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Blow-up to a p-Laplacian parabolic equation with a general nonlinear source

effondrement d'une équation parabolique p-laplacienne avec une source non-linéaire générale

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Abstract. A p-Laplacian parabolic equation with a general nonlinear source term is considered. It is shown that the solution may blow up in finite time at positive initial energy. Moreover, under some suitable assumptions about the nonlinear source term, the solution is proved to blow up in finite time at arbitrarily high initial energy. These results generalize the previous ones.

Résumé. Une équation parabolique p-laplacienne avec un terme source non linéaire général est considérée. On montre que la solution peut exploser en temps fini pour une énergie initiale positive. De plus, sous certaines hypothèses appropriées concernant le terme source non linéaire, il est prouvé que la solution explose en temps fini pour une énergie initiale arbitrairement élevée. Ces résultats généralisent des résultats antérieurs.

Keywords. p-Laplacian parabolic equation, general nonlinear source term, blow-up.

Mots-clés. équation parabolique p-laplacienne, terme source non linéaire général, explosion.

2020 Mathematics Subject Classification. 35K92, 35B44.

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1. Introduction and main results

We investigate the p-Laplacian parabolic equation:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \overline{\Omega}, \\ u = 0, & x \in \partial\Omega,\ t \geq 0, \end{cases} \tag{1}$$

where the domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is bounded, the boundary $\partial\Omega$ is smooth, $p \geq 2$, and $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is non-trivial and non-negative. Moreover, the locally Lipschitz continuous function f satisfies $f(0) = 0$, $f(s) > 0$ for $s > 0$ and

$$\alpha F(s) \leq s f(s) + \beta s^p + \gamma \ (s > 0) \tag{2}$$

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for some $\alpha > p$, $\gamma > 0$ and $0 < \beta < (\alpha - p)\lambda_1/p$, where

$$F(s) = \int_0^s f(\tau)\,d\tau$$

and λ_1 is the first eigenvalue of the p-Laplacian operator, namely,

$$\lambda_1 = \inf_{\phi \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\nabla\phi\|_p^p}{\|\phi\|_p^p} > 0. \tag{3}$$

In what follows, the inner product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$ and the norm of $L^\sigma(\Omega)$ $(1 \le \sigma \le \infty)$ is denoted by $\|\cdot\|_\sigma$.

As is well-known, the p-Laplacian parabolic equation often appears in the theory of non-Newtonian fluids, see [1]. Therefore, the model (1) has been widely studied by many researchers, see [2–9]. For example, Fujii and Ohta [4] investigated the initial boundary value problem (IBVP) of the equation

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2}u, \quad p > 2$$

and established some blow-up results.

Li and Xie [7] considered IBVP of the equation

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda |u|^{q-2}u,$$

where $p > 1$, $\lambda > 0$ and $q > 2$. Using the comparison principle and concavity argument, the authors studied the blow-up properties of solutions.

Le et al. [5] dealt with IBVP of the equation

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2}u\log|u|, \quad p > 2.$$

Applying the potential well theory, the global existence and blow-up of solutions were analysed.

In particular, when $f(u)$ satisfies the general assumption (2), Chung and Choi [3] proved the nonnegative solution to (1) blows up in finite time with negative initial energy (i.e., $J(u_0) < 0$), where

$$J(u) := \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega F(u)\,dx + \gamma|\Omega|. \tag{4}$$

Considering the blow-up result obtained in [3], there are two natural questions:

(QS1): Whether the nonnegative solution to (1) can be proved to blow up in finite time at positive initial energy?

(QS2): Is it possible for the nonnegative solution of (1) to blow up in finite time at arbitrarily high initial energy?

The main purpose of the present paper is to answer the above questions.

The local existence of the weak solution to (1) can be found in [3]. Now, we give the blow-up results of this paper.

Theorem 1. *Assume the nonnegative function $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and* (2) *holds. If*

$$J(u_0) < \max\{0, M(u_0)\}, \tag{5}$$

where

$$M(u_0) := \frac{\lambda_1(\alpha - p) - \beta p}{2\alpha}\left(\|u_0\|_2^2 - \frac{(p-2)|\Omega|}{p}\right) + \frac{\gamma(\alpha-1)|\Omega|}{\alpha}, \tag{6}$$

then the nonnegative weak solution u to (1) *blows up at the time $T < \infty$ in the sense of*

$$\lim_{t \to T^-} \|u\|_2^2 = \infty.$$

Remark 2. Under some suitable assumptions about the nonlinear term f, we will prove that there exists an initial value u_0 such that

$$0 < J(u_0) < M(u_0). \tag{7}$$

Assume that

$$c_1 s^{\alpha-1} \le f(s) \le c_2\left(s^{\alpha-1}+1\right), \quad s>0, \tag{8}$$

where $c_1, c_2 > 0$ are constants and $c_1\alpha > c_2$. Let

$$u_0 := \kappa\varpi(x), \tag{9}$$

where $\kappa > 0$ is a constant to be specified later and $\varpi(x)$ is a function that satisfies $0 < \varpi(x) \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

