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Géry de Saxcé and Dina Razafindralandy

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Guest editors: Francisco Chinesta (PIMM, UMR CNRS 8006, Arts et Métiers Institute of Technology, Paris, France) and Aziz Hamdouni (LaSIE, UMR CNRS 7356, La Rochelle Université, France)

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Lie groups and continuum mechanics: where do we stand today?

Groupes de Lie et mécanique des milieux continus : où en sommes-nous aujourd'hui ?

Géry de Saxcé [*, a] **and Dina Razafindralandy** [b]

[a] Univ. Lille, CNRS, Centrale Lille, UMR 9013 - LaMcube - Laboratoire de mécanique multiphysique multiéchelle, F59000, Lille, France

[b] Laboratoire des Sciences de l'Ingénieur pour l'Environnement (LaSIE) - UMR CNRS 7356, Pôle Sciences et Technologie, Avenue Michel Crépeau, F17042 La Rochelle Cedex 1, France

E-mails: gery.de-saxce@univ-lille.fr (G. de Saxcé), drazafin@univ-lr.fr (D. Razafindralandy)

Abstract. The geometric methods have experienced a fast growth in the past few decades. In this survey, we discuss the use of Lie groups in continuum mechanics. We address both the theoretical and numerical aspects. We explore the classical symmetry groups of the mechanics, the covariant form of the equations and the symmetry group of constitutive laws. We consider the Lie symmetry group of the equations of a mechanical problem and investigate how to take advantage of them in developping analytical models (self-similar solutions, conservation laws, turbulence, . . .) of the physical phenomena encoded in these equations. Lastly, we present a method of constructing robust numerical integrators from the knowledge of the Lie symmetry group of the equations.

Résumé. Les méthodes géométriques ont connu une croissance rapide au cours des dernières décennies. Dans cette étude, nous discutons de l'utilisation des groupes de Lie en mécanique des milieux continus. Nous abordons à la fois les aspects théoriques et numériques. Nous explorons les groupes de symétrie classiques de la mécanique, la forme covariante des équations et le groupe de symétrie des lois constitutives. Nous considérons le groupe de symétrie de Lie des équations d'un problème de mécanique et montrons comment en tirer profit dans le développement de modèles analytiques (solutions auto-similaires, lois de conservation, turbulence, etc.) des phénomènes physiques encodés dans ces équations. Enfin, nous présentons une méthode de construction d'intégrateurs numériques robustes à partir de la connaissance du groupe de symétrie de Lie des équations.

Keywords. Lie groups, continuum mechanics, symmetry groups, geometric integrators, turbulence modelling.

* Corresponding author

Mots-clés. Groupes de Lie, mécanique des milieux continus, groupes de symétrie, intégrateurs géométriques, modélisation de la turbulence.

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1. Introduction

The Ancients, Greeks after Empedocles at fifth century BC and Roman people but also Persian, Indian and Japonese, explained the complexity of all matter in terms of simple substances: the *Earth, water, air* and *fire*. Today our method is the same as Ancients' one but the simple substances, the elementary bricks of the geometric approach of the mechanics and physics of continua are different and are *manifolds, groups* and *fibre bundles*.

A Lie group is a group that is also a differentiable manifold [1,2]. Its theory was developed from 1873 by Marius Sophus Lie, a Norwegian mathematician (see [3–5]). His interest in the geometry of differential equations was first motivated by the work of Carl Gustav Jacobi, on the theory of partial differential equations of first order and on the equations of classical mechanics. Some of Lie's early ideas were developed in close collaboration with Felix Klein. It was Lie who introduced Klein to the concept of group, which was to have a major role in his later work. In the famous Erlangen program, Klein synthesized in 1872 the geometry as the study of the properties of a space that is invariant under a given group of transformations, called today a *symmetry group*. In modern terminology, there is a group action on a manifold. Lie's set idea was to develop a theory of symmetries of differential equations that would accomplish for them what Évariste Galois had done for algebraic equations [6–8]. An important idea introduced by Lie was to work with the set of infinitesimal perturbations of the identity of the group called its *Lie algebra*. Next, new and strong developments of Lie's theory are the work of his successors, especially Wilhelm Killing, Élie Cartan and Hermann Weyl.

As already mentioned, the links between Lie groups and mechanics are deep since problems of mechanics are among others at the origin of Lie's concern. From a theoretical point of view, two kinds of issues can be addressed. At first, a symmetry group is given and we are interested in finding the invariant objects, the invariant properties or the invariant functions of these objects. It was already the case in Klein's approach of the geometry. In mechanics, the objects at issue are naturally tensors. The most fundamental aspects concern the covariance of the theory. To be general, the basic principles and equations of balance must be covariant, i.e. their form must be conserved by every transformation of the group. Another important aspect concerns the constitutive laws. A relevant challenge is to find the invariant functions of the elasticity tensor, or today the *constitutive tensors* representing the coupling with other phenomena such as thermoelasticity, piezoelasticity, magnetoelasticity and so on.

For the second issue, in contrast, the objects are known and we are searching the symmetry group which let invariant these objects, especially the systems of algebraic equations or partial differential equations.

By acting on the manifolds, the group gives a structure to the equations of the mechanical problem. Since the advent of numerical approximations and overall the computational methods, a large spectrum of methods have been developed that proved their efficiency but also their weaknesses. Worsening performance have been observed when these structure is not conserved by the approximations. In dynamics, odd numerical schemes suffer from numerical dissipation then the developments of symplectic integrators. These spurious effects can be eliminated by using symplectic integrators that respect the structure of the equations of the dynamics and conserve the energy.

In the present article, we propose a journey through selected topics where Lie groups and mechancis work together. More precisely, we make an overview of different concepts of Lie symmetry groups in continuum mechanics and present some applications. In Section 2, we revisit the fundamental symmetry groups of the mechanics. Many areas from statics to thermodynamics, including classical and relativistic approaches, will be covered. The symmetry group of constitutive laws will also be discussed. Section 3 is devoted to the symmetry group of the differential equations of a mechanical problem and their use as a modelling tool. Most applications are oriented towards fluid mechanics. It is shown how to obtain analytical (self-similar) solutions, conservation laws through an extension of Nœther's theorem, and physics-preserving turbulence models from the symmetry group of the equations. In the same spirit as symplectic schemes, Section 4 presents a way to eliminate spurious effects in numerical integrations thanks to the knowledge of the symmetry group admitted by the equations.

2. Symmetry groups of the mechanics

In a nutshell, the leading idea of Erlangen program can be expressed as "a geometry is a group". By selecting a group, we define a geometry. All stems back to the symmetry group. Paraphrasing Klein's idea, "a physics is a group". For instance, by choosing Lorentz–Poincaré group, we define the relativistic mechanics. By the way, which is the symmetry group of the classical mechanics? As we shall see in the sequel, the response is not unique but before tackling the subject, let us discuss which are the main aspects concerned by the action of a symmetry group: the coordinate systems and local charts associated to the group, the transformation laws of tensors, the connections linked to the group.

Generally speaking, the elements of a symmetry group of the mechanics are coordinate changes which are linear or affine and can be represented by a matrix. From a physical point of view, the corresponding coordinate systems are those in which the observers measure lengths, durations, velocities and other related physical quantities. The space $\mathbb{R}^n$ on which the symmetry group G acts is a simple vector affine space. On the contrary, the universe has an underlying structure of differentiable manifold and the question arises as to whether there exists at every point changes of local charts of which the Jacobean matrix is a transformation of the symmetry group. This leads to a system of partial derivative equations of first order that can be solved by Frobenius method. If the equations are compatible, the corresponding local charts are said admissible and the G-structure is integrable [9]. As we shall see, this is not always possible. An important property of a G-structure is that it is integrable if and only there exist one or more tensor fields of which the components in the admissible local charts are invariant ([9, Proposition 1.1]). These *structural tensors* equip the manifold with an underlying structure.

The behaviour of the continuum is represented by tensor fields that are sections of fiber bundles over the universe manifold. In the frames associated to the aforementioned coordinate systems or local charts associated to the group (called *G-frames*), these tensors are represented by components that are modified according to a transformation law where the group acts by representation. If we see tensors as orbits of the group, we define a class of tensors called *G-tensors* (for instance, as there are Euclidean tensors, there are Galilean tensors, Poincaréan tensors, . . .). In fact the group plays a central role in the sense that it reveals the physical meaning of the components. A fruitful method consists in determining the minimal number of elements of a functional basis (able to generate all the invariants). It is constructive in the sense that it allows to determine the invariants themselves and a *normal form* (or reduced form) of the system of components. For instance, the normal form of a symmetric tensor is the diagonal form where the eigenvalues are invariant, but this method is general for tensors of arbitrary ranks. Next, starting from the normal form and applying a transformation of the group, we obtain the *generic form* which is *covariant* in the sense that its expression is the same in every coordinate system or local chart associated to the group.

As there are G-tensors, there are *G-connections*, i.e. symmetric connections of which the connection matrix in the G-frames belongs to the Lie algebra of the group. In the modern approach, they are defined as connections on a principal bundle of structural G of which the orbits are the fibers. As tensors, connections acquire a physical meaning. For instance, in relativity, the connection represents the gravitation.

2.1. *Statics*

This is the most simple situation since the equilibrium is maintained (as long as it is not perturbed by a new loading), then the time does not matter and the universe is reduced to the 3D physical space. The transformations are changes of coordinate systems in which the observers measure lengths, then they are isometries. The symmetry group is *Euclid's group*. It is the group of the Euclidean geometry, according to Erlangen program, but it is also the group of Statics.

