge. In the form $\frac{\pi|C_K|}{2\pi\gamma Y^*}$, this number appears as the quotient of the contributions of the bending and liquid surface tension forces at the distorted bridge neck. Strictly speaking, the formulas obtained retain the value $\frac{\pi|C_K|}{2\pi\gamma Y^* - \Delta p_0 \pi Y^{*2}}$ as the most accurate criterion, taking then into account the contribution of the hydrostatic pressure.

By placing oneself out of gravity for a simple illustration, it appears that the bending effects will be of little importance when the dimensional number $\frac{|C_K|}{2\gamma Y^*}$ is small compared to 1, i.e. since y^* or the characteristic length of the capillary bridge is of the order of a few millimeters. This effect will be enhanced for synclastic capillary bridge surfaces[4] for which the meridian in concave, so that y^* is larger than for anticlastic surfaces[5]. Hence, the common horizontal axis nodoid with convex upper meridian is certainly sensitive to bending effects. In the modeling of membranes and vesicles [15], when the thickness of the fluid membrane is of the order of a few micrometers or even less, the contribution of the bending energy may become important.

In the case of minimal surfaces such as catenoids[6] where the mean curvature is zero, the Gaussian curvature has the specificity of being determined by the direct relationship at established equilibrium:

$$\frac{1}{\rho_c\, N} = -\frac{1}{\rho_c{}^2} = \frac{1}{C_K}\Delta p_0, \tag{2}$$

Δp_0 being here an unknown spontaneous value to be identified by the data of an additional boundary condition. This implicit unknown value, a priori non-zero, would highlight the significant interest in considering the bending effects after experimental verifications in microgravity to inter in the framework of Eq. (2).

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[3]It must be noted that, in respect of certain theoretical issues, a capillary bridge may be considered as an optic system because it is composed of two interfaces [24]. In particular, the Gullstrand equation of geometrical optics involves the gravitational bending angle of light for finite distance. It presents also strong correspondances with geometrical approachs to gravitational lensing theory in the astrophysical context.

[4]Synclastic surfaces are those in which the centres or curvature are on the same side of the surface (dome-shape or elliptic surface). The Gaussian curvature is everywhere strictly positive; for examples among the Delaunay constant mean curvature surfaces of revolution (see *in* [18] a synoptic table for identifying the capillary bridges of revolution): a portion of unduloid, catenoid or nodoid with concave upper meridian, the axis of the bridge being horizontal.

[5]Anticlastic surfaces are those in which the centers of curvature are located on opposing sides of the surface (saddle shape or hyperbolic surface for the confined liquid). The Gaussian curvature is then everywhere strictly negative; for example: a portion of unduloid, catenoid, nodoid, or sphere with convex upper meridian, the axis of the bridge being horizontal.

It is then not mathematically correct to say without further information that a nodoid is an anticlastic surface.

[6]Surface with strictly negative Gaussian curvature and therefore, a priori, really subject to bending effects.

3. Analytical evaluation of weak capillary distortions by gravity effects

3.1. *Generalized Young–Laplace equation for axisymmetric vertical liquid bridge*

As a benchmark to be used for comparative purposes, consider, first in the classical theory, an axisymmetric vertical liquid bridge (i.e. the $x-$ axis is vertical and Δp_0 is the pressure difference through the interface at the neck level $x = 0$). I is an open interval on which we can define by Cartesian representation, say $x \to y(x)$, a portion of the Delaunay roulette strictly containing the convex profile of the bridge considered without taking into account the gravity (a zero or low gravity environment) [18]. So the shear stress is zero in the y direction and at first, we place ourselves in the relevant cases in which $y''(0) > 0$.