By (4) and (8), we obtain

$$J(u_0) \ge \frac{1}{p}\|\nabla u_0\|_p^p - \frac{c_2}{\alpha}\int_\Omega u_0^\alpha\, dx - c_2\int_\Omega u_0\, dx + \gamma|\Omega|$$

and

$$J(u_0) \le \frac{1}{p}\|\nabla u_0\|_p^p - \frac{c_1}{\alpha}\int_\Omega u_0^\alpha\, dx + \gamma|\Omega|.$$

Then we deduce from (6) and (9) that

$$\begin{aligned}
M(u_0) &= \frac{\lambda_1(\alpha-p)-\beta p}{2\alpha}\left(\kappa^2\|\varpi(x)\|_2^2 - \frac{(p-2)|\Omega|}{p}\right) + \frac{\gamma(\alpha-1)|\Omega|}{\alpha},\\
J(u_0) &\ge \frac{\kappa^p}{p}\|\nabla\varpi(x)\|_p^p - \frac{c_2\kappa^\alpha}{\alpha}\int_\Omega (\varpi(x))^\alpha\, dx - c_2\kappa\int_\Omega \varpi(x)\, dx + \gamma|\Omega|,\\
J(u_0) &\le \frac{\kappa^p}{p}\|\nabla\varpi(x)\|_p^p - \frac{c_1\kappa^\alpha}{\alpha}\int_\Omega (\varpi(x))^\alpha\, dx + \gamma|\Omega|.
\end{aligned}$$

Therefore, to prove (7), we only need to show that

$$\frac{\kappa^p}{p}\|\nabla\varpi(x)\|_p^p - \frac{c_2\kappa^\alpha}{\alpha}\int_\Omega (\varpi(x))^\alpha\, dx - c_2\kappa\int_\Omega \varpi(x)\, dx + \gamma|\Omega| > 0 \tag{10}$$

and

$$\begin{aligned}
&\frac{\kappa^p}{p}\|\nabla\varpi(x)\|_p^p - \frac{c_1\kappa^\alpha}{\alpha}\int_\Omega (\varpi(x))^\alpha\, dx + \gamma|\Omega|\\
&\qquad < \frac{\lambda_1(\alpha-p)-\beta p}{2\alpha}\left(\kappa^2\|\varpi(x)\|_2^2 - \frac{(p-2)|\Omega|}{p}\right) + \frac{\gamma(\alpha-1)|\Omega|}{\alpha}.
\end{aligned} \tag{11}$$

In fact, when

$$\gamma > \frac{1}{|\Omega|}\left[\frac{c_2\kappa^\alpha}{\alpha}\int_\Omega (\varpi(x))^\alpha\, dx + c_2\kappa\int_\Omega \varpi(x)\, dx - \frac{\kappa^p}{p}\|\nabla\varpi(x)\|_p^p\right] \tag{12}$$

and

$$\begin{aligned}
\gamma < \frac{c_1\kappa^\alpha}{|\Omega|}\int_\Omega (\varpi(x))^\alpha\, dx - \frac{\alpha\kappa^p}{p|\Omega|}\|\nabla\varpi(x)\|_p^p + \frac{\kappa^2[\lambda_1(\alpha-p)-\beta p]}{2|\Omega|}\|\varpi(x)\|_2^2&\\
- \frac{(p-2)[\lambda_1(\alpha-p)-\beta p]}{2p}&,
\end{aligned} \tag{13}$$

it is easy to see that (10) and (11) hold.

In order to show that there is a γ that makes (12) and (13) hold, we only need to prove that

$$\begin{aligned}\frac{1}{|\Omega|}\left[\frac{c_2\kappa^\alpha}{\alpha}\int_\Omega(\varpi(x))^\alpha dx+c_2\kappa\int_\Omega\varpi(x)dx-\frac{\kappa^p}{p}\|\nabla\varpi(x)\|_p^p\right]&\\ <\frac{c_1\kappa^\alpha}{|\Omega|}\int_\Omega(\varpi(x))^\alpha dx-\frac{\alpha\kappa^p}{p|\Omega|}\|\nabla\varpi(x)\|_p^p+\frac{\kappa^2[\lambda_1(\alpha-p)-\beta p]}{2|\Omega|}\|\varpi(x)\|_2^2&\\ -\frac{(p-2)[\lambda_1(\alpha-p)-\beta p]}{2p},&\quad(14)\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{|\Omega|(p-2)[\lambda_1(\alpha-p)-\beta p]}{2p}<\kappa\Bigg\{\kappa\Bigg[\frac{\lambda_1(\alpha-p)-\beta p}{2}\|\varpi(x)\|_2^2&\\ +\kappa^{p-2}\left(\frac{\kappa^{\alpha-p}(c_1\alpha-c_2)}{\alpha}\int_\Omega(\varpi(x))^\alpha dx-\frac{\alpha-1}{p}\|\nabla\varpi(x)\|_p^p\right)\Bigg]-c_2\int_\Omega\varpi(x)dx\Bigg\}.&\quad(15)\end{aligned}$$

