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Bornes sur le temps d'explosion pour une équation intégro-différentielle de type parabolique avec un terme réactif variable

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Abstract. This paper is concerned with the blow-up time of the solutions to an integro-differential problem of parabolic type with variable growth if blow-up occurs. By using the differential inequality technique, we obtain lower bounds for the blow-up time and some global existence results under some conditions to variable exponent of reaction, memory kernel, and initial value.

Résumé. Cet article traite du temps d'explosion des solutions d'un problème intégro-différentiel de type parabolique avec une croissance variable si l'explosion se produit. En utilisant la technique de l'inégalité différentielle, nous obtenons des bornes inférieures pour le temps d'explosion et des résultats d'existence globale sous certaines conditions sur l'exposant variable de réaction, le noyau de mémoire et la valeur initiale.

Keywords. Integro-differential problem, Parabolic, Variable reaction, Global existence, Blow-up, Lower bounds.

Mots-clés. Problème intégro-différentiel, Parabolique, Réaction variable, Existence globale, Explosion, Limites inférieures.

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1. Introduction

In this paper, we study the lower bounds for the blow-up time of a class integro-differential problem of parabolic type with variable reaction

$$\begin{cases} u_t - \Delta u + \displaystyle\int_0^t g\,(t-s)\,\Delta u\,(x,s)\,\mathrm{d}s = |u|^{p(x)-2}\,u, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ ($N \geq 1$), $T \in (0,\infty]$, the initial value $u_0 \in H_0^1(\Omega)$, $p : \Omega \to (1,\infty)$ measurable function, $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing C^1 function (memory kernel) satisfying some additional conditions to be specified later.

  In mathematical point of view, equations of the types (1) with variable exponent (or variable source) are usually referred to as equations with nonstandard growth conditions. Under certain conditions on the initial data and certain ranges of exponents, the existence, uniqueness, and other qualitative properties of solutions for parabolic and hyperbolic equations with variable nonlinearity have been studied by many authors (see [1–10] and references therein). These types of problems appear in many modern physical and engineering models such as electrorheological fluids (smart fluids), fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media, image processing, nuclear science, chemical reactions, heat transfer, population dynamics, biological sciences, etc., and have attracted a great deal of attention in the literature. More applications and details on the subject can be found in [11–17] and the references therein.

  Problem (1) arises from many important mathematical models in engineering and physical sciences. For example, in the study of heat conduction in materials with memory, the classical Fourier's law is replaced by the following form (cf. [18])

$$q = -d\nabla u - \int_{-\infty}^t g\,(x,t-s)\,u\,(x,s)\,\mathrm{d}s, \tag{2}$$

where q is proportional to the temperature differences per unit length, u is the temperature, d is the diffusion coefficient and g is a relaxation kernel. Here (2) means that q is not linearly dependent on ∇u. Then substituting Fourier's law (2) into the conservation of heat law, we can deduce

$$u_t - \Delta u + \int_0^t g\,(t-s)\,\Delta u\,(x,s)\,\mathrm{d}s = |u|^{p-2}u, \quad \text{in } \Omega \times (0,T), \tag{3}$$

which is the linear form of (1) for $p(x) = p$ for all $x \in \overline{\Omega}$. Results concerning the controllability of Equation (3) are given in [19,20]. In recent years, semilinear parabolic problems

$$\begin{cases} u_t - \Delta u + \displaystyle\int_0^t g\,(t-s)\,\Delta u\,(x,s)\,\mathrm{d}s = |u|^{p-2}\,u, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega \end{cases} \tag{4}$$

with a memory term associated with the Laplace operator and source term with Dirichlet type condition in the different cases of the values of the memory kernel, specifically when $g = 0$ or $g > 0$, have been studied by many authors. By assuming suitable conditions on g, p, and u_0, using some known theorems in the mathematical literature, the global existence in time, blow-up in finite time, the asymptotic behavior, and a lower bound for the blow-up time of the unique weak solution have been discussed. When $g = 0$, problem (4) has been studied by various authors and several results of global and nonglobal existence, have been established. For instance, in the early 1970s, Levine [21] introduced the concavity method and proved that solutions with negative energy blow-up in finite time. Later, this method was improved by Kalantarov and

