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Frequency response of certain temporal and complex transfer functions for the Interfacial 1D temperature wave

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Abstract. This note is concerned with the dynamics of 1D temperature waves generated by time modulation of a boundary heat flux. It demonstrates how a certain temporal transfer function of both parabolic and low frequency hyperbolic interfacial temperature waves happens to be frequency semi-invariant with a vibrating boundary heat flux. It is proved that only high frequency hyperbolic interfacial temperature waves can have a fully frequency-invariant temporal transfer function relative to such a vibrating boundary. The frequency response of an associated complex transfer function is also studied and demonstrated to behave, at low frequencies, as fixed lag compensator. Only according to hyperbolic theory of heat conduct, this compensator converts, at high frequencies, to a fixed gain amplifier.

Keywords. Temporal transfer function,Temperature wave, Frequency invariance, Complex transfer function, Interfacial solutions, Hyperbolic heat conduction.

2020 Mathematics Subject Classification. 35L51, 20A20.

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1. Introduction

A transfer function (TF) is a compact description of the output/input relation for a linear dynamical system governed by time-invariant ordinary differential equations (ODEs). Its key advantage is that it allows engineers to use simple complex-variable algebraic equations instead of ODEs to analyze, design and control a dynamical technical process in block diagram form, see e.g. [1–4] and references therein. A temperature wave can be conceived as the cosinusoidal response of a linear heat conducting medium. Temperature waves (TWs) have widely been in use, [5–9], during the 20$^{\text{th}}$ century for determination of thermophysical properties of solids, especially at low temperatures. More recent additional applications of them have been found in areas like nondestructive testing [10] and medicare [11, 12]. In particular, based on a phase-shift principle, an improved sinusoidal heating method has been developed in [13] to estimate the tissue blood perfusion to replace an original non-oscilliatory heating technique. In this respect, like diffusional neutron density waves which are transverse [14], TWs experience strong spatial attenuation and significant dispersion during propagation [14–17].

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The concept of a TF can also be generalized, in a variety of different ways [18–20], to time-invariant dynamical systems governed by partial differential equations (PDEs). This work is a contribution to these generalizations by development of a new *temperature wave/boundary heat flux* temporal transfer function (TTF) and its analysis (in the next section) as a new duplexed simultaneous hierarchy. The paper is further structured as follows: in the Section 3, the frequency invariance of this TTF is studied for the case of dynamical parabolic heat conduction. Here, the concept of frequency semi-invariance is introduced. The Section 4 extends this analysis to hyperbolic dynamical heat conduction. The complex TF associated with this TTF is demonstrated in Section 5 to behave, at low frequency, like a fixed lag compensation block in a thermal control engineering cascade. Only according to hyperbolic theory of heat conduction, this compensator converts, at high frequencies, to a fixed gain amplifier. The last Section 6 summarizes the TTF frequency-invariance results for both theories of heat conduction.

2. Duplexed hierarchal analysis

We consider in this work the generation of temperature waves, as an output signal $\mathfrak{O}(x,t)$, in a linear heat conducting medium by time modulation of a boundary condition, as an input signal $\mathfrak{I}(x,t;\omega)$. In such a setting, it is always possible to conceive a frequency-dependent transfer function

$$\mathfrak{Z}(x,t;\omega) = \mathfrak{O}(x,t;\omega)/\mathfrak{I}(x,t;\omega)$$

where the common frequency ω represents a synchronization guaranteed by the linearity of this system, while x and t represent respectively spatial and temporal variables.

Frequency-invariance of this transfer function occurs when

$$\mathfrak{Z}(x,t;\omega) \Longrightarrow \mathfrak{Z}(x,t),$$

i.e. when the terms containing ω in $\mathfrak{O}(x,t;\omega)$ and $\mathfrak{I}(x,t;\omega)$ happen to cancel out.

In a similar fashion a $t-$ or $x-$invariant transfer function can also be conceived.

It is our purpose in this note to reveal the tenuously existing constraints on this frequency-invariance and its possibly accompanying $t-$invariance. The analysis is reported for the 1D case and compares both Fourier and non-Fourier heat conduction theories.

The temperature wave we address here is a conventional parabolic (or hyperbolic) temperature oscillation generated by a time-dependent boundary condition [7, 15–21]; with a constitutive law, temporal relaxation time γ, excitation frequency ω, thermal conductivity k, thermal diffusivity α and temperature fluctuation spatial relaxation length λ. It does not cover temperature waves in ballistic heat conduction of thermo-mechanical phenomena. Furthermore, it is well known that a parabolic heat conduction equation (PHCE) is characterized by $\gamma = 0$,which leads to an infinite heat disturbance propagation speed $\varsigma = \sqrt{\frac{\alpha}{\gamma}}$. Accordingly, boundary value problems (BVPs) based on it can generate TWs that travel with a phase speed $V_p = \sqrt{2\alpha\omega}$ which can unrelativistically reach ∞ when $\omega \to \infty$. Despite that, the expression for a hyperbolic TW turns out to be qualitatively identical, and quantitatively quite similar, to that of a parabolic TW. This should not be a surprise since in all common materials, at ambient temperatures, γ is quite short (of the order of 10^{-14}-10^{-10} sec) [21, 22], i.e. ≈ 0. Such a satisfactory performance of the PHCE happens to hold for almost all heat engineering applications.