Obviously, if $\kappa>0$ is large enough, then (15) holds, i.e., (14) holds.

To prove the existence of the finite time blow-up solution at arbitrarily high initial energy by using the fountain theorem (see [10]), we assume the nonlinear term f has a concrete expression. Let

$$f(s)=\gamma s^{\alpha-1}+\frac{\beta p}{\alpha-p}s^{p-1},\quad s>0,\tag{16}$$

then it is obvious that

$$F(s)=\frac{\gamma}{\alpha}s^\alpha+\frac{\beta}{\alpha-p}s^p,$$

where

$$p<\alpha<\begin{cases}\infty, & \text{if } n\le p;\\ \dfrac{np}{n-p}, & \text{if } n>p.\end{cases}\tag{17}$$

Clearly, the nonlinear term f given in (16) satisfies (2). Moreover, if (16) holds, then we obtain from (4) that

$$J(u)=\frac{1}{p}\|\nabla u\|_p^p-\frac{\gamma}{\alpha}\int_\Omega u^\alpha dx-\frac{\beta}{\alpha-p}\int_\Omega u^p dx+\gamma|\Omega|.$$

Theorem 3. *Assume $u_0\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and (2) holds. If (16) and (17) hold, then for any constant $\mathscr{B}\ge 0$, there exists a nonnegative function*

$$u_{\mathscr{B}}\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$$

satisfying (5) and $J(u_{\mathscr{B}})=\mathscr{B}$, and the nonnegative weak solution u to (1) with the initial value $u_{\mathscr{B}}$ blows up in finite time in the sense of

$$\lim_{t\to T^-}\|u\|_2^2=\infty.$$

Remark 4. In Theorem 3, the choice of the nonlinear term f is not unique. For instance, when $f(s)=\gamma s^{\alpha-1}$ for $s>0$ and α satisfies (17), we can also obtain the same blow-up result as Theorem 3.

Remark 5. When f is a general nonlinear term satisfying (2), we cannot find an effective method to show the finite time blow-up result at arbitrarily high initial energy, thus we leave it as an open question.

The rest of this paper is to prove Theorems 1 and 3.

2. Proofs of the theorems

Proof of Theorem 1. Let $u = u(t)$, $t \in [0,T)$ be the nonnegative weak solution to (1) mentioned in [3] with u_0 satisfying (5), where T represents the maximum existence time. If $M(u_0) \le 0$, then the blow-up result follows from [3, Theorems 1.1 and 1.2]. Hence, in what follows, we assume $M(u_0) > 0$. Then from (5), we know $J(u_0) < M(u_0)$.

If there is a $t_0 \in [0,T)$ such that $J(u(t_0)) < 0$, then we deduce from [3, Theorems 1.1 and 1.2] that u blows up in finite time. Therefore, in the remaining proof, we assume $J(u) \ge 0$ for $t \in [0,T)$.

By contradiction, we suppose u exists globally. Then we obtain from Hölder's inequality, [3, (11) and (18)] and $J(u) \ge 0$ for $t \ge 0$ that

$$\begin{aligned} \|u\|_2 &= \left\| \int_0^t u_\tau d\tau + u_0 \right\|_2 \\ &\le t^{\frac{1}{2}} \left(\int_0^t \|u_\tau\|_2^2 d\tau \right)^{\frac{1}{2}} + \|u_0\|_2 \\ &\le (J(u_0))^{\frac{1}{2}} t^{\frac{1}{2}} + \|u_0\|_2. \end{aligned} \tag{18}$$

Moreover, we deduce from [3, (14) and (21)], (2), (4), (3), [3, (11) and (18)], Hölder's and Young's inequalities that