Taking then, if necessary into account the effects of gravity, *via* an over-pressure [19, 28, 29], results conventionally in the modified nonlinear differential equation for the distorted profile $x \to Y(x)$, according to the volumic mass densities difference between the liquid and the surrounding fluid

$$\Delta\rho = \rho_{int} - \rho_{ext}$$

a quantitated balance between the surface tension and gravity forces:

$$\frac{Y''(x)}{\left(1+Y'^2(x)\right)^{3/2}} - \frac{1}{Y(x)\sqrt{1+Y'^2(x)}} = -\frac{\Delta p_0}{\gamma} + \frac{g\,\Delta\rho}{\gamma}x =: H + Bx\,,\ x \in I. \tag{3}$$

In (3), the only parameter of the disturbance is the apparent density $\Delta\rho = \rho_{int} - \rho_{ext}$. The bridge fluid is not necessarily completely embedded in the surrounding fluid as for a wall-bound pendant drop without frictional contact constraints on the low boundary, possibly strongly distorting[7]. In continuum mechanics, this equation is obtained in the absence of motion when gravity is the only body force present. It is counterintuitive that the sign and the order of magnitude of the Gaussian curvature do not come into consideration for defining the distorted shape of the free capillary surface. This implicitly assumes that bending effects are neglected and that we are *de facto* limited here to studying rather moderate distortions. The question of bending and its impact on the deformation will be thoroughly discussed below.

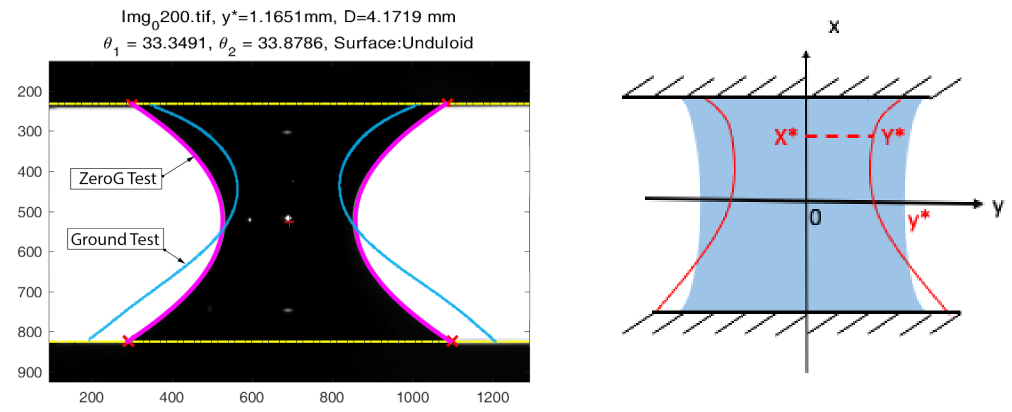
3.2. *A first integral of the Young–Laplace equation for distorted bridges with gravity*

We still find ourselves in the framework and notations of the previous subsection, concerning essentially any bridge with strictly negative Gaussian curvature K (the product of the two principal curvatures). The free surface is then saddle shaped, precisely the case mainly concerned by the bending effects.

It is particularly proposed to provide a theoretical justification for an extension of the conventional gorge method in order to evaluate the interparticle capillary force under gravitational perturbation at the neck level as a special case of an energy conservation principle. Unlike the situation of axisymmetric bridges with constant mean curvature, the capillary force is no longer constant at all points of the distorted profile. The analytic expression of the interparticle force $\mathscr{F}_{cap}(x)$ is given with exactness at the generic level x; it can be used by direct calculation from observed data and takes into account the gravitational forces versus the upward buoyancy forces. For other approaches, we can consult the Russian authors about low-gravity fluid mechanics [30].

First, we introduce (X^*, Y^*) the coordinates of the moved neck (i.e. the point such that $Y(X^*) = Y^*$, $Y'(X^*) = 0$) and the two branches x^+ and x^- of Y^{-1} in the set-theoretical sense,

[7] See in [10, Figure 15 p. 780], a stable pendent water drop in a bath of castor oil exhibiting inflection on the profile, also neck and bulge (artificial low gravity: $\Delta\rho = 39\ kg/m^3$).



(a) Capillary bridge of pure water between two parallel planes of glass. Distorsion observed due to Earth gravity in comparison with the same experiment done without gravity (zeroG experiment) during a parabolic flight campaign with CNES and Novespace.

(b) Paramaterization adopted.

Figure 1. Distorded capillary bridge and paramaterization adopted.

respectively defined on $\{x \geq X^*\}$ and $\{x \leq X^*\}$, subsets of the vertical $x-$ axis, in the Cartesian coordinate system linked to the neck level of the distorted bridge (Figure 1).