The practical setup for generating a 1D temperature wave $\theta(x,t)$, [6, 7, 15–17, 21, 23], in a semi-infinite heat conducting wedge $\Re = [0.\infty)$, assumes that the infinite surface $x = 0$ of $\Re$ is uniformly illuminated by a laser heating beam of periodically time-modulated intensity,

$$\mathscr{E}(0,t) = (\mathscr{E}_0/2) + (\mathscr{E}_0/2)\cos\,\omega t,$$

and that $\mathscr{E}_0$ is proportional to an amplitude J_0 of an associated similar heat flux $E(0,t)$. The constant level $(\mathscr{E}_0/2) \sim \frac{J_0}{2}$ is anticipated to support near $x = 0$ a steady state temperature $T(0,t) = T_s$,while an interfacial temperature fluctuation $\theta(0,t) = T(0,t) - T_s$ is supported, via Fourier's Law $I(0,t) = -k\,\nabla_x\,\theta(0,t)$, by an oscillating boundary heat flux

$$I(0,t) = \mathrm{Re}\left[\frac{J_0}{2}\,\mathrm{e}^{i\omega t}\right]\;; I(0,0) = \frac{J_0}{2}, \tag{1}$$

as sketched in Figure 1.

In a control dynamical system governed by a partial differential equation (PDE), like that of heat conduction, the output is obviously the solution to the associated boundary value problem (BVP). The basic problem with defining a t-domain (or $s-$domain; $s = \sigma + i\omega$) TF for such a system lies in the versatility of possible input functions in its associated BVP. The input could happen to be a source term, any of the active boundary conditions (BCs), or even one of their several parameters. A considerable relaxation of this problem occurs, however, when

(i) The PDE is of minimal order
(ii) The $x-$domain is singular (of infinite dimension); which reduces the number of effectively active BC's.
(iii) Time-invariance of the PDE and its BCs.
(iv) Partial linearity of the PDE and its BCs when the $x-$variable is fixed.

For an output temperature wave $\theta(x,t)$, taking into consideration the principal role played by the constitutive law for the boundary heat flux $I(0,t)$, as an input ([18, Neumann actuation]), and the validity of the preceding four aspects, motivate the definition for the TTF that follows.

Definition 1. *A temporal transfer function for a temperature wave could be*

$$Z(x,t;\omega) = \frac{\phi(x,t)}{I(0,t)}. \tag{2}$$

$Z(x,t;\omega)$*, for any fixed* x_o*, can be paired to a conventional complex transfer function*

$$\mathbb{W}_{x_o}(s;\omega) = \Phi(x_o,s)/\mathbb{I}(0,\,s) \neq \mathscr{L}[Z(x_o,t;\omega)]\;, \tag{3}$$

where $\Phi(x_o,s) = \mathscr{L}[\theta(x_o,t)]$ *and* $\mathbb{I}(0,\,s) = \mathscr{L}[I(0,t)]$ *are individual Laplace transforms.*

Our dynamical analysis of the previous $\theta(x,t)$ shall follow a novel duplexed simultaneous hierarchy. In one phase of this hierarchy, which is of first-order nature, $I(0,t)$ acts on the institutional law [21], alone to yield a "rudimentary" temperature wave $\phi(x,t)$. The other, second-order, phase witnesses an additional action of $I(0,t)$ on the dynamical heat conduction equation [7, 15–17, 21], in a complete BVP. This should transform $\phi(x,t)$ to $\theta(x,t)$. Simultaneity of these hierarchal phases, i.e. of the $\phi(x,t) \to \theta(x,t)$ transition, turns out to facilitate the decomposition of $\theta(x,t)$ into its basic components, for both parabolic and hyperbolic situations [7].

Definition 2. *Frequency-invariance of the TTF* $Z(x,t;\omega) = \frac{\phi(x,t)}{I(0,t)}$ *for the rudimentary temperature wave* $\phi(x,t)$ *occurs if* $Z(x,t;\omega) \Longrightarrow Z(x,t)$*, i.e. is independent of* ω*,while possibly depending on* t*. Furthermore, time-invariance of this transfer function can occur only when* $Z(x,t;\omega) \equiv Z(x)$*, i.e. independent of* t*.*

Obviously frequency-invariance has little to do with time-invariance of $Z(x,t;\omega)$. Accordingly, analysis of $Z(x,t;\omega)$ for frequency-invariance, or for its time-invariance, can rely heavily on possible separability of variables of $\phi(x,t)$,in the BVP, whenever that is possible. Incidentally, the novel hierarchal structure of $\theta(x,t)$ is probably the reason for the lack of any previously published literature on $\phi(x,t)$.