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\|u\|_2^2 - \Xi\right) &= -\int_\Omega |\nabla u|^p dx + \int_\Omega u f(u) dx \\ &\ge \frac{\alpha - p}{p} \int_\Omega |\nabla u|^p dx - \beta \int_\Omega u^p dx + \gamma(\alpha-1)|\Omega| - \alpha J(u) \\ &\ge \frac{\lambda_1(\alpha-p) - \beta p}{p} \int_\Omega u^p dx + \gamma(\alpha-1)|\Omega| - \alpha J(u_0) \\ &\ge \frac{\lambda_1(\alpha-p) - \beta p}{2} \int_\Omega u^2 dx - \frac{(p-2)\left[\lambda_1(\alpha-p) - \beta p\right]|\Omega|}{2p} + \gamma(\alpha-1)|\Omega| - \alpha J(u_0) \\ &= \left[\lambda_1(\alpha-p) - \beta p\right]\left(\frac{1}{2}\|u\|_2^2 - \Xi\right), \end{aligned} \tag{19}$$

where

$$\Xi = \frac{(p-2)\left[\lambda_1(\alpha-p) - \beta p\right]|\Omega| - 2p\gamma(\alpha-1)|\Omega| + 2p\alpha J(u_0)}{2p\left[\lambda_1(\alpha-p) - \beta p\right]}.$$

Since

$$\underbrace{\lambda_1(\alpha-p) - \beta p > 0}_{\text{by (2)}}, \qquad \underbrace{\frac{1}{2}\|u_0\|_2^2 - \Xi > 0}_{\text{by } J(u_0) < M(u_0)},$$

we infer from (19) that

$$\frac{1}{2}\|u(t)\|_2^2 - \Xi > 0, \quad t \ge 0.$$

Integrating (19) from 0 to t, we arrive at

$$\|u\|_2^2 \ge \left(\|u_0\|_2^2 - 2\Xi\right) e^{[\lambda_1(\alpha-p)-\beta p]t} + 2\Xi. \tag{20}$$

The combination of (18) and (20) yields

$$\left(\|u_0\|_2^2 - 2\Xi\right) e^{[\lambda_1(\alpha-p)-\beta p]t} + 2\Xi \le \left((J(u_0))^{\frac{1}{2}} t^{\frac{1}{2}} + \|u_0\|_2\right)^2, \quad t \ge 0. \tag{21}$$

Clearly, (21) cannot hold for sufficiently large t, a contradiction. The proof is complete. □

To prove Theorem 3, the following three lemmas are required.

Lemma 6 ([10, Theorem 3.6] Fountain Theorem). *Suppose $\mathscr{H}$ is a Banach space with the norm $\|\cdot\|$ and $\mathscr{H}_j$ is a subspace of $\mathscr{H}$ with $\dim\mathscr{H}_j<\infty$ for each $j\in\mathbb{N}:=\{1,2,\cdots\}$. Let $\mathscr{H}=\overline{\bigoplus_{j\in\mathbb{N}}\mathscr{H}_j}$ be the closure of the direct sum of all $\mathscr{H}_j$. Let*

$$\mathscr{W}_k=\bigoplus_{j=1}^{k}\mathscr{H}_j,\quad \mathscr{V}_k=\overline{\bigoplus_{j=k}^{\infty}\mathscr{H}_j}.$$

Assume $\Psi\in C^1(\mathscr{H},\mathbb{R})$ is an even functional. For each $k\in\mathbb{N}$, if there exist $\rho_k>r_k>0$ such that

(i) $m_k:=\max\limits_{\psi\in\mathscr{W}_k,\|\psi\|=\rho_k}\Psi(\psi)\le 0$;

(ii) $z_k:=\inf\limits_{\psi\in\mathscr{V}_k,\|\psi\|=r_k}\Psi(\psi)\to\infty$ *as* $k\to\infty$;

(iii) Ψ *satisfies the* $(PS)_c$ *condition for every* $c>0$,

then Ψ has an unbounded sequence of critical values.

Lemma 7. *Let* (16) *and* (17) *hold. There exist functions $\{\psi_k\}_{k=1}^{\infty}\subset W_0^{1,p}(\Omega)$ satisfying*

$$\widetilde{J}(\psi_k):=\frac{1}{p}\|\nabla\psi_k\|_p^p-\frac{\gamma}{\alpha}\|\psi_k\|_\alpha^\alpha-\frac{\beta}{\alpha-p}\|\psi_k\|_p^p+\gamma|\Omega|\to\infty \text{ as } k\to\infty. \tag{22}$$

