INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mécanique

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Stochastic control with state constraints via the Fokker–Planck equation. Application to renewable energy plants with batteries

Published online: 23 February 2024

Part of Special Issue: The scientific legacy of Roland Glowinski
Guest editors: Gregoire Allaire (CMAP, Ecole Polytechnique, Institut Polytechnique de Paris, Palaiseau, France),
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https://doi.org/10.5802/crmeca.236

The Comptes Rendus. Mécanique *are a member of the Mersenne Center for open scientific publishing*
www.centre-mersenne.org — e-ISSN : 1873-7234

Comptes Rendus
Mécanique
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Research article / *Article de recherche*

The scientific legacy of Roland Glowinski / *L'héritage scientifique de Roland Glowinski*

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Contrôle stochastique avec contraintes d'état via l'équation de Fokker–Planck. Application aux centrales d'énergie renouvelable avec batteries

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Abstract. Although renewable energies are beneficial to reduce carbon emissions, its intermittent characteristics may result in power-supply issues in distribution grid. Battery energy storage system is generally regarded as an effective tool to deal with them. On the other hand mathematical modelling, numerical simulation, optimization and control theory are nowadays of paramount importance to handle this kind of problems and related issues. In this paper we present a methodology for the development of bidding strategies and real-time control for electricity producers in a competitive electricity marketplace. Firstly, a stochastic model of a wind power plant with battery storage is stated in the framework of stochastic differential equations (SDE). Then, a stochastic control problem with state constraints is introduced and the corresponding optimality conditions involving the Hamilton–Jacobi–Bellman equation are deduced. For this purpose, advantage is taken from the fact that optimal control problems for *stochastic ordinary* differential equations (SDE) can be equivalently formulated as optimal control problems for *deterministic partial* differential equations (PDE), namely, the corresponding Fokker–Planck equation.

Résumé. Bien que les énergies renouvelables permettent de réduire les émissions de carbone, leurs caractéristiques intermittentes peuvent entraîner des problèmes d'approvisionnement en électricité dans les réseaux de distribution. Le système de stockage d'énergie par batterie est généralement considéré comme un

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ISSN (electronic) : 1873-7234 https://comptes-rendus.academie-sciences.fr/mecanique/

outil efficace pour y remédier. D'autre part, la modélisation mathématique, la simulation numérique, l'optimisation et la théorie du contrôle sont aujourd'hui d'une importance capitale pour traiter ce type de problèmes et les questions connexes. Dans cet article, nous présentons une méthodologie pour le développement de stratégies de soumission et de contrôle en temps réel pour les producteurs d'électricité sur un marché de l'électricité concurrentiel. Tout d'abord, un modèle stochastique d'une centrale éolienne avec stockage sur batterie est présenté dans le cadre des équations différentielles stochastiques (EDS). Ensuite, un problème de contrôle stochastique avec des contraintes d'état est introduit et les conditions d'optimalité correspondantes impliquant l'équation de Hamilton–Jacobi–Bellman sont déduites. À cette fin, on tire parti du fait que les problèmes de contrôle optimal pour les équations différentielles ordinaires stochastiques peuvent être formulés de manière équivalente comme des problèmes de contrôle optimal pour les équations aux dérivées partielles déterministes, à savoir l'équation de Fokker–Planck correspondante.

Keywords. Renewable energy plant, optimal energy biddings, stochastic control, Fokker–Planck equation, Hamilton–Jacobi–Bellman equation.

Mots-clés. Installations d'énergie renouvelable, offres d'énergie optimales, contrôle stochastique, équation de Fokker–Planck, équation de Hamilton–Jacobi–Bellman.

Funding. This work has been partially supported by MCIN /AEI/10.13039/501100011033/FEDER, UE through grant PID2021-122625OB-I00 and by Xunta de Galicia through grant 2021GRC GI-1563-ED431C 2021/15.

Manuscript received 20 November 2023, accepted 21 November 2023.

1. Introduction

Renewable energy sources are growing quickly all over the world thanks to both environmental and geopolitical concerns [1]. Many renewable power sources (wind, solar, tidal) are naturally controlled and hence, being intermittent or variable, it is often difficult to predict their output [2]. This means that, in areas where there is a high proportion of generation capacity provided by renewable sources, those tasked with balancing supply and demand are faced to new problems to the solution of which stochastic mathematical modelling and optimization can significantly contribute [3]. Indeed, the stochastic production obtained from some of these renewable sources poses technical and economic challenges due to the introduction of great uncertainties into the operation and planning of the power systems. Although a traditional solution to the intermittency of wind generation has been primarily based on improving grid interconnection, as wind penetration becomes more and more important new solutions have had to be proposed. Energy storage is identified as one of the potential solutions to handle this issue [4]. In fact, power plants including wind farms and energy storage systems are playing an increasing role which makes their offering nonnegligible in some markets. From the perspective of wind farm-energy storage systems, this paper proposes an integrated mathematically based strategy of day-ahead offering and real-time operation policies to maximize their overall profit [5, 6].

For this purpose, stochastic control [7, 8] is the chosen mathematical framework. Firstly, the wind power plant is modelled by a system of three stochastic differential equations (SDE). The state variables are the instantaneous values of wind velocity, electricity price and energy in the battery, and the control is the discharge/charge power of the battery. An objective function is introduced taking into account the total revenue along the time interval of optimization, and the energy in the battery at the end of the process. Since the energy in the battery needs to be kept within certain limits depending on its capacity, the optimal control problem involves constraints on the state which makes it more difficult as the Hamilton–Jacobi–Bellman equation is not enough for optimality conditions. In order to analyze and numerically solve this problem we take advantage of the fact that it can be equivalently formulated as an optimal control problem with state constraints for a deterministic partial differential equation, namely, the Fokker–Planck equation (see, for instance, [9, 10]).

2. Statement of the problem

The penetration of renewable energy source(RES) have been growing globally, encouraged by environmental and low-carbon energy policies. Due to the volatility and uncertainty of renewable energy output, it is very difficult to bid in the market. To solve this problem, combining wind power(WP) (and/or photovoltaic(PV)) with power storage units to form a virtual power plant(VPP) is an effective path to stabilize the output deviation and promote renewable energy consumption. The final goal of this paper is twofold:

- to determine the optimal offers in the electricity market auctions in order to maximize the revenue of the plant,
- to control the plant in real time.

In what follows, a stochastic model for a wind power plant with a generic storage device like a battery system will be introduced. Then, based on that model, a stochastic control problem will be stated and solved.

2.1. *Notations*

Firstly, let us introduce the following notations:

- $s \in [0, T]$ is time (h).
- Π_s is the spot price of electricity at time s (e.g., €/MWh) (*stochastic process*).
- v_s is the velocity of wind (m/s) (*stochastic process*).
- $\mathscr{E}_s$ is the energy in the battery at time s (e.g., MWh) (*stochastic process*).
- $P_{bat,s}$ is the power delivered by the battery at time s (MW). For convenience, it is taken as the *control function* in the mathematical formulation. *Sign convention*: $P_{bat,s}$ is positive in discharge and negative in charge.
- $P_{wind,s}$ is the power produced by the wind park at time s (MW)(*stochastic process*). We have
$$P_{wind,s} = \widehat{P}(v_s),$$
where $\widehat{P}(v)$ is a *deterministic* function. In [11] one can find the following formula:
$$\widehat{P}(v) = P_r \begin{cases} 0 & \text{if } v \le v_{cut-in}, \\ \dfrac{v^3 - v_{cut-in}^3}{v_r^3 - v_{cut-in}^3} & \text{if } v_{cut-in} \le v \le v_r, \\ 1 & \text{if } v_r \le d \le v_{cut-off}, \\ 0 & \text{if } v \ge v_{cut-off}, \end{cases} \tag{1}$$
where
  - P_r (W) is the rated power,
  - v_{cut-in} (m/s) is the cut-in wind velocity
  - v_r (m/s) is the rated wind velocity
  - $v_{cut-off}$ (m/s) is the cut-off wind velocity
- The rated power can be computed by the formula
$$P_r = \frac{1}{2} C_{per} \rho \pi R^2 v_r^3 10^{-6} \text{ (MW)}, \tag{2}$$
where R is the radius of the turbine (m), ρ is the air density (kg/m^3) and C_{per} is a coefficient of performance of the turbine (nondimensional).
- $P_{grid,s}$ is the power sold (positive) or bought (negative) at time s (MW) (*stochastic process*). It satisfies
$$P_{grid,s} = P_{wind,s} + P_{bat,s} = \widehat{P}(v_s) + P_{bat,s} \tag{3}$$

2.2. *Mathematical model*

The following stochastic model for a wind power plant is a slight modification of that introduced in [12]. It consists of two *mean-reversion-like* SDE for wind velocity and spot electricity price, respectively, and a *bucket* model for the energy contain in the battery system:

(1) *State equation*:

$$dv_s = \kappa_v \left(\theta_v(s) + \frac{1}{\kappa_v} \frac{d\theta_v}{ds}(s) - v_s \right) ds + \sigma_{v,v} v_s dB_{v,s} + \sigma_{v,\Pi} v_s dB_{\Pi,s}, \tag{4}$$

$$d\Pi_s = \kappa_\Pi \left(\theta_\Pi(s) + \frac{1}{\kappa_\Pi} \frac{d\theta_\Pi}{ds}(s) - \Pi_s \right) ds + \sigma_{\Pi,v} \Pi_s dB_{v,s} + \sigma_{\Pi,\Pi} \Pi_s dB_{\Pi,s}, \tag{5}$$

$$d\mathcal{E}_s = -P_{bat,s} ds + \sigma_{\mathcal{E},\mathcal{E}} \mathcal{E}_s dB_{\mathcal{E},s} \tag{6}$$

- Functions of time, θ_v (m/s) and θ_Π (€/MWh) are given
- Constant parameters κ_v (h^{-1}) and κ_Π (h^{-1}) are given
- $B_{v,s}$ and $B_{\Pi,s}$ and $B_{\mathcal{E},s}$ are standard Brownian motions
- Let us define matrix $\mathbf{\Sigma}$ as

$$\mathbf{\Sigma}(v, \Pi, \mathcal{E}) = \begin{pmatrix} \sigma_{v,v} v & \sigma_{v,\Pi} v & 0 \\ \sigma_{\Pi,v} \Pi & \sigma_{\Pi,\Pi} \Pi & 0 \\ 0 & 0 & \sigma_{\mathcal{E},\mathcal{E}} \mathcal{E} \end{pmatrix}, \tag{7}$$

  where constants $\sigma_{i,j}$, $i, j = v, \Pi, \mathcal{E}$ are measured in $h^{1/2}$.
- *Initial conditions:* $v_0 = v^0$, $\Pi_0 = \Pi^0$, $\mathcal{E}_0 = \mathcal{E}^0$, where v^0 (m/s), Π^0 (€/MWh) and $\mathcal{E}^0$ (MW) are given random variables.
- *Objective function:* it is the expected revenue (€), namely,

$$J(P_{bat}) = \mathrm{E}\left[\int_0^T \Pi_s \left(\widehat{P}(v_s) + P_{bat,s} \right) ds + g(\mathcal{E}_T) \right], \tag{8}$$

  where $g : \mathbb{R} \to \mathbb{R}$ is a given deterministic function (€). The last term allows us, for instance, to penalize the loss of energy in the battery at the end of the process, i.e., at time T. For instance, if $g(x) = \beta x$ then it is the price of the energy contained in the battery at terminal time T, assuming that the unit cost at this time is β (€/MWh).