Ladyzhenskaya [22] to accommodate more general situations. Ball [23] also studied (4) with $f(u, \nabla u)$ instead of $|u|^{p-2}u$ and established a nonglobal existence result in bounded domains. Recently, for problem (4), Payne *et al.* [24, 25] obtained both the lower and the upper bounds for blow-up time when the blow-up occurs.

When the viscoelastic term g is positive, in [26], Messaoudi concerned with the finite-time blow-up of solutions for the initial boundary value problem (4), where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1 function, $p > 2$, and Ω is a bounded domain of $\mathbb{R}^N$ ($N \geq 1$), with a smooth boundary $\partial\Omega$. Under suitable conditions on g and p, the author proved a blow-up result for certain solutions with positive initial energy.

Tian [27] considered the problem (4) with $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1 function, $p > 2$, and $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary, $T \in (0, +\infty]$, the initial value $u_0 \in H_0^1(\Omega)$. For $N \geq 1$, by using the differential inequality technique, the author obtained a lower bound for blow-up time of the solution if blow-up occurs. Also, established a new blow-up criterion and gave an upper bound for blow-up time of the solution under some conditions on p, g, u_0.

In [28], the author generalized the results obtained by Tian in [27] using $|u|^{p(x)-2}u$ instead of $|u|^{p-2}u$. Using energy methods, the author obtained a lower bound for blow-up time of the solution if blow-up occurs. Furthermore, assuming the initial energy is negative established a new blow-up criterion and gave an upper bound for blow-up time of the solution to the problem (1).

In our this paper, we study some global existence results and give sufficient conditions on the variable reaction $p(\cdot)$, memory term g for the lower bounds at the time blow-up in L^2 of the problem (1). We extend the value range of the p in the conditions given in [27] and [28].

2. Main results and proofs

In this section, we list and recall some well-known results and facts from the theory of Sobolev spaces with a variable exponent (for details, see [29, 30]).

Let $p : \Omega \to (1, \infty)$ be a measurable function. We introduce p^- and p^+ such that

$$1 < p^- := \operatorname*{ess\,inf}_{x \in \Omega} p(x) \leq p(x) \leq p^+ := \operatorname*{ess\,sup}_{x \in \Omega} p(x) < \infty. \tag{5}$$

Throughout this paper, Ω is considered to be a bounded domain of $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$, assuming that $p(\cdot)$ is a measured function on $\overline{\Omega}$ and satisfies the following logarithmic continuity condition (see [29]):

$$\left| p(x) - p(y) \right| \leq \frac{C_{\log}}{\left| \log |x - y| \right|} \quad \text{for all } x, y \in \Omega, \, |x - y| < \frac{1}{2}, \, C_{\log} > 0. \tag{6}$$

We denote $L^{p(\cdot)}(\Omega)$ the set of measurable real-valued functions u on Ω such that

$$\rho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} \, \mathrm{d}x < \infty.$$

The variable exponent space $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf \left\{ a > 0, \rho_{p(\cdot)} \left(\frac{u}{a} \right) \leq 1 \right\},$$

is a Banach space and it is called variable exponent Lebesgue space.

Next, we denote $L^p(\Omega)$ by L^p and $H_0^1(\Omega)$ by H_0^1, the usual Sobolev space. The norm and inner of $L^p(\Omega)$ are denoted by $\|u\|_{L^p(\Omega)} := \|u\|_p \left(1 \leq p \leq \infty \right)$ and $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the equivalent norm $\|u\|_{H_0^1} = \|\nabla u\|_2$.