Proof. To prove the lemma, it is sufficient to show $\widetilde{J}$ satisfies the assumptions of Lemma 6. Because $W_0^{1,p}(\Omega)$ is separable, one can select $\{e_j\}_{j=1}^{\infty}$ as a base of $W_0^{1,p}(\Omega)$ and $\{l_j\}_{j=1}^{\infty}\subset W^{-1,p'}(\Omega)$ such that $\|\nabla e_j\|_p=1$, $\|l_j\|_{W^{-1,p'}(\Omega)}=1$, and $l_j(e_i)=1$ if $i=j$ and $l_j(e_i)=0$ if $i\ne j$, where $W^{-1,p'}(\Omega)$ represents the dual space of $W_0^{1,p}(\Omega)$. For $j=1,2,\cdots$, we set

$$\mathscr{H}_j:=\operatorname{span}\{e_j\}=\{ce_j:c\in\mathbb{R}\}.$$

Then $\mathscr{H}_j\perp\mathscr{H}_i$ for $i\ne j$, i.e., $l_i(ce_j)=0$ and $l_j(ce_i)=0$ for any $c\in\mathbb{R}$. With this sense, for $k=1,2,\cdots$, we set

$$\mathscr{W}_k:=\bigoplus_{j=1}^{k}\mathscr{H}_j,\qquad \mathscr{V}_k:=\overline{\bigoplus_{j=k}^{\infty}\mathscr{H}_j}.$$

Then

$$\mathscr{V}_{k+1}=\mathscr{W}_k^{\perp},\qquad W_0^{1,p}(\Omega)=\mathscr{W}_k\bigoplus\mathscr{V}_{k+1},$$

and $\mathscr{W}_k\subset W_0^{1,p}(\Omega)$ with $\dim\mathscr{W}_k<\infty$.

Firstly, one can easily verify $\widetilde{J}\in C^1(W_0^{1,p}(\Omega),\mathbb{R})$ is an even functional.

Secondly, we prove $\widetilde{J}$ satisfies Lemma 6 (ii). Let

$$\delta_k:=\sup_{\psi\in\mathscr{V}_k,\,\|\nabla\psi\|_p=1}\|\psi\|_\alpha, \tag{23}$$

then it holds $0<\delta_{k+1}\le\delta_k$. Thus, there is a $\delta\ge 0$ such that

$$\delta_k\to\delta \text{ as } k\to\infty.$$

For every k, there is $\psi_k\in\mathscr{V}_k$ with $\|\nabla\psi_k\|_p=1$ such that

$$\|\psi_k\|_\alpha>\frac{\delta}{2}\ge 0.$$

It follows from the definition of $\mathscr{V}_k$ that

$$\psi_k\rightharpoonup 0 \text{ weakly in } W_0^{1,p}(\Omega) \text{ as } k\to\infty.$$

Due to $W_0^{1,p}(\Omega)\hookrightarrow L^\alpha(\Omega)$ compactly (see (17)), we know

$$\psi_k\to 0 \text{ strongly in } L^\alpha(\Omega) \text{ as } k\to\infty,$$

which means $\lim_{k\to\infty}\delta_k = 0$. By (3) and (23), one has, for $\psi\in\mathcal{V}_k$,

$$\begin{aligned}\widetilde{J}(\psi) &\geq \frac{1}{p}\|\nabla\psi\|_p^p - \frac{\gamma}{\alpha}\|\psi\|_\alpha^\alpha - \frac{\beta}{\alpha-p}\|\psi\|_p^p\\ &\geq \left(\frac{1}{p}-\frac{\beta}{\lambda_1(\alpha-p)}\right)\|\nabla\psi\|_p^p - \frac{\gamma\delta_k^\alpha}{\alpha}\|\nabla\psi\|_p^\alpha.\end{aligned}$$

Take

$$r_k = \left(\frac{\lambda_1(\alpha-p)-\beta p}{\lambda_1\gamma\delta_k^\alpha(\alpha-p)}\right)^{\frac{1}{\alpha-p}}.$$

If $\psi\in\mathcal{V}_k$ and $\|\nabla\psi\|_p = r_k$, then we obtain

$$\widetilde{J}(\psi) \geq \frac{\lambda_1(\alpha-p)-\beta p}{\alpha p\lambda_1}\left(\frac{\lambda_1(\alpha-p)-\beta p}{\lambda_1\gamma\delta_k^\alpha(\alpha-p)}\right)^{\frac{p}{\alpha-p}},$$

which implies that $\widetilde{J}$ satisfies Lemma 6 (ii).

Thirdly, we prove $\widetilde{J}$ satisfies Lemma 6 (i). For any $\rho_k>0$ and $\widehat{\psi}\in\mathcal{W}_k$ with $\|\nabla\widehat{\psi}\|_p = 1$, one has

$$\widetilde{J}(\rho_k\widehat{\psi}) = \rho_k^p\left(\frac{1}{p}\left\|\nabla\widehat{\psi}\right\|_p^p - \frac{\gamma}{\alpha}\rho_k^{\alpha-p}\left\|\widehat{\psi}\right\|_\alpha^\alpha - \frac{\beta}{\alpha-p}\left\|\widehat{\psi}\right\|_p^p\right) + \gamma|\Omega|. \tag{24}$$