(2) *State constraints:* at each time $s \in [0, T]$, the value of the energy contained in the battery has to be kept between two extreme values:

$$\mathcal{E}_{min} \le \mathcal{E}_s \le \mathcal{E}_{max}. \tag{9}$$

Usually, $\mathcal{E}_{min} = 0$ and $\mathcal{E}_{max}$ is the *capacity* of the battery. This state constraint is reformulated in the style of *chance constraint programming*: it is imposed that this constraint holds with a probability greater than some parameter $\alpha \in (0, 1]$:

$$\mathcal{P}\left(\mathcal{E}_{min} \le \mathcal{E}_s \le \mathcal{E}_{max} \right) \ge \alpha. \tag{10}$$

Another alternative is

$$\mathcal{E}_{min} \le \mathrm{E}[\mathcal{E}_s] \le \mathcal{E}_{max}. \tag{11}$$

The optimal control problem consists in finding a control $P_{bat,s}$ maximizing the objective function (8) under the constraints

- *On the control:* $P_{bat,min} \le P_{bat,s} \le P_{bat,max}$,
- *On the state:* $\mathcal{P}(\mathcal{E}_{min} \le \mathcal{E}_s \le \mathcal{E}_{max}) \ge \alpha$, (or $\mathcal{E}_{min} \le \mathrm{E}[\mathcal{E}_s] \le \mathcal{E}_{max}$).

  Notice that, in practice, $P_{bat,min} < 0$ so $|P_{bat,min}| > 0$ is the maximum battery charge power while $P_{bat,max} > 0$ is the maximum battery discharge power.

2.3. *Remarks*

By using the stochastic control theory we shall determine the optimal control in a *feedback* form. For this purpose, the Hamilton–Jacobi–Bellman(HJB) equation of Dynamic Programming, which is a nonlinear backward partial differential equation, needs to be stated and solved. More precisely, by solving the HJB equation one can find *off-line* a deterministic feedback function

$$\check{P}_{bat} : (s, v, \Pi, \mathscr{E}) \in [0, T] \times \mathbb{R}^3 \to \check{P}_{bat}(s, v, \Pi, \mathscr{E}) \in \mathbb{R}$$

such that the optimal control at time s is given by

$$P_{bat,s} = \check{P}_{bat}(s, v_s, \Pi_s, \mathscr{E}_s) \tag{12}$$

This feedback function, $\check{P}_{bat}$, allows us to determine and implement the optimal control in real time, s, from the values of the state at this time. Notice that while $\check{P}_{bat}(s, v, \Pi, \mathscr{E})$ is a *deterministic* function, $P_{bat,s}$ as defined in (12) is a *stochastic* process, since it is the composition of a deterministic function and a vector stochastic process.

Regarding the bids at electricity auctions, what should we do? Firstly, we notice that the feedback function above can be computed *off-line* before real time. Thus, it can be known before the time at which the auction takes place. The problem is that, at that time we do not know the values of the stochastic variables at later time s, namely, v_s, Π_s and $\mathscr{E}_s$ so we cannot take

$$P_{grid,s} = \widehat{P}(v_s) + \check{P}_{bat}(s, v_s, \Pi_s, \mathscr{E}_s)$$

as the optimal bid in the electricity auction. However, a reasonable and straightforward choice is the following: the bid for time s is taken as the expectation of the stochastic process $P_{grid,s}$. Recall that this expectation can be computed by the formula,

$$P_{grid}(s) = \mathrm{E}\left[P_{grid,s}\right] = \int_{\mathbb{R}^3} \left(\widehat{P}(y_1) + \check{P}_{bat}(s, \mathbf{y})\right) \varphi(s, \mathbf{y})\, d\mathbf{y} \tag{13}$$

where $\varphi(s, \mathbf{y})$ denotes the joint *probability density function* of the state $(v_s, \Pi_s, \mathscr{E}_s)$ which can be computed by solving the Fokker–Planck equation (see below).

3. The Fokker–Planck equation

The *Fokker*(1914)–*Planck*(1917) (also called *forward Kolmogorov*(1931)) equation (see, for instance, [13, 14]) is a linear partial differential equation whose solution is the *joint probability density function (pdf)* of the vector stochastic process $\mathbf{X}_s^{t,\mathbf{x}}$ which satisfies the following Itô's stochastic differential equation (SDE):

$$d\mathbf{X}_s^{t,\mathbf{x}} = \mathbf{b}\left(s, \mathbf{X}_s^{t,\mathbf{x}}\right) ds + \mathbf{\Sigma}\left(s, \mathbf{X}_s^{t,\mathbf{x}}\right) d\mathbf{B}_s, \tag{14}$$

$$\mathbf{X}_t^{t,\mathbf{x}} = \mathbf{x}. \tag{15}$$

Let us denote by

$$\varphi(s, .) : \mathbf{y} \in \mathbb{R}^n \to \varphi(s, \mathbf{y}) \in \mathbb{R}$$

the *joint probability density function* (pdf) of the random vector $\mathbf{X}_s^{t,\mathbf{x}}$, $s \in [t, T]$. Then φ is a solution of the following *linear* partial differential equation, called the *Fokker–Planck (or forward Kolmogorov) equation*:

$$\frac{\partial \varphi}{\partial s} + \operatorname{div}\Big(\mathbf{b}(s, \mathbf{y})\varphi\Big) - \frac{1}{2} \operatorname{div} \mathbf{div}\Big(\mathbf{\Sigma}(s, \mathbf{y})\mathbf{\Sigma}^{\mathrm{T}}(s, \mathbf{y})\varphi\Big) = 0, \;\; \mathbf{y} \in \mathbb{R}^n, \; s \in [t, T]. \tag{16}$$

It can be obtained by using the Euler–Bernstein approximation of (14) and the Itô's lemma (see, for instance, [13, Th. 11.6.1] where some results on existence of solution are given).

An initial condition at time t is needed to solve this equation, namely, the *pdf* $\varphi(t, \mathbf{y})$ of the initial random vector $\mathbf{X}_t^{t,\mathbf{x}}$.

Let us recall that if $\mathbf{Y}_s$ is a d-dimensional stochastic process defined by $\mathbf{Y}_s = \mathbf{F}(s,\mathbf{X}_s^{t,\mathbf{x}})$, $\mathbf{F}$ being a given deterministic function from $[t,T]\times\mathbb{R}^n \to \mathbb{R}^d$, then the *expectation* of $\mathbf{Y}_s$ is given by

$$\mathrm{E}[\mathbf{Y}_s] = \int_{\mathbb{R}^n} \mathbf{F}(s,\mathbf{y})\varphi(s,\mathbf{y})\,d\mathbf{y}.$$

On the other hand, we notice that the optimization problem for the renewable energy plant described above falls within the following general stochastic control framework:

- The *state* of the system, $\mathbf{X}_s^{t,\mathbf{x}}$ is governed by the SDE valued in $\mathbb{R}^n$:
$$d\mathbf{X}_s^{t,\mathbf{x}} = \mathbf{b}\left(s,\mathbf{X}_s^{t,\mathbf{x}},u_s\right)ds + \mathbf{\Sigma}\left(s,\mathbf{X}_s^{t,\mathbf{x}}\right)d\mathbf{B}_s,$$
$$\mathbf{X}_t^{t,\mathbf{x}} = \mathbf{x},$$
where u_s is the *control*, $u\in\mathscr{U}$, $\mathscr{U}$ being a set of functions,
$$\mathscr{U} = \{u:[t,T]\to \mathrm{U}|\ u \text{ measurable}\},$$
and U is a compact set in $\mathbb{R}$.
- $\mathbf{B}_s$ is a d-dimensional Brownian motion.
- $\mathbf{b}$ is a vector field and $\mathbf{\Sigma}$ is a tensor field; both are given.
- The *gain* or *objective* function (sometimes also called *cost to go*) is
$$J(t,\mathbf{x},u) = \mathrm{E}\left[\int_t^T f\left(s,\mathbf{X}_s^{t,\mathbf{x}},u_s\right)ds + g\left(\mathbf{X}_T^{t,\mathbf{x}}\right)\right]$$

The goal is to maximize the objective function over the admissible set of controls $\mathscr{U}$ and under the *chance state constraint*,

$$\mathscr{P}\left(\mathbf{X}_s^{t,\mathbf{x}} \in \mathscr{A}\right) \geq \alpha \text{ a.e. } s\in[t,T],$$

where $\mathscr{A}$ is a convex subset of $\mathbb{R}^n$, or alternatively,

$$\mathrm{E}\left[\mathbf{X}_s^{t,\mathbf{x}}\right] \in \mathscr{A}.$$

Remark 1. In the present paper we follow the approach of the Dynamic Programming Principle leading to the Hamilton–Jacobi–Bellman equation. However, as it is well-known, in deterministic control theory there is another classical approach to characterize the optimality, namely, the so-called *maximum principle*, introduced by L. Pontryagin and collaborators in the case of ordinary differential equations and later extended to partial differential equations by J.L. Lions [15]. It can be formally obtained from the HJB equation, also for optimal control of SDE, by using Itô's Lemma and defining

$$\mathbf{Y}_s^{t,\mathbf{x}} := \operatorname{grad} v\left(s,\mathbf{X}_s^{t,\mathbf{x}}\right) \quad \text{and} \quad \mathbf{Z}_s^{t,\mathbf{x}} := \left(\mathbf{grad}\operatorname{grad} v\right)\mathbf{\Sigma}\left(s,\mathbf{X}_s^{t,\mathbf{x}}\right),$$

where v is the solution of the HJB equation. For the state constraint (20) it consists of the following equations:

$$\begin{aligned}
&\mathrm{d}\mathbf{X}_s^{t,\mathbf{x}} = \mathbf{b}\left(s,\mathbf{X}_s^{t,\mathbf{x}},u_s\right)ds + \mathbf{\Sigma}\left(s,\mathbf{X}_s^{t,\mathbf{x}}\right)d\mathbf{B}_s,\\
&\mathbf{X}_t^{t,\mathbf{x}} = \mathbf{x},\\
&\mathrm{d}\mathbf{Y}_s^{t,\mathbf{x}} = -\Big(\operatorname{grad} f\left(s,\mathbf{X}_s^{t,\mathbf{x}},u_s\right) + \mathbf{grad}\,\mathbf{b}\left(s,\mathbf{X}_s^{t,\mathbf{x}},u_s\right)^{\mathrm{T}}\mathbf{Y}_s^{t,\mathbf{x}} + \operatorname{tr}\Big(\mathbf{grad}\,\mathbf{\Sigma}\left(s,\mathbf{X}_s^{t,\mathbf{x}}\right)^{\mathrm{T}}\mathbf{Z}_s^{t,\mathbf{x}}\Big) - \boldsymbol{\mu}(s)\Big)\mathrm{d}s\\
&\qquad + \mathbf{Z}_s^{t,\mathbf{x}}\mathrm{d}\mathbf{B}_s,\\
&\qquad \boldsymbol{\mu}(s) \in \partial\chi_{\mathscr{A}}\left(\mathrm{E}\left[\mathbf{X}_s^{t,\mathbf{x}}\right]\right),\\
&\mathbf{Y}_T^{t,\mathbf{x}} = \operatorname{grad} g\left(T,\mathbf{X}_T^{t,\mathbf{x}}\right),\\
&u_s = \arg\min_{u\in\mathrm{U}}\left\{f\left(s,\mathbf{X}_s^{t,\mathbf{x}},u\right) + \mathbf{b}\left(s,\mathbf{X}_s^{t,\mathbf{x}},u\right)\cdot\mathbf{Y}_s^{t,\mathbf{x}}\right\}
\end{aligned}$$

This forward-backward system of SDE can be solved by combining Montecarlo with deep learning methods similar to those recently introduced (see [16]) for solving high-dimensional nonlinear PDEs. These methods are an interesting alternative to those based on the Fokker–Planck and

Hamilton–Jacobi–Bellman equations proposed below in the present paper. Their main advantage is that they do not suffer from the *curse of dimensionality*.

