INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mécanique

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Volume 351, Special Issue S2 (2023), p. 125-137

Published online: 29 September 2023

https://doi.org/10.5802/crmeca.200

Part of Special Issue: Physical Science in Microgravity within the Thematic Group Fundamental and Applied Microgravity

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Les Comptes Rendus. Mécanique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1873-7234

Comptes Rendus
Mécanique
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https://doi.org/10.5802/crmeca.200

Physical Science in Microgravity within the Thematic Group Fundamental and Applied Microgravity / *Sciences physiques en microgravité au sein du GDR Micropesanteur Fondamentale et Appliquée*

A direct relation between bending energy and contact angles for capillary bridges

Une relation directe entre l'énergie de flexion et les angles de contact pour les ponts capillaires

Olivier Millet [a] **and Gérard Gagneux** [*, a]

[a] LaSIE, UMR-CNRS 7356, Université de La Rochelle, avenue Michel Crépeau, 17042 La Rochelle cedex 1, France.

E-mails: olivier.millet@univ-lr.fr, gerard.maryse.gagneux@gmail.com

Abstract. The didactic object of these developments on differential geometry of curves and surfaces is to present fine and convenient mathematical strategies, adapted to the study of capillary bridges. The common thread is to be able to calculate accurately in any situation the bending stress over the free surface, represented mathematically by the integral of the Gaussian curvature over the surface (called the total curvature) involved in the generalized Young–Laplace equation. We prove in particular that the resultant of the bending energy is directly linked to the wetting angles at the contact line.

Résumé. L'objet didactique de ces développements basés sur la géométrie différentielle des courbes et des surfaces est de présenter des stratégies mathématiques adaptées à l'étude des ponts capillaires. Le fil conducteur est de pouvoir calculer avec précision, dans n'importe quelle situation, la contrainte de flexion de la surface libre d'un pont capillaire, représentée mathématiquement par l'intégrale de courbure de Gauss (courbure totale) de la surface libre intervenant dans l'équation de Young–Laplace généralisée. Nous établissons en particulier un résultat très général suivant lequel la résultante de l'énergie de flexion est directement liée aux angles de mouillage au niveau de la ligne de contact.

Keywords. Distortion of nonaxisymmetric capillary bridges, Mean and Gaussian curvatures impact, Euler characteristic, Generalized Young–Laplace equation, Bending effects, Fenchel's theorem in differential geometry, Gauss–Bonnet Theorem, Geodesic curvature, Bending stress, Influence of the contact angles.

Mots-clés. Distorsion des ponts capillaires non axisymétriques, impact des courbures moyennes et gaussiennes, caractéristique d'Euler, équation de Young–Laplace généralisée, effets de flexion, théorème de Fenchel en géométrie différentielle, théorème de Gauss–Bonnet, courbure géodésique, contrainte de flexion, influence des angles de contact.

2020 Mathematics Subject Classification. 49N45, 53A10, 58E12, 74F10, 74G05, 74G15, 53Z05.

Manuscript received 13 July 2022, revised 8 February 2023 and 23 May 2023, accepted 24 May 2023.

[*] Corresponding author.

1. Introduction

Under boundaryless manifold condition, the most common formulation of the Gauss–Bonnet integration theorem indicates that for a spherical drop or a soap bubble without contact, with or without bump, the integral of the Gaussian curvature[1] over the surface, proportional to the bending energy, is invariant if one bends and deforms the surface (this value is a topological invariant).

Indeed, according to the Gauss–Bonnet integration theorem, for a closed free surface, the energy contribution of the Gaussian curvature during deformation is directly proportional to the Euler constant (see Eq. (2)), and therefore is constant as long as the topology of the surface, described by the Euler characteristic of the surface, does not change [1], and can be ignored when determining the shape of such a membrane.

This has probably favored the in-depth study of constant mean curvature surfaces, excluding gravity effects [2–8]. In the case of capillary bridges, the presence of contact surfaces does not allow this simplification (the total geodesic curvature of the boundary is to be taken into account to apply the Gauss–Bonnet integration theorem) and makes it a priori necessary to take into account the Gaussian curvature, to establish a hierarchy of the various configurations with regard to the bending effects and to introduce the generalized Young–Laplace equation. Some other works proposed a general law for continuum media with interface or a derivation of thermo balance equation for systems with interface [9]. A second order thermodynamical approach has been used to model surface tension of bubbles leading to a generalization of Young–Laplace theory [10], but without involving the Gaussian curvature of the interface.