Additionally, we obtain from $\dim\mathcal{W}_k<\infty$ and $\|\nabla\widehat{\psi}\|_p = 1$ that, for some $\varsigma_1,\varsigma_2>0$,

$$\varsigma_1 \leq \left\|\widehat{\psi}\right\|_\alpha \leq \varsigma_2, \quad \varsigma_1 \leq \left\|\widehat{\psi}\right\|_p \leq \varsigma_2,$$

which, along with (24), yields

$$\widetilde{J}(\rho_k\widehat{\psi}) \leq \rho_k^p\left(\frac{1}{p} - \frac{\gamma\varsigma_1^\alpha}{\alpha}\rho_k^{\alpha-p} - \frac{\beta\varsigma_1^p}{\alpha-p}\right) + \gamma|\Omega| \to -\infty \text{ as } \rho_k\to\infty.$$

Let $\psi = \rho_k\widehat{\psi}$, then

$$\left\|\nabla\psi\right\|_p = \rho_k\left\|\nabla\widehat{\psi}\right\|_p = \rho_k \text{ and } \psi\in\mathcal{W}_k.$$

Thus, for sufficiently large $\rho_k > r_k$, we know $\widetilde{J}$ satisfies Lemma 6 (i).

Finally, we prove $\widetilde{J}$ satisfies Lemma 6 (iii). For any $c>0$, we assume $\{\psi_j\}_{j=1}^\infty \subset W_0^{1,p}(\Omega)$ satisfies

$$\widetilde{J}(\psi_j)\to c \quad\text{and}\quad \left\|\widetilde{J}'(\psi_j)\right\|_{W^{-1,p'}(\Omega)}\to 0 \quad\text{as}\quad j\to\infty.$$

Then there are $\mathscr{C}_1 > c$ and $\mathscr{C}_2 > 0$ independent of j such that

$$\widetilde{J}(\psi_j)\leq\mathscr{C}_1 \quad\text{and}\quad \left\|\widetilde{J}'(\psi_j)\right\|_{W^{-1,p'}(\Omega)}\leq\mathscr{C}_2 \quad\text{for}\quad j=1,2,\cdots.$$

Thus, we infer from (3) that

$$\begin{aligned}\mathscr{C}_1 + \frac{\mathscr{C}_2}{\alpha}\left\|\nabla\psi_j\right\|_p &\geq \widetilde{J}(\psi_j) - \frac{1}{\alpha}\left\langle\widetilde{J}'(\psi_j),\psi_j\right\rangle\\ &= \frac{\alpha-p}{p\alpha}\left\|\nabla\psi_j\right\|_p^p - \frac{\beta}{\alpha}\left\|\psi_j\right\|_p^p + \gamma|\Omega|\\ &\geq \frac{\lambda_1(\alpha-p)-\beta p}{p\alpha\lambda_1}\left\|\nabla\psi_j\right\|_p^p + \gamma|\Omega|, \quad j=1,2,\cdots,\end{aligned} \tag{25}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$, which implies there exists a $\mathscr{C}_3>0$ independent of j such that

$$\left\|\nabla\psi_j\right\|_p \leq \mathscr{C}_3, \quad j=1,2,\cdots. \tag{26}$$

Then there is a $\psi\in W_0^{1,p}(\Omega)$ and a subsequence of $\{\psi_j\}_{j=1}^\infty$ (still denoted by $\{\psi_j\}_{j=1}^\infty$) such that

$$\psi_j \rightharpoonup \psi \quad\text{weakly in } W_0^{1,p}(\Omega) \quad\text{as}\quad j\to\infty. \tag{27}$$

From (26) and (27), we arrive at

$$\left\|\nabla\psi\right\|_p \le \liminf_{j\to\infty}\left\|\nabla\psi_j\right\|_p \le \mathscr{C}_3. \tag{28}$$

For any $\phi \in W_0^{1,p}(\Omega)$, one can see

$$\langle \widetilde{J}'(\psi),\phi\rangle = \int_\Omega |\nabla\psi|^{p-2}\nabla\psi\nabla\phi dx - \gamma\int_\Omega |\psi|^{\alpha-2}\psi\phi dx - \frac{\beta p}{\alpha-p}\int_\Omega |\psi|^{p-2}\psi\phi dx.$$

Consequently, it holds

$$\begin{aligned}\langle \widetilde{J}'(\psi_j)-\widetilde{J}'(\psi),\psi_j-\psi\rangle = &\int_\Omega \left(\left|\nabla\psi_j\right|^{p-2}\nabla\psi_j - \left|\nabla\psi\right|^{p-2}\nabla\psi\right)\nabla\left(\psi_j-\psi\right)dx\\ &-\gamma\int_\Omega\left(\left|\psi_j\right|^{\alpha-2}\psi_j-|\psi|^{\alpha-2}\psi\right)\left(\psi_j-\psi\right)dx\\ &-\frac{\beta p}{\alpha-p}\int_\Omega\left(\left|\psi_j\right|^{p-2}\psi_j-|\psi|^{p-2}\psi\right)\left(\psi_j-\psi\right)dx.\end{aligned} \tag{29}$$