Remark 2. As mentioned before for the particular case of the renewable plant, the stochastic control theory gives us the optimal control in *feedback* form, i.e., as a function of the state of the system:

$$u_s = \check{u}\left(s, \mathbf{X}_s^{t,\mathbf{x}}\right) \tag{17}$$

where $\check{u}$ is a *deterministic* function that can be computed *off-line*. Thus, u_s is also a stochastic process and therefore, from the point of view of the real time implementation, that is, in order to determine the control to be applied at time s, one needs not only to compute the deterministic function $\check{u}$, but also to measure the state of the system at time s.

Now, consider the case where we want to determine the optimal control a priori, i.e., before the process starts. This is the case, for instance, if we try to determine the optimal biddings to the electricity auctions for the day-ahead. Then, a logical choice is to take the bid for time s as the expectation of the optimal stochastic control, i.e, the following deterministic function:

$$u(s) = \mathrm{E}[u_s] = \int_{\mathbb{R}^n} \check{u}(s,\mathbf{y})\varphi(s,\mathbf{y})\mathrm{d}\mathbf{y},$$

where $\varphi(s,\mathbf{y})$ is the pdf of the stochastic process $\mathbf{X}_s^{t,\mathbf{x}}$ which, as said above, can be computed by solving the Fokker–Planck equation (16) associated to the SDE that models the plant.

4. Reformulating the stochastic optimal control via the Fokker–Planck equation

Now, we will see that, by using the *pdf*, $\varphi(s,\mathbf{y})$, as state of the system, the above stochastic optimal control problem can be rewritten as a deterministic optimal control problem for a system governed by a *linear* partial differential equation, namely, the Fokker–Planck equation.

Indeed, firstly we notice that by using $\varphi(s,\mathbf{y})$ one can compute the expectation appearing in the cost function. Secondly, we look for a deterministic optimal control function, which is nothing but the *feedback function*:

$$\check{u} : [t,T] \times \mathbb{R}^n \to \mathbb{R}.$$

Then, the cost function J can be rewritten as follows:

$$\mathrm{J}(\check{u}) = \int_t^T \int_{\mathbb{R}^n} f\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\varphi(s,\mathbf{y})\,d\mathbf{y}ds + \int_{\mathbb{R}^n} g(\mathbf{y})\varphi(T,\mathbf{y})\,d\mathbf{y}. \tag{18}$$

Thirdly, by using again function φ, the chance constraint can be written as one of the following two alternatives:

Constraint 1:

$$\int_{\mathscr{A}} \varphi(s,\mathbf{y})\,d\mathbf{y} \geq \alpha,\ s \in [t,T], \tag{19}$$

Constraint 2:

$$\int_{\mathbb{R}^n} \mathbf{y}\varphi(s,\mathbf{y})\mathrm{d}\mathbf{y} \in \mathscr{A}. \tag{20}$$

Therefore, we are led to analyze the following *deterministic* optimal control problem with state constraints:

$$\max_{\check{u}\in\mathscr{U}} \mathrm{J}(\check{u}), \tag{21}$$

$$\int_{\mathscr{A}} \varphi(s,\mathbf{y})\,d\mathbf{y} \geq \alpha, \left(\text{or} \int_{\mathbb{R}^n} \mathbf{y}\varphi(s,\mathbf{y})\mathrm{d}\mathbf{y} \in \mathscr{A}\right),\ s \in [t,T]. \tag{22}$$

where $\mathscr{U} = \{\check{u} : [t,T] \times \mathbb{R}^n \to \mathbb{R}, \text{measurable and such that } \check{u}(s,\mathbf{y}) \in \mathrm{U} \text{ a.e. }\}$.

Recall that the *state of the system* is the scalar function $\varphi(s,\mathbf{y})$ and the *state equation* is the Fokker–Planck equation which is now written in a weak form:

$$\int_{\mathbb{R}^n} \frac{\partial\varphi}{\partial s}(s,\mathbf{y})\,v(\mathbf{y})\,\mathrm{d}\mathbf{y} - \int_{\mathbb{R}^n} \varphi(s,\mathbf{y})\mathbf{b}\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\cdot \operatorname{grad} v(\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_{\mathbb{R}^n} \frac{1}{2}\mathbf{div}\Big(\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\varphi(s,\mathbf{y})\Big)\cdot \operatorname{grad} v(\mathbf{y})\,\mathrm{d}\mathbf{y} = 0, \quad (23)$$

$$\varphi(t,\mathbf{y}) = \varphi_t(\mathbf{y}). \quad (24)$$

The initial condition, φ_t, is the joint *pdf* of the initial random vector $\mathbf{X}_t^{t,\mathbf{x}}$.

Two features of this problem are worth emphasizing, as they have important consequences on its first-order optimality conditions:

(1) The state equation (i.e., the Fokker–Planck equation) is linear with respect to its main unknown, the *pdf* φ.
(2) The cost function depends linearly on the state φ.

However, we notice that the mapping giving the state, φ, from de control, $\check{u}$, is in general nonlinear. Therefore, in general the problem (21), (22) is not a standard linear-quadratic optimal control problem with state constraints.

5. Optimality conditions

The goal is to write first order optimal conditions for the above deterministic optimal control problem. In this article, they will be *formally* obtained by introducing a Lagrangian function involving Lagrange multipliers associated with state constraints. The latter will not only correspond to the constraint on the state but also to the state equation itself. These two constraints have associated Lagrange multipliers that will be denoted by $\mu(s)$ (19) or $\boldsymbol{\mu}(s)$ (20), and $v(s,\mathbf{y})$, respectively. Indeed, as usual, it turns out that the Lagrange multiplier $v(s,\mathbf{y})$ associated to the state equation is the *adjoint state* of the deterministic optimal control problem. Firstly, the state equation (i.e., Fokker–Planck) together with its initial condition is rewritten in an equivalent weak form by integrating (23) in time from t to T:

$$\int_t^T \int_{\mathbb{R}^n} \frac{\partial\varphi}{\partial s}(s,\mathbf{y})\,v(s,\mathbf{y})\mathrm{d}\mathbf{y}\,\mathrm{d}s - \int_t^T \int_{\mathbb{R}^n} \varphi(s,\mathbf{y})\mathbf{b}\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\cdot \operatorname{grad} v(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s + \int_t^T \int_{\mathbb{R}^n} \frac{1}{2}\mathbf{div}\Big(\boldsymbol{\Sigma}(s,\mathbf{y},)\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\varphi(s,\mathbf{y})\Big)\cdot \operatorname{grad} v(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s = 0,$$

Integrating by parts in time the leftmost term and using the initial condition we get,

$$\int_{\mathbb{R}^n} v(T,\mathbf{y})\varphi(T,\mathbf{y})\,\mathrm{d}\mathbf{y} - \int_{\mathbb{R}^n} v(t,\mathbf{y})\varphi_t(\mathbf{y})\,\mathrm{d}\mathbf{y} - \int_t^T \int_{\mathbb{R}^n} \frac{\partial v}{\partial s}(s,\mathbf{y})\varphi(s,\mathbf{y})\mathrm{d}\mathbf{y}\,\mathrm{d}s - \int_t^T \int_{\mathbb{R}^n} \varphi(s,\mathbf{y})\mathbf{b}(s,\mathbf{y},\check{u}(s,\mathbf{y}))\cdot \operatorname{grad} v(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s + \int_t^T \int_{\mathbb{R}^n} \frac{1}{2}\mathbf{div}\Big(\boldsymbol{\Sigma}(s,\mathbf{y},)\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\varphi(s,\mathbf{y})\Big)\cdot \operatorname{grad} v(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s = 0.$$

Finally, for the state constraint (19), the Lagrangian function is defined as

$$
\begin{aligned}
\mathscr{L}(\check{u},\varphi,v,\mu) = & \int_t^T \int_{\mathbb{R}^n} f\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\varphi(s,\mathbf{y})\, d\mathbf{y} dt + \int_{\mathbb{R}^n} g(\mathbf{y})\varphi(T,\mathbf{y})\,\mathrm{d}\mathbf{y} \\
& - \int_{\mathbb{R}^n} v(T,\mathbf{y})\varphi(T,\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_{\mathbb{R}^n} v(t,\mathbf{y})\varphi(t,\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_t^T \int_{\mathbb{R}^n} \frac{\partial v}{\partial s}(s,\mathbf{y})\varphi(s,\mathbf{y})\mathrm{d}\mathbf{y}\mathrm{d}s \\
& + \int_t^T \int_{\mathbb{R}^n} \varphi(s,\mathbf{y})\mathbf{b}\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\cdot \operatorname{grad} v(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\mathrm{d}s \\
& - \int_t^T \int_{\mathbb{R}^n} \frac{1}{2}\mathbf{div}\Big(\mathbf{\Sigma}(s,\mathbf{y},)\mathbf{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\varphi(s,\mathbf{y})\Big)\cdot \operatorname{grad} v(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\mathrm{d}s \\
& - \int_t^T \mu(s)\left(\int_{\mathscr{A}} \varphi(s,\mathbf{y})\, d\mathbf{y} - \alpha\right) ds.
\end{aligned}
\tag{25}
$$

Recall that $\varphi(t,\mathbf{y}) = \varphi_t(\mathbf{y})$ is given.