2.1. *Global existence*

In this section, we first show the global existence result for $N \geq 1$. The main idea of this section is the comparison principle. Let $\varphi(x)$ satisfies the following elliptic problem:

$$-\Delta\varphi = 1, \ x \in \Omega; \quad \varphi(x) = 1, \ x \in \partial\Omega.$$

By using the result in [31], we can see that the above nonlinear problem has an unique solution, and the following inequalities hold:

$$M := \max_{x \in \overline{\Omega}} \varphi(x) < +\infty, \quad \varphi(x) > 1, \ x \in \Omega; \quad \nabla\varphi \cdot v < 0, x \in \partial\Omega.$$

We assume that:
  (G) The memory kernel $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^1\left(\mathbb{R}^+\right)$ function satisfying

$$g(t) \geq 0, g'(t) \leq 0 \quad \text{and} \quad 1 - \int_0^\infty g(s)\,\mathrm{d}s = l > 0.$$

Theorem 1. *Let u be a nonnegative solution of the problem (1) , memory kernel g satisfies (G), and function p satisfies the conditions (5) and (6). If $1 < p(x) < \infty$ for all $x \in \overline{\Omega}$, u remains globally bounded.*

Proof. Let $1 < p(x) < \infty$ for all $x \in \overline{\Omega}$. Define

$$\overline{u} = A\varphi(x),$$

where

$$A \geq \max\left\{\left(\frac{l}{M^{p^+}}\right)^{\frac{1}{p^+-1}}, \max_{x\in\overline{\Omega}}u_0(x), 1\right\}.$$

It can be checked that

$$\Delta\overline{u} + \overline{u}^{p(x)} - \int_0^t g(t-s)\Delta\overline{u}(x,s)\,\mathrm{d}s \leq -A + M^{p^+}A^{p^+} + A\int_0^\infty g(\xi)\,\mathrm{d}\xi$$

$$\leq -A + M^{p^+}A^{p^+} + A(1-l) = -lA + M^{p^+}A^{p^+}$$

$$\leq \overline{u}_t = 0,$$

in $\Omega \times (0, T)$ and $\overline{u} \geq 0$ on $\partial\Omega \times (0, T)$, and $\overline{u}(x,0) \geq u_0(x)$ in Ω. By the comparison principle, $\overline{u}(x)$ is a bounded supersolution of (1). The proof of Theorem 1 is completed. $\qquad\square$

2.2. *Blow-up in finite time for any initial data*

In this section, we derive a lower bound for T if the weak solution $u(x,t)$ of (1) blows up in finite time T. We start with a local existence result for the problem (1), which is a direct result of the existence theorem by [32,33].

Theorem 2. *For all $u_0 \in H_0^1(\Omega)$, problem (1) possesses a weak solution u on $[0, T_0]$ satisfying:*

$$u \in C_\omega\left([0, T_0]; W^{1,2}(\Omega)\right) \cap C\left([0, T_0]; L^{p(\cdot)}(\Omega)\right) \cap W^{1,2}\left([0, T_0]; L^2(\Omega)\right),$$

where $T_0 \in (0, T]$ is a suitable number and $C_\omega(X)$ the set of all real functions defined on X such that their growths are tempered by the modulus of continuity ω with $\omega(0) = 0$, $\omega(\varepsilon) > 0$ for $\varepsilon > 0$.

The energy functional corresponding to problem (1) is

$$E(t) = \frac{1}{2}\left(g \circ \nabla u\right)(t) + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,\mathrm{d}x, \tag{7}$$

where

$$\left(g \circ \nabla u\right)(t) = \int_0^t g\left(t-s\right) \int_\Omega |\nabla u\left(t\right) - \nabla u\left(s\right)|^2 \, dx ds$$

$$= \int_0^t g\left(t-s\right) \|\nabla u\left(t\right) - \nabla u\left(s\right)\|_2^2 \, ds. \tag{8}$$

Lemma 3. *Let $u(x,t)$ be a nonnegative solution of the problem (1), then E is a nonincreasing function on $[0, T]$.*