By (27), $W_0^{1,p}(\Omega)\hookrightarrow L^\alpha(\Omega)$ compactly, and $W_0^{1,p}(\Omega)\hookrightarrow L^p(\Omega)$ compactly, we deduce

$$\begin{aligned}\left|\langle \widetilde{J}'(\psi),\psi_j-\psi\rangle\right| \le &\left|\int_\Omega \left|\nabla\psi\right|^{p-2}\nabla\psi\nabla\left(\psi_j-\psi\right)dx\right| + \gamma\left\|\,|\psi|^{\alpha-1}\right\|_{\frac{\alpha}{\alpha-1}}\left\|\psi_j-\psi\right\|_\alpha\\ &+\frac{\beta p}{\alpha-p}\left\|\,|\psi|^{p-1}\right\|_{\frac{p}{p-1}}\left\|\psi_j-\psi\right\|_p \to 0 \text{ as } j\to\infty.\end{aligned} \tag{30}$$

Furthermore, it follows from (26) and (28) that

$$\left|\langle \widetilde{J}'\left(\psi_j\right),\psi_j-\psi\rangle\right| \le 2\mathscr{C}_3\left\|\widetilde{J}'\left(\psi_j\right)\right\|_{W^{-1,p'}(\Omega)} \to 0 \quad \text{as} \quad j\to\infty,$$

which, along with (29) and (30), yields

$$\begin{aligned}2^{2-p}\left\|\nabla\psi_j-\nabla\psi\right\|_p^p \le &\int_\Omega\left(\left|\nabla\psi_j\right|^{p-2}\nabla\psi_j-\left|\nabla\psi\right|^{p-2}\nabla\psi\right)\nabla\left(\psi_j-\psi\right)dx\\ \le &\langle \widetilde{J}'\left(\psi_j\right)-\widetilde{J}'(\psi),\psi_j-\psi\rangle + \gamma\left\|\left|\psi_j\right|^{\alpha-2}\psi_j-|\psi|^{\alpha-2}\psi\right\|_{\frac{\alpha}{\alpha-1}}\left\|\psi_j-\psi\right\|_\alpha\\ &+\frac{\beta p}{\alpha-p}\left\|\left|\psi_j\right|^{p-2}\psi_j-|\psi|^{p-2}\psi\right\|_{\frac{p}{p-1}}\left\|\psi_j-\psi\right\|_p \to 0 \text{ as } j\to\infty.\end{aligned}$$

Thus, $\widetilde{J}$ satisfies Lemma 6 (iii).

According to the above analysis and Lemma 6, (22) holds. □

Lemma 8. *Let* (16) *and* (17) *hold. For any constant* $\mathscr{D}\ge 0$, *there is a nonnegative function* $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ *such that* $J(\varphi)=\mathscr{D}$.

Proof. Let

$$v_k = |\psi_k| \in W_0^{1,p}(\Omega),$$

where $\{\psi_k\}$ is the sequence given in Lemma 7. From Lemma 7, we arrive at

$$J(v_k)\to\infty \text{ as } k\to\infty.$$

Therefore, for any constant $\mathscr{D}\ge 0$, there is a v_k such that $J(v_k)\ge 2\mathscr{D}$.

Choose a sequence of nonnegative functions $\{\phi_j\}_{j=1}^\infty\subset C_0^\infty(\Omega)$, then one can verify that

$$\left|J(\phi_j)-J(v_k)\right|\to 0 \quad \text{as} \quad j\to\infty \quad \text{if} \quad \phi_j\to v_k \quad \text{in} \quad W_0^{1,p}(\Omega) \quad \text{as} \quad j\to\infty.$$

Thus, there exists a function $\phi_j\in C_0^\infty(\Omega)$ such that $J(\phi_j)\ge\mathscr{D}$.