The first order optimality conditions are,

$$\frac{\partial\mathscr{L}}{\partial v}\left(\check{u},\varphi,v,\mu\right)(\widehat{v}) = 0 \quad \forall\,\widehat{v}, \tag{26}$$

$$\frac{\partial\mathscr{L}}{\partial \varphi}\left(\check{u},\varphi,v,\mu\right)(\widehat{\varphi}) = 0 \quad \forall\,\widehat{\varphi}, \tag{27}$$

$$\frac{\partial\mathscr{L}}{\partial \mu}\left(\check{u},\varphi,v,\mu\right)(\widehat{\mu}-\mu) \le 0 \quad \forall\,\widehat{\mu}\le 0, \tag{28}$$

$$\frac{\partial\mathscr{L}}{\partial \check{u}}\left(\check{u},\varphi,v,\mu\right)(\widehat{u}-\check{u}) \le 0 \quad \forall\,\widehat{u}\in\mathscr{U}. \tag{29}$$

The equation (26) yields the state equation, that is the Fokker–Planck equation (23), and the initial condition (24). Equation (27) characterizes the Lagrange multiplier v (that is, the *adjoint state*):

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{\partial v}{\partial s}(s,\mathbf{y})\widehat{\varphi}(\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_{\mathbb{R}^n} \mathbf{b}\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\cdot \operatorname{grad} v(s,\mathbf{y})\widehat{\varphi}(\mathbf{y})\,\mathrm{d}\mathbf{y} & \\
- \frac{1}{2}\int_{\mathbb{R}^n} \operatorname{grad} v(s,\mathbf{y})\cdot \mathbf{div}\Big(\mathbf{\Sigma}(s,\mathbf{y})\mathbf{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\widehat{\varphi}(\mathbf{y})\Big)\, d\mathbf{y} & \\
+ \int_{\mathbb{R}^n} f\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\widehat{\varphi}(\mathbf{y})\, d\mathbf{y} - \mu(s)\int_{\mathscr{A}} \widehat{\varphi}(\mathbf{y})\, d\mathbf{y} = 0, &
\end{aligned}
\tag{30}
$$

$$v(T,\mathbf{y}) = g(\mathbf{y}). \tag{31}$$

Equation (28) characterizes the Lagrange multiplier associated with the chance constraint. It is equivalent to

$$\mu(s) \in \partial\chi_{[0,\infty)}\left(\int_{\mathscr{A}} \varphi(s,\mathbf{y})\, d\mathbf{y} - \alpha\right) \quad \text{a.e. } s \text{ in } [t,T], \tag{32}$$

where, in general, $\chi_{\mathscr{K}}$ denotes the indicator function of the closed convex set $\mathscr{K}$ and $\partial\chi_{\mathscr{K}}$ denotes its subdifferential (see, for instance, [17]). In its turn, equation (32) is equivalent to the three equations,

(1) $\int_{\mathscr{A}} \varphi(s,\mathbf{y})\, d\mathbf{y} \ge \alpha$ (*primal constraint*)
(2) $\mu(s) \le 0$ (*dual constraint*)
(3) $\mu(s)(\int_{\mathscr{A}} \varphi(s,\mathbf{y})\, d\mathbf{y} - \alpha) = 0$ (*complementary slackness condition*).

In the case of chance constraint (20), the last term in the Lagrangian function (25) has to be replaced by

$$-\int_t^T \left(\boldsymbol{\mu}(s)\cdot \int_{\mathbb{R}^n} \mathbf{y}\varphi(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\right)\mathrm{d}s$$

and, accordingly, the last term in the adjoint state equation (30) by

$$-\boldsymbol{\mu}(s)\cdot \int_{\mathbb{R}^n} \mathbf{y}\widehat{\varphi}(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

Moreover, in that case, equation (28) is equivalent to

$$\boldsymbol{\mu}(s) \in \partial\chi_{\mathscr{A}}\left(\int_{\mathbb{R}^n} \mathbf{y}\varphi(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\right). \tag{33}$$

Finally, equation (29) is the first order optimality condition for the maximization problem,

$$\max_{\hat{u}\in\mathscr{U}} \mathscr{L}(\hat{u},\varphi,v,\mu).$$

Notice that

$$\check{u} = \arg\max_{\hat{u}\in\mathscr{U}} \mathscr{L}(\hat{u},\varphi,v,\mu)$$

if and only if

$$\check{u} = \arg\max_{\hat{u}\in\mathscr{U}} \Phi(\hat{u},\varphi,v)$$

where

$$\Phi(\hat{u},\varphi,v) = \int_t^T \int_{\mathbb{R}^n} f\left(s,\mathbf{y},\hat{u}(s,\mathbf{y})\right)\varphi(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s + \int_t^T \int_{\mathbb{R}^n} \mathbf{b}\left(s,\mathbf{y},\hat{u}(s,\mathbf{y})\right)\cdot \operatorname{grad} v(s,\mathbf{y})\varphi(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s.$$

Moreover, since there are no derivatives of $\hat{u}$ in the above function Φ and φ is a postive function, the optimization problem can be solved as a set of uncoupled optimization problems, one for each $(s,\mathbf{y}) \in [t,T]\times\mathbb{R}^n$. In other words, $\check{u} = \arg\max_{\hat{u}\in\mathscr{U}} \Phi(\hat{u},\varphi,v)$ if and only if

$$\check{u}(s,\mathbf{y}) = \arg\max_{u\in\mathrm{U}} \zeta\left(s,\mathbf{y},u,\operatorname{grad} v(s,\mathbf{y})\right)$$

where $\zeta(s,\mathbf{y},u,\mathbf{p}) := f(s,\mathbf{y},u) + \mathbf{b}(s,\mathbf{y},u)\cdot\mathbf{p}$. Therefore, the adjoint state equation can be rewritten as follows:

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\partial v}{\partial s}(s,\mathbf{y})\hat{\varphi}(\mathbf{y})\,\mathrm{d}\mathbf{y} - \frac{1}{2}\int_{\mathbb{R}^n} \operatorname{grad} v(s,\mathbf{y})\cdot \mathbf{div}\left(\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\hat{\varphi}(\mathbf{y})\right)d\mathbf{y}\\ -\mu(s)\int_{\mathscr{A}}\hat{\varphi}(\mathbf{y})\,d\mathbf{y} + \int_{\mathbb{R}^n}\max_{u\in\mathrm{U}}\left\{f(s,\mathbf{y},u) + \mathbf{b}(s,\mathbf{y},u)\cdot\operatorname{grad} v(s,\mathbf{y})\right\}\hat{\varphi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = 0,\end{aligned} \tag{34}$$

$$v(T,\mathbf{y}) = g(\mathbf{y}). \tag{35}$$

Notice that for $\mu(s) = 0$ (i.e., without state constraints), equation (34) is nothing but the second order *Hamilton–Jacobi–Bellman* equation for the stochastic control problem. The mathematical analysis of these equations relies upon the notion of viscosity solution (see, for instance, [18]). In the case with state constraints, its unknowns are functions $v(s,\mathbf{y})$ and $\mu(s)$. We also notice that it is a *backward* nonlinear partial differential equation. Its strong form is

$$\begin{aligned}\frac{\partial v}{\partial s}(s,\mathbf{y}) + \max_{\hat{u}\in\mathscr{U}}\left\{f(s,\mathbf{y},\hat{u}) + \mathbf{b}(s,\mathbf{y},\hat{u})\cdot\operatorname{grad} v(s,\mathbf{y})\right\}\\ + \frac{1}{2}\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\cdot\mathbf{grad}\operatorname{grad} v(s,\mathbf{y}) - \mu(s)I_{\mathscr{A}}(\mathbf{y}) = 0 \quad \text{in } \mathbb{R}^n,\ s\in[t,T],\end{aligned} \tag{36}$$

$$v(T,\mathbf{y}) = g(\mathbf{y}) \text{ in } \mathbb{R}^n. \tag{37}$$

For the chance constraint (20), the term with the Lagrange multiplier in (34) has to be replaced by $-\boldsymbol{\mu}(s)\cdot\int_{\mathbb{R}^n}\mathbf{y}\hat{\varphi}(\mathbf{y})\,\mathrm{d}\mathbf{y}$ and in (36) by $-\boldsymbol{\mu}(s)\cdot\mathbf{y}$.

On the other hand, let us recall that in the theory of the Hamilton–Jacobi–Bellman equation the function

$$H(s,\mathbf{y},\mathbf{p}) := \max_{u\in\mathrm{U}}\zeta\left(s,\mathbf{y},u,\mathbf{p}\right),$$

where $\zeta(s,\mathbf{y},u,\mathbf{p}) := f(s,\mathbf{y},u) + \mathbf{b}(s,\mathbf{y},u)\cdot\mathbf{p}$, is called the *Hamiltonian.*

Remark 3. If $\mu(s)$ (or $\boldsymbol{\mu}(s)$) were known (of course, this is the case when there are no state constraints), then equation (36) with (37) could be solved for v *independently* of the other optimality conditions, since the *pdf* φ does not appear in it. In other words, the Fokker–Planck equation would not need to be solved.

However, if there are state constraints, then the Lagrange multiplier is not known a priori so that either equation (32) or equation (33) have to be solved. The drawback is that both equations involve the *pdf*, φ, and hence the Fokker–Planck equation must also be solved. Summarizing, under state constraints the whole set of optimality conditions has to be solved.

On the other hand, sometimes the above maximization problem for u can be explicitly solved in terms of grad v, in which case the *adjoint state* equation, that is, the Hamilton–Jacobi–Bellman equation becomes an *usual* partial differential equation, in general fully nonlinear. This is the case of the application considered in the present paper.

We summarize the optimality system for the stochastic optimal control problem with chance constraint (19) by writing the state and adjoint state equations in strong form

$$\frac{\partial\varphi}{\partial s}+\operatorname{div}\Big(\mathbf{b}\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\varphi(s,\mathbf{y})\Big)-\frac{1}{2}\operatorname{div}\mathbf{div}\Big(\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\varphi(s,\mathbf{y})\Big)=0, \tag{38}$$

$$\varphi(t,\mathbf{y})=\varphi_t(\mathbf{y}),\ \mathbf{y}\in\mathbb{R}^n, \tag{39}$$

$$\begin{aligned}&\frac{\partial v}{\partial s}(s,\mathbf{y})+\max_{u\in \mathrm{U}}\left\{\mathbf{b}(s,\mathbf{y},u)\cdot\operatorname{grad}v(s,\mathbf{y})+f(s,\mathbf{y},u)\right\}\\&\qquad+\frac{1}{2}\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\cdot\mathbf{grad}\operatorname{grad}v(s,\mathbf{y})-\mu(s)I_{\mathscr{A}}(\mathbf{y})=0,\end{aligned} \tag{40}$$

$$v(T,\mathbf{y})=g(\mathbf{y}),\ \mathbf{y}\in\mathbb{R}^n, \tag{41}$$

$$\mu(s)\in\partial\chi_{[0,\infty)}\left(\int_{\mathscr{A}}\varphi(s,\mathbf{y})d\mathbf{y}-\alpha\right), \tag{42}$$

$$\check{u}(s,\mathbf{y})=\arg\max_{u\in\mathrm{U}}\left\{\mathbf{b}(s,\mathbf{y},u)\cdot\operatorname{grad}v(s,\mathbf{y})+f(s,\mathbf{y},u)\right\}. \tag{43}$$

In the case of chance constraint (20), (40) and (42) have to be replaced by

$$\begin{aligned}&\frac{\partial v}{\partial s}(s,\mathbf{y})+\mathbf{b}\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\cdot\operatorname{grad}v(s,\mathbf{y})+f\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\\&\qquad\qquad+\frac{1}{2}\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\cdot\mathbf{grad}\operatorname{grad}v(s,\mathbf{y})-\boldsymbol{\mu}(s)\cdot\mathbf{y}=0,\end{aligned}$$

and $\boldsymbol{\mu}(s)\in\partial\chi_{\mathscr{A}}(\int_{\mathbb{R}^n}\mathbf{y}\varphi(s,\mathbf{y})d\mathbf{y})$, respectively.