In this work, we present various results and complementary strategies of mathematical analysis that can be applied to concrete capillary bridges problems, concerning in a new way, the Gauss–Bonnet and Fenchel's theorems to establish various analytical formulas easy to use for capillary bridges. In a first step, we will focus on surfaces of revolution (circular boundaries, which makes these cases much easier, especially for the explicit calculation of the total geodesic curvatures of the boundaries, rarely possible in practice by the integral calculus). We prove that, in the general case including particular axisymmetric capillary bridges, the resultant of the bending energy is directly linked to the wetting angles at the contact line. We also highlight the determining parameters and their respective influence in the bending energy and its variation.

Then the approach is extended to the rather delicate modelling of nonaxisymmetric capillary bridges distortions. The key to achieving generalization is a direct consequence of the Fenchel's theorem in differential geometry which avoids a lot of dead-end integration calculations. These developments relate to surfaces of revolution on the basis of an unit speed reparameterization (or by arc length) for a regular curve, in this case, the semi-meridian. For detailed presentations of the subject, the reader may refer to [11, p. 161-164], [12, p. 161-162] and also to [13, 14].

2. Generalized Young–Laplace equation and associated

The generalized Young–Laplace equation concerns the strong distortions for which the bending effects are modeled by an additional curvature-related term involving the Gaussian curvature K though a multiplier coefficient C_K which has the dimension of a force and stands for the bending stress [15–17]. According to generalized Young–Laplace equation, the downward vertical measurement x in relation with the value Δp_0 at $x = 0$ (a spontaneous unknown value), may be linked to the mean an Gaussian curvature according to [7, 8, 16, 18, 19]:

$$\gamma\left(\frac{1}{\rho_c}+\frac{1}{N}\right)+C_K\frac{1}{\rho_c\, N}=\Delta p_0-\Delta\rho\, gx, \tag{1}$$

[1] The product of the main curvatures.

where C_K has a dimension of a force[2]. Therefore the term $C_K/\rho_c N$ stands *via* a pressure for the local bending stress, where ρ_c and N denote the principal radii of curvature (evaluated algebraically, positively when the curvature is turned into the interior of the capillary bridge). Finally, Δp_0 is the pressure deficiency at $x = 0$ and $\Delta\rho$ the difference of the densities between the fluid and the gas.

It is assumed that the different cefficients, implicit unknown a priori, as Δp_0, resulting from the final equilibrium, have been previously identified *in situ* from exprerimental data, by solving a linear system, well posed and numerically stable (for example, thanks to a first integral and a principle of conservation) [20–22], [23–28].

It is extremely noteworthily [29], that this strongly nonlinear differential equation is mathematically isomorphic (the same structure) but with different variables and physical units, to the Gullstrand equation of geometrical optics, which relates the optic power P'_{op} of a thick lens (in diopters, the reciprocal of the equivalent focal length) to its geometry and the properties of the media. For example, the superficial tension γ is equivalent to the refractivity $\frac{n_1}{n_2} - 1$, where n_i is a refractive index, C_K is analogous to the expression $-(\frac{n_1}{n_2} - 1)^2 \frac{n_2}{n_1} d$, d the lens thickness and Δp_0 corresponds to P'_{op}.

Shear or free energy problems and the longitudinal bending stress of ship hulls have an analogy with the subject [30–34]. The mathematical modeling and simulations of the petroleum engineering are also concerned by this theoretical topic, in order to obtain for media with periodic microstructure an "equivalent" macroscopic representation, by some statistical or homogenization methods [35, Chapter 1].

The bending stress over the free surface Σ may be represented in the following integral form, at the dimension of a force:

$$\mathscr{E}_{bending\,stress} = C_K \int_\Sigma K \, d\Sigma,$$

where K is the Gaussian curvature of the free surface Σ, intrinsic value, in particular independent of the choice of the unit normal vector, and the nondimensional integral is the total curvature[3].

Concerning the capillary tension forces, by term by term integrating over the free surface Σ the generalized Young–Laplace equation, we have for example the relationship between various forces:

$$\gamma \int_\Sigma \left(\frac{1}{\rho_c} + \frac{1}{N}\right) d\Sigma = -C_K \int_\Sigma K \, d\Sigma + \int_\Sigma \left(\Delta p_0 - \Delta\rho \, g x\right) d\Sigma,$$

with the particular situation:

$$\gamma \int_\Sigma \left(\frac{1}{\rho_c} + \frac{1}{N}\right) d\Sigma = -C_K \int_\Sigma K \, d\Sigma + \Delta p_0 \text{ area } (\Sigma)$$

when neglecting gravity effects.