The vast literature on theory of isotropic and anisotropic heat transport contains many models. It has evolved from the classical (Fourier) parabolic model, through the model based on the Maxwell–Cattaneo–Vernotte (MCV) hyperbolic equation [21]. Then to a model based on the

Table 1. Selected subset of used symbols

Symbol	Meaning	Symbol	Meaning
$1D$	One-dimensional	$Z(x_o, t; \omega)$	TTF
$\theta(x,t)$	Temperature wave	α	Thermal diffusivity
$\phi(x,t)$	Rudimentary TW	γ	Relaxation time
$I(0,t)$	Boundary heat flux	λ	Relaxation length
k	Thermal conductivity	μ, ε, β	Coefficients
J_0	Constant	ω	Frequency
t	Time variable	Λ	Constant
$U(t)$	Quotient factor	Ω	Function
x	Spatial variable	$\Phi(\omega), \Psi(\Omega)$	Functions
s	complex variable	$\mathbb{W}_{x_o}(s;\omega)$	Complex TF

Guyer–Krumnhansl equation [24], which is outside the scope of the present note, that has found application in anisotropic heat transport. Each of these models incorporates a specific constitutive law.

3. Parabolic temperature wave

Despite the importance of temperature modulation in the field of micro-thermal analysis [25,26], it was only in 2014 when transfer functions were first addressed in this area by Hong & Chou [27], but not in the present context.

Principle 3. *By Fourier's constitutive law, if the rudimentary temperature wave $\phi(x,t)$ is multiplicatively separable in variables, $\phi(x,t) = Y(x)\ y(t)$, then its transfer function $Z(x,t)$ is not only frequency-invariant, but also time-invariant, and has a ω-independent $Y(x)$, despite its λ-dependence, viz*

$$Y(x) = Y(x;\lambda) = -Y'(0)\lambda e^{-\frac{x}{\lambda}}. \tag{4}$$

Proof. Invoke Fourier's [6], constitutive law

$$I(x,t) = -k\nabla_x\phi(x,t) = -kY'(x)y(t). \tag{5}$$

Then $\mathrm{Re}[\frac{J_0}{2}\mathrm{e}^{i\omega t}] = -kY'(0)y(t)$ implies that

$$y(t) = -\frac{J_0}{2kY'(0)}\mathrm{Re}\left[e^{i\omega t}\right],$$

which demonstrates that, for any $x \in \Re$, the wave $\phi(x,t)$ has a transfer function, relative to $I(0,t)$,that is time-invariant with a real scaling factor of

$$-\frac{1}{kY'(0)}.$$

Moreover

$$Z(x,t) = -\frac{Y(x)}{kY'(0)} = Z(x) \tag{6}$$

is not only independent of ω but also independent of t,which asserts both claimed frequency-invariance and time-invariance of $Z(x,t)$.

Integration of $Y'(x)$ near $x = 0$ leads to

$$Y(x) = Y'(0)x + \Lambda. \tag{7}$$

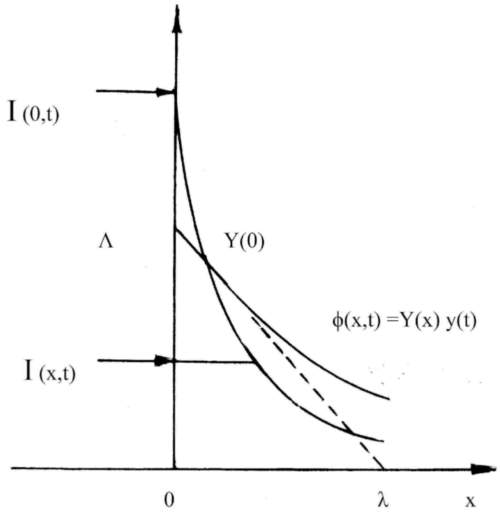

Figure 1. Sketch to illustrate the boundary heat flux $I(0,t)$ and rudimentary temperature wave $\phi(x,t)$.

Then $Y(\lambda) = 0$ means that $\Lambda = -Y'(0)\lambda$. Therefore

$$Y(x) = Y'(0)(x-\lambda) = -Y'(0)\lambda\left(1-\frac{x}{\lambda}\right), \tag{8}$$

which is the same as (4) when $\frac{x}{\lambda} \to 0$.

Finally, extrapolation of this $Y(x)$ away from $x = 0$ yields (4), for which

$$Y(0) = \Lambda = -Y'(0)\lambda. \tag{9}$$

To fully determine $Z(x)$, substitute (4) in (6) to obtain

$$Z(x) = \frac{\lambda}{k}e^{-\frac{x}{\lambda}}. \tag{10}$$

□

Remark 4. Independence of the $Y(x)$ in (4) on ω is a further verification of the validity, in Fourier's theory, of the assumption on separation of variables of the temperature wave $\phi(x,t)$ near its $x = 0$ generating boundary.