2.2. *Dynamics*

To take into account the evolution of the systems, we need an extra coordinate, the time. The universe is modelized by the 4D space-time. It is a relativistic idea but it turns out to be also relevant for the classical mechanics. Anyway, the space-time is not empty. There are objects (single particles or continuum bodies) moving inside, and among them observers. Writing the equations of the dynamics as a differential system of order one (the canonical equations) on the cotangent bundle, there is a group that conserves the structure of these equations, the *symplectic group* $Sp(2n)$ (with $n = 4$) that conserves the *canonical symplectic form* $\omega = dx^1 \wedge dp_1 + \cdots + dx^n \wedge dp_n$ [10]. More generally, a group is symplectic if it let invariant a closed non-degenerate 2-covariant tensor called the *symplectic form* and *presymplectic* if the invariant 2-covariant tensor is degenerate. The symmetry group of any theory of dynamics is expected to be symplectic or presymplectic. Although at the beginning the symplectic geometry was applied mainly to the dynamics of particles and rigid bodies [11, 12], it is now spreading within the continuum mechanics. It is also widely applied to other physical theories such that optics, electromagnetism, quantum mechanics, statistical mechanics [12–14].

According to Darboux theorem, if ω is a presymplectic form of rank k, at every point of the manifold, there exists a local chart in which it is represented by $\omega = dx^1 \wedge dx^{r+1} + \cdots + dx^r \wedge dx^{2r}$. If the dimension of the manifold is $2n$ and $r = n$, the local charts are said symplectic or canonical. The corresponding change of local charts define an integrable $Sp(2n)$-structure.

An important tool introduced by Jean-Marie Souriau [12] is the *momentum map* from the manifold on which the group acts into the dual of its Lie algebra which is constant on the trajectories (first integral). It is a modern version of Noether's theorem.

There are two challenges: how to generalize the symplectic form and the momentum map to continuum mechanics. A natural idea is to introduce a *multisymplectic* form of degree > 2 [15]. Although there are various attempts in the literature, no significant breakthrough has been achieved until now that opens promising perspectives for the continuum mechanics.

2.3. *Classical dynamics*

The space-time is the set of events. An event occurring at time t and position $\boldsymbol{x}$ is represented in a local chart by a system of four components

$$\boldsymbol{X} = \begin{bmatrix} t \\ \boldsymbol{x} \end{bmatrix} \in \mathbb{R}^4 ,$$

In absence of gravitation, the particles are moving in Uniform Straight Motion (USM). The transformations of $\mathbb{R}^4$ conserving the USM, the durations, the distances and the space orientations are affine of the form

$$\boldsymbol{X} = \boldsymbol{C} + \mathsf{P}\,\boldsymbol{X}', \qquad \boldsymbol{C} = \begin{bmatrix} \tau_0 \\ \boldsymbol{k} \end{bmatrix}, \qquad \mathsf{P} = \begin{bmatrix} 1 & \mathbf{0} \\ \boldsymbol{v}_T & \mathsf{R} \end{bmatrix}$$

where $\boldsymbol{v}_T \in \mathbb{R}^3$ is the velocity of transport or Galilean boost, $\mathsf{R} \in \mathbb{SO}(3)$ is a rotation, $\boldsymbol{k} \in \mathbb{R}^3$ is a spatial translation and $\tau_0 \in \mathbb{R}$ is a clock change. Their set $\mathbb{GAL}$ is *Galileo's group*, a Lie group of dimension 10. The set $\mathbb{GAL}_0$ of the linear Galilean transformations P is a Lie subgroup of dimension 6. The usual velocity $\boldsymbol{v}$ is extended in the space-time as the 4-velocity

$$\boldsymbol{U} = \frac{d\boldsymbol{X}}{dt} = \frac{d}{dt}\begin{bmatrix} t \\ \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 1 \\ \boldsymbol{v} \end{bmatrix}.$$

The transformation law of the vectors $\boldsymbol{U} = \mathsf{P}\,\boldsymbol{U}'$ leads to the additive composition of velocities of the classical mechanics $\boldsymbol{v} = \boldsymbol{v}_T + \mathsf{R}\,\boldsymbol{v}'$.

One of the properties of Galileo's group is that its G-structure is integrable. More precisely, every change of local chart of the G-structure $\boldsymbol{X}' \mapsto \boldsymbol{X}$ is compound of a clock change $t = t' + \tau_0$, and a rigid body motion

$$\boldsymbol{x} = \mathsf{R}(t)\boldsymbol{x}' + \boldsymbol{x}_0(t)$$

this last relation being often used in the statement of the principle of objectivity and material indifference but rather presented as an axiom, although this is a spin-off of the structure given by Galileo's group. By the way, it equips the space-time with a structure based on two structural tensors, a semi-metric (the metric of the space slice) and a 1-form $\tau = dt$ (the clock form). This neoclassic modelling, introduced by Toupin [16] and taken up later on by Noll [17] and Künzle [18], offers a theoretical framework for the universal (or absolute) time and space. In modern literature, it is often called Newton–Cartan structure.

The compatible local chart of Galileo's group are called *Galilean charts*. In the geometric language, *Galileo's principle of relativity* claims that the statement of the physical laws of the classical mechanics is the same in all the Galilean charts. To respect this principle in practice, the laws are expressed in a *covariant form*, using a $\mathbb{GAL}$-connection ∇ called *Galilean connection*. In the Galilean charts (where the index 0 is related to time and Latin indices run from 1 to 3), the non vanishing Christoffel's symbols are $g^j = -\Gamma^j_{00}$, identified to the gravity, and $\Omega^i_j = \Gamma^i_{j0} = -\Gamma^j_{i0}$ is interpreted as representing Coriolis' effects [19].

The matter and its evolution can be modelized by a line bundle $\mathscr{M} \to \mathscr{M}_0$ where $\mathscr{M}$ is the space-time of dimension 4, the base $\mathscr{M}_0$ is a manifold of dimension 3 and the fibers are the trajectories of the material particles [20]. The coordinates in a local chart of $\mathscr{M}_0$ are Lagrangian while the coordinates of the position x are Eulerian.

In statics, the key object to modelize internal forces in a continuum is Cauchy's stress tensor. Tackling the dynamics is simply a matter of introducing an extra dimension, the time. We start with a space-time symmetric 2-contravariant tensor represented in a Galilean chart of coordinates $\boldsymbol{X}'$ by a symmetric 4×4 matrix T'. To determine the number of independent invariants, we consider the action of $\mathbb{GAL}_0$, *i.e.* the transformation law of the tensor $\mathsf{T} = \mathsf{P}\,\mathsf{T}'\mathsf{P}^\mathsf{T}$. We calculate first of all the dimension of the orbit of T' as the difference 6 between the dimension 6 of the group and the dimension 0 of its isotropy subgroup; next the number of independent invariants as the difference between the number 10 of independent components of T' and the dimension 6 of the orbit. They are 4 independent invariants that can be gathered in the normal form:

$$\mathsf{T}' = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & -\sigma_1 & 0 & 0 \\ 0 & 0 & -\sigma_2 & 0 \\ 0 & 0 & 0 & -\sigma_3 \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & -\sigma' \end{bmatrix}$$

Next, using once again the transformation law $\mathsf{T} = \mathsf{P}\,\mathsf{T}'\mathsf{P}^\top$ with a Galilean boost v and a rotation R, we obtain the generic form

$$\mathsf{T} = \begin{bmatrix} \rho & \rho\,\boldsymbol{v}^\top \\ \rho\,\boldsymbol{v} & \rho\,\boldsymbol{v}\,\boldsymbol{v}^\top - \sigma \end{bmatrix}$$

that reveals the physical meaning of the components: the density of mass ρ, the linear momentum $\boldsymbol{p} = \rho\,\boldsymbol{v}$, the Cauchy stresses $\sigma = \mathsf{R}\sigma'\mathsf{R}^\top$ and the dynamical stresses $\rho\boldsymbol{v}\boldsymbol{v}^\top - \sigma$. These particular quantities, that are introduced drop by drop in the course of standard textbooks, are gathered here in a *big tensor* of higher dimension. They are working together as a team. We call T the *stress-mass tensor.*

Generalizing the equation of internal equilibrium of the statics ($\mathrm{Div}\,\sigma = 0$), the behaviour of the continuum is modelized by a conservation equation $\mathrm{Div}\,\mathsf{T} = 0$, or in tensor notations,

$$\nabla_\beta T^{\alpha\beta} = 0$$

where the Greek indices run from 0 to 3, that gives rise to

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x^j}\left(\rho\,v^j\right) = 0, \qquad \rho\,\frac{\partial v^i}{\partial t} + \rho\,v^j\,\frac{\partial v^i}{\partial x^j} = \frac{\partial\sigma^{ij}}{\partial x^j} + \rho\left(g^i - 2\Omega^i_k\,v^k\right).$$

We can recognize Euler's equations of continua in a covariant form (according to Galileo's principle of relativity) where occurs in the right hand side of the latest equation the sum of the gravity force and Coriolis' force. Then we recover the equation of balance of mass and linear momentum. However, the balance of energy is missing.