This would allow to have a reasoned opinion on the relative importance of the bending forces; according to an objective criterion, either by relative value or by intrinsic value.

3. Homotopic surfaces, Euler characteristic and Gauss–Bonnet theorem

3.1. *General theory*

Recall that the Euler characteristic (or Euler–Poincaré characteristic) is a topological invariant, an integer that describes, according to precise axiomatic principles, the shape or a structure of a topological space regardless of how it is bent according to the formula: number of

[2] C_K results from the physics at the interface molecular scale. Its sign a priori depends on the dynamics of the wetting (advancing or recessing wetting angle) and on the value of the wetting angle (see Eq. (7)).

[3] For example, the total curvature of the catenoid whose axis is of infinite length is -4π, the total curvature of the sphere of radius r is 4π and the torus 0.

vertices−number of edges+ number of faces with the property of invariance by homeomorphy. It is commonly denoted by χ or $\chi(M)$. As examples for surfaces in homological algebra, we have $\chi(M) = 2$ for a sphere, $\chi(M) = 4$ for two spheres (not connected), $\chi(M) = 0$ for a torus and $\chi(M) = -2$ for a two-holed torus.

To speak very figuratively, quite approximately, the Euler–Poincaré characteristic is an integer, invariant when the size and the shape of a geometrical object change by an effect of a "plastic" deformation.

This invariance property makes it a providential tool in the context of this study on the bending effects, associated to the Gauss–Bonnet theorem, a deep relationship between surfaces in differential geometry, connecting the Gaussian curvature of a surface to its Euler characteristic.

The Euler characteristic of the right cylinder is zero, thus so is that of the cylinder with one or two boundaries. These following free surfaces with two circular boundaries and whose meridian is an arc of Delaunay roulette are considered topologically equivalent (same common topological genus), because it is possible to continuously move one to obtain the other: portion of concave or convex, catenoid or unduloid (the right cylinder being the transition case). Accordingly, these axisymmetric surfaces have in common the same Euler characteristic, in this case, the value zero. It is the same for their continuous axisymmetric smooth deformations by distorting effect of bending or gravity [19, 36].

The Gauss–Bonnet theorem is reputed to be one of the most profound and elegant results of the study of surfaces [1, 11, 13, 37]. It has no surprisingly many applications in Physics. It is used in sectors of activity where the problems of bending beams surely arises (civil engineering, naval architecture, shell theory to predict the stress and the displacement arising in an elastic shell, [37–40], etc. . .).

In fact, it unexpectedly links two completely different ways of studying a surface: one geometric, the other topological. Indeed, for any compact, boundaryless two-dimensional Riemannian manifold Σ, the integral of the Gaussian curvature K over the entire manifold with respect to area measure is 2π times the Euler characteristic of Σ, also called the Euler number of the manifold, i.e.

$$\int_\Sigma K\, d\Sigma = 2\pi\chi(\Sigma)\,. \tag{2}$$

For example, for a sphere Σ of radius R in $\mathbb{R}^3$, it comes:

$$\int_\Sigma K\, d\Sigma = \frac{1}{R^2} 4\pi R^2 = 4\pi \text{ and here } \chi(\Sigma) = 2 \tag{3}$$

Suppose now that M is a compact two-dimensional Riemannian manifold with a boundary δM and let k_g the signed geodesic curvature of δM. Then, in nondimensional writing,

$$\int_M K\, dM + \int_{\delta M} k_g ds = 2\pi\chi(M)\,. \tag{4}$$

We recall that the geodesic curvature k_g, of an arbitrary curve at a point P on a smooth surface, is defined as the curvature at P of the orthogonal projection of the curve onto the plane tangent to the surface at P and we have:

$$k_g = k\cos\theta_g, \tag{5}$$

where θ_g is the angle between the osculating plane of C and the tangent plane Q at point P, which corresponds to the local contact angle for capillary bridges wetting a plane surface. In particular, when the curve C representing δM is a circle of radius R, we have $k = \frac{1}{R}$ and therefore:

$$\int_{\delta M} k_g ds = 2\pi\cos\theta_g \tag{6}$$

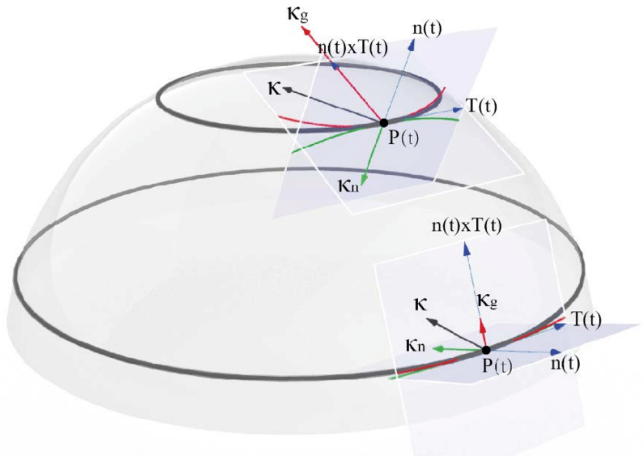


Figure 1. Representation of the geodesic curvature.

3.2. *Application to droplets and capillary bridges*

As explained in the introduction, the case of a spherical droplet (closed surface) of radius R has no interest, as the contribution of the Gaussian curvature is directly proportional to the Euler constant $\chi(M)$ whose value $\chi(M) = 2$ for a sphere[4]. In that case, relation (4) gives no supplementary information, as no wetting area exists.

Let us consider now the case of a spherical droplet lying on a plane surface as represented on Fig. 2(a). The application of Gauss–Bonnet theorem leads to calculate both contributions in the right hand side of equation (4). The first one is directly link the external surface area of the spherical cap representing the droplet and is equal to $2\pi(1-\cos\alpha)$, where α is the opening angle (Fig. 2(a)). The contribution of the geodesic curvature is equal to $2\pi\cos\theta$, according to (6), since the contact line is circular and the droplet lies on a plane surface (2(a)). In that case, (4) leads to the obvious and known geometric relation $\theta = \alpha$. For a droplet lying on the sphere, as the wetted surface is not plane, the geodesic curvature is $k_g = \frac{1}{r_c}\cos\theta_g$ with $\theta_g \neq \theta$. In that case, the Gauss–Bonnet theorem leads to the relation $\theta_g = \frac{\pi}{2} - \delta$.

For axisymmetric capillary bridges whose contact lines are circles, typically capillary bridges between two parallel planes (Fig. 3), we obtain a general expression of the bending stress for a surface of revolution[5]

$$\mathscr{E}_{bending\ stress} = C_k \int_M K\, dM = -2\pi C_k (\cos\theta_1 + \cos\theta_2), \tag{7}$$

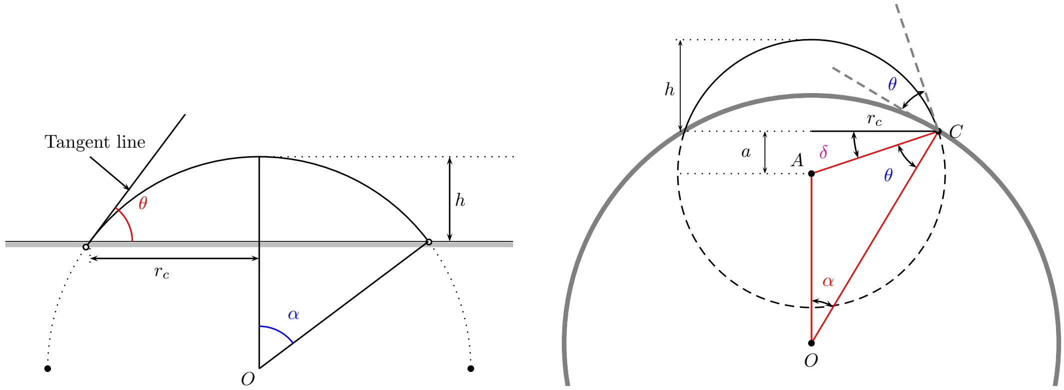
with here the Euler characteristic $\chi(M) = 0$ and when the total geodesic curvature at the boundaries is $2\pi(\cos\theta_1 + \cos\theta_2)$ according to (6).