Corollary 5. *An oscillatory boundary heat flux $I(0,t) = \frac{J_0}{2}\cos\,\omega t$ has inside the $\Re$ wedge a frequency-invariant and time-invariant transfer function for the diffusional rudimentary Fourier temperature wave*

$$\phi(x,t) = \frac{J_0}{2}\frac{\lambda}{k}e^{-\frac{x}{\lambda}}\cos\,\omega t. \tag{11}$$

At this instance, it is possible to make use of (9) in (11) to write

$$Y'(0) = -\frac{1}{k} \& Y(0) = \frac{\lambda}{k}, \tag{12}$$

then to transform (4) to

$$Y(x) = \frac{\lambda}{k}e^{-\frac{x}{\lambda}}. \tag{13}$$

To complete the present hierarchal analysis, we consider now the, generated by $I(0,t)$, diffusional temperature wave $\theta(x,t)$. This should satisfy the BVP of (5) coupled with the parabolic heat conduction equation

$$\frac{1}{\alpha}\nabla_t\theta - \triangle_x\theta = 0. \tag{14}$$

The analytical expression for $\theta(x,t)$, is well-known, see e.g. [5–9], as

$$\theta(x,t) = \frac{J_0}{2\varepsilon\sqrt{\omega}}\, e^{-\frac{x}{\mu}} cos\left(\frac{x}{\mu} - \omega t + \frac{\pi}{4}\right), \tag{15}$$

with

$$\mu = \mu(\omega;\alpha) = \sqrt{\frac{2\alpha}{\omega}}, \varepsilon = \frac{k}{\sqrt{\alpha}}. \tag{16}$$

A comparison of $\phi(x,t)$, of (11), and $\theta(x,t)$, of (15), indicates that they differ essentially by an $(\frac{x}{\mu} + \frac{\pi}{4})$ phase shift and by a $\frac{1}{\sqrt{\omega}}$ multiplicative factor. Accordingly, the transfer function of $\theta(x,t)$, relative to $I(0,t)$, cannot be frequency-invariant. Moreover, it is clear that while ε does not influence $Z(x,t)$, μ is decisive in it. At the $x = 0$ interface, however, $Z(0,t)$ is not impacted by μ. Here it is clear [7], that $\lambda = \mu$.

Definition 6. *Frequency semi-invariance of the TTF for the wave $\theta(0,t)$ (of period P), relative to the boundary heat flux $I(0,t) = \frac{J_0}{2}\cos\,\omega t$ (of period $\mathrm{T} = \frac{2\pi}{\omega}$), occurs if, when $\cos\omega t = \Omega$, $\exists$ single-variable functions $\Phi(\omega)$ and $\Psi(\Omega)$ such that*

$$U(t) = \frac{\theta(0,t)}{I(0,t)} = \Phi(\omega)\Psi(\Omega). \tag{17}$$

Since $\int_0^P U(t)dt = \Psi_0\Phi(\omega)$, where $\Psi_0 = \Psi_0(\omega) = \int_0^P \Psi(\Omega)dt$, then $U(t) = \theta(0,t)/I(0,t)$ is necessarily time-invariant, on a zoomed out (when T > P) time scale of $\frac{T}{P}$ units, as sketched in Figure 2 (b), for a possible realization of this situation.

Principle 7. *The TTF of the parabolic interfacial temperature wave $\theta(0,t)$ is frequency-semi-invariant despite its frequency-non-invariance.*

Proof. Consider (15) for $x = 0$, in (17), to write

$$U(t) = \frac{1}{\varepsilon\sqrt{\omega}}\frac{cos\left(\omega t - \frac{\pi}{4}\right)}{cos\omega t} = \frac{1}{\varepsilon\sqrt{2\omega}}\left[1 + \frac{\sqrt{1-\Omega^2}}{\Omega}\right] = \Phi(\omega)\Psi(\Omega). \tag{18}$$

The presence of ω in $U(t)$ is a verification of the claimed frequency-non-invariance. Also $\int_0^P U(t)dt = \Psi_0\Phi(\omega)$, where $\Psi_0 = \Psi_0(\omega) = \int_0^P (1 + \tan\omega t)dt = P + \frac{1}{\omega}\ln|\sec\omega P|$ and $\Phi(\omega) = \frac{1}{\varepsilon\sqrt{2\omega}}$. □

4. Hyperbolic temperature wave

Regardless of the insignificant general difference between hyperbolic and parabolic TWs, there are specific heat conduction applications where the hyperbolic TW is expected to perform better than the parabolic TW. This should particularly be true when γ can be significant [28, 29], and in periodic heat fluxes of laser heating [30, 31], or other phenomena on the nanoscale.

By a result that follows, Principle 3 turns out not to hold in case of MCV heat transport [17, 32, 33]. This non-Fourier conduction is characterized by a temporal relaxation time, γ, for a heat flux fluctuation, i.e. time between $\nabla_x\,\theta$ and I.