2.4. *Relativistic dynamics and thermodynamics*

By opposition to Galilean relativity, Einstein's Relativity is based on the experimental fact that the speed of the light has a huge but finite value c for every observer. In absence of gravitation, the light rays are straight lines and the particles of light –the photons– move at the constant velocity c in any coordinate system X where the observer takes measures to identify the events. The corresponding symmetry group is composed of the transformations preserving the USM (then it is affine) and Minkowski's metrics $G_{\alpha\beta}dX^\alpha dX^\beta = c^2dt^2 - \|dx\|^2$. It is called *Lorentz–Poincaré group* and it is a Lie group of dimension 10 (as Galileo's group). The 4-velocity satisfying the normalization condition $G_{\alpha\beta}U^\alpha U^\beta = c^2$ is of the form

$$\boldsymbol{U} = \frac{d\boldsymbol{X}}{dt} = \gamma\begin{bmatrix} 1 \\ \boldsymbol{v} \end{bmatrix}$$

where $\gamma = (1 - \|\boldsymbol{v}\|^2/c^2)^{-1/2}$. An important difference with the classical mechanics is that the corresponding G-structure *is not integrable*, the obstruction being the curvature, i.e. the gravitation. In Relativity, it is simpler to manipulate the tensors than in classical mechanics because of the existence of the metrics that allows to raise and lower the indices. For instance, $T^0_0 = G_{00}T^{00}$ is no more than $e = \rho\,c^2$ stating the equivalence between the mass and the energy. Then the 2-rank tensor T is called *stress-energy tensor.* The behaviour of the continuum is always modelized by a conservation equation $\nabla_\beta T^{\alpha\beta} = 0$ allowing to recover the equations of balance of the energy and the linear momentum.

The idea to extend this *geometrization process* to the Thermodynamics of continua is natural. In [21], Eckart proposed to generalize the *first principle* of the Thermodynamics in terms of the stress-energy tensor in the form $\nabla_\beta T^{\alpha\beta} = 0$, the projection onto the 4-velocity allowing to recover the heat transfert equation $U_\alpha\nabla_\beta T^{\alpha\beta} = 0$. For the dissipative processes, Landau and Lifshitz propose in their relativistic mechanics of fluids [22] an additive decomposition of the stress-energy tensor into reversible and irreversible parts $T^{\alpha\beta} = T^{\alpha\beta}_R + T^{\alpha\beta}_I$ and introduce the 4-flux of entropy s given by $S^\alpha = T^{\alpha\beta}b_\beta$ where $\boldsymbol{b} = \frac{1}{\theta}\boldsymbol{U}$ is Planck's temperature 4-vector built from the inverse of the absolute temperature and the 4-velocity. In his relativistic thermodynamics ([23, 24]), Souriau proposed to generalize the *second principle* as $\nabla_\alpha S^\alpha \geq 0$ and introduced the *friction tensor* $f_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha b_\beta + \nabla_\beta b_\alpha)$ that merges the temperature gradient and the strain rate tensor, allowing to extend Fourier's conduction law and viscous flow rules to Relativity, as proposed by Vallée [25, 26]. By the way, Relativity is a consistent framework for the hypoelastic, hyperelastic and dissipative constitutive laws [27–29].

2.5. *Classical thermodynamics*

It is tempting to apply a geometrization process to the classical Thermodynamics by changing the symmetry group, that is possible with suitable modifications. In order to recover directly the equation of balance of energy, it is convenient to work in a manifold $\widehat{\mathscr{M}}$ of dimension 5 by adding an extra coordinate z of which the meaning is the action per unit mass. For a free particle of mass m and velocity $\boldsymbol{v}$, using the transformation law of the velocity, we have

$$dz = \frac{\mathscr{L}}{m}\,dt = \frac{1}{2}\,\|\boldsymbol{v}\|^2\,dt = \frac{1}{2}\,\left\|\boldsymbol{v}_T + \mathsf{R}\,\boldsymbol{v}'\right\|^2\,dt = dz' + (\boldsymbol{v}_T)^\top \mathsf{R}\,dx' + \frac{1}{2}\,\|\boldsymbol{v}_T\|^2\,dt'$$

that leads to introduce the affine transformations of $\mathbb{R}^5$ composed of a translation and a linear transformation of the form

$$\widehat{\boldsymbol{C}} = \begin{bmatrix} \tau_0 \\ \boldsymbol{k} \\ \eta \end{bmatrix}, \qquad \widehat{\mathsf{P}} = \begin{bmatrix} 1 & 0 & 0 \\ \boldsymbol{v}_T & \mathsf{R} & 0 \\ \frac{1}{2}\,\|\boldsymbol{v}_T\|^2 & (\boldsymbol{v}_T)^\top \mathsf{R} & 1 \end{bmatrix}$$

where the bottom row of $\widehat{\mathsf{P}}$ is given by the transformation law of z above. Their set is a Lie group of dimension 11 (because of the new parameter η of translation in the z direction). This group, called *Bargmann's group*, was introduced to solve problems of group quantization [30]. In addition to the invariants of Galileo's group, Bargmann's group let invariant a metric $\widehat{G}_{\alpha\beta}\,d\widehat{X}^\alpha\,d\widehat{X}^\beta = 2\,dt\,dz - \|d\boldsymbol{x}\|^2$. Plank's temperature 4-vector $\boldsymbol{b}$ is extended in the fifth dimension

$$\widehat{\boldsymbol{b}} = \begin{bmatrix} \boldsymbol{b} \\ \zeta \end{bmatrix} = \begin{bmatrix} \frac{1}{\theta}\boldsymbol{U} \\ \zeta \end{bmatrix},$$

where ζ is Planck's potential. The friction tensor is now simply $f_\alpha^\beta = \nabla_\alpha b^\beta$ (without symmetrization because the space-time is not Riemannian in classical mechanics). What about the stress-mass-tensor T? As the temperature vector, it can be also extended as a linear mapping from the tangent space to $\widehat{\mathscr{M}}$ onto the tangent space to the space-time $\mathscr{M}$. Introducing the mass 4-flux $\boldsymbol{N} = \rho\,\boldsymbol{U}$, it is represented by a 4×5 matrix

$$\widehat{\mathsf{T}} = [\mathsf{T}, \boldsymbol{N}] = \begin{bmatrix} e & -\boldsymbol{p}^\top & \rho \\ \boldsymbol{k} & \sigma_* & \boldsymbol{p} \end{bmatrix}$$

where e is the energy, $\boldsymbol{p}$ the linear momentum, σ_* the dynamic stresses and

$$\boldsymbol{k} = \boldsymbol{h} + e\,\boldsymbol{v} - \sigma\,\boldsymbol{v}$$

decomposed into the energy flux by *conduction* $\boldsymbol{h}$, by *convection* $e\,\boldsymbol{v}$ and by stress $\sigma\,\boldsymbol{v}$. The Landau–Lifshitz decomposition is still valid in the extended form $\widehat{\mathsf{T}} = \widehat{\mathsf{T}}_R + \widehat{\mathsf{T}}_I$. The first principle is extended as $\nabla_\alpha \widehat{T}^\alpha_\beta$. The second principle reads

$$\nabla_\alpha S^\alpha - \left(f_\alpha^\beta\,\tau_\beta\,U^\alpha\right)\left((T_I)_\alpha^\beta\,\tau_\beta\,U^\alpha\right) \geq 0$$

where occurs the clock form τ. It is the geometrization of the Clausius–Duhem inequality

$$\rho\frac{ds}{dt} - \frac{\rho}{\theta}\,\frac{dq_I}{dt} + \nabla_\alpha\left(\frac{h^\alpha}{\theta}\right) \geq 0$$

where s is the entropy, q_I is the specific irreversible heat source (present in T_I) and $\boldsymbol{h}$ the conduction flux.

2.6. *Affine mechanics*

Despite of several infringements to the relativistic model, the classical mechanics can be cast into the mould of the Relativity. It works both ways. It is the epistemological reversal. For instance, as the relativistic 4-velocity U_α approaches $c^2\tau_\alpha$ when c approaches the infinity, it is easy to guess the relativistic version of the second principle

$$\nabla_\alpha S^\alpha - \frac{1}{c^2}\left(f_\alpha^\beta U_\beta U^\alpha\right)\frac{1}{c^2}\left((T_I)_\alpha^\beta U_\beta U^\alpha\right) \geq 0$$

Taking the reasoning further, we can observe that Galileo's group, Lorentz–Poincaré group and Bargmann's group are all Lie subgroups of the affine group, hence the idea to develop an affine mechanics based on the concept of affine tensors [31]. The most simple types of affine tensors are the points of the tangent space to the manifold and the real-valued affine functions on this space. Analysing the underlying structure of the mechanics, we can identify three kinds of affine tensors relevant for the mechanics: the *torsor* (2-contravariant, anti-symmetric), the *co-torsor* (2-covariant, anti-symmetric), objects naturally in duality, and the *momentum tensor* (1-contravariant and 1-covariant), in duality with the infinitesimal generators of the Lie algebra of the symmetry group. Then the component system of the momentum tensor belongs to the dual of the Lie algebra, hence the name since the momentum map takes its values in this space.

On this ground, we can modelize curvilinear media of dimension p embedded in an ambient space of dimension n. The torsor is vector-valued in the tangent space to a matter manifold of dimension p. Its components $({}^\gamma T^\alpha, {}^\gamma J^{\alpha\beta})$ have several indices, the left-hand one being related to the matter manifold. The J components allow to treat *generalized continua* such as *Cosserat media*. The behaviour of the curvilinear medium is governed by equations of the form

$$_\gamma\widetilde{\nabla}\ {}^\gamma T^\alpha = 0, \qquad {}_\gamma\widetilde{\nabla}\ {}^\gamma J^{\alpha\beta} = 0$$

where $\widetilde{\nabla}$ is an affine connection [32, 33]. These equations are very general and may have declined with respect to the chosen curvilinear medium and the ambient space. The case $p = 1, n = 3$, corresponds to the statics of arches, $p = 1, n = 4$, to the dynamics of the rigid body, $p = 2, n = 4$, to the dynamics of arches (if solid), water jets, flows in a pipe, in a hose (if fluid) [19]. For $p = n = 4$, we recover Euler's equation of a 3D continuum.