For symmetric profiles with $\theta_1 = \theta_2 = \theta$, we obtain a relationship between the contact angle θ and the bending stress:

$$\theta = \arccos \frac{\int_M -K\, dM}{4\pi}.$$

[4] $\chi(M) = 1$ for a half-sphere or a cap.

[5] We have denoted θ_1 and θ_2 the upper and lower contact angles assumed to be constant. In the case of a plane wetted surface, θ_g corresponds to the wetting angle θ.



(a) Representation of a droplet on a plane surface.

(b) Representation of a droplet on a sphere.

Figure 2

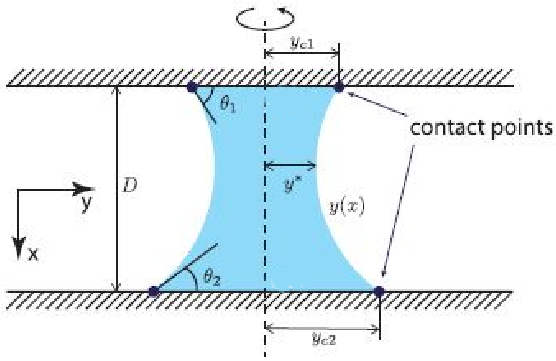


Figure 3. Capillary bridge between two parallel planes.

Coming back to (7) with the same contact angle θ, the relative finite variation of the bending stress, as function of the contact angle θ, is then given by the formula:

$$\frac{\delta\left(\mathscr{E}_{bending\,stress}\right)}{\mathscr{E}_{bending\,stress}} = -\tan\theta\;\delta\theta\;,\;\theta \neq \frac{\pi}{2}. \tag{8}$$

In summary, it should be kept in mind that the value of the bending stress depends, besides physical constants, only on the observed values of the contact angles, whereas these angles result in part implicitly from the final equilibrium of the device.

In the general case of capillary bridges, using the classification and the associated parameterization of [20], the Gauss–Bonnet theorem leads to a supplementary relation linking the geometric properties of the capillary bridge. In the particular case of a catenoid[6], where the parameterization of the meridian is given by

$$y(x) = y^* \cosh\left(\frac{x}{y^*}\right),$$

y^* denoting the gorge radius (Fig. 3), we may obtain an explicit and useful relation between between θ, y^* and the length D of the capillary bridge.

[6] Transition between unduloid or nodoid shape.

In the next, we want to generalize these expressions in any situation, for capillary bridges with non necessarily circular contact lines in non-axisymmetric cases (this is the case for instance when coalescence of capillary bridges occurs [21]).

3.3. *The prevaling roles of the contact angles values and of the convexity/nonconvexity of the boundaries.*

For didactic purposes, we mainly limited ourselves to the detailed case of axisymmetric capillary bridges to show the interest of the simultaneous use of the Gauss–Bonnet formula and of the topological notion of Euler characteristic to evaluate the importance of the bending stress. The obtained result clearly shows in an explicit way, the major role of the contact angles values after distortion effects, eventually distinct (Eqs. (7) and (8)). Therefore, all the factors determining the contact angle have consecutively an influence on the bending stress (surface roughness and heterogeneity, influence of gravity, contact angle hysteresis [4, 30, 41–43]).

It is well known that the contact angle value is determined by the balance between adhesive and cohesive forces on the rigid supports. As the tendency of a drop to spread out over a flat, solid surface increases, the contact angle decreases. Thus, the contact angle provides an inverse measure of wettability. In this context, the case of the right cylinders is still a borderline case.

A contact angle less than $\frac{\pi}{2}$ (low contact angle) usually indicates that wetting of the surface is very favorable, and the fluid will spread over a large area of the surface. Contact angles greater than $\frac{\pi}{2}$ (high contact angle) generally mean that wetting of the surface is unfavorable. It should be quoted that a certain number of terms of the generalized Young–Laplace equation are spontaneous values, resulting from instantaneous equilibrium, and are therefore implicit unknowns. This is a difficulty for the mathematical resolution of this nonlinear differential boundary problem.

In addition, the Fenchel's theorem sheds light on the importance of the convexity or the nonconvexity of the outer edges in calculating exactly the value of the total curvatures.

Consequently, even a limited displacement of the surface boundary can modify the bending stress, by local modifications of the contact angles or affecting the local curvature of the outline curve and then, the total geodesic curvature. The contact angle hysteresis can also be significant.