6. Numerical solution: algorithm of multipliers

From now on we consider the general stochastic optimal control problem including the state constraint. In this case, the full optimality system has to be solved. Numerical solution of this system first requires discretizations both in time and in space. Then, an optimization algorithm has to be used. We propose two nested iterations:

- the outermost concerns the *state constraint*, i.e., updating the Lagrange multiplier $\mu(s)$ (or $\boldsymbol{\mu}(s)$)
- the innermost involves the solution of a *nonlinear programming problem* with only bound constraints on the control.

For the outer loop a *Lagrange multipliers* algorithm is introduced below. This kind of algorithms have been extensively used for solving nonlinear partial differential equations (see [19], the review paper [20] and references therein). In the present paper, the approach followed is more directly inspired by [21]. It is based on the following result, the proof of which is straightforward:

Lemma 4. *Let G be a (possibly multivalued) maximal monotone operator in a Hilbert space $\mathscr{H}$. The following statements are equivalent*

(1) $\mathbf{g} \in G(\mathbf{h})$,
(2) $\mathbf{g} = G_\lambda(\mathbf{h} + \lambda\mathbf{g})$,

for any positive real number λ, where G_λ is the Yosida approximation of G, that is,

$$G_\lambda = \frac{I - J_\lambda^G}{\lambda},$$

$J_\lambda^G := (I - \lambda G)^{-1}$ being the resolvent operator of G.

In the particular case where $G = \partial\chi_{\mathscr{K}}$, i.e., G is the subdifferential of the indicator function of a convex set $\mathscr{K}$, we have $J_\lambda^G = P_{\mathscr{K}}$, where $P_{\mathscr{K}}$ is the projection on the convex set $\mathscr{K}$; then,

$$G_\lambda = \frac{I - P_{\mathscr{K}}}{\lambda}.$$

In order to introduce the algorithm of multipliers, the first step is to use the previous Lemma to replace the inclusion

$$\mu(s) \in \partial\chi_{[0,\infty)}\left(\int_{\mathscr{A}} \varphi(s,\mathbf{x})\,d\mathbf{x} - \alpha\right)$$

by the equivalent equality

$$\mu(s) = \left(\partial\chi_{[0,\infty)}\right)_\lambda\left(\int_{\mathscr{A}} \varphi(s,\mathbf{x})\,d\mathbf{x} - \alpha + \lambda\mu(s)\right).$$

We have

$$\left(\partial\chi_{[0,\infty)}\right)_\lambda(y) = \frac{I - P_{[0,\infty)}}{\lambda} = -\frac{1}{\lambda}y^-,$$

where $y^- = \max\{-y, 0\}$. Notice that λ *needs not to be a small number.*

Hence, the optimality system can be rewritten as

$$\begin{aligned}
&\frac{\partial\varphi}{\partial s}(s,\mathbf{y}) + \operatorname{div}\Big(\mathbf{b}\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\varphi(s,\mathbf{y})\Big) - \frac{1}{2}\operatorname{div}\mathbf{div}\Big(\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\varphi(s,\mathbf{y})\Big) = 0,\\
&\varphi(t,\mathbf{y}) = \varphi_t(\mathbf{y}),\\
&\frac{\partial v}{\partial s}(s,\mathbf{y}) + \mathbf{b}\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\cdot \operatorname{grad} v(s,\mathbf{y}) + f\left(s,\mathbf{y},\check{u}(s,\mathbf{y})\right)\\
&\qquad + \frac{1}{2}\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\cdot \mathbf{grad}\operatorname{grad} v(s,\mathbf{y}) - \mu(s)I_{\mathscr{A}}(\mathbf{y}) = 0,\\
&v(T,\mathbf{y}) = g(\mathbf{y}),\\
&\mu(s) = -\left[\frac{1}{\lambda}\left(\int_{\mathscr{A}}\varphi(s,\mathbf{y})\,d\mathbf{y} - \alpha\right) + \mu(s)\right]^-,\\
&\check{u}(s,\mathbf{y}) = \arg\max_{u\in \mathrm{U}}\left\{\mathbf{b}(s,\mathbf{y},u)\cdot \operatorname{grad} v(s,\mathbf{y}) + f(s,\mathbf{y},u)\right\},\\
&\qquad\qquad\qquad\qquad s \in [t,T],\ \mathbf{y}\in\mathbb{R}^n
\end{aligned}$$

which leads to the following iterative algorithm:

(1) $k = 0$, $\mu^0(s) \le 0$ is given

(2) $k > 0$, $\mu^{k-1}(s)$ is known, then compute v^k, $\check{u}^k$ by solving the optimal control problem *without state constraints* characterized by the following Hamilton–Jacobi–Bellman equation:

$$\begin{aligned}
&\frac{\partial v^k}{\partial s}(s,\mathbf{y}) + \Big[\mathbf{b}\left(s,\mathbf{y},\check{u}^k(s,\mathbf{y})\right)\cdot \operatorname{grad} v(s,\mathbf{y}) + f\left(s,\mathbf{y},\check{u}^k(s,\mathbf{y})\right) \\
&\qquad + \frac{1}{2}\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\cdot \mathbf{grad}\operatorname{grad} v^k(s,\mathbf{y})\Big] - \mu^{k-1}(s) I_{\mathscr{A}}(\mathbf{y}) \\
&v^k(T,\mathbf{y}) = g(\mathbf{y}),\ \mathbf{y}\in\mathbb{R}^n, \\
&\check{u}^k(s,\mathbf{y}) = \arg\max_{\hat{u}\in \mathrm{U}}\left[\mathbf{b}\left(s,\mathbf{y},\hat{u}\right)\cdot \operatorname{grad} v^k(s,\mathbf{y}) + f\left(s,\mathbf{y},\hat{u}\right)\right]
\end{aligned}$$

(3) Compute φ^k by solving the Fokker–Planck equation

$$\frac{\partial \varphi^k}{\partial s}(s,\mathbf{y}) + \operatorname{div}\Big(\mathbf{b}\left(s,\mathbf{y},\check{u}^k(s,\mathbf{y})\right)\varphi^k(s,\mathbf{y})\Big) - \frac{1}{2}\operatorname{div}\mathbf{div}\Big(\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\varphi^k(s,\mathbf{y})\Big) = 0,$$
$$\varphi^k(t,\mathbf{y}) = \varphi_t(\mathbf{y}).$$

(4) Update $\mu(s)$:

$$\mu^k(s) = -\left[\frac{1}{\lambda}\left(\int_{\mathscr{A}}\varphi^k(s,\mathbf{y})d\mathbf{y} - \alpha\right) + \mu^{k-1}(s)\right]^-$$

In order to solve the optimal control problem in the previous item (2), an optimization algorithm allowing bound constraints on the control variables should be used (the *inner loop*).

In the case, of the state constraint (20), the iterative algorithm is the following one:

(1) $k = 0$, $\boldsymbol{\mu}^0(s) \le 0$ is given

(2) $k > 0$, $\boldsymbol{\mu}^{k-1}(s)$ is known, then compute φ^k, v^k, $\check{u}^k$ by solving the optimal control problem *without state constraints* corresponding to the following Hamilton–Jacobi–Bellman equation:

$$\begin{aligned}
&\frac{\partial v^k}{\partial s}(s,\mathbf{y}) + \Big[\mathbf{b}\left(s,\mathbf{y},\check{u}^k(s,\mathbf{y})\right)\cdot \operatorname{grad} v^k(s,\mathbf{y}) + f\left(s,\mathbf{y},\check{u}^k(s,\mathbf{y})\right) \\
&\qquad + \frac{1}{2}\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\cdot \mathbf{grad}\operatorname{grad} v^k(s,\mathbf{y})\Big] - \boldsymbol{\mu}^{k-1}(s)\cdot\mathbf{y}, \\
&v^k(T,\mathbf{y}) = g(\mathbf{y}),\ \mathbf{y}\in\mathbb{R}^n, \\
&\check{u}^k(s,\mathbf{y}) = \arg\max_{\hat{u}\in \mathrm{U}}\left[\mathbf{b}\left(s,\mathbf{y},\hat{u}\right)\cdot \operatorname{grad} v^k(s,\mathbf{y}) + f\left(s,\mathbf{y},\hat{u}\right)\right].
\end{aligned}$$

(3) Compute φ^k by solving the Fokker–Planck equation

$$\frac{\partial \varphi^k}{\partial s}(s,\mathbf{y}) + \operatorname{div}\Big(\mathbf{b}\left(s,\mathbf{y},\check{u}^k(s,\mathbf{y})\right)\varphi^k(s,\mathbf{y})\Big) - \frac{1}{2}\operatorname{div}\mathbf{div}\Big(\boldsymbol{\Sigma}(s,\mathbf{y})\boldsymbol{\Sigma}(s,\mathbf{y})^{\mathrm{T}}\varphi^k(s,\mathbf{y})\Big) = 0,$$
$$\varphi^k(t,\mathbf{y}) = \varphi_t(\mathbf{y}).$$

(4) Update $\mu(s)$:

$$\begin{aligned}
\boldsymbol{\mu}^k(s) &= \frac{(I - P_{\mathscr{A}})}{\lambda}\left(\int_{\mathbb{R}^n}\mathbf{y}\varphi^k(s,\mathbf{y})d\mathbf{y} + \lambda\boldsymbol{\mu}^{k-1}(s)\right) \\
&= \boldsymbol{\mu}^{k-1}(s) + \frac{1}{\lambda}\int_{\mathbb{R}^n}\mathbf{y}\varphi^k(s,\mathbf{y})d\mathbf{y} - \frac{1}{\lambda}P_{\mathscr{A}}\left(\int_{\mathbb{R}^n}\mathbf{y}\varphi^k(s,\mathbf{y})d\mathbf{y} + \lambda\boldsymbol{\mu}^{k-1}(s)\right),
\end{aligned}$$

where $P_{\mathscr{A}}$ is the Euclidean projection on the closed convex set $\mathscr{A}\subset\mathbb{R}^n$.