Keep in mind that the capillary bridge profile loses its symmetry: the gravitational perturbation modifies the localization of the contact points and hence, also the domain of definition for the modified nonlinear differential Young-Laplace equation; the associated boundary value problem does not admit locally symmetric solutions that are physically relevant.

Due to gravity, the mass of water is displaced toward the lower solid.

Moreover, the upper boundary of the liquid thus slides over a wetted part of the solid, while the lower part spreads over a dry part, which should substantially affect the resulting values of the wetting angles. As it is well known, the observed contact angle hysteresis depends on whether the liquid is advancing or receding on the surface. Let us add that the capillary phenomena are known to be highly sensitive to all types of microscopic non-uniformity (canthotaxis effects). The main result is stated as follows:

Result 1. *Whatever the shape taken by the distorted axisymmetric bridge due to gravity, we have in relation to the case where the effects of gravity are neglected, the following relationship which is a generalization of an energy conservation principle:*
along each concerned branch of the profile, the two following functional expressions are constant and equal, at the dimension of a force.

For any $x \geq X^*$,
we have:

$$\mathscr{F}^+ = 2\pi\gamma\left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2}Y^2(x) + B\int_{Y^*}^{Y(x)} x^+(y)\,y\,dy\right) \tag{4}$$

and, if $x \leq X^*$,

$$\mathscr{F}^- = 2\pi\gamma\left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2}Y^2(x) + B\int_{Y^*}^{Y(x)} x^-(y)\,y\,dy\right). \tag{5}$$

Moreover, the common value is

$$\begin{aligned}\mathscr{F}^+ = \mathscr{F}^- &= 2\pi\gamma Y^* + \pi\gamma H Y^{*2}\\ &= -\pi\Delta p_0 Y^{*2} + 2\pi\gamma Y^*.\end{aligned}$$

Proof. The key to understanding how to get this first integral is to rewrite locally the modified nonlinear differential Young–Laplace equation (3) in the following local form, separately in the two branches related to $\{x \geq X^*\}$ and $\{x \leq X^*\}$, H being rigorously evaluated as the mean curvature at the neck of the distorted bridge:

$$-\frac{1}{Y}\frac{d}{dY}\frac{Y}{\sqrt{1+Y'^2}} = H + B\,x^{\pm}(Y). \tag{6}$$

Hence, by quadrature, quantities $\mathscr{F}^+$ and $\mathscr{F}^-$ are constant respectively on $\{x \geq X^*\}$ and $\{x \leq X^*\}$; write $\mathscr{F}^+ = C^+$ and $\mathscr{F}^- = C^-$.

A continuity argument at (X^*, Y^*) implies

$$C^+ = C^- = 2\pi\gamma Y^* + \pi\gamma H Y^{*2}.$$

The demonstration lends itself to various easy generalizations, especially when a surface inflection exists and the Gaussian curvature changes sign. These analytical expressions are generalizations of formulas obtained in [18, equation (14)], and used in several other works [31–33]. □

4. Generalized Young–Laplace equation with Gaussian curvature at strong distortions

4.1. *A generalization of an exact energy invariant related to strongly distorted bridges*

In the case of strong capillary distortions, the meridian $x \to Y(x)$ of an axisymmetric capillary bridge is given by the generalized Young–Laplace equation (1) taking then into account simultaneously the combined effects of gravity and flexure, that can be rewritten as:

$$\frac{Y''(x)}{\left(1+Y'^2(x)\right)^{3/2}} - \frac{1}{Y(x)\sqrt{1+Y'^2(x)}} - \frac{C_K}{\gamma}\frac{Y''(x)}{Y(x)\left(1+Y'^2(x)\right)^2} = -\frac{\Delta p_0}{\gamma} + \frac{g\,\Delta\rho}{\gamma}x \;=: H + Bx\;,\; x \in I. \tag{7}$$

It must be borne in mind that for very distorted profiles, the surface tension γ may be surface temperature and curvature-dependent, which severely complicates the mathematical treatment. It follows the specified formulation, with suitably dimensioned coefficients c_T and c_J, c_K ([26, equation (27) p. 9] here, interfacial tension equals local Gibbs free energy per non-planar surface area for chemically pure fluids):

$$\gamma = \gamma_0 + c_T T + c_J\left(\frac{1}{\rho_c} + \frac{1}{N}\right) + c_K\frac{1}{\rho_c\,N}.$$