4. The aim and effective convenience of the arc length reparameterization strategy

The generalization of expression (7) for non necessarily circular contact lines and possibly non closed will be performed using Fenchel's theorem based on an unit speed reparameterization (or by arc length) of the contact line to calculate the total geodesic curvature.

4.1. *Surface of revolution*

The rather paradoxical aspect of this method is that it is only very rarely easy to get in practice an explicit calculation formula. However, it leads to general quantitative results in the form of analytical formulas very easy to use from the experimental data, via a very convenient expression of the Gaussian curvature of a surface of revolution (the speed and acceleration vectors are then orthogonal).

To illustrate what we are talking about, let us consider a smooth curve of the half-plan $\{y > 0, z = 0\}$ parametrized by the arc length. The surface of revolution resulting in $\mathbb{R}^3$ from the rotation of the curve around the x-axis, ψ being the angle of rotation, is parametrized by:

$$M\left((s,\psi)\right) = \left(x(s),\ y(s)\cos\psi,\ y(s)\sin\left(\psi\right)\right)\ 0 \le \psi \le 2\pi, 0 \le s \le L.$$

As the meridian portion is parameterized by arc length, we have *ipso facto* the following remarkable and convenient relations and convenient expressions for angular (in radians) and trigonometric values as well as for the Gaussian curvature of the surfaces of revolution:

$$x'^2 + y'^2 = 1 \text{ at any point,} \tag{9}$$

and therefore, by differentiating, the orthogonality relationship

$$x'x'' + y'y'' = 0,$$

that is to say that $T.\frac{dT}{ds} = 0$ where the dot denotes the scalar product of $\mathbb{R}^3$ and where $T(s) = \frac{dM(s)}{ds}$ is the unit tangent vector to the curve $M(s) = M(s,0)$.

The Gaussian curvature K of the surface of revolution has then the very convenient expression (see [13, p. 162, eq. (9)]):

$$K(s,\theta) = -\frac{y''(s)}{y(s)}.$$

In this context, the expression of the mean curvature H of a surface of revolution is less attractive (see [13, p. 162, eq. (11)]).

A remarkable illustrative example of the arc length reparameterization strategy is the determination of the axisymmetric surfaces of constant Gaussian curvature[7]. We consider then the classical differential equation, linear, of the second order, homogeneous:

$$y''(s) + Ky(s) - 0,\ 0 \leq s \leq L,$$

with the three following cases: $K < 0$, $K = 0$, $K > 0$. Then, introducing the general form of the corresponding solutions in y, we consider the resulting differential equations

$$x'^2 = 1 - y'^2, 0 \leq s \leq L \tag{10}$$

resulting from (9).

Moreover, φ being the angle that the tangent to the profile curve makes with the x-axis, we have the following relationships:

$$sin\varphi = \frac{y'(s)}{\sqrt{x'^2(s) + y'^2(s)}}, \text{i.e. } \sin\varphi = y'(s) \text{ and } \cos\varphi = x'(s).$$

To compute the global bending stress in this context, we have to consider succesively:

$$\mathscr{E}_{bending\,stress} = C_k \int_M K\,dM = C_k \int\int -\frac{y''(s)}{y(s)}\, y(s)\,d\psi\,ds$$

and therefore

$$C_k \int\int -y''(s)\,d\psi\,ds = -2\pi C_k\,\left(y'(L) - y'(0)\right)$$

so that

$$\mathscr{E}_{bending\,stress} = -2\pi(C_k\left(\sin\left(\varphi(L)\right) - \sin\left(\varphi(0)\right)\right),$$

φ being the angle that the tangent to the profile curve makes with the $x-$axis, the axis of rotation.

Let us quote that this case of a surface of revolution around x-axis may correspond to a capillary bridge between two parallel planes at $x = 0$ and $x = L$. With the notation of Fig. 3, we have $\varphi(L) = \pi/2 - \theta_1$ and $\varphi(0) = \theta_2 - \pi/2$ and we recover the general expression (7) for axisymmetric capillary bridges.

[7] Problem studied more extensively by Gaston Darboux 1890. Among the solutions, surfaces are found that look like a hyperboloid.

4.2. *The conclusive Fenchel's theorem for the general case*

In the case of non circular contact lines, for instance portions of an ellipse, the parameterization involves elliptic integrals, rarely possible to explain in practice, so that the parameters would have to be sought numerically (spline interpolation) [44, 45]. This computational difficulty is overcame by knowing the Fenchel's theorem [13, 46], which shows the complementarity of the three methods leading to a generalization of expression (7) for possibly non convex or even non closed contact lines.