Proof. We define as $f(x,s) = |s|^{p(x)-2} s$ and $F(x,s) = \int_0^s f(x,\zeta) \, d\zeta$ for all $x \in \Omega$. Multiplying $u_t(x,t)$ on both sides of (1) and integrating over Ω, we can obtain

$$\int_\Omega u_t^2(x,t) \, dx = \int_\Omega \Delta u(x,t) u_t(x,t) \, dx - \int_\Omega u_t(x,t) \left(\int_0^t g(t-s) \Delta u(x,s) \, ds\right) dx$$

$$+ \int_\Omega u_t(x,t) f(x,u) \, dx$$

$$= \int_\Omega f(x,u) u_t dx - \int_\Omega \nabla u \nabla u_t dx + \int_\Omega \left(\int_0^t g(t-s) \nabla u(s) \nabla u_t(t) \, ds\right) dx$$

$$= -\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |\nabla u|^2 \, dx\right) + \frac{d}{dt} F(x,u) dx$$

$$+ \frac{d}{dt}\left(\int_0^t g(t-s) \int_\Omega \nabla u_t(t) \nabla u(s) \, dx ds\right).$$

The last term of the right-hand side of the above equality can be rewritten as

$$\int_0^t g(t-s) \int_\Omega \nabla u_t(t) \nabla u(s) \, dx ds = -\frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-s) \int_\Omega |\nabla u(t) - \nabla u(s)|^2 \, dx ds\right)$$

$$+ \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s) \, ds \int_\Omega |\nabla u(t)|^2 \, dx\right)$$

$$+ \frac{1}{2}\frac{d}{dt}\left(\int_0^t g'(t-s) \int_\Omega |\nabla u(t) - \nabla u(s)|^2 \, dx ds\right)$$

$$- \frac{1}{2} g(t) \int_\Omega |\nabla u(t)|^2 \, dx.$$

Combining the above two equalities with (7) and (8), we see that

$$E'(t) = -\frac{1}{2} g(t) \|\nabla u(t)\|^2 + \frac{1}{2}\left(g' \circ \nabla u\right)(t) - \int_\Omega u_t^2 dx \le 0,$$

for smooth solutions. Thus $E(t)$ is a nonincreasing function on $[0, T]$, which implies that $E(t) \le E(0)$. The same result can be established for strong solutions for almost every t, by a standard density argument. The proof of the Lemma 3 is completed. $\qquad\square$

Now, we give a sufficient condition for the nonnegative solution of problem (1) to blow-up in L^2-norm and establish a lower bound for the blow-up time.

Theorem 4. *Let $u(x,t)$ be a nonnegative solution of problem (1) in a bounded domain $\Omega \subset \mathbb{R}^N$ and $u_0 \in H_0^1(\Omega)$ such that $\|u_0\|_2 \ne 0$, function p satisfies the condition (5) and g satisfies condition (G). We define $\varphi(t)$ as following*

$$\varphi(t) = \int_\Omega u^2 dx.$$

(i) If $N \ge 3$ and

$$1 < p^-, \quad p^+ < \frac{2N-3}{N-2},$$

then a lower bound for the time of blow-up for any solution which blows up in L^2 norm is given by

$$\int_{\|u_0\|^2}^{+\infty} \frac{\mathrm{d}\xi}{K_4 \xi^{\frac{3(N-2)}{3N-8}} + K_3} \le T, \tag{9}$$

where $\|u_0\|^2 = \int_\Omega u_0^2 \mathrm{d}x$ and K_3, K_4 are positive constants which will be stated later.
   (ii) If $N = 1, 2$ and

$$1 < p^-, \quad p^+ < 4,$$

then a lower bound for the time of blow-up for any solution which blows up in L^2 norm is given by

$$\int_{\|u_0\|^2}^{+\infty} \frac{\mathrm{d}\xi}{K_5 \xi^{\frac{2}{4-p^+}} + K_3} \le T,$$

where K_3 is the constant in (9) and K_5 is a positive constant which will be stated later.