Remark 5. Since the approach considered in this paper is based on partial differential equations with as many space variables as components of the state of the system, it suffers from the *curse of dimensionality*, a concept introduced by Richard Bellman in the 1950s. Indeed, the use of standard deterministic numerical discretization methods becomes prohibitive even for quite small dimensions. Fortunately, Monte Carlo techniques based on representations of the solution of semilinear PDEs as expectations of the solution of backward stochastic differential equations (BSDE), see [22], avoid the curse of dimensionality. Furthermore, recent developments [23] that combine these representations with the use of artificial neural networks, have completely changed the landscape. Nowadays, high-dimensional PDEs with thousands of variables can be solved, something unthinkable less than a decade ago. This opens the way to a multitude of new applications, in particular to optimal control problems.

7. Application to the wind power plant

In the previous section we have described an algorithm of multipliers to solve the equivalent deterministic control problem with state constraints. Now, the goal is to apply this algorithm to the example of the renewable plant with battery. Let us recall the stochastic control problem for the plant:

- State: $\mathbf{X}_s = (v_s, \Pi_s, \mathscr{E}_s)^{\mathrm{T}}$. Control: $u_s = P_{bat,s}$.
- Initial time: $t = 0$. Initial state: the random vector $\mathbf{X}_t = (v_0, \Pi_0, \mathscr{E}_0)^{\mathrm{T}}$.
  (Recall that this initial condition is needed only to solve the Fokker–Planck equation for the *pdf*, φ)
- Vector field $\mathbf{b}$:

$$\mathbf{b}(s,\mathbf{y},u) = \begin{pmatrix} \kappa_v \left(\theta_v(s) + \frac{1}{\kappa_v}\frac{d\theta_\Pi}{ds}(s) - y_1\right) \\ \kappa_\Pi \left(\theta_\Pi(s) + \frac{1}{\kappa_\Pi}\frac{d\theta_\Pi}{ds}(s) - y_2\right) \\ -u \end{pmatrix}.$$

- Noise: $\mathbf{B}_s = (B_{v,s}, B_{\Pi,s}, B_{\mathscr{E},s})^{\mathrm{T}}$.
- Tensor field $\mathbf{\Sigma}$:

$$\mathbf{\Sigma}(s,\mathbf{y}) = \begin{pmatrix} \sigma_{v,v} y_1 & \sigma_{v,\Pi} y_1 & 0 \\ \sigma_{\Pi,v} y_2 & \sigma_{\Pi,\Pi} y_2 & 0 \\ 0 & 0 & \sigma_{\mathscr{E},\mathscr{E}} y_3 \end{pmatrix}.$$

- Objective function: $f(s, v_s, \Pi_s, \mathscr{E}_s, u) = \Pi_s(\widehat{P}(v_s) + P_{bat,s}), \quad g(\mathscr{E}_T) = \beta\mathscr{E}_T$.
  That is, $f(s,\mathbf{y},u) := y_2(\widehat{P}(y_1) + u), \quad g(\mathbf{y}) = \beta y_3$.
- Admissible control set: $\mathrm{U} = \{u : \mathscr{P}_{bat,min} \le u \le \mathscr{P}_{bat,max}\}$.
- $\mathscr{A} = \{\mathbf{y} \in \mathbb{R}^3 : \mathscr{E}_{min} \le y_3 \le \mathscr{E}_{max}\}$.
- The Hamiltonian function can be easily obtained analytically:

$$\begin{aligned} H(s,\mathbf{y},\widehat{u},\mathbf{p}) &= \max_{\widehat{u}\in \mathrm{U}} \left[\mathbf{b}\left(s,\mathbf{y},\widehat{u}\right)\cdot \mathbf{p} + f\left(s,\mathbf{y},\widehat{u}\right)\right] \\ &= \max_{\widehat{u}\in \mathrm{U}} \left[b_1 p_1 + b_2 p_2 - \widehat{u} p_3 + y_2\left(\widehat{P}(y_1) + \widehat{u}\right)\right] \\ &= \max_{\widehat{u}\in \mathrm{U}} \left[(-p_3 + y_2)\widehat{u}\right] + b_1 p_1 + b_2 p_2 + y_2 \widehat{P}(y_1) \\ &= \Psi(y_2, p_3) + b_1 p_1 + p_2 b_2 + y_2 \widehat{P}(y_1), \end{aligned}$$

  where

$$\Psi(y_2,p_3) = \begin{cases} P_{bat,min}(-p_3 + y_2) & \text{if } -p_3 + y_2 < 0, \\ 0 & \text{if } -p_3 + y_2 = 0, \\ P_{bat,max}(-p_3 + y_2) & \text{if } -p_3 + y_2 > 0. \end{cases}$$

Notice that Ψ is a nonlinear function of p_3. Finally, the HJB equation to be solved is

$$\begin{aligned}
&\frac{\partial v}{\partial s}(s,\mathbf{y})+\kappa_v\left(\theta_v(s)+\frac{1}{\kappa_v}\frac{d\theta_v}{ds}(s)-y_1\right)\frac{\partial v}{\partial y_1}(s,\mathbf{y})\\
&\quad+\kappa_\Pi\left(\theta_\Pi(s)+\frac{1}{\kappa_\Pi}\frac{d\theta_\Pi}{ds}(s)-y_2\right)\frac{\partial v}{\partial y_2}(s,\mathbf{y})\\
&\quad+\frac{1}{2}\left(\sigma_{v,v}^2+\sigma_{v,\Pi}^2\right)y_1^2\frac{\partial^2 v}{\partial y_1^2}(s,\mathbf{y})+\left(\sigma_{v,v}\sigma_{\Pi,v}+\sigma_{v,\Pi}\sigma_{\Pi,\Pi}\right)y_1y_2\frac{\partial^2 v}{\partial y_1\partial y_2}(s,\mathbf{y})\\
&\quad+\frac{1}{2}\left(\sigma_{\Pi,v}^2+\sigma_{\Pi,\Pi}^2\right)y_2^2\frac{\partial^2 v}{\partial y_2^2}(s,\mathbf{y})+\frac{1}{2}\sigma_{\mathscr{E},\mathscr{E}}^2y_3^2\frac{\partial^2 v}{\partial y_3^2}(s,\mathbf{y})+\Psi\left(y_2,\frac{\partial v}{\partial y_3}(s,\mathbf{y})\right)\\
&\quad+y_2\widehat{P}(y_1)-\mu(s)I_{\mathscr{A}}(\mathbf{y})=0,\\
&v(T,\mathbf{y})=\beta y_3.
\end{aligned}$$

Notice that if

$$\text{meas}\left\{(s,\mathbf{y})\in[0,T]\times\mathbb{R}^n:-\frac{\partial v}{\partial y_3}(s,\mathbf{y})+y_2=0\right\}=0,$$

then the optimal control is *bang-bang*:

$$\begin{aligned}
P_{bat,s}&=\check{P}_{bat}(s,\mathbf{y})\\
&=\arg\max_{\hat{u}\in U}\left[(-p_3+y_2)\hat{u}\right]=\begin{cases}P_{bat,min} & \text{if } -p_3+y_2<0,\\ \left[P_{bat,min},P_{bat,max}\right] & \text{if } -p_3+y_2=0,\\ P_{bat,max} & \text{if } -p_3+y_2>0.\end{cases}
\end{aligned}$$

In general, since $\mathbf{p}(s,\mathbf{y})=\text{grad}\,v(s,\mathbf{y})$, then

$$P_{bat,s}\in F\left(-\frac{\partial v}{\partial \mathscr{E}}(s,v_s,\Pi_s,\mathscr{E}_s)+\Pi_s\right),$$

where F is the maximal monotone graph,

$$F(r)=\begin{cases}P_{min} & \text{if } r<0,\\ [P_{min},P_{max}] & \text{if } r=0,\\ P_{max} & \text{if } r>0.\end{cases}$$

Remark 6. Assuming $g(\mathbf{y})=\beta y_3$, we have $v(T,\mathbf{y})=\beta\mathscr{E}_T$ at terminal time $s=T$ and then $\frac{\partial v}{\partial y_3}(T,\mathbf{y})=\beta$. Therefore, given β which is a data of the control problem we have two possibilities (recall that Π_T is the price of electricity at time T):

(1) Either $\Pi_T<\beta$, i.e., the price of the electricity in the market at time T is lower than the a priori estimated price of the electricity contained in the battery at time T. Then $P_{bat}(T)=P_{bat,min}$, that is, the optimal strategy is to charge the battery at the highest rate (recall that $-P_{bat,min}$ is the maximum charge power of the battery).

(2) Or $\Pi_T>\beta$, i.e., the price of the electricity in the market at time T is higher than the a priori estimated price of the electricity contained in the battery at time T. Then $P_{bat}(T)=P_{bat,max}$ that is, the optimal strategy is to discharge the battery at the highest rate (recall that $P_{bat,max}$ is the maximum discharge power of the battery).

7.1. *The algorithm of multipliers*

In the case of the state constraint (19) the *algorithm of multipliers* becomes,

(1) $k=0$, $\mu^0(s)$ is given

(2) $k>0$, $\mu^{k-1}(s)$ is known, then compute φ^k, v^k, $\check{u}^k$ by solving the optimal control problem *without state constraints:*

(a) Solve for $v^k(s,\mathbf{y})$ the HJB equation,

$$\begin{aligned}
&\frac{\partial v^k}{\partial s}(s,\mathbf{y})+\kappa_v\left(\theta_v(s)+\frac{1}{\kappa_v}\frac{d\theta_v}{ds}(s)-y_1\right)\frac{\partial v^k}{\partial y_1}(s,\mathbf{y})\\
&\quad+\kappa_\Pi\left(\theta_\Pi(s)+\frac{1}{\kappa_\Pi}\frac{d\theta_\Pi}{ds}(s)-y_2\right)\frac{\partial v^k}{\partial y_2}(s,\mathbf{y})\\
&\quad+\frac{1}{2}\left(\sigma^2_{v,v}+\sigma^2_{v,\Pi}\right)y_1^2\frac{\partial^2 v}{\partial y_1^2}(s,\mathbf{y})+\left(\sigma_{v,v}\sigma_{\Pi,v}+\sigma_{v,\Pi}\sigma_{\Pi,\Pi}\right)y_1y_2\frac{\partial^2 v}{\partial y_1\partial y_2}(s,\mathbf{y})\\
&\quad+\frac{1}{2}\left(\sigma^2_{\Pi,v}+\sigma^2_{\Pi,\Pi}\right)y_2^2\frac{\partial^2 v}{\partial y_2^2}(s,\mathbf{y})+\frac{1}{2}\sigma^2_{\mathscr{E},\mathscr{E}}y_3^2\frac{\partial^2 v}{\partial y_3^2}(s,\mathbf{y})+\Psi\left(y_2,\frac{\partial v}{\partial y_3}(s,\mathbf{y})\right)\\
&\quad+y_2\widehat{P}(y_1)-\mu^{k-1}(s)I_{\mathscr{A}}(\mathbf{y})=0,\\
&\quad v^k(T,\mathbf{y})=\beta y_3.
\end{aligned}$$

(b) Compute the feedback control: $\check{u}^k(s,\mathbf{y})\in F(-\frac{\partial v^k}{\partial y_3}(s,\mathbf{y})+y_2)$.