To well illustrate the interest of Fenchel's theorem associated with the theorems of Gauss and Bonnet, let us consider a reparameterization by arc length of the curve Γ.

4.2.1. *Close plane curve*

According to Fenchel's theorem[8] (1929), the value of the total curvature

$$\int_\Gamma k(s)ds$$

of any smooth closed space curve Γ is at least 2π, *i.e* $\int_\Gamma k(s)ds \geq 2\pi$. The equality holds if and only if the curve is a convex plane curve. In other words, the average curvature of a closed convex plane curve equals $2\pi/L$, where L is the length (the perimeter) of the curve[9].

By the Fenchel's theorem, without calculation of primitive functions, often tedious or ineffective, we deduce directly, for any closed convex plane curve Γ (i.e. the curve is the boundary of a convex set in the Euclidean plane), that

$$\int_\Gamma k(s)ds = 2\pi.$$

This case is certainly the most encountered in practice when the boundaries of the capillary bridge (the contact lines) are two closed plane convex curves, not necessarily circular. We then recover expression (7) which is still valid in this more general case[10]:

$$\mathscr{E}_{bending\,stress} = C_k \int_M K\, dM = -2\pi C_k\ (\cos\theta_1 + \cos\theta_2). \tag{11}$$

In the case of a closed nonconvex plane curve, we are led to conclude by defining of the notion of the winding index, a topological argument, in what follows.

4.2.2. *Open plane curve*

Let us give some classical preliminary elements of differential geometry related to smooth boundaries of surfaces, parametrized by arc length. The curvature of a plane curve parametrized by arc length is the rate of turning of the tangent line with respect to an *ad hoc* frame along the curve.

Let $\varphi(s)$ be the angle of inclination of the unit tangent vector $T = T(s)$ with respect to a fixed frame of reference, for instance $x-$axis. Considered then as a rate of turning for the tangent line when one moves along the curve at unit speed, the curvature $k(s)$ becomes

$$k(s) = \frac{d\varphi}{ds}(s) = \varphi'(s).$$

[8] The Fary–Milnor theorem concerning the total curvature of the knotted closed curves does not seem appropriate for the subject of this study.

[9] For a given arc of a plane curve, the local average curvature quantifies the ratio of the change in inclination of the tangent to the curve over the arc length.

[10] When the wet solid surface is not plane, relation (11) involves θ_{g1} and θ_{g2} which corresponds to θ_1 and θ_2 to within a constant linked to the geometry of the wet solid interface.

It follows that the total curvature of a smooth curve C is then given by the formula depending only of the initial and final states:

$$\int_C k(s)\,ds = \varphi(ending) - \varphi(starting), \text{ (in radians)}. \tag{12}$$

For a piecewise smooth curve parametrized by arc length, then we need to deal with the exterior angles at the corners according to the orientation of the curve in the turning motion. However, up to now, to our knowledge such capillary bridges with non convex or open contact line have are not considered in literature.

5. The general case and its implementation

In the general cases of nonaxisymmetric capillary bridges between two supports, possibly of distinct natures, the method remains applicable in principle. The difficulty is not conceptual in dealing with the general case but rather calculative. We must then, in any given case, engage in a delicate exercise in differential and analytical geometries to explicitly calculate the total signed geodesic curvature of the boundaries by the classical methods of analytical geometry.

5.1. *The calculation procedure is as follows*

At any point P of the border liquid-solid, one considers the tangent plane in P to the free surface (that supposes an adequate local regularity). One then considers the orthogonal projection of each edge into this tangent plane. The curvature in P of the projected curve is then calculated, what introduces the important role of the cosine of the local contact angle and leads to expression (5) of the the curvature k which related related to the geodesic curvature k_g at P by the relationship:

$$k_g = k\cos\theta_g$$

where θ_g is linked to the local contact angle θ to within a constant depending on the geometry of the wet solid surface. When it is plane, θ_g corresponds to the wetting angle θ, that will be considered in the next to simplify the developments.