Remark 5. Let g satisfies condition (G). Parameter p had given as

$$2 < p < 2 + \frac{4}{N}, \quad N \ge 3 \text{ or } 2 < p < 4, \ N = 1, 2$$

in [27] and function $p(\cdot)$ had given as

$$2 < p^- \le p^+ < 2 + \frac{4}{N}, \quad N \ge 3 \text{ or } 2 < p^- \le p^+ < 4, \ N = 1, 2$$

in [28] to obtain blow-up in finite time T for problem (4) and (1) respectively.

Our in this paper, we extend the value range of the p in the conditions given in [27] and [28].

Proof of Theorem 4. (i) Let $N \ge 3$. By multiplying the equation in (1) by $u(x, t)$, integrating by parts over Ω, one obtains that

$$\begin{aligned}
\varphi'(t) &= 2\int_\Omega u(x,t)\, u_t(x,t)\, \mathrm{d}x \\
&= 2\int_\Omega u\left(\Delta u - \int_0^t g(t-s)\,\Delta u(x,s)\,\mathrm{d}s + |u|^{p(x)-2}u\right)\mathrm{d}x \\
&= -2\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + 2\int_\Omega |u|^{p(x)}\,\mathrm{d}x + I, \tag{10}
\end{aligned}$$

where

$$I = 2\int_0^t g(t-s)\int_\Omega \nabla u(t)\,(\nabla u(s) - \nabla u(t))\,\mathrm{d}x\mathrm{d}s.$$

Furthermore, by the condition (5), we derive

$$\begin{aligned}
\int_\Omega u^{p(x)}\,\mathrm{d}x &= \int_{\Omega \cap \{x:u\ge 1\}} u^{p(x)}\,\mathrm{d}x + \int_{\Omega \cap \{x:u<1\}} u^{p(x)}\,\mathrm{d}x \\
&\le \int_\Omega u^{p^+}\,\mathrm{d}x + |\Omega|. \tag{11}
\end{aligned}$$

Then by (10) and (11), we obtain

$$\begin{aligned}
\varphi'(t) &\le -2\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + 2\int_\Omega u^{p^+}\,\mathrm{d}x + I + K_0 \\
&\le -2l\,\|\nabla u(t)\|_2^2 + 2\int_\Omega u^{p^+}\,\mathrm{d}x + I + K_0,
\end{aligned}$$

where

$$K_0 = 2|\Omega| > 0.$$

By Hölder's inequality and the condition (G), we know

$$|I| \le 2l \int_0^t g(s)\, ds \, \|\nabla u(t)\|_2^2 + \frac{1}{2l}\left(g \circ \nabla u\right)$$

$$\le 2l(1-l)\|\nabla u(t)\|_2^2 + \frac{1}{2l}\left(g \circ \nabla u\right).$$

From (7), (8) and Lemma 3, we have

$$\frac{1}{2l}\left(g \circ \nabla u\right) = \frac{1}{l}\left[E(t) - \frac{1}{2}\|\nabla u(t)\|_2^2\left(1 - \int_0^t g(s)\, ds\right) + \int_\Omega \frac{1}{p(x)} u^{p(x)}\, dx\right]$$

$$\le \frac{1}{l}E(0) - \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{lp^-}\int_\Omega u^{p^+}\, dx + K_1, \tag{12}$$

where

$$K_1 = \frac{|\Omega|}{lp^-} > 0.$$

From (10) and (12), we obtain

$$\varphi'(t) \le -2l\|\nabla u(t)\|_2^2 + 2\int_\Omega u^{p^+}\, dx + 2l(1-l)\|\nabla u(t)\|_2^2$$

$$+ \frac{1}{l}E(0) - \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{lp^-}\int_\Omega u^{p^+}\, dx + K_2$$