The qualitative results elaborated in the framework of the constant mean curvature theory are essentially based on the existence of an exact invariant (in fact, a first integral for the second order nonlinear differential equation which reveals the conservation of the total energy of the free surface). With minor adaptations, they are immediately applicable to the situation where the Gaussian curvature and bending effects are taken into account. Indeed, as we will see, we still highlight in this case a first integral for the generalized Young–Laplace equation by limiting ourselves to a presentation concerning essentially any bridge with strictly negative Gaussian curvature.

It is possible to deduce from Eq. (7) a generalization of an exact energy invariant related to strongly distorted bridges taking into account Gauss curvature and gravity. We then have the following result:

Result 2. *For the spontaneous but a priori unknown value of H, the generalized Young–Laplace equation can be rewritten, with the previous notations, in the differential form:*

$$-\frac{1}{Y}\frac{d}{dY}\left(\frac{Y}{\sqrt{1+Y'^2}}-\frac{C_K}{2\gamma}\frac{1}{1+Y'^2}\right)=H+Bx^{\pm}(Y). \tag{8}$$

Hence, along each concerned branch of the strongly distorted profile, the two following functional expressions are constant and equal, at the dimension of a force. Moreover, we have for any $x \geq X^*$, (X^*, Y^*) being the coordinates of the moved neck,

$$\mathscr{F}^{+}_{C_K}=2\pi\gamma\left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}}-\frac{C_K}{2\gamma}\frac{1}{1+Y'^2(x)}+\frac{H}{2}Y^2(x)+B\int_{Y^*}^{Y(x)}x^{+}(y)\,y\,dy\right) \tag{9}$$

and, if $x < X^*$,

$$\mathscr{F}^{-}_{C_K}=2\pi\gamma\left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}}-\frac{C_K}{2\gamma}\frac{1}{1+Y'^2(x)}+\frac{H}{2}Y^2(x)+B\int_{Y^*}^{Y(x)}x^{-}(y)\,y\,dy\right). \tag{10}$$

By highlighting a continuous connection at the neck, the common value is

$$\begin{aligned}\mathscr{F}^{+}_{C_K}=\mathscr{F}^{-}_{C_K}&=2\pi\gamma Y^*-\pi C_K+\pi\gamma H {Y^*}^2\\&=-\pi\Delta p_0 {Y^*}^2-\pi C_K+2\pi\gamma Y^*,\end{aligned}$$

Proof. The proof of this result is very similar to those of Result 1 and is left to the reader. It results from quadrature of Eq. (8) and then a continuity argument at the neck Y^*. □

4.2. *The special case of only bending effects without gravity*

It is interesting to note that when considering the only bending effects (i.e. $C_K \neq 0, B = 0$), then for any axisymmetric capillary bridge, the interparticle capillary force

$$\mathscr{F}^{cap}_{C_K}=2\pi\gamma\left(\frac{Y}{\sqrt{1+Y'^2}}-\frac{C_K}{2\gamma}\frac{1}{1+Y'^2}+\frac{H}{2}Y^2\right) \tag{11}$$

is constant at all points of the profile. It constitutes a generalization of [18, Proposition 1] on the conservation of the total energy of the liquid bridge free surface.

The evaluation of $\mathscr{F}^{cap}_{C_K}$ at the gorge radius Y^* leads to a generalization of the classical "gorge method"[8]:

$$\mathscr{F}^{cap}_{C_K}=2\pi\gamma Y^*-\pi C_K+\pi\gamma H {Y^*}^2.$$

As explicited at the beginning of the paper, this exact formula allows to assess the relative importance of bending effects linked to C_K. Of course equivalently, this expression may be evaluated at one or the other triple line. More generally, the capillary force may be calculated at any point of the profile of the capillary bridge according to

$$\mathscr{F}^{cap}_{C_K}=2\pi\gamma Y(x)\cos\Theta(x)-\pi C_K\cos^2\Theta(x)+\pi\gamma H Y^2(x)$$

where $\Theta(x)$ is the easily calculable angle made by the tangent vector to the meniscus with the x−axis at the generic point $(x, Y(x))$. It constitutes a generalization of the classical "gorge method" only valid when the bending effects are negligible or not taken into account.