2.7. *Symmetry group of constitutive laws*

A topical issue which is becoming increasingly important for the improvement of material properties and the design of new materials (architectured materials) is the study of their symmetries. The most simple constitutive relation is the elastic behaviour characterized by the elasticity tensor, an Euclidean tensor of rank four. The group $\mathbb{O}(3)$ acts by linear representation onto its components. The materials are anisotropic by nature (rocks, wood, ...) or by construction (composite materials). The symmetry group of a material is the isotropy group of its elasticity tensor. The set of materials with the same isotropy subgroup is called a symmetry class. It may be a Lie subgroup of $\mathbb{O}(3)$ but also a finite subgroup. The relevant mathematical tools are also representation theory and algebric geometry. If the isotropy group is $\mathbb{O}(3)$ itself, the material is isotropic.

Otherwise, if its components have been measured, determiming the symmetry class of the material can be a difficult task because its components depend not only on the elasticity tensor but also on the frame of reference in which they have been determined, that is the orthonormal basis in which the components are given. A tool to classify the elastic materials is to define an integrity basis of invariants, that is a set of independent invariant functions of the components. Another tools is the harmonic decomposition of the elasticity tensor. In 3D, The road ahead is long and full of obstacles [34–36]. In 2D, the problems are more affordable. This topic has been investigated recently, for instance in [37, 38]. It is worth to mention a very elegant and efficient method making use of complex numbers, the so-called 'polar method' by Verchery [39, 40]. This approach, developed further by Vannucci, is useful for the design of composite materials [41]. Another point of view consists in evaluating the distance to a given symmetry class [42]. In recent years, a new trend appears, the study of the coupling of the elasticity with other phenomena, in particular the piezoelectricity (appearance of a polarization in a dielectric material when it is subjected to an uniform mechanical strain) and the flexoelectricity (linear response of a polarization to a non-uniform strain or a strain-gradient) [43]. In [44], use is made of Curie principle [45]: the symmetry group of the consequences (the elasticity tensor) includes the symmetry group of the causes (the microstructure).

After this overview of the fundamental symmetry groups of the mechanics, we now look at another class of Lie symmetry groups, which are the symmetry groups of the equations of a mechanical problem, and their applications.

3. Symmetry group of the equations of a mechanical problem

An efficient tool in analyzing a mechanical problem is the Lie symmetry group of its evolution equation, that is the maximal Lie group of transformations which leave the set of solutions invariant [1, 2, 46]. One first use of Lie symmetry groups in mechanics is the reduction of the equations. Indeed, knowing a Lie symmetry group, one can find a suitable ansatz which reduces the equations. With successive reductions, one may obtain semi- or entirely analytical (self-similar) solutions. More abstractly, the Lie symmetry group approach has been used to classify mechanical problems according to the class of self-similar solutions that they may share [47–54].

Self-similar solutions are particularly helpful to build models of the evolution of a mechanical system, to study its stability or to validate numerical software. When self-similar solutions do not verify boundary conditions, they can be thought as asymptotic solutions in some regions far from boundary. This approach has been exploited to find solutions of equations in mathematical physics (see for example [55–59]). In mechanics, the Lie group reduction has been used to establish models of cosmology, to study the equilibrium of a membrane, viscoelasticity, plasticity, to classify beam equations, ... [59–68]. In fluid mechanics, self-similar solutions of the Navier–Stokes equations have been computed [69–72] some decades ago. In [71], self-similar vortex-like solutions have been particularly studied. In Section 3.1, the reduction process will be illustrated on an anisothermal model of fluid flow.

Another use of Lie symmetry in mechanics is to derive conservation laws. In the case of a variational problem, it is known from the first Nœther's theorem that to each variationnal symmetry group of the Lagrangian action corresponds a conservation law [73–75]. Recall that a variationnal symmetry is a transformation which leave the Lagrangian action invariant. It is a particular symmetry of the Euler–Lagrange equations. Nœther's theorem establishes for example a link between the invariance of the Lagrangian function under the time translation group and the conservation of energy, or between the spatial homogeneity of the Lagrangian and the conservation of linear momentum. Table 1 summarizes some very classical consequences of Nœther's theorem in classical and quantum mechanics. In structure mechanics, conservation of Eshelby-like tensors and the path invariance of J-integral in crack mechanics, for instance, can be explained by Nœther's theorem [76–84]. A detailed view of the derivation and application of conservation laws in elasticity can be found in the books [85, 86]. In inviscid fluid mechanics, it is now known

Table 1. Some variational symmetries and the corresponding conserved currents.

Symmetry/Symmetry group	Conserved current
Time translation	Energy
Space translation	Linear momentum
Rotation	Angular momentum
Lorentz boost	Boost 3-vector [102]
Permutation of identical particles	Fermi-Dirac or Bose-Einstein statistics [103]
Circle group $U(1)$	Electric charge
Unitary matrix group $SU(3)$	Color charge

that classical conservation (energy, momentum, vorticity, helicity) laws are consequences of symmetries. In addition, Nœther's theorem permitted to find a new conservation law in [87] from a scaling symmetry. Moreover, Nœther's theorem was used to study schocks [88] or to show the non-existence of time-periodic solutions with finite nonzero energy [87]. Some other recent applications in fluid dynamics are proposed in [89–94]. Exploitation of Nœther's theorem for modelling and stability analysis in cosmology can be found in [60, 61]. Applications in relativistic and in quantum mechanics can be found in [95–99]. Conservation laws also enable to analyse the quality of a numerical scheme. Some schemes, such as (multi-)symplectic integrators, are even based on conservation laws [100, 101]. In Section 3.2, a short illustrative application of Nœther's theorem is given, followed by a generalisation to the non-Lagrangian case.

The Lie symmetry group theory can also serve as a guide in modelling mechanical phenomena. In this spirit, a group theoretical approach has been used to unify laminar thin layer flows in [104] in the field of fluid mechanics. In turbulence, a link between the symmetry group of the Navier–Stokes equations and Kolmogorov's $-5/3$ law has been exposed [105]. At the same time, scaling laws of various turbulent flows have been obtained in [106–111]. In these article, the classical scaling laws in the literature ([112–115]) has also be obtained, but also new ones. The fact that they can be obtained from a Lie symmetry group approach suggests that the Lie symmetry group contains, in some extents, the scaling laws of the equations.

Finding scaling laws is a way of modelling a fluid flow in some specific regions. Another direction to modelize turbulent flows is to run a numerical simulation. One of the mostly used turbulence simulation approach is the large eddy simulation. It permits to speed up significantly the calculation but necessitates the modelling of subgrid terms. The Lie group theory has been used to analyze classical turbulence models in [116–119]. These works conclude that many turbulence models used in litterature break the symmetries of the equations. An approach developped in [120–123] enables to build a class of turbulence models which preserve the Lie symmetry group of the Navier–Stokes equations. The works have latter been extended to the anisothermal case [108, 124–126]. This approach will be recalled in Section 3.3.

3.1. *Lie symmetry group and equation reduction*

In this subsection, the reduction process using the Lie symmetry group theory is illustrated very briefly through an example.

As said, a symmetry of a differential equation is a transformation which transforms any solution into another solution. A Lie symmetry group of an equation is a set of symmetries carrying a Lie group structure. Under suitable regularity conditions on the equation, the Lie algebra of the maximal Lie symmetry group (and then the Lie group itself by exponentiation) of the equation can be computed algorithmically from Lie's theory (see [127–129]). For instance, consider a Newtonian incompressible anisothermal fluid flow. Under Boussinesq's assumption, the flow is governed by the anisothermal Navier–Stokes equations

$$\begin{cases} \operatorname{div} \boldsymbol{v} = 0, \\ \dfrac{\partial \boldsymbol{v}}{\partial t} + \operatorname{div}(\boldsymbol{v}\boldsymbol{v}^{\top}) + \dfrac{1}{\rho} \operatorname{grad} p - \nu \Delta \boldsymbol{v} + \beta g \theta \mathrm{e}_2 = 0, \\ \dfrac{\partial \theta}{\partial t} + \operatorname{div}(\boldsymbol{v}\theta) - \kappa \Delta \theta = 0. \end{cases} \tag{1}$$

where $\boldsymbol{v}$ is the Eulerian velocity field, p the pressure, ρ the density, assumed constant, β the thermal expansion, g the gravity acceleration, ν the kinematic viscosity and κ the thermal diffusivity. e_2 is the unit ascending vertical vector. Table 2 lists the Lie (one-point) symmetry groups of equations (1) obtained from Lie's theory. It shows that equations (1) admit a 4-dimensional and four infinite-dimensional Lie symmetry groups.

Table 2. Lie Symmetry groups of equations (1). ϵ is a real scalar, ζ (resp. $\boldsymbol{\alpha}$) is an arbitrary scalar (resp. vectorial) function of time and R is a constant horizontal rotation matrix.