Principle 8. *According to the MCV non-Fourier constitutive law, if the rudimentary temperature wave $\phi(x,t)$ is multiplicatively separable in variables, then its transfer function can neither be frequency-invariant, nor be time-invariant, while having the same ω-independent $Y(x)$ as in (13).*

Proof. Invoke the MCV non- Fourier's constitutive law [32, 33],

$$\gamma\nabla_t I(x,t)+I(x,t)=-k\nabla_x\phi(x,t)=\ -kY'(x)y(t) \tag{19}$$

This, on one hand, can be conceived as an initial-value problem (IVP),

$$\begin{gathered}\text{Solve}: \ \frac{d}{dt}I(0,t)+\frac{1}{\gamma}\ I(0,t)=-\frac{k}{\gamma}\ Y'(0)y(t),\\ \text{Subject to}\ \ I(0,0)=\frac{J_0}{2}.\end{gathered} \tag{20}$$

Alternatively, on the other hand, $\mathrm{Re}[i\gamma\omega\frac{J_0}{2}e^{i\omega t}+\frac{J_0}{2}e^{i\omega t}]=-k\ Y'(0)y(t)$ implies that

$$\begin{aligned} y(t)&=\mathrm{Re}\left[-\frac{J_0}{2kY'(0)}(1+i\gamma\omega)e^{i\omega t}\right]\\ &=-\frac{J_0}{2kY'(0)}\sqrt{\gamma^2\omega^2+1}\,cos\left(\omega t+\tan^{-1}\gamma\omega\right)\\ &=-\frac{J_0}{2kY'(0)}\left[cos\omega t-\gamma\omega\sin\omega t\right].\end{aligned} \tag{21}$$

This demonstrates that, for any $x\in\Re$, the wave $\phi(x,t)$ follows $I(0,t)$ with the effectively complex scaling factor of

$$\begin{aligned}&-\frac{1}{kY'}(0)\sqrt{\gamma^2\omega^2+1}\left[cos\left(\tan^{-1}\gamma\omega\right)-(\tan\ \omega t)\ \sin\left(\tan^{-1}\ \gamma\omega\right)\right]\\ &\qquad=-\frac{1}{kY'(0)}\left[cos\ \omega t-\ \gamma\omega\ \sin\ \omega t\right].\end{aligned} \tag{22}$$

In view of (6) and (21), the structure of (22) suggests, moreover, that

$$\begin{aligned} Z(x,t)&=\mathrm{Re}\left[-\frac{Y(x)}{kY'(0)}(1+i\gamma\omega)\right]=Z(x;\gamma\omega)\\ &=\frac{\lambda}{k}e^{-\frac{x}{\lambda}}\sqrt{\gamma^2\omega^2+1}\left[cos\left(\tan^{-1}\gamma\omega\right)-(\tan\omega t)\sin\left(\tan^{-1}\gamma\omega\right)\right]\\ &=\frac{\lambda}{k}e^{-\frac{x}{\lambda}}\left[cos\omega t-\gamma\omega\sin\omega t\right],\end{aligned} \tag{23}$$

is dependent on both ω and t, which negates both frequency-invariance and time-invariance.

Integration of $Y'(x)$ near $x=0$, as in the proof of Principle 3, leads nevertheless again to (4). Finally substitute (4) in (22) to obtain

$$Z(x,t)=Z(x;\gamma\omega)=\frac{\lambda}{k}(1+i\gamma\omega)e^{-\frac{x}{\lambda}} \tag{24}$$

which is a complex multiple of $Z(x)$,in (10), of the earlier Fourier's heat conduction theory. □

Incidentally, the same $y(t)$ of Principle 8 can be obtained by solving an inverse problem of the IVP (20): given $I(0,t)=\frac{J_0}{2}\cos\ \omega t$, what is the the corresponding $y(t)$? Indeed, this happens to be equivalent to the first-kind convolution-type Volterra integral equation

$$\int_0^t \mathrm{e}^{-\frac{(t-\tau)}{\gamma}}\ y(\tau)d\tau=\frac{\gamma J_0}{2kY'(0)}\left(e^{-\frac{t}{\gamma}}-cos\ \omega t\right),$$

which represents the analytic general solution to the first-order IVP (20). This integral equation is solvable for $y(t)$ by Laplace transformation as (21).

Within the framework of the Principle 7, combination of (21) with (2) can be restated as follows.

Corollary 9. *The boundary heat flux* $I(0,t)=\frac{J_0}{2}\cos\ \omega t$ *generates in* $\Re$ *the non-Fourier transport temperature wave*

$$\phi(x,t)=\frac{J_0}{2}\frac{\lambda}{k}e^{-\frac{x}{\lambda}}\left[cos\omega t-\gamma\omega\sin\omega t\right] \tag{25}$$

with a transfer function that is neither frequency-invariant nor time-invariant.

Moreover, when $\gamma = 0$, relation (25) coincides with (11) and both Principles 3 & 8 become identical.

Remark 10. The transport temperature wave (25) can be conceived as a sum of two terms viz

$$\phi(x,t) = [Y_1(x) + Y_2(x)]\,\frac{J_0}{2} cos\omega t$$

where $Y_1(x) = Y(x) = \frac{\lambda}{k} e^{-\frac{x}{\lambda}}$ of (13) is a diffusional component and

$$Y_2(x) = -\frac{\lambda}{k}\gamma\omega\, e^{-\frac{x}{\lambda}} \tan\,\omega t,$$

is a transport component.