Let

$$g(\xi)=J\left(\xi\phi_j\right)=\frac{\xi^p}{p}\left\|\nabla\phi_j\right\|_p^p-\frac{\gamma\xi^\alpha}{\alpha}\int_\Omega\phi_j^\alpha dx-\frac{\beta\xi^p}{\alpha-p}\int_\Omega\phi_j^p dx+\gamma|\Omega|, \quad \forall\,\xi\ge 1.$$

Clearly, the function $g(\xi)$ is continuous and $\lim_{\xi\to\infty} g(\xi) = -\infty$. Let $R(g(\xi))$ denote the range of $g(\xi)$, then we know $R(g(\xi)) \supset (-\infty, g(1)]$. Owing to $g(1) = J(\phi_j)$, we obtain from $J(\phi_j) \geq \mathscr{D}$ that there is a $\widehat{\xi} \geq 1$ such that $\varphi := \widehat{\xi}\phi_j \in C_0^\infty(\Omega) \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies $J(\varphi) = \mathscr{D}$. □

Proof of Theorem 3. Assume Ω_1 and Ω_2 are two arbitrary disjoint open subsets of Ω. Let

$$v \in \left(W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\right) \setminus \{0\}$$

be an arbitrary nonnegative function satisfying

$$\mathrm{supp}(v) = \overline{\{x \in \Omega : v(x) \neq 0\}} \subset \Omega_1.$$

Then for any constant $\mathscr{B} \geq 0$, one can choose $\epsilon > 0$ large enough such that

$$J(\epsilon v) \leq 0, \qquad \frac{\lambda_1(\alpha - p) - \beta p}{2\alpha}\left(\|\epsilon v\|_2^2 - \frac{(p-2)|\Omega|}{p}\right) + \frac{\gamma(\alpha - 1)|\Omega|}{\alpha} > \mathscr{B}. \tag{31}$$

For such ϵ, from Lemma 8, one can select a nonnegative function

$$\mu \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$$

such that

$$\mathrm{supp}(\mu) \subset \Omega_2 \text{ and } J(\mu) = \gamma|\Omega| + \mathscr{B} - J(\epsilon v).$$

Then for $u_{\mathscr{B}} = \epsilon v + \mu$, we infer from (31) that

$$J(u_{\mathscr{B}}) = J(\epsilon v) + J(\mu) - \gamma|\Omega| = \mathscr{B}$$

and

$$\begin{aligned} &\frac{\lambda_1(\alpha - p) - \beta p}{2\alpha}\left(\|u_{\mathscr{B}}\|_2^2 - \frac{(p-2)|\Omega|}{p}\right) + \frac{\gamma(\alpha - 1)|\Omega|}{\alpha} \\ &\qquad \geq \frac{\lambda_1(\alpha - p) - \beta p}{2\alpha}\left(\|\epsilon v\|_2^2 - \frac{(p-2)|\Omega|}{p}\right) + \frac{\gamma(\alpha - 1)|\Omega|}{\alpha} \\ &\qquad > \mathscr{B} = J(u_{\mathscr{B}}). \end{aligned}$$

Taking $u_{\mathscr{B}}$ as the initial value, the blow-up result follows from Theorem 1. □

Ethical Approval

Not applicable.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

Authors' contributions

Both authors prepared and reviewed all the contents of the manuscript.

Availability of data and materials

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] N. D. Alikakos and L. C. Evans, "Continuity of the gradient for weak solutions of a degenerate parabolic equation", *J. Math. Pures Appl.* **62** (1983), pp. 253–268.

[2] Y. Cao and C. Liu, "Initial boundary value problem for a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity", *Electron. J. Differ. Equ.* **2018** (2018), article no. 116.

[3] S.-Y. Chung and M.-J. Choi, "A new condition for the concavity method of blow-up solutions to p-Laplacian parabolic equations", *J. Differ. Equations* **265** (2018), no. 12, pp. 6384–6399.

[4] A. Fujii and M. Ohta, "Asymptotic behavior of blowup solutions of a parabolic equation with the p-Laplacian", *Publ. Res. Inst. Math. Sci., Ser. A* **32** (1996), no. 3, pp. 503–515.

[5] C. N. Le and X. T. Le, "Global solution and blow-up for a class of p-Laplacian evolution equations with logarithmic nonlinearity", *Acta Appl. Math.* **151** (2017), no. 1, pp. 149–169.

[6] K.-A. Lee, A. Petrosyan and J. L. Vázquez, "Large-time geometric properties of solutions of the evolution p-Laplacian equation", *J. Differ. Equations* **229** (2006), no. 2, pp. 389–411.

[7] Y. Li and C. Xie, "Blow-up for p-Laplacian parabolic equations", *Electron. J. Differ. Equ.* **2003** (2003), article no. 20.

[8] Y. Tian and C. Mu, "Extinction and non-extinction for a p-Laplacian equation with nonlinear source", *Nonlinear Anal., Theory Methods Appl.* **69** (2008), no. 8, pp. 2422–2431.

[9] J. N. Zhao, "Existence and nonexistence of solutions for $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + f(\nabla u, u, x, t)$", *J. Math. Anal. Appl.* **172** (1993), no. 1, pp. 130–146.

[10] M. Willem, *Minimax theorems*, Birkhäuser, 1996.