[8]The "gorge method" (see for instance [4]) consists in calculating at the gorge Y^* the first integral of Young–Laplace equation or of generalized Young–Laplace equation (11) which is directly linked to the capillary force.

5. Conclusions

In this work, we have studied distorsion of capillary bridges due to bending effects and gravity. The true shape of the static bridge surfaces can be described by parametric equations, generalizing Delaunay formulas. Moreover, we showed that the related generalized Young–Laplace boundary value system can be solved as an inverse problem from experimental data for the unknown parameters identification.

On the other hand, we have establish generalized energy conservation laws than enable to obtain theoretical expressions of the varying inter-particle force, quantified effects of flexural strength. These expressions, involving the Gaussian curvature and gravity effect, constitute a generalization of the classical "gorge method" to calculate accurately the capillary force of a profile subjected to weak or strong distorsions.

It must of course be kept in mind that conduct such a predictive modeling for the motion of the contact lines by gravity and flexure effects is a problem considerably more difficult than to model the static distorted case, observed *in situ*. The isomorphic structure between the Gullstrand and generalized Young–Laplace equations may be thought to allow experimenters to consider a capillary bridge as an optical system. Although the two physical phenomena seem a priori disjoined but intellectually close, it could be deduced new practices for curvature measurements and fast, effective parameters identification. Nevertheless, the combined effect of volume, bending and axial gravity on the axisymmetric liquid bridge stability is a broad research subject to explore. The considerations on the numerical treatment of the distortion problem are given here as an indication of a research direction necessary to the advancement of the topic.

Conflicts of interest

The authors have no conflict of interest to declare.

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References

[1] P. G. de Gennes, "Wetting: statics and dynamics", *Rev. Mod. Phys.* **57** (1985), no. 3, p. 827-863.

[2] J. P. Gras, J. Y. Delenne, M. S. El Youssoufi, "Study of capillary interaction between two grains: A new experimental device with suction control", *Granul. Matter* **15** (2013), no. 1, p. 49-56.

[3] R. E. Johnson, R. H. Dettre, D. A. Brandreth, "Dynamic contact angles and contact angle hysteresis", *J. Colloid Interface Sci.* **62** (1977), no. 2, p. 205-212.

[4] G. Lian, C. Thornton, M. J. Adams, "A theoretical study of the liquid bridge forces between two rigid spherical bodies", *J. Colloid Interface Sci.* **161** (1993), no. 1, p. 138-147.

[5] F. M. Orr, L. E. Scriven, A. P. Rivas, "Pendular rings between solids: meniscus properties and capillary force", *J. Fluid Mech.* **67** (1975), no. 104, p. 723-742.

[6] C. D. Willett, S. A. Johnson, M. J. Adams, J. P. Seville, "Chapter 28 Pendular capillary bridges", in *Granulation*, Handbook of Powder Technology, vol. 11, Elsevier, 2007, p. 1317-1351.

[7] P. G. Ciarlet, "An Introduction to Differential Geometry. Differential geometry: theory and applications", Series in contemporary applied mathematics, vol. 9, p. 1-93, Series in contemporary applied mathematics, World Scientific, 2008.

[8] J. C. Berg, *An introduction to interfaces colloids: the bridge to nanoscience*, World Scientific, 2010.

[9] S. R. Brown, "Fluid flow through rock joints: the effect of surface roughness", *J. Geophys. Res. Solid Earth* **92** (1987), p. 1337-1347.

[10] R. Finn, "Capillary surface interfaces", *Notices Am. Math. Soc.* **46** (1999), no. 7, p. 770-781.

[11] E. Giusti, G. H. Williams, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, vol. 80, Birkhäuser, 1984.

[12] A. de Lazzer, M. Dreyer, H. J. Rath, "Particle-surface capillary forces", *Langmuir* **15** (1999), no. 13, p. 4551-4559.

[13] A. D. Rey, "Capillary models for liquid crystal fibers, membranes, films, and drops", *Soft Matter* **3** (2007), no. 11, p. 1349-1368.