This is an indication that in MCV theory, separation of variables is intrinsically additive [34], and not multiplicative, as anticipated. The dependence of $Y_2(x)$ on both ω and t is a further manifestation of the contradiction with our assumed multiplicative separation of variables for this theory.

Finally we consider the, generated by $I(0,t)$, hyperbolic temperature wave $\theta(x,t)$.This should satisfy the BVP of (20) coupled with the hyperbolic heat conduction equation [15, 17, 32],

$$\frac{1}{\varsigma^2}\triangle_t\theta + \frac{1}{\alpha}\nabla_t\theta = \triangle_x\theta, \tag{26}$$

in which $\varsigma = \sqrt{\frac{\alpha}{\gamma}}$. The solution to this BVP is usually approximated [15, 16], over two distinct domains for ω, viz

(i) when $\gamma\omega << 1$,

$$\theta(x,t) = \frac{J_0}{2\varepsilon\sqrt{\omega}}\frac{1}{\sqrt{\gamma^2\omega^2+1}} e^{-\frac{x}{\mu}}\, cos\left(\frac{x}{\mu} - \omega t + \frac{\pi}{4}\right). \tag{27}$$

Its associated $U(t)$ is

$$U(t) = \frac{1}{\varepsilon\sqrt{\omega}}\frac{1}{\sqrt{\gamma^2\omega^2+1}}\frac{cos\left(\omega t - \frac{\pi}{4}\right)}{cos\omega t} = \frac{1}{\varepsilon\sqrt{2\omega}}\frac{1}{\sqrt{\gamma^2\omega^2+1}}\left[1 + \frac{\sqrt{1-\Omega^2}}{\Omega}\right] = \Phi(\omega)\Psi(\Omega). \tag{28}$$

(ii) when $\gamma\omega >> 1$ [15, 17],

$$\theta(x,t) = \frac{J_0\sqrt{\gamma}}{2\varepsilon}\, e^{-\frac{1}{2\sqrt{\alpha\gamma}}x}\, cos\left(\frac{\omega}{\varsigma}x - \omega t\right), \tag{29}$$

with a remarkable

$$U(t) = \sqrt{\gamma}/\varepsilon, \tag{30}$$

for which $\Phi(\omega) = 1, \Psi(\Omega) = \sqrt{\gamma}/\varepsilon$ and U is independent of both ω and t.

Principle 11. *The low ω hyperbolic interfacial temperature wave $\theta(0,t)$ (like the parabolic wave) has a TTF that is frequency-semi-invariant despite its having a TTF that is frequency-invariant and time-invariant. Distinctively, the high ω hyperbolic interfacial temperature wave $\theta(0,t)$ has a transfer function that is neither frequency-invariant nor time-invariant.*

A summary of results stated by Principles 7 & 11 on frequency-invariance of the transfer function for $I(0,t)$ of parabolic and hyperbolic interfacial temperature waves $\theta(0,t)$, is provided in Figure 2.

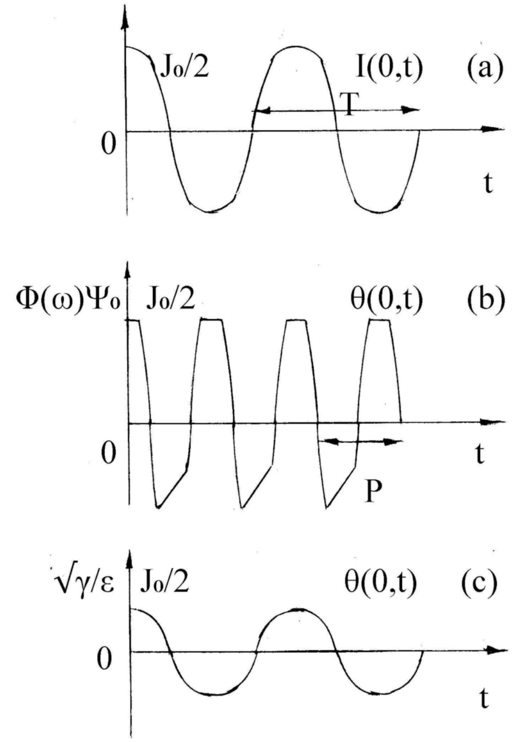

Figure 2. Sketch to represent the oscillating boundary heat flux (a), the parabolic (or low-ω hyperbolic) interfacial temperature waves (b), and high-ω hyperbolic $\theta(0,t)$ (c).

5. The complex transfer function

This section starts with recalling that for a control system governed by ODEs, the TF, $\mathbb{P}(s)$, is universal in representing the system impulse response when all its initial conditions are zeroes. This universality may not be valid, however, for the present $\mathbb{W}_o(s;\omega)$, of (3), whose

$$\mathscr{L}^{-1}\left[\mathbb{W}_{x_o}(s;\omega)\right] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left[\Theta(x_o,s)/\mathbb{I}(0,s)\right]e^{st}ds \neq Z(x_o,t;\omega),$$

despite the fact that

$$\mathscr{L}^{-1}[\mathbb{W}_{x_o}(s;\omega)] \approx Z(x_o,t;\omega),$$

within reason.