$$\le -2l\|\nabla u(t)\|_2^2 + 2l(1-l)\|\nabla u(t)\|_2^2 + \frac{1}{l}E(0)$$

$$- \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{2lp^- + 1}{lp^-}\int_\Omega u^{p^+}\, dx + K_2$$

$$= -\frac{4l^2 + 1}{2}\|\nabla u(t)\|_2^2 + K\int_\Omega u^{p^+}\, dx + K_3, \tag{13}$$

where

$$K = \frac{2lp^- + 1}{lp^-} > 0,$$

$$K_2 = K_0 + K_1,$$

and

$$K_3 = K_2 + \frac{1}{l}E(0) \quad \text{with } \max\left\{\frac{1}{l}E(0), 0\right\} \ge 0.$$

By using the Hölder and Young inequalities we get

$$K\int_\Omega u^{p^+}\, dx \le K\left(\int_\Omega u^{\frac{2N-3}{N-2}}\, dx\right)^{a_1} |\Omega|^{a_2}$$

$$\le a_1 K\int_\Omega u^{\frac{2N-3}{N-2}}\, dx + a_2 |\Omega|, \tag{14}$$

where

$$a_1 = \frac{(N-2)p^+}{2N-3}, \quad a_2 = 1 - \frac{2(N-2)p^+}{2N-3}.$$

We now make use of Schwarz's inequality to the first term on the right hand side of (14) as follows:

$$a_1 K\int_\Omega u^{\frac{2N-3}{N-2}}\, dx \le a_1 K\left(\int_\Omega u^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega u^{\frac{2(N-1)}{N-2}}\, dx\right)^{\frac{1}{2}}$$

$$\le a_1 K\left(\int_\Omega u^2 dx\right)^{\frac{3}{4}}\left(\int_\Omega u^{2^*}\, dx\right)^{\frac{1}{4}}, \tag{15}$$

where $2^* = (2N/N-2)$. Next, by using the Sobolev inequality (see [34]), for $N \ge 3$, we get

$$\|u\|_{2^*} \le B\|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega), \tag{16}$$

where

$$B = \left(\frac{1}{N(N-2)\pi} \right)^{1/2} \left(\frac{N!}{2\Gamma\left(\frac{N}{2}+1\right)} \right)^{1/N} > 0,$$

is the best constant in the Sobolev inequality. By inserting the (16) to the last inequality in (15), we have

$$a_1 K \int_\Omega u^{\frac{2N-3}{N-2}} \, dx \le a_1 K B^{\frac{N}{2(N-2)}} \varphi^{\frac{3}{4}}(t) \|\nabla u(t)\|_2^{\frac{N}{2(N-2)}}. \tag{17}$$

Now, we can use the Young inequality to inequality (17) and we get

$$a_1 K \int_\Omega u^{\frac{2N-3}{N-2}} \, dx \le \frac{a_1 K (3N-8)}{4(N-2)} B^{\frac{N}{2(N-2)}} \left(\frac{4\varepsilon(N-2)}{3N-8} \right)^{-\frac{N}{3N-8}} \varphi^{\frac{3(N-2)}{3N-8}}(t)$$
$$+ a_1 K \varepsilon \|\nabla u(t)\|_2^2. \tag{18}$$

Combining (13), (14) and (18), we have ε is a positive constant to be determined later:

$$\varphi'(t) \le -\frac{4l^2+1}{2} \|\nabla u(t)\|_2^2 + \frac{a_1(3N-8)K}{4(N-2)} B^{\frac{N}{2(N-2)}} \left(\frac{4\varepsilon(N-2)}{3N-8} \right)^{-\frac{N}{3N-8}} \varphi^{\frac{3(N-2)}{3N-8}}(t)$$
$$+ a_1 K \varepsilon \|\nabla u(t)\|_2^2 + K_3. \tag{19}$$