[14] R. N. Wenzel, "Surface roughness and contact angle", *J. Phys. Chem.* **53** (1949), no. 9, p. 1466-1467.

[15] U. Seifert, "Configurations of fluid membranes and vesicles", *Adv. Phys.* **46** (1997), no. 1, p. 13-137.

[16] E. Hartmann, "G^2 interpolation and blending on surfaces", *Vis. Comput.* **12** (1996), no. 4, p. 181-192.

[17] C. H. Delaunay, "Sur la surface de révolution dont la courbure moyenne est constante", *J. Math. Pures Appl.* **6** (1841), p. 309-315.

[18] G. Gagneux, O. Millet, "Analytic Calculation of Capillary Bridge Properties Deduced as an Inverse Problem from Experimental Data", *Transp. Porous Med.* **105** (2014), no. 1, p. 117-139.

[19] G. Gagneux, O. Millet, B. Mielniczuk *et al.*, "Theoretical and experimental study of pendular regime in unsaturated granular media", *Eur. J. Environ. Civ. Eng.* **21** (2016), no. 7-8, p. 840-853.

[20] H. N. G. Nguyen, O. Millet, G. Gagneux, "Exact calculation of axisymmetric capillary bridge properties between two unequal-sized spherical particles", *Math Mech Solids* **24** (2019), no. 9, p. 2767-2784.

[21] H. N. G. Nguyen, O. Millet, G. Gagneux, "Liquid bridges between a sphere and a plane-classification of meniscus profiles for unknown capillary pressure", *Math Mech Solids* **24** (2019), no. 10, p. 3042-3060.

[22] H. N. G. Nguyen, O. Millet, G. Gagneux, "On the capillary bridge between spherical particles of unequal size: analytical and experimental approaches", *Contin. Mech. Thermodyn.* **31** (2019), no. 1, p. 225-237.

[23] H. N. G. Nguyen, O. Millet, C. Zhao, G. Gagneux, "Theoretical and experimental study of capillary bridges between two parallel planes", *Eur. J. Environ. Civ. Eng.* (2020), p. 1-11.

[24] M. A. Rodriguez-Valverde, M. A. Cabrerizo-Vilchez, R. Hidalgo-Alvarez, "The Young–Laplace equation links capillarity with geometrical optics", *Eur. J. Phys.* **24** (2003), no. 2, article no. 159.

[25] L. Boruvka, A. W. Neumann, "Generalization of the classical theory of capillarity", *J. Chem. Phys.* **66** (1977), no. 12, p. 5464-5476.

[26] J. Gaydos, L. Boruvka, Y. Rotenberg *et al.*, "The Generalized Theory of Capillarity", in *Applied Surface Thermodynamics* (A. W. Neumann, J. K. Spelt, eds.), Surfactant science series, vol. 63, Marcel Dekker, 1996, p. 1-52.

[27] J. W. Van Honschoten, N. Brunets, N. R. Tas, "Capillarity at the nanoscale", *Chem. Soc. Rev.* **39** (2010), no. 3, p. 1096-1114.

[28] P. G. De Gennes, F. Brochard-Wyart, D. Quéré, "Capillarity and gravity", in *Capillarity and Wetting Phenomena*, Springer, 2004, p. 33-67.

[29] B. Mielniczuk, O. Millet, G. Gagneux *et al.*, "Characterisation of pendular capillary bridges derived from experimental data using inverse problem method", *Granul. Matter* **20** (2018), no. 1, article no. 14.

[30] A. D. Myshkis, V. G. Babskii, N. D. Kopachevskii *et al.*, *Low-gravity fluid mechanics*, Springer, 2012.

[31] G. Gagneux, O. Millet, "An analytical framework for evaluating the cohesion effects of coalescence between capillary bridges", *Granul. Matter* **18** (2016), no. 2, article no. 16.

[32] H. N. G. Nguyen, C. F. Zhao, O. Millet *et al.*, "An original method for measuring liquid surface tension from capillary bridges between two equal-sized spherical particles", *Powder Technol.* **363** (2020), p. 349-359.

[33] H. N. G. Nguyen, C. F. Zhao, O. Millet *et al.*, "Effects of surface roughness on liquid bridge capillarity and droplet wetting", *Powder Technol.* **378** (2021), p. 487-496.