Lie symmetry group	$\left(t, \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{v}}, \widehat{p}, \widehat{\theta}\right)$
Time translation group	$(t+\epsilon, \boldsymbol{x}, \boldsymbol{v}, p, \theta)$,
Pressure translation group	$(t, \boldsymbol{x}, \boldsymbol{v}, p+\zeta(t), \theta)$,
Pressure-temperature translation group	$\left(t, \boldsymbol{x}, \boldsymbol{v}, p+\epsilon\, \beta g\, x_3, \theta+\epsilon/\rho\right)$,
Plane rotation group	$(t, \mathsf{R}\boldsymbol{x}, \mathsf{R}\boldsymbol{v}, p, \theta)$,
Generalized Galilean boost group	$\left(t, \boldsymbol{x}+\boldsymbol{\alpha}(t), \boldsymbol{v}+\dot{\boldsymbol{\alpha}}, p-\rho\,(\boldsymbol{x}+\boldsymbol{\alpha})\cdot\ddot{\boldsymbol{\alpha}}, \theta\right)$
Scale transformation group	$\left(\mathrm{e}^{2\epsilon}\, t, \mathrm{e}^{\epsilon}\, \boldsymbol{x}, \mathrm{e}^{-\epsilon}\, \boldsymbol{v}, \mathrm{e}^{-2\epsilon}\, p, \mathrm{e}^{-3\epsilon}\, \theta\right)$.

As illustration, let us compute a self-similar solution under the scale transformation group in Table 2. To this end, one can check that the following quantities are invariants of this group:

$$\eta^i = \frac{x^i}{\sqrt{t}}, \qquad V^i = \sqrt{t}\, u^i, \qquad P = tp, \qquad \Theta = t^{3/2}\theta, \qquad i = 1, \ldots 3.$$

The symmetry group theory guarantees then that the following change of variables reduces equations (1) [46]:

$$\boldsymbol{v}(t,\boldsymbol{x}) = \frac{1}{\sqrt{t}} \boldsymbol{V}(\boldsymbol{\eta}), \qquad p(t,\boldsymbol{x}) = \frac{1}{t} P(\boldsymbol{\eta}), \qquad \theta(t,\boldsymbol{x}) = \frac{1}{t^{3/2}} \Theta(\boldsymbol{\eta}) \qquad \text{where} \qquad \boldsymbol{\eta} = \frac{\boldsymbol{x}}{\sqrt{t}}.$$

Indeed, inserting these relations into equations (1), we get the following reduced system:

$$\begin{cases} \dfrac{\partial V^j}{\partial \eta^j} = 0 \\ V^j \dfrac{\partial V^i}{\partial \eta^j} - \dfrac{\eta^j}{2} \dfrac{\partial V^i}{\partial \eta^j} - \dfrac{V^i}{2} + \dfrac{1}{\rho} \dfrac{\partial P}{\partial \eta^i} - \nu \dfrac{\partial^2 V^i}{\partial \eta^j \partial \eta^j} - \beta g \delta^i_2 \Theta = 0, \qquad i = 1,2,3 \\ V^j \dfrac{\partial \Theta}{\partial x^j} - \dfrac{\eta^j}{2} \dfrac{\partial \Theta}{\partial \eta^j} - \dfrac{3}{2} \Theta - \kappa \dfrac{\partial^2 \Theta}{\partial \eta^j \partial \eta^j} = 0 \end{cases} \tag{2}$$

These equations have one less independent variable than the original equations (1). Equations (2) can further be reduced, knowing that they admit (among others) the following Lie groups:

$$\left\{\left(\, \boldsymbol{\eta},\, \boldsymbol{V},\, P,\, \Theta \,\right) \longmapsto \left(\, \mathsf{R}\left(\epsilon, \eta^2\right) \boldsymbol{\eta},\, \mathsf{R}\left(\epsilon, \eta^2\right) \boldsymbol{V},\, P,\, \Theta \,\right),\, \epsilon \in \mathbb{R}\right\}$$

where $\mathsf{R}(\epsilon, \eta^2)$ is a rotation matrix with angle ϵ about the axis $(O\eta^2)$. It suggests the following ansatz:

$$V^1(\boldsymbol{\eta}) = W^{\varrho}(\varrho, y)\cos\varphi - W^{\varphi}(\varrho, y)\sin\varphi, \qquad V^3(\boldsymbol{\eta}) = W^{\varrho}(\varrho, y)\sin\varphi + W^{\varphi}(\varrho, y)\sin\varphi,$$

$$V^2(\boldsymbol{\eta}) = W^y(\varrho, y), \qquad P(\boldsymbol{\eta}) = W^p(\varrho, y), \qquad \Theta(\boldsymbol{\eta}) = W^\theta(\varrho, y),$$

where (ϱ, φ, y) are cylindrical coordinates verifying

$$\eta^1 = \varrho\cos\varphi, \qquad \eta^3 = \varrho\sin\varphi, \qquad y = \eta^2.$$

The functions $W^\bullet$ satisfy the following system of equations with only two independent variables:

$$\begin{cases}
\dfrac{1}{\varrho}\dfrac{\partial\left(\varrho W^\varrho\right)}{\partial\varrho} + \dfrac{\partial W^y}{\partial y} = 0 \\
\left(W^\varrho - \dfrac{\varrho}{2}\right)\dfrac{\partial W^\varrho}{\partial\varrho} + \left(W^y - \dfrac{y}{2}\right)\dfrac{\partial W^\varrho}{\partial y} - \dfrac{W^\varrho}{2} - \dfrac{(W^\varphi)^2}{\varrho} + \dfrac{1}{\rho}\dfrac{\partial W^p}{\partial\varrho} = \nu\left(\dfrac{1}{\varrho}\dfrac{\partial}{\partial\varrho}\left(\varrho\dfrac{\partial W^\varrho}{\partial\varrho}\right) - \dfrac{W^\varrho}{\varrho^2} + \dfrac{\partial^2 W^\varrho}{\partial y^2}\right) \\
\left(W^\varrho - \dfrac{\varrho}{2}\right)\dfrac{\partial W^\varphi}{\partial\varrho} + \left(W^y - \dfrac{y}{2}\right)\dfrac{\partial W^\varphi}{\partial y} - \dfrac{W^\varphi}{2} + \dfrac{W^\varrho W^\varphi}{\varrho} = \nu\left(\dfrac{1}{\varrho}\dfrac{\partial}{\partial\varrho}\left(\varrho\dfrac{\partial W^\varphi}{\partial\varrho}\right) - \dfrac{W^\varphi}{\varrho^2} + \dfrac{\partial^2 W^\varphi}{\partial y^2}\right) \\
\left(W^\varrho - \dfrac{\varrho}{2}\right)\dfrac{\partial W^y}{\partial\varrho} + \left(W^y - \dfrac{y}{2}\right)\dfrac{\partial W^y}{\partial y} - \dfrac{W^y}{2} + \dfrac{1}{\rho}\dfrac{\partial W^p}{\partial y} = \nu\left(\dfrac{1}{\varrho}\dfrac{\partial}{\partial\varrho}\left(\varrho\dfrac{\partial W^y}{\partial\varrho}\right) + \dfrac{\partial^2 W^y}{\partial y^2}\right) - \beta g W^\theta \\
\left(W^\varrho - \dfrac{\varrho}{2}\right)\dfrac{\partial W^\theta}{\partial\varrho} + \left(W^y - \dfrac{y}{2}\right)\dfrac{\partial W^\theta}{\partial y} - \dfrac{3W^\theta}{2} = \kappa\left(\dfrac{1}{\varrho}\dfrac{\partial}{\partial\varrho}\left(\varrho\dfrac{\partial W^\theta}{\partial\varrho}\right) + \dfrac{\partial^2 W^\theta}{\partial y^2}\right)
\end{cases}$$

If we impose for example that W^ϱ depends only on ϱ, then the solution of this twice reduced system is

$$\begin{aligned}
W^\varrho =& \frac{4a_1\nu}{\varrho} + \frac{1-a_2}{2}\varrho, \\
W^\varphi =& \frac{a_3}{\varrho} + a_4\nu^2\exp\left(-\frac{a_2\varrho^2}{8\nu}\right)\varrho^{2a_1-3} \\
&\cdot\left[2(1+a_1)^2\,\mathrm{WM}_{1+a_1,\frac{3}{2}+a_1}\left(\frac{a_2\varrho^2}{4\nu}\right) + \left(1+a_1+\frac{a_2\varrho^2}{8\nu}\right)\mathrm{WM}_{a_1,\frac{3}{2}+a_1}\left(\frac{a_2\varrho^2}{4\nu}\right)\right] \\
W^y =& (a_2-1)y + a_5 \\
W^\Theta =& a_6\,\mathrm{KM}_{\frac{3}{6-4a_2},\frac{1}{2}}\left(\frac{\left((2a_2-3)y+2a_5\right)^2}{4\kappa(2a_2-3)}\right) + a_7\,\mathrm{KU}_{\frac{3}{6-4a_2},\frac{1}{2}}\left(\frac{\left((2a_2-3)y+2a_5\right)^2}{4\kappa(2a_2-3)}\right) \\
W^P =& \rho W^P_\varrho(\varrho) + \rho W^P_y(y)
\end{aligned}$$

where WM, KM and KU are the Whittaker M, the Kummer M and the Kummer U functions (see [130]). a_1, …, a_7 are arbitrary constants. The radial and longitudinal parts W^P_ϱ and W^p_y of W^P can be obtained by integration from the following ordinary differential equations:

$$\frac{\mathrm{d}W^P_\varrho}{\mathrm{d}\varrho} = \frac{(W^\varphi)^2}{r} + \frac{16a_1^2\nu^2}{\varrho^3} + \frac{\varrho}{4} - \frac{a_2^2\varrho}{4}, \qquad \frac{\mathrm{d}W^P_y}{\mathrm{d}y} = -\beta g W^\Theta - \left(a_2^2 - 2a_2 + 2\right)y - \frac{a_5}{2}(2a_2-3).$$