(c) Solve the Fokker–Planck equation for φ^k

$$\begin{aligned}
&\frac{\partial\varphi^k}{\partial s}(s,\mathbf{y})+\frac{\partial}{\partial y_1}\left(\kappa_v\left(\theta_v(s)+\frac{1}{\kappa_v}\frac{d\theta_v}{ds}(s)-y_1\right)\varphi^k\right)\\
&+\frac{\partial}{\partial y_2}\left(\kappa_\Pi\left(\theta_\Pi(s)+\frac{1}{\kappa_\Pi}\frac{d\theta_\Pi}{ds}(s)-y_2\right)\varphi^k\right)-\frac{\partial}{\partial y_3}\left(\check{u}^k(s,\mathbf{y})\varphi^k\right)-\frac{1}{2}\frac{\partial^2}{\partial y_1^2}\left(\left(\sigma^2_{v,v}+\sigma^2_{v,\Pi}\right)y_1^2\varphi^k\right)\\
&-\frac{\partial^2}{\partial y_1\partial y_2}\left(\left(\sigma_{v,v}\sigma_{\Pi,v}+\sigma_{v,\Pi}\sigma_{\Pi,\Pi}\right)y_1y_2\varphi^k\right)-\frac{1}{2}\frac{\partial^2}{\partial y_2^2}\left(\left(\sigma^2_{\Pi,v}+\sigma^2_{\Pi,\Pi}\right)y_2^2\varphi^k\right)-\frac{1}{2}\frac{\partial^2}{\partial y_3^2}\left(\sigma^2_{\mathscr{E},\mathscr{E}}y_3^2\varphi^k\right)=0,\\
&\varphi^k(t,\mathbf{y})=\varphi_t(\mathbf{y}),\ \mathbf{y}\in\mathbb{R}^n.
\end{aligned}$$

(3) Update $\mu(s)$ as $\mu^k(s)=-[\frac{1}{\lambda}(\int_{\mathscr{A}}\varphi^k(s,\mathbf{y})d\mathbf{y}-\alpha)+\mu^{k-1}(s)]^-$.

In the case of the state constraint (20), the term involving the Lagrange multiplier in the HJB equation should be replaced by -$\boldsymbol{\mu}^{k-1}(s)\cdot\mathbf{y}$ while in step (3), the update of the multiplier should be

$$\boldsymbol{\mu}^k(s)=\boldsymbol{\mu}^{k-1}(s)+\frac{1}{\lambda}\int_{\mathbb{R}^n}\mathbf{y}\varphi^k(s,\mathbf{y})d\mathbf{y}-\frac{1}{\lambda}P_{\mathscr{A}}\left(\int_{\mathbb{R}^n}\mathbf{y}\varphi^k(s,\mathbf{y})d\mathbf{y}+\lambda\boldsymbol{\mu}^{k-1}(s)\right).$$

8. Numerical results

In this section we show some numerical results obtained for the stochastic control problem defined in § 2.2 by using the algorithm of multipliers described in § 7.1.

8.1. *Data*

Firstly, the employed data are listed. Some of them have been taken from [12].

- Parameters for functions θ_v and θ_Π: $\kappa_v=0.1\ \mathrm{h}^{-1}$ and $\kappa_\pi=0.04\ \mathrm{h}^{-1}$
- $\theta_v(s)=\bar{\theta}_v(1+\alpha_v\sin(\gamma(s+\psi_v)))$
  with $\bar{\theta}_v=8$ m/s, $\alpha_v=0.375$, $\gamma=\pi/12\ \mathrm{h}^{-1}$, $\psi_v=2\ \mathrm{h}^{-1}$
- Function $\theta_\Pi(s)$ has been taken as the hourly electricity price for December 21, 2022 of the Iberian market (MIBEL) (see https://www.omie.es/es/market-results/daily/daily-market/daily-hourly-price?scope=daily&date=2022-12-21). In fact, we have taken an interpolation spline of this step function of hourly prices. Plots of curves θ_v and θ_Π can be seen in Fig. 1.
- Parameters for function $\widehat{P}_v$ in (1): $v_r=13$ m/s, $v_{cut-off}=25$ m/s, $v_{cut-in}=4$ m/s, $C_{per}=0.5$, $\rho=1.29\ \mathrm{kg/m^3}$, $R=24$ m.

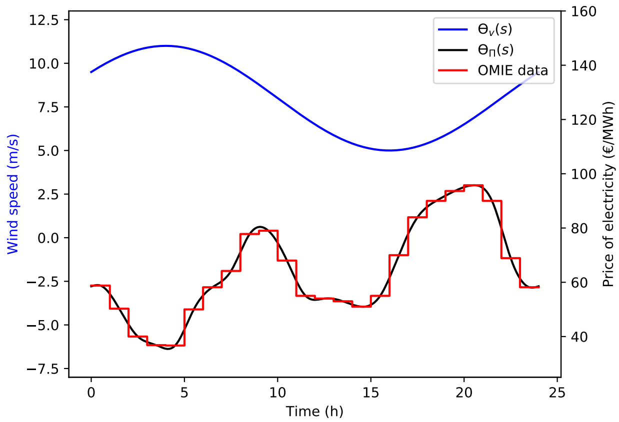


Figure 1. Functions θ_v and θ_Π. The latter is a spline interpolation of the step function of hourly prices of electricity on December 21, 2022, in the Iberian market.

- Constants in matrix Σ (in $\mathrm{h}^{-1/2}$): $\sigma_{v,v} = 0.2$, $\sigma_{v,\Pi} = 0$, $\sigma_{\Pi,v} = 0.01$, $\sigma_{\Pi,\Pi} = 0.075$, $\sigma_{\mathcal{E},\mathcal{E}} = 0.1$.
- Bounds for energy content in battery: $\mathcal{E}_{min} = 0$ MWh, $\mathcal{E}_{max} = 4$ MWh.
- Bounds for battery charge/discharge: $\mathcal{P}_{bat,min} = -1$ MW, $\mathcal{P}_{bat,max} = 1$ MW.
- In chance constraint (19): $\alpha = 0.8$
- In cost function: $g(x) = \beta x$ with $\beta = \theta_\Pi(T)$ €/MWh.
- Initial conditions for the SDE model: joint normal distribution of mean vector and variance matrix

$$\bar{\mathbf{x}} = \begin{pmatrix} \theta_v(0) \\ \theta_\Pi(0) \\ 2 \end{pmatrix} \text{ and } V_0 = \begin{pmatrix} 0.226 & 0 & 0 \\ 0 & 8.560 & 0 \\ 0 & 0 & 0.01 \end{pmatrix},$$

respectively. Variance matrix is taken so that each state variable has an initial standard deviation equal to the 5% of the correspondent initial mean value.

8.2. *Results*

The problem has been solved by using the algorithm of multipliers described in § 7.1 with initial null value for the Lagrange multiplier.

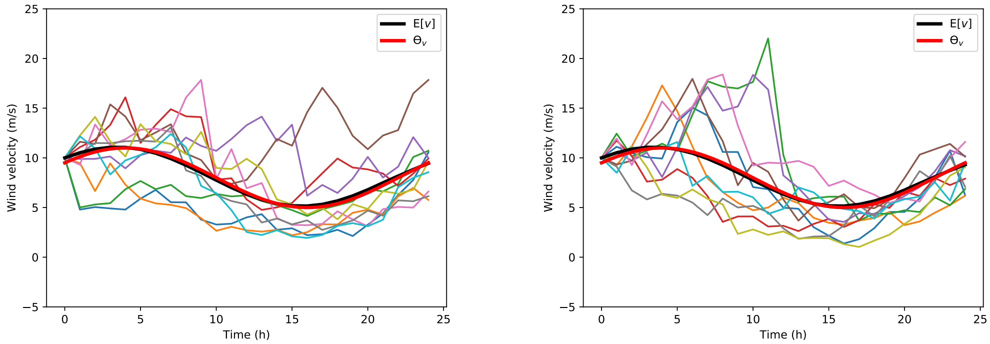
Both the Fokker–Planck and the Hamilton–Jacobi–Bellman equations have been discretized in time by the Euler implicit scheme, and in space using continuous piecewise linear finite elements on a tetrahedral mesh.

The computational domain is $(-30,50) \times (-40,220) \times (-30,40)$. It has been adjusted for the integral of the solution of the Fokker–Planck equation has integral equal to 1. We have taken a structured mesh of $21 \times 66 \times 36$ nodes. The time interval is $(0,24)$ (one day) and the time step is 1 h.

We include several figures to illustrate the numerical results. We distinguish two sets corresponding to the two formulations of the constraints, namely, (10) and (11).

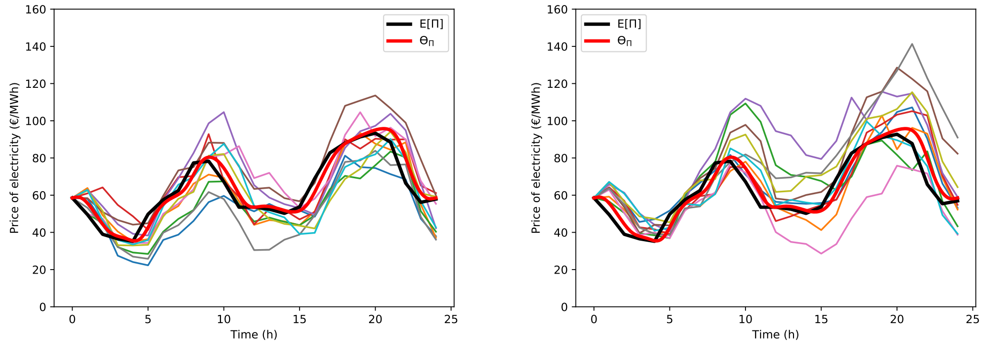
All figures concern the optimal solution. The left (resp. the right) of each figure corresponds to the state constraint (10) (respectively, (11)). Firstly, Fig. 2, Fig. 3, and Fig. 4 show ten realizations and the expectations of the three components of the process state: wind velocity, price, and energy content of the battery. Moreover, Fig. 2 and Fig. 3 include their respective θ functions.

Similarly, in Fig. 5 the ten optimal controls corresponding to the above realizations, that is the discharge/charge power of the battery, can be seen. We emphasize its *bang-bang* character. The expectation of the control is also shown.



(a) **(b)**

Figure 2. Ten realizations and expectation of wind velocity.



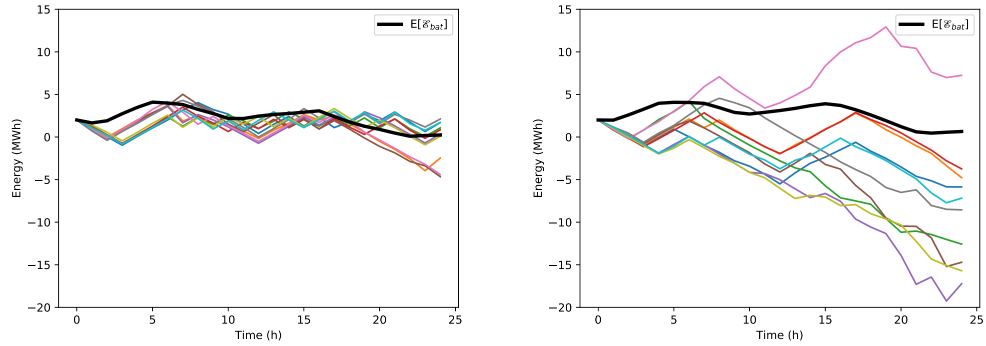
(a) **(b)**

Figure 3. Ten realizations and expectation of the price.