If we choose $\varepsilon > 0$ stated in (19) such that

$$\varepsilon = \frac{1+4l^2}{2a_1 K},$$

then we obtain the ordinary differential inequality

$$\varphi'(t) \le K_4 \varphi^{\frac{3(N-2)}{3N-8}}(t) + K_3, \tag{20}$$

where

$$K_4 = \frac{a_1(3N-8)K}{4(N-2)} B^{\frac{N}{2(N-2)}} \left(\frac{2\left(1+4l^2\right)(N-2)}{a_1 K (3N-8)} \right)^{-\frac{N}{3N-8}} > 0.$$

An integration of the differential inequality (20) from 0 to t, we obtain the following inequality

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\xi}{K_4 \xi^{\frac{3(N-2)}{3N-8}} + K_3} \le t,$$

which with $\lim_{t \to T^-} \varphi(t) = +\infty$ implies that

$$\int_{\varphi(0)}^{+\infty} \frac{d\xi}{K_4 \xi^{\frac{3(N-2)}{3N-8}} + K_3} \le T,$$

where $\varphi(0) = \int_\Omega u_0^2(x) \, dx$. Note that $3(N-2)/(3N-8) > 1$ $(N \ge 3)$, hence the left-hand side of the above inequality is finite.

(ii) Let $N = 1, 2$. Now, recall the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$ which provide the inequality

$$\|u\|_\infty \le B \|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega), \tag{21}$$

where $B > 0$ is the best constant of the Sobolev embedding. Using (21) and Hölder's inequality to $\|u\|_{p^+}^{p^+}$ which is in (13), we show that

$$\|u\|_{p^+}^{p^+} = \|u\|^2 \|u\|_\infty^{p^+-2} = \varphi(t) \|u\|_\infty^{p^+-2}$$
$$\le \varphi(t) B^{p^+-2} \|\nabla u\|_2^{p^+-2}. \tag{22}$$

Now, by using Young's inequality to (22), we have for all $\varepsilon > 0$,

$$\|u\|_{p^+}^{p^+} \le \frac{4-p^+}{2} \left(\frac{2\varepsilon}{p^+-2} \right)^{-\frac{p^+-2}{4-p^+}} \varphi(t)^{\frac{2}{4-p^+}} + \varepsilon \|\nabla u\|_2^2. \tag{23}$$

From (13) and (23), we have

$$\varphi'(t) \leq -\frac{4l^2+1}{2}\|\nabla u(t)\|_2^2 + K\int_\Omega u^{p^+}\,\mathrm{d}x + K_3$$

$$= -\frac{4l^2+1}{2}\|\nabla u(t)\|_2^2 + \frac{(4-p^+)K}{2}\left(\frac{2\varepsilon}{p^+-2}\right)^{-\frac{p^+-2}{4-p^+}}\varphi(t)^{\frac{2}{4-p^+}}$$

$$+ \varepsilon K\|\nabla u\|_2^2 + K_3.$$

Let us choose $\varepsilon > 0$ such that

$$\varepsilon = \frac{4l^2+1}{2K}.$$

Then we obtain the ordinary differential inequality

$$\varphi'(t) \leq K_5\varphi(t)^{\frac{2}{4-p^+}} + K_3, \tag{24}$$

where

$$K_5 = \frac{(4-p^+)K}{2}\left(\frac{4l^2+1}{K(p^+-2)}\right)^{-\frac{p^+-2}{4-p^+}} > 0.$$

By integration of the differential inequality (24) from 0 to t, we obtain the following inequality

$$\int_{\varphi(0)}^{\varphi(t)} \frac{\mathrm{d}\xi}{K_5\zeta^{\frac{2}{4-p^+}} + K_3} \leq t,$$

which with $\lim_{t\to T^-}\varphi(t) = +\infty$ implies

$$\int_{\varphi(0)}^{+\infty} \frac{\mathrm{d}\zeta}{K_5\zeta^{\frac{2}{4-p^+}} + K_3} \leq T$$

with $p^+ \in (2,4)$ and $N = 1, 2$. Thus the Theorem 4 is proved. $\qquad\square$

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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