Going back to the original variables, the velocity field in cylindrical frame writes

$$\boldsymbol{v} = \left(\frac{4a_1\nu}{r} + \frac{1-a_2}{2t}r\right)\mathrm{e}_r + v^\varphi\mathrm{e}_\varphi + \left(\frac{a_2-1}{\sqrt{t}}x^2 + a_5\right)\mathrm{e}_2 \tag{3}$$

where the azimutal component is

$$\begin{aligned}
v^\varphi = & \frac{a_3}{r} + a_4\nu^2\frac{r^{2a_1-3}}{t^{a_1-1}}\exp\left(-\frac{a_2r^2}{8\nu t}\right) \\
&\cdot\left[2(1+a_1)^2\,\mathrm{WM}_{1+a_1,\frac{3}{2}+a_1}\left(\frac{a_2r^2}{4\nu t}\right) + \left(1+a_1+\frac{a_2r^2}{8\nu t}\right)\mathrm{WM}_{a_1,\frac{3}{2}+a_1}\left(\frac{a_2r^2}{4\nu t}\right)\right]
\end{aligned} \tag{4}$$

and the temperature field is

$$\theta = a_6 t^{\frac{3}{2}}\,\mathrm{KM}_{\frac{3}{6-4a_2},\frac{1}{2}}\left(\frac{\left((2a_2-3)x^2+2a_5\right)^2}{4\kappa(2a_2-3)}\right) + a_7 t^{\frac{3}{2}}\,\mathrm{KU}_{\frac{3}{6-4a_2},\frac{1}{2}}\left(\frac{\left((2a_2-3)x^2+2a_5\right)^2}{4\kappa(2a_2-3)}\right) \tag{5}$$

Figure 1 sketches the projection of the velocity field onto (x^1Ox^3) plane at $t = 1$ and $x^2 = 0$, with $\nu = 1$, $\kappa = 1$ and for some values of the arbitrary constants. As can be seen, the solution may represent different configurations of a vertex and may be used to study vertex dynamics. The temperature is a linear combination of two Kummer functions of the altitude. These two functions are plotted in Figure 2 for $a_5 = 0$ and some values of a_2.

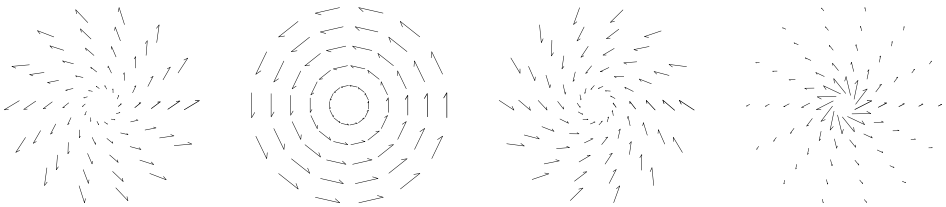


Figure 1. (xOz)-velocity field. From left to right: $(a_1, a_2, a_3, a_4) = (0, 0.75, 0, 1)$, $(0, 1, 0, 1)$, $(0, 2, 0, 1)$ and $(1, 1, 0, 1)$.

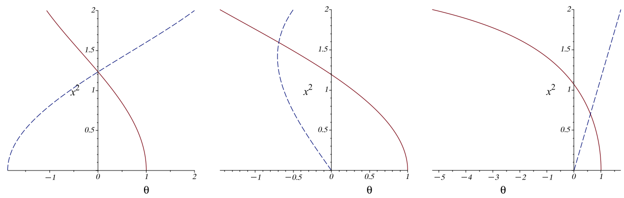


Figure 2. Temperature as function of the altitude x^2. From left to right: $a_2 = 0.75$, $a_2 = 1$, $a_2 = 2$. Solid line: $a_6 = 1, a_7 = 0$, dashed line: $a_6 = 0, a_7 = 1$.

To calculate the previous self-similar solution, we chose arbitrarily two symmetries of the equations. Other choices yield other solutions. Note also that applying any of the symmetries listed in Table 2, except those already used to reduce the equations, to a self-similar solution yields another solution. To avoid redundancy, self-similar solutions can be optimally classified [52, 131].

In this subsection, we gave an overview of the Lie symmetry group reduction of an equation. In the next subsection, we present a glimpse of the application of Lie symmetry group in finding conservation laws.

3.2. *Conservation laws*

Consider an equation

$$\boldsymbol{E}\left(\boldsymbol{y}, \boldsymbol{u}_{(n)}\right) = 0 \tag{6}$$

$\boldsymbol{y}$ being the independent variable, $\boldsymbol{u}$ the dependent one and $\boldsymbol{u}_{(n)}$ denoting the derivatives of $\boldsymbol{u}$ up to order n. A conservation law of equation (6) is an expression

$$\mathrm{Div}\,\boldsymbol{C} = 0, \qquad \text{that is} \qquad \frac{\mathrm{d}C^i}{\mathrm{d}y^i} = 0, \tag{7}$$

which holds on the space of solutions of the equation. In this expression, $\boldsymbol{C}$ is called the conserved current and depends on $\boldsymbol{y}$, $\boldsymbol{u}$ and the derivatives of $\boldsymbol{u}$. Div is the total divergence operator. In the case where $\boldsymbol{y} = (t, \boldsymbol{x})$ is composed of the time and space variables, conservation law (7) can be formulated as a balance over an arbitrary spatial domain $\Omega_{\boldsymbol{x}}$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}} C^0 \mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega_{\boldsymbol{x}}} \boldsymbol{C}^{\boldsymbol{x}} \cdot \boldsymbol{n} \mathrm{d}\Gamma \qquad \text{with} \qquad \boldsymbol{C}^{\boldsymbol{x}} = \left(C^1, C^2, C^3\right)^{\top} \tag{8}$$

where $\mathrm{d}\Gamma$ is the surface element on the boundary $\partial\Omega_{\boldsymbol{x}}$ and $\boldsymbol{n}$ the exterior unitary normal to $\partial\Omega_{\boldsymbol{x}}$. If C^0 is thought as a density then relation (8) states that the variation of the total density over $\Omega_{\boldsymbol{x}}$ is equal to the flux of $-\boldsymbol{C}^{\boldsymbol{x}}$ over the boundary.

Nœther's theorem permits to calculate conservation laws when the problem is variational, that is when the solution is a local extremum of a Lagrangian action

$$\mathscr{L} = \int_{\Omega} L\left(\boldsymbol{y}, \boldsymbol{u}_{(k)}\right) \mathrm{d}\boldsymbol{y} \tag{9}$$

for some k-order Lagrangian function L, Ω being a domain in $\boldsymbol{y}$-space. It states that to each Lie group of variational symmetries of the action (9) corresponds a conservation law of the Euler–Lagrange equations [46, 73–75, 132]. Recall that a variational symmetry is a transformation on $(\boldsymbol{y}, \boldsymbol{u})$ which leaves action (9) invariant. It is automatically a symmetry of the corresponding Euler–Lagrange equation.

As an example, consider an elastic medium described by a first-order Lagrangian $L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})$ where $\boldsymbol{x}$ is the position and $\boldsymbol{u}$ the displacement, and, from now on, ∇ is the gradient operator. If the medium is homogeneous in a direction j of space then the group of space translations in j-direction is a variationnal symmetry of the Lagrangian action. Nœther's theorem then implies that

$$\frac{\mathrm{d}}{\mathrm{d}x^i}\left(L\,\delta^i_j - \frac{\partial L}{\partial u^a_i} u^a_j\right) = 0$$

on the solution of the Euler–Lagrange equation. If the medium is homogeneous in the three directions then the impulsion-energy Eshelby tensor

$$L\,\mathsf{I}_{\mathsf{d}} - \left(\frac{\partial L}{\partial \nabla \boldsymbol{u}} \nabla \boldsymbol{u}\right)^{\top}$$

is a conserved tensor. Similarly, if the material is isotropic then there is a conserved angular momentum tensor. This analysis has been extended to more complex cases such as second gradient Lagrangian and micropolar elasticity [76, 79–81]. Note that Nœther's theorem has also recently been extended to variationnal problems involving Lagrangian with fractionnal derivatives [133, 134].