When the contact angle is constant on the considered contact surface, we have the particularly simple relationship:

$$\int_{\Gamma_i} k_g(s)\,ds = cos\theta \int_{\Gamma_i} k(s)\;ds.$$

5.2. *The special situation of heterogeneous contact surfaces*

When the contact angles are separately variable on each of the contact surfaces, i.e. $\theta = \theta_1(M)$ and $\theta = \theta_2(M)$ according to the physical conditions of the two surfaces (non ideal smooth surfaces), the integral along each boundary, corresponding to the total geodesic curvature, in fact, of the kind

$$\int_{\Gamma_i} cos\theta_i(M(s))\; k(s)\; ds, \tag{13}$$

is more complicated to calculate with computational prediction of wetting (at our knowledge, an open problem for the probably most realistic case). The use of a mean theorem would likely be imprecise (effects of surface roughness).

5.3. *The general case of homogeneous contact surfaces*

When multiplied by the coefficient $(-C_K)$ at the dimension of a force, the dimensionless integral of these curvature values along the reunion of the two contact edges gives finally the value of the resulting bending stress by the Fenchel's theorem (the cornerstone of the method).

The three possible scenarios then arise according to the geometry of the boundaries (closed plane convex or nonconvex curves) are the following, the surfaces having in common, without loss of generality, the same Euler characteristic, in this illustrative case, the value zero.

By introducing the contact angles θ_1 and θ_2 (in radians) on each outline of contact surfaces, we proved that, at least theoretically, the wettability being evaluated, here, by constant contact angles, separately on each contact support.

Case 1: The boundaries are two closed plane convex curves. Then,

$$\mathcal{E}_{bending\,stress} = C_k \int_M K\, dM = -2\pi C_k\ (\cos\theta_1 + \cos\theta_2)\,.$$

Case 2: The boundaries are two closed plane curves, one convex and the other nonconvex. Then,

$$\mathcal{E}_{bending\,stress} = C_k \int_M K\, dM = -C_k\ (2k_1\pi\cos\theta_1 + 2\pi\cos\theta_2)\,,$$

the observed integer k_1, $k_1 \geq 2$, being the winding number of the nonconvex curve (the winding index in algebraic topology).

Case 3: The boundaries are two closed disjoint plane curves, nonconvex. Then,

$$\mathcal{E}_{bending\,stress} = C_k \int_M K\, dM = -C_k\ (2k_1\pi\cos\theta_1 + 2k_2\pi\cos\theta_2)\,,$$

k_1 and k_2 being the integers, ≥ 2, winding numbers of the curves, observed and known *in situ.*

In the rather theoretical case, where the value of the Euler characteristic is non-zero, it should be necessary to write:

$$\mathcal{E}_{bending\,stress} = C_k \int_M K\, dM = 2\pi\chi(M) - C_k\ (2k_1\pi\cos\theta_1 + 2k_2\pi\cos\theta_2)\,.$$

It must be emphasized that, when the contact angles are separately variable on each of the contact surfaces according to the physical conditions of the two surfaces (non ideal smooth surfaces), the integral along each boundary, corresponding to the total geodesic curvature of the plane and closed boundaries, seems a serious difficulty to explain. The question might interest specialists in differential geometry.

6. Conclusion

The developments obtained here for surfaces of revolution and their generalization to more general surfaces representing the shape of capillary bridges, result from concepts in differential geometry and geometric analysis with applications to Lagrangian Mechanics, without resorting to differential calculus and integral calculus. The methods of Euler's characteristic, associated to the Gauss–Bonnet–Binet theorem and the strongly complementary Fenchel's theorem apply immediately to the cases of the nonaxisymmetric surfaces, with explicit, easy-to-use, results formulations.

We proved that in the general way, the value of the bending stress depends, besides physical constants, only on the observed values of the contact angles, whereas these angles result in part implicitly from the final equilibrium of the device. Therefore, all the factors determining the contact angle have an influence on the bending stress (surface roughness and heterogeneity, influence of gravity, contact angle hysteresis.

It would be interesting to reconsider, in taking into account these new results concerning the bending effects, the important role of the contact curves geometry and the Gauss–Bonnet and Fenchel theorems, an analytical framework for reassessing the cohesion effects of coalescence between saddle shaped capillary bridges [21]. Finally, by creating a support material having a nonconvex region with high wettability and a complementary region with very low wettability, the experimenter could illustrate the theory by experimentation.

Conflicts of interest

The authors have no conflict of interest to declare.

Acknowledgments

The authors would like to express their sincere gratitude to the French National Centre for Space Studies (CNES) for having supported this work. They also thank Antoine Moreau (LaSIE) and Marc Medale (IUSTI) for valuable discussions.

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