Fig. 6 shows the expectations of the stochastic processes $\widehat{P}(v_s)$ (instantaneous wind power), $P_{bat,s}$ (instantaneous battery discharge/charge power), $P_{grid,s}$ (instantaneous power to the grid) and Π_s (instantaneous electricity price).

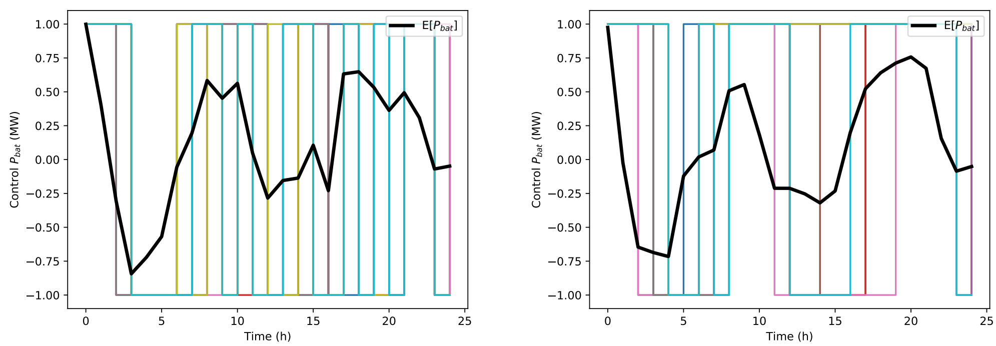
Fig 7 includes the expectation along the time, s, of the energy content in the battery, $\mathscr{E}_s$, and the electricity price Π_s. Notice that the energy content in the battery is kept between the stated bounds, namely, 0 and 4 MWh. Moreover, the bounds for the control $P_{bat,s}$, i.e., -1 and 1 are also respected.

Fig. 8 shows the hourly power bid for the day-ahead electricity auction. In order to offer *capacity firming*, i.e., constant power along each hour, the bid has been obtained by computing the average power from the expectation of the control (see Remarks in § 2.3, particularly formula (13)). We can observe that the plant buys energy from the grid to charge the battery during some time intervals in which the price of electricity is the lowest. Moreover, it sells electricity when the price is high.



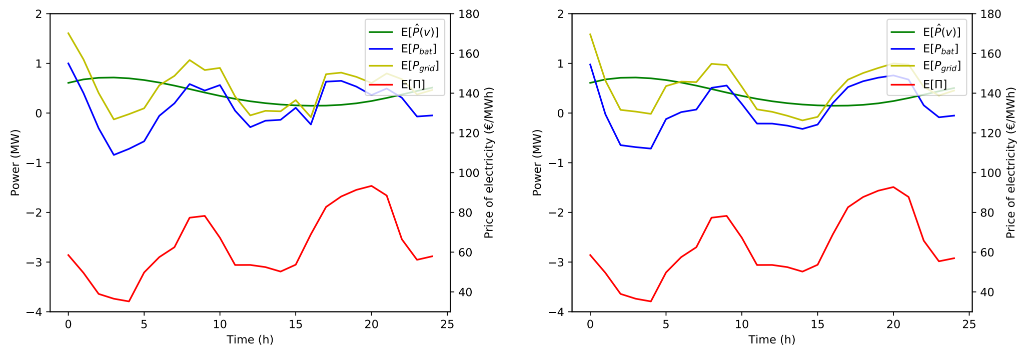
(a) **(b)**

Figure 4. Ten realizations and expectation of energy content in the battery.



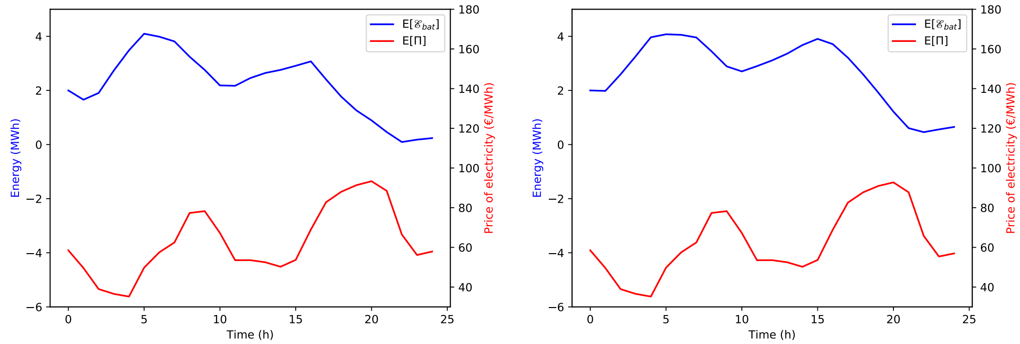
(a) **(b)**

Figure 5. Ten realizations and expectation of the control function (battery discharge/-charge power).



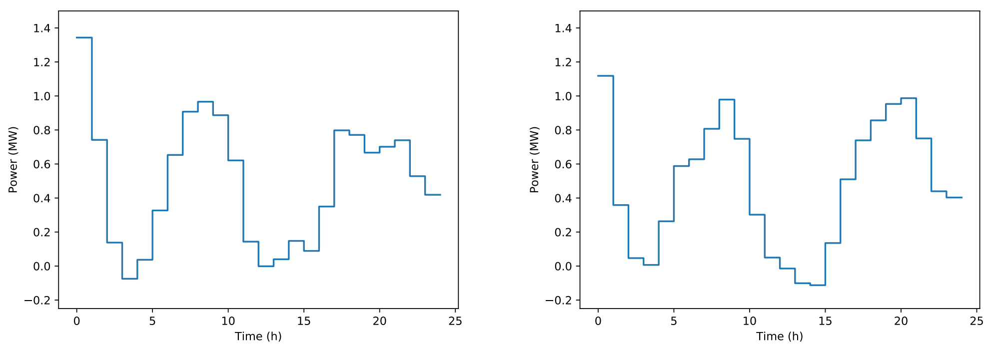
(a) **(b)**

Figure 6. Expectations along the time of stochastic processes $\widehat{P}(v_s)$ (instantaneous wind power), $P_{bat,s}$ (instantaneous battery discharge/charge power), $P_{grid,s}$ (instantaneous power to the grid) and Π_s (instantaneous electricity price).



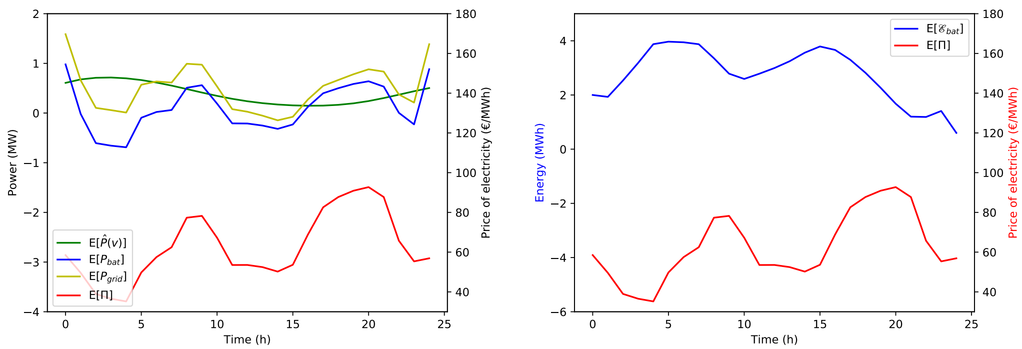
(a) **(b)**

Figure 7. Expectations along the time of the energy content in the battery, $\mathscr{E}_s$, and the spot electricity price Π_s.



(a) **(b)**

Figure 8. Optimal hourly power bid to the day-ahead electricity auction (*capacity firming*).



(a) **(b)**

Figure 9. Case $\beta = 0$. **(a)**: Expectations along the time of stochastic processes $\widehat{P}(v_s)$ (instantaneous wind power), $P_{bat,s}$ (instantaneous battery discharge/charge power), $P_{grid,s}$ (instantaneous power to the grid) and Π_s (instantaneous electricity price). **(b)**: Expectations along the time of the energy content in the battery, $\mathscr{E}_s$, and the spot electricity price Π_s.

Finally, the optimal control problem has been solved for $\beta = 0$, that is, we do not care about the energy content of the battery at the end of the process, i.e., at time T. The state constraint (20) has been considered, namely, (11). Results are shown in Fig. 9 and Fig. 10. Note that, contrary to what happens in the case where β is not null (see Fig. 7b), the expectation of the energy content in the battery at time T is null since there is no incentive to not to sell all the available energy. We also notice the difference in the optimal hourly power bids in the last hours (compare Fig. 10 to Fig. 8b).

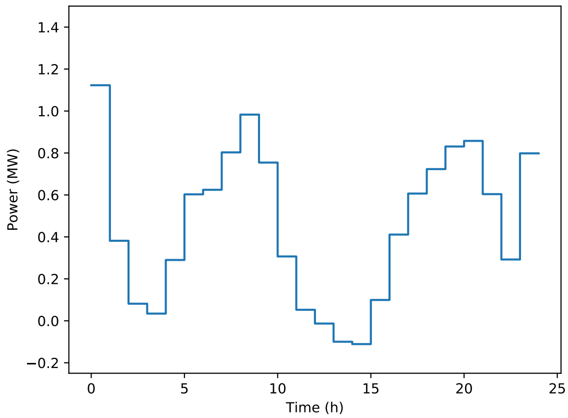


Figure 10. Case $\beta = 0$. Optimal hourly power bid to the day-ahead electricity auction (*capacity firming*).

9. Conclusion

This paper deals with the use of stochastic control theory to determine the optimal bids to the day-ahead electricity auctions and also to find the optimal feedback control for the plant in real time. Since the problem involves state constraints due to the limited battery capacity, it has been solved by reformulating it as a deterministic optimal control problem for the Fokker–Planck PDE. The adjoint state equation of this control problem is a nonlinear Hamilton–Jacobi–Bellman equation of the second order including a Lagrange multiplier to handle the state constraint. The equation to determine this Lagrange multiplier involves the probability density function of the state of the original stochastic problem, which is solution of the Fokker-Planck equation. As a consequence, the optimality conditions involve a coupled forward-backward system of PDEs and not only the usual Hamilton–Jacobi–Bellman equation. A realistic numerical test is included which shows a good qualitative behaviour of the solution.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

Dedication

This paper is dedicated to the memory of Professor Roland Glowinski.

Acknowledgments

The authors would like to thank an anonymous referee for his/her careful reading of the manuscript and valuable suggestions that have improved the readability of the manuscript.

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