For non-variational problems, described by equation (6), few methods of finding conservation laws emmerged [135, 136]. One of them, called the direct method, consists in finding multipliers Λ^i depending on $\boldsymbol{y}$, $\boldsymbol{u}$ and derivatives of $\boldsymbol{u}$ such that the scalar function $\Lambda^i E_i$ is the total divergence of some function $\boldsymbol{C}$, which then becomes automatically a conserved current [137–139]. A second method, in fact closely related to the first one, consists in finding another equation

$$\boldsymbol{E}^*\left(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{u}^*\right) = 0 \tag{10}$$

involving not only on $\boldsymbol{y}$ and $\boldsymbol{u}$ but also on an adjoint dependent variable $\boldsymbol{u}^*$ such that the augmented system (6)-(10) has a variational principle [56, 140–143]. Nœther's theorem applies then to this augmented system. The adjoint function $\boldsymbol{E}^*$ is the adjoint of the Frechet derivative of $\boldsymbol{E}$. For instance, the adjoint equation of the anisothermal Navier–Stokes equations (1) are

$$\begin{cases} \operatorname{div} \boldsymbol{w} = 0 \\ \dfrac{\partial \boldsymbol{w}}{\partial t} + (\nabla \boldsymbol{w})\boldsymbol{u} - (\nabla \boldsymbol{u})^{\top} \boldsymbol{w} + \nabla q + \nu \Delta \boldsymbol{w} - (\nabla \theta)\vartheta = 0 \\ \dfrac{\partial \vartheta}{\partial t} + \boldsymbol{u} \cdot \nabla \vartheta + \kappa \Delta \vartheta - \beta g \boldsymbol{w} \cdot \mathrm{e}_2 \end{cases} \tag{11}$$

where $\boldsymbol{w}$, q and ϑ are respectively the adjoint of the velocity, the pressure and the temperature. The dot $\cdot$ symbolizes the Euclidean inner product. It can be checked that equations (1)-(11) are the Euler–Lagrange equations associated to the Lagrangian function

$$\begin{aligned} L = \boldsymbol{w}^{\top} \frac{\boldsymbol{u}_t + (\nabla \boldsymbol{u})\boldsymbol{u}}{2} - \boldsymbol{u}^{\top} \frac{\boldsymbol{v}_t + (\nabla \boldsymbol{w})\boldsymbol{u}}{2} + \frac{\theta_t + \boldsymbol{u}^{\top} \nabla \theta}{2}\vartheta - \frac{\vartheta_t + \boldsymbol{u}^{\top} \nabla \vartheta}{2}\theta + \beta g \theta \boldsymbol{w} \cdot \mathrm{e}_2 \\ + \frac{\theta\vartheta - \boldsymbol{u}^{\top} \boldsymbol{w} + 2q}{2} \operatorname{div} \boldsymbol{u} - \frac{p}{\rho} \operatorname{div} \boldsymbol{w} + \nu \operatorname{tr}\big((\nabla \boldsymbol{u})^{\top} \nabla \boldsymbol{w}\big) + \kappa \nabla\theta^{\top} \nabla \vartheta. \end{aligned} \tag{12}$$

Knowing that any variational symmetry of the Lagrangian action is also a symmetry of the Euler–Lagrange equations, one can seek among the Lie groups admitted by system (1)-(11) those which are also variational symmetry groups and then apply Nœther's theorem. For instance, the group of scale transformations

$$\big(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta, \boldsymbol{w}, q, \vartheta\big) \longmapsto \big(\mathrm{e}^{2\epsilon}\, t, \mathrm{e}^{\epsilon}\, \boldsymbol{x}, \mathrm{e}^{-\epsilon}\, \boldsymbol{u}, \mathrm{e}^{-2\epsilon}\, p, \mathrm{e}^{-3\epsilon}\, \theta, \mathrm{e}^{-2\epsilon}\, \boldsymbol{w}, \mathrm{e}^{-3\epsilon}\, q, \vartheta\big)$$

is a symmetry group of system (1)-(11). It leaves the Lagrangian action (not the Lagrangian function itself) associated to Lagrangian (12) invariant and is then a variational symmetry group. Nœther's theorem yields a conservation law where the time-component of the conserved current is

$$C^t = \frac{\boldsymbol{u}^{\top} \boldsymbol{w} - 3\theta\vartheta}{2} + t\big(2L + \boldsymbol{w}_t^{\top} \boldsymbol{u} - \boldsymbol{u}_t^{\top} \boldsymbol{w} + \vartheta_t \theta - \theta_t \vartheta\big) + \frac{\boldsymbol{u}^{\top} \nabla \boldsymbol{w} - \boldsymbol{w}^{\top} \nabla \boldsymbol{u} + \theta \nabla \vartheta^{\top} - \vartheta \nabla \theta^{\top}}{2} \boldsymbol{x}$$

and the space component is

$$\begin{aligned} \boldsymbol{C}^{\boldsymbol{x}} = \boldsymbol{x} L - \frac{\boldsymbol{u}\boldsymbol{w}^{\top} + 2\nu \nabla \boldsymbol{w}^{\top}}{2}\big(\boldsymbol{u} + 2t\boldsymbol{u}_t + (\nabla \boldsymbol{u})\boldsymbol{x}\big) - \frac{\boldsymbol{u}\vartheta + 2\kappa \nabla \vartheta}{2}\big(3\theta + 2t\theta_t + (\nabla\theta)^{\top} \boldsymbol{x}\big) \\ + \frac{\boldsymbol{u}\boldsymbol{u}^{\top} - 2\nu \nabla \boldsymbol{u}^{\top}}{2}\big(2\boldsymbol{w} + 2t\boldsymbol{w}_t + (\nabla \boldsymbol{w})\boldsymbol{x}\big) - \frac{\boldsymbol{u}\theta + 2\kappa \nabla \theta}{2}\big(2t\vartheta_t + (\nabla\vartheta)^{\top} \boldsymbol{x}\big). \end{aligned} \tag{13}$$

This conservation is non-local since it contains the adjoint variables which depend non-locally on $(\boldsymbol{u}, p, \theta)$ through the adjoint equations.

3.3. *Modelling*

In this last subsection, we illustrate the use of Lie symmetry groups in turbulence modelling, and more precisely in the large-eddy simulation approach [144]. Consider a fluid flow, governed by equations (1). When the flow is turbulent, the presence of very small eddy scales necessitates a very fine grid and makes the numerical resolution of these equations computationally unaffordable. In order to reduce the computational cost, only large scales, those larger than the grid size, are resolved. The effect of the small scales on the larger one are modelized. Concretely, instead of equation (1), one solves the following equations

$$
\begin{cases}
\operatorname{div} \overline{\boldsymbol{v}} = 0, \\
\dfrac{\partial \overline{\boldsymbol{v}}}{\partial t} + \operatorname{div}(\overline{\boldsymbol{v}}\,\overline{\boldsymbol{v}}^{\top}) + \dfrac{1}{\rho}\operatorname{grad}\overline{p} - \nu\Delta\overline{\boldsymbol{v}} + \beta g \overline{\theta} \mathrm{e}_2 = \operatorname{div} \mathsf{T}_s, \\
\dfrac{\partial \overline{\theta}}{\partial t} + \operatorname{div}(\overline{\boldsymbol{v}}\overline{\theta}) - \kappa\Delta\overline{\theta} = \operatorname{div} \boldsymbol{h}_s,
\end{cases}
\tag{14}
$$

where $\overline{\boldsymbol{v}}$, $\overline{p}$ and $\overline{\theta}$ are the resolved velocity, pressure and temperature. T_s and $\boldsymbol{h}_s$ are the subgrid stress tensor and temperature flux. They represent the effect of the unresolved scales and have to be modelled.

An approach in [108, 122, 126] suggests to construct T_s and $\boldsymbol{h}_s$ such that the symmetry groups of equations (1), summarized in Table 2, are also symmetry groups of equations (1). With this approach, the resolved variables have the same properties as $\boldsymbol{v}$, p and θ, regarding the consequences of symmetries (self-similar solutions, conservation laws, ...). The symmetry preservation leads to the the following model:

$$
\begin{aligned}
\mathsf{T}_s^d &= \nu F_1 \overline{\mathsf{S}}^d + \nu F_2 \frac{\operatorname{Adj}^d \overline{\mathsf{S}}}{\left\|\overline{\mathsf{S}}\right\|} + \nu F_3 \frac{\left(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}}\right)^d}{\left\|\overline{\mathsf{S}}\right\|^3} + \nu F_4 \frac{\left[\overline{\mathsf{S}}\left(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}}\right)\right]^d}{\left\|\overline{\mathsf{S}}\right\|^4} + \nu F_5 \frac{\overline{\mathsf{S}}\left[\left(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}}\right)\overline{\mathsf{S}}\right]^d}{\left\|\overline{\mathsf{S}}\right\|^5}, \\
\boldsymbol{h}_s &= \kappa F_6 \overline{\mathbb{T}} + \kappa F_7 \frac{\overline{\mathsf{S}}\,\overline{\mathbb{T}}}{\left\|\overline{\mathsf{S}}\right\|} + \kappa F_8 \frac{\overline{\mathsf{S}}^2 \overline{\mathbb{T}}}{\left\|\overline{\mathsf{S}}\right\|^2}.
\end{aligned}
\tag{15}
$$

In these expressions, the superscript ^{d} designates the deviatoric part, $\overline{\mathsf{S}}$ is the resolved strain rate tensor, $\operatorname{Adj}\overline{\mathsf{S}}$ its adjugate (or comatrix), $\overline{\mathbb{T}}$ is the gradient of the resolved temperature and F_i are arbitrary scalar functions of the following variables:

$$
\frac{\det \overline{\mathsf{S}}}{\left\|\overline{\mathsf{S}}\right\|^3}, \qquad \frac{\overline{\mathbb{T}}^2}{\left\|\overline{\mathsf{S}}\right\|^4}, \qquad \frac{\overline{\mathbb{T}} : \overline{\mathsf{S}}}{\left\|\overline{\mathsf{S}}\right\|^5}, \qquad \frac{\left(\overline{\mathsf{S}}\,\overline{\mathbb{T}}\right) : \left(\overline{\mathsf{S}}\,\overline{\mathbb{T}}\right)}{\left\|\overline{\mathsf{S}}\right\|^6}.
\tag{16}
$$

These variables are invariants of the symmetry groups of the equations and appear naturally. In the isothermal case, only the two first terms are present in T_s^d, and F_1 and F_2 are functions of the first invariant in equation (16). The presence of ν and κ in the model is uncommon. It is due to the fact that the model preserves not only the Lie point-symmetry group of the equations but also an equivalent transformation (symmetry transformation acting also on ν and κ) group.