Indeed, any $t-$variant and $\omega-$variant $Z(x_o,t;\omega)$ generates a ω-variant $\mathbb{W}_{x_o}(s;\omega)$. Moreover, for any thermal flux transform $\mathbb{I}(0,s)$, we can always hypothesize that, as with $\mathbb{P}(s)$, the pole-zero configuration of $\mathbb{W}_{x_o}(s;\omega)$ is decisive for the stability of the heat conductin block supporting its output $\Theta(x_o,s)$ transform. Namely, if all the poles of $\mathbb{W}_{x_o}(s;\omega)$ have negative real parts, the block should be stable.

Motivated by [35, 36] or [37], let us explore the applicability of $\mathbb{W}_{x_o}(s;\omega)$ for some realistic situation. Consider therefore a network of dynamical heat conducting blocks with a TW block at its end, as illustrated in Figure 3.

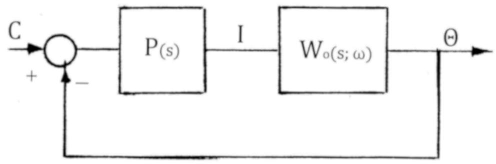

Figure 3. Block diagram for a control system with a generated interfacial temperature wave.

The block diagram of this figure is for a standard negative unit feedback control system containing a modulated lazar heater with a $\mathbb{P}(s)$ for its TF. $\mathbb{C}$ is the Laplace transform of the lazar power supply, while

$$\mathbb{W}_o(s;\omega) = \Theta(0,s)/\mathbb{I}(0,s), \tag{31}$$

is for $x_o = 0$.

5.1. *Parabolic system*

Taking the Laplace transforms for the parabolic interfacial $\theta(0,t)$, based on (15), and for $I(0,t)$ of (1), in (31) leads to

$$\mathbb{W}_o(s;\omega) = A(\omega)\frac{s+\omega}{s} = A(\omega)\mathbb{Q}(s), \tag{32}$$

in which the amplitude

$$A(\omega) = \frac{\sqrt{2}}{2\varepsilon}\frac{1}{\sqrt{\omega}}, \tag{33}$$

blows-up to ∞ when $\omega \to 0$.

This $\mathbb{W}_o(s;\omega)$ turns out to be a TF for a *lag compensator*, having a zero at $s = -\omega$ and a pole at $s = 0$. Moreover, by the Routh stability criterion [1–3], this block is only marginally stable. Furthermore, the block reads, algebraically, as $(\mathbb{C} - \Theta)\mathbb{P}(s)\mathbb{W}_o(s;\omega) = \Theta$. Then the closed-loop TF of this system,

$$\Theta/\mathbb{C} = A\mathbb{P}(s)\mathbb{Q}(s)/\left[1 + A\mathbb{P}(s)\mathbb{Q}(s)\right] = A\mathbb{P}(s)(s+\omega)/\left[s + A\mathbb{P}(s)(s+\omega)\right], \tag{34}$$

has the characteristic equation

$$s + A\mathbb{P}(s)(s+\omega) = 0. \tag{35}$$

Clearly, the magnitude of the amplitude $A = A(\omega)$, with its effects on location of the system poles, becomes decisive in the stability of the closed-loop control system. Hence the amplitude (i.e. ω) is remarkably usable for the entire system stabilization via a root-locus or Nyquist design [1,2].

Replace now s by $i\omega$ in (31), see e.g. [2] or [3], to obtain

$$\mathbb{W}_o(i\omega) = A(\omega)\frac{i+1}{i} = \sqrt{2}A(\omega)e^{-i\frac{\pi}{4}}. \tag{36}$$

This means that the amplitude $|\mathbb{W}_o(i\omega)|$ falls monotonically as $A(\omega)$ with increasing ω, while $\arg(\mathbb{W}_o(i\omega))$ is a constant lag of $\frac{\pi}{4}$. Moreover, at $\omega = 0$ the compensator turns out to loose its lagging property. For that reason, this compensator is practically employable only for $\omega > 0$.

Alternatively, for a block supporting a hyperbolic temperature wave, the properties of $\mathbb{W}_o(s;\omega)$ tun out to exhibit a different variability, with a varying ω, over two distinct domains as follows.

Table 2. Highlights of frequency-invariance of the TTF $Z(0,t;\omega)$ and associated complex TF $\mathbb{W}_o(s;\omega)$ for interfacial temperature waves by the two theories of heat conduction.

Theory	$\phi(0,t)\ /\ I(0,t)$	$\theta(0,t)\ /\ I(0,t)$	Self-organ.	$\mathbb{W}_o(s;\omega)$
Parabolic	Perfect ω-invar.	Semi- ω-invariance	$\swarrow$	Complex; $A(\omega)$; $\frac{\pi}{4}$ lag
Hyperbolic: low ω	No ω-invar.	Semi- ω-invariance	$\nearrow$	Complex; $\widetilde{A}(\omega)$; $\frac{\pi}{4}$ lag
Hyperbolic: high ω	No ω-invar.	Perfect ω-invariance	$\upuparrows$	Real; $\beta-$gain

5.2. *Low-frequency hyperbolic system*

Here the hyperbolic interfacial $\theta(0,t)$ is defind by (27) when $\gamma\omega << 1$, and $\mathbb{W}_o(s;\omega)$ is given by the same complex (32) but with a diffrent amplitude of

$$\widetilde{A}(\omega) = \frac{\sqrt{2}}{2\varepsilon}\frac{1}{\sqrt{\omega}}\frac{1}{\sqrt{\gamma^2\omega^2+1}} = \frac{1}{\sqrt{\gamma^2\omega^2+1}}A(\omega), \tag{37}$$

which also blows-up to ∞ when $\omega \to 0$.

This $\mathbb{W}_o(s;\omega)$ is also TF for a *lag compensator* but with a differnt $A(\omega)$ and

$$\mathbb{W}_o(i\omega) = \sqrt{2}A(\omega)e^{-i\frac{\pi}{4}}. \tag{38}$$

Again here the amplitude $|\mathbb{W}_o(i\omega)|$ falls monotonically as $A(\omega)$ with increasing ω, while $\arg(\mathbb{W}_o(i\omega))$ is a constant lag of $\frac{\pi}{4}$, and this compensator is practically employable only for $\omega > 0$, when $\gamma\omega << 1$.

5.3. *High-frequency hyperbolic system*

The situation for high frequencies, i.e. when $\gamma\omega >> 1$, happens to be entirely different. To demonstrate this fact, we substitute $\theta(0,t)$ of (29) in (31) to obtain

$$\mathbb{W}_o(s) = \mathbb{W}_o = \sqrt{\gamma}/\varepsilon\ = \beta, \tag{39}$$

a fixed constant, independent of both s and ω. Here the block supporting the hyperbolic TW, with this $\mathbb{W}_o(s) = \mathbb{W}_o$, behaves simply like a fixed gain amplifier for which (34) converts to

$$\Theta/\mathbb{C} = \beta\mathbb{P}(s)/\left[1+\beta\mathbb{P}(s)\right].$$

The characteristic equation for this TF is

$$1+\beta\mathbb{P}(s) = 0,$$

and β in it can make the closed-loop control system of Figure 3 either stable or unstable, pending to the nature of $\mathbb{P}(s)$. Clearly then the block with $\mathbb{W}_o$ is not usable for compensation control of this closed-loop system when $\gamma\omega >> 1$.

6. Summary

The results of the reported hierarchal analysis on frequency-invariance of the TTF for temperature waves with $I(0,t)$ of (1) are summarized in Table 2 for both Fourier and non-Fourier heat conduction.

The table indicates that, in the realm of parabolic heat conduction, the $\phi(x,t) \to \theta(x,t)$ transition is accompanied by deteriorated (-) ω-invariance of the TTF at $x = 0$, or "self-disorganization", perhaps. A reversed (+) behavior, $\nearrow$, takes place, however, with low-ω hyperbolic theory; and complete "self-organization" can only be reached in the high$-\omega$ limit of this theory. The symbols

$\swarrow$, $\nearrow$ and $\upuparrows$ refer to probably different mechanisms. The underlying hierarchal analysis can suggest therefore an existence of some correlation between this self-organization and consistency of the employed theory.

Due to the strong spatial attenuation of $\theta(x,t)$, this ω-invariance of the TTF behavior at the $x=0$ boundary is not expected to hold deep inside the wedge. It is expected to vary though both with frequency and theory for heat conduction.

The nature of the $\mathbb{W}_o(s;\,\omega)$ complex TFs, analyzed in the preceding section and summarized in Table 2, seems to reflect, in some consistent sense, the above results for their associated real $Z(0,t;\omega)$ TTFs. Indeed, at low frequencies ($\gamma\omega << 1$) there is little difference between the fixed lag compensator $\mathbb{W}_o(s;\,\omega)$ of parabolic and hyperbolic theories. The difference is restricted only to their $A(\omega)$ and $\widetilde{A}(\omega)$ amplitudes of (33) and (37), respectively. At high frequencies ($\gamma\omega >> 1$), however, only hyperbolic heat conduction predicts a transition $\mathbb{W}_o(s;\,\omega) \Longrightarrow \mathbb{W}_o$, from a lag compensator to a $\mathbb{W}_o = \beta$ fixed gain amplifier.

Finally it should be noted that similar techniques have also been used in thermoelasticity [37, 38], using time harmonic heating sources. This fact supports the hope that the present results may become in the future possibly valuable for microwave heating systems, especially in the manufacture and design of their actuators and/or dynamical temperature sensors.

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Conflict of interest

The author declares no known competing financial interests.

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