Figure 3 shows a result of a simulation in an isothermal ventilated room (see [122] for details). It compares a very simple symmetry preserving model (where F_1 is linear and F_2 is constant), called invariant model herein, to two other popular LES models. It can be noticed that the velocity profile obtained by the invariant model is closer to the experimental value compared to the other models. It is especially verified close to the wall. This can be explained by the fact that wall laws are contained in the Lie symmetry group and that the model preserves this group.

In the anisothermal case, the same behaviour, namely that the invariant model is particularly efficient close to the wall, was also observed in [108]. Figure 4 shows simulation results extracted from [108] in a differentially heated room (see details in [108]). The temperature profile given by the invariant model also approximates the experimental results better than that obtained with the popular Smagorinsky model. However, a notable difference with the experimental results can still be seen. One reason which may explain this difference is that radiation has not been

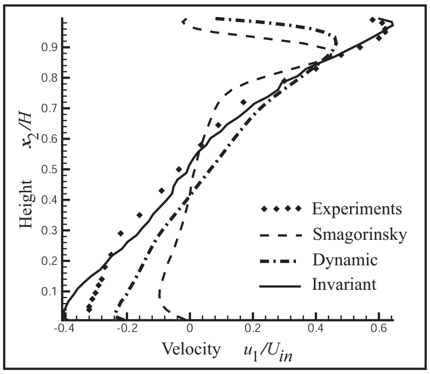


Figure 3. Vertical velocity profile.

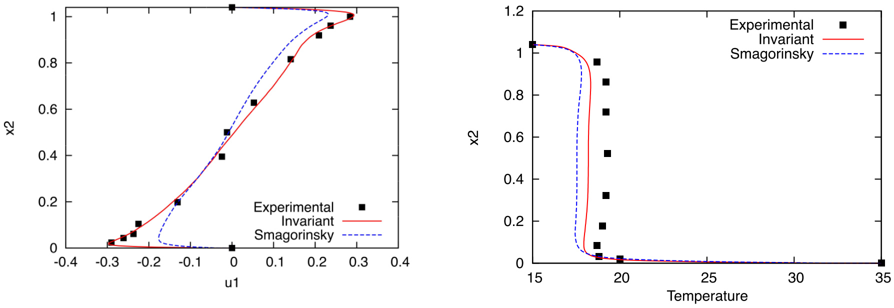


Figure 4. Horizontal velocity and temperature profile along the vertical direction in a differentially heated room.

considered in the equations. Moreover, temperature boundary conditions are not easy to deal with in an experimentation.

The simulations above were carried out with codes based on classical finite difference/volume schemes which are generally not symmetry preserving. In fact, it is also possible to design symmetry preserving numerical schemes. It is the object of the next section.

4. Lie groups and numerical integrators

Many mechanical systems evolve on a (manifold acted upon by a) Lie group [145–147]. It is for example the case of the heavy top, of which the space of configuration is $\mathbb{SO}3$, or a rigid multibody systems which evolves on multiple copies of $\mathbb{SE}3$. The use of classical (vector space) solvers to integrate the equation of these systems may lead to approximate solutions which does not lie on the Lie group and may result in numerical issues. Algorithms that preserve the Lie group structure have been proposed and generally have better stability and robustness properties [100, 148–155]. One of them consists in formulating and solving the differential equation of the system in the corresponding Lie algebra. An exponentiation is then carried out to obtain the solution on the Lie group.

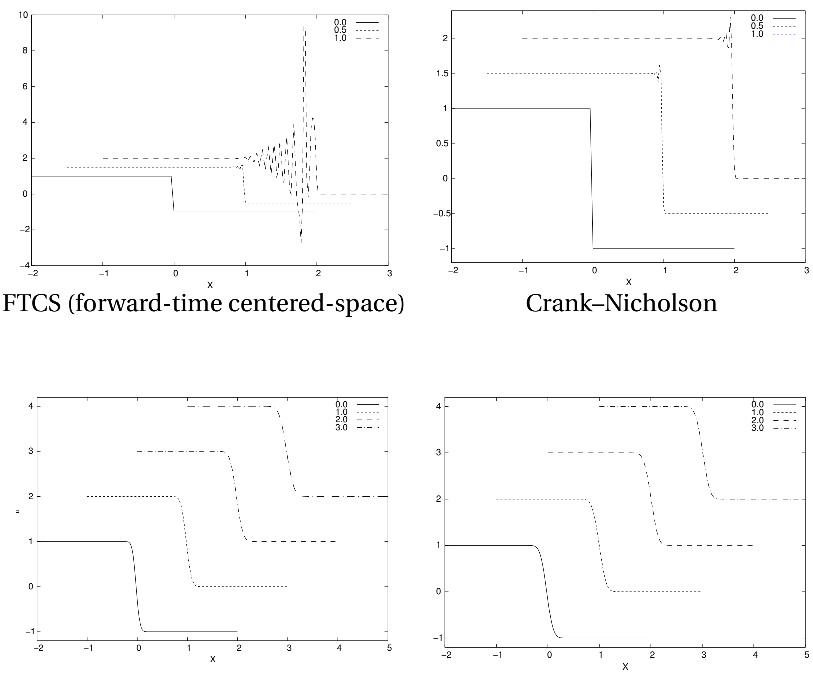
Lie group integrators are generally time integrators. For mechanical systems described by partial differential equations, one can consider Lie symmetry group preserving schemes. Indeed, as seen in the previous sections, the Lie symmetry group admitted by the equations of a mechanical system encode fundamental properties (invariance, conservation laws, ...). When representing the evolution of the system by a numerical model, it is then advisable to preserve these properties during the simulation. Few approaches exist to tackle this problem. One of them is to build numerical schemes based on the discrete invariants of the symmetry group [156–160]. This is done through the extension of the action of the group to difference equations. Another technique relies on the concept of differential approximations. It consists in finding a numerical scheme such that its differential approximation is invariant under the Lie group admitted by the equation [161–163]. It generally leads to only partially invariant schemes. A third method to build Lie symmetry-preserving schemes is the invariantization process. It consists in transforming a classical scheme into an invariant one using Cartan's moving frame approach [162,164–166]. To illustrate this method, consider a numerical scheme of equation (6):

$$\begin{cases} \Omega_h(\boldsymbol{m}) = 0, \\ \boldsymbol{E}_h(\boldsymbol{m}) = 0 \end{cases}$$

where Ω_h represents the grid equation, $\boldsymbol{E}_h$ is the discrete equation and $\boldsymbol{m} = (\boldsymbol{y}^1, \dots, \boldsymbol{y}^J, \boldsymbol{u}^1, \dots, \boldsymbol{u}^J)$ represents the lattice and the discreate approximate solution. Denote $N = (\Omega_h, \boldsymbol{E}_h)$ and $\mathbb{G}$ the symmetry group of equation (6). Scheme N is said invariant under $\mathbb{G}$ if [1]

$$N(\boldsymbol{m}) = 0 \qquad \Longrightarrow \qquad N(g \cdot \boldsymbol{m}) = 0 \qquad \forall\, g \in \mathbb{G}. \tag{17}$$

Under some conditions on $\mathbb{G}$, if $N(\boldsymbol{m})$ is a discretisation scheme of equation (6) then $\widetilde{N}(\boldsymbol{m}) = N(\rho(\boldsymbol{m}) \cdot \boldsymbol{m})$ is an invariant discretisation scheme of (6), where ρ is a moving frame relative to $\mathbb{G}$.



FTCS (forward-time centered-space) Crank–Nicholson

Invariantized FTCS Invariantized Crank–Nicholson

Figure 5. Burgers solution at $t = 1$ with $\lambda = 0, \lambda = 0.5$ and $\lambda = 1$

[1] Each $g \in \mathbb{G}$ extends into a transformation, also denoted g, acting on the space described by the variable $\boldsymbol{m}$.

Figure 5, in part extracted from [165], compares the numerical compatibility of some classical discretization schemes of Burgers equation and their invariantized versions with the Galilean invariance. In this simulation, a Galilean boost is applied to the original referential frame. The initial solution is a pseudo-shock. Since the Galilean boost is a symmetry of Burgers equation, the analytical solution at each time is simply a shift of the initial condition. However, spurious oscillation near the shock location appears when classical schemes are used in the numerical resolution. These oscillations grows with the boost velocity λ. With invariantized schemes, no such oscillation appears; the solution at each discrete time remains a shift of the initial one.

Before concluding, note that some concepts related to Lie groups have been adapted to the discrete case. For example, the notion of Lie symmetry group of differential equations, their computation and the symmetry reduction method have been formalized for difference equations [157, 167–171]. Nœther's theorem has also been formulated for discrete equations and discrete conservation laws were computed [172–176].

5. Conclusion

In this paper, we presented a quick survey of the today use of Lie group techniques in continuum mechanics that, although not exhaustive, gives a glimpse of the benefits that these methods can bring both from a theoretical and numerical point of view. As geometric approaches spread in the mechanics community, there is no doubt that the link between Lie groups and continuous mechanics is bound to develop in the future. Through this work, we hope to have contributed to raise the reader's interest in several facets of this topic.

Conflicts of interest

The authors declare no competing financial interest.

Dedication

The manuscript was written through contributions of all authors. All authors have given approval to the final version of the manuscript.

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