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Asymptotic modeling of viscoelastic thin plates and slender beams, a unifying approach

Analyse asymptotique de plaques et de poutres viscoélastiques minces, une approche unitaire

Yotsawat Terapabkajornded [a, b], Somsak Orankitjaroen [a, c], Christian Licht [a, b, c] and Thibaut Weller [*, b]

[a] Department of Mathematics, Faculty of Science, Mahidol University, Bangkok,Thailand

[b] LMGC, Université de Montpellier, CNRS, Montpellier, France

[c] Centre of Excellence in Mathematics, CHE, Bangkok, Thailand

E-mails: ongyosawat@gmail.com (Y. Terapabkajornded), somsak.ora@mahidol.ac.th (S. Orankitjaroen), christian.licht@umontpellier.fr (C. Licht), thibaut.weller@umontpellier.fr (T. Weller)

Abstract. A dimension reduction problem is tackled using Trotter's theory of convergence of semi-groups of operators acting on variable spaces [1, 2]. We show that this framework makes it possible to perform the asymptotic analysis for both viscoelastic thin plates and slender beams in a unifying manner. Several models are provided for the dynamic behavior of such structures in bilateral contact with a rigid body on a part of their boundaries with Norton or Tresca friction.

Résumé. Nous nous penchons sur la modélisation mathématique asymptotique de structures minces viscoélastiques dans le cadre de la théorie de Trotter de convergence de semi-groupes d'opérateurs agissant sur des espaces variables [1, 2]. Nous montrons que dans ce contexte, il est possible d'effectuer d'une manière unitaire l'analyse asymptotique des plaques et des poutres minces. Nous mettons en évidence divers modèles de comportements dynamiques de telles structures en contact bilatéral avec frottement de type Norton ou Tresca le long d'une partie de leur surface latérale avec un corps rigide.

Keywords. Thin plates, slender beams, viscoelasticity of non-linear Kelvin-Voigt type, Norton or Tresca friction, transient problems, asymptotic analysis, dimension reduction, maximal-monotone operators, approximation of semi-groups in the sense of Trotter.

Mots-clés. Plaques minces, poutres élancées, viscoélasticité de type Kelvin-Voigt non linéaire, frottement de type Norton ou Tresca, problèmes d'évolution, analyse asymptotique, réduction de dimension, opérateurs maximaux-monotones, approximation de semi-groupes au sens de Trotter.

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* Corresponding author

1. Introduction

This paper intends to provide a unifying presentation of the asymptotic mathematical modeling of the mechanical behavior of thin plates and slender beams. How we go about this task is related to our previous works, so our aim here is manifold. Indeed we first carried out the asymptotic analysis of the *quasi-static* response of a linearly viscoelastic plate of Kelvin–Voigt type in [3] before addressing the case of the *transient* response of a thin linearly elastic plate with Norton or Tresca friction in [4]. This background led us to the present, more demanding study, which aims to combine (1) asymptotic mathematical modeling of slender beams, (2) Kelvin–Voigt viscoelasticity and (3) Norton or Tresca friction in (4) the dynamic case. But our intention is to do so in two radical ways. As stated before, one is to embrace both plates and beams modeling in the same framework. The other is to bypass the classical step of transforming the initial problem into an equivalent scaled one. This short-circuit allows to provide more refined information about the displacements in the real thin structures than in traditional and separate studies like [5–7]. Such approach is made possible by a suitable non-linear extension [8] of Trotter's theory of convergence of semi-groups of linear operators acting on variable spaces [1]. This theory is particularly well suited to the mathematical modeling in Physics of continuous media (see in particular the introduction in [2] which contains an overview of the details of this method, which we did not consider relevant to reproduce here ; the reader may find it useful to refer to it), particularly because almost all boundary value problems stemming from Physics are parameterized by the domain where the problem is posed and/or by the physical coefficients which may be very large, very small or strongly oscillating. Throughout this paper the letter r will be used as *upper left index* in our notations for both a thin flat plate ($r = 1$) and a slender beam ($r = 2$). Indeed our approach through the theory of Trotter of approximation of semi-groups of operators acting on *variable* Hilbert spaces works in the same way in the case of thin plates and slender beams. The reader not acquainted with this topic is invited to refer first to the Appendix. There, from (66) onward, is given a concise exposition of how this innovative unifying treatment is carried out in the basic case of linearly elastic slender beams and thin plates. Because the limit models obviously depend on the mathematical objects whose asymptotic behavior is being studied, a careful reading of this short section will make our strategy clear : *we aim at introducing a notion of convergence that provides an energetically sound equivalent to the solution to the genuine physical problem.* This is achieved by a careful and comprehensive study of the asymptotic behavior of the strain tensor. The technical apparatus required to achieve the asymptotic analysis in a more demanding framework, as it is the case in the main text of the article, will then be easier to digest.

The thin structure evolution problem is set in Section 2 where it is stated that the thin plate or the slender beam occupies a domain deduced from the same cylindrical domain by a shrinkage either in the direction of its axis or normally to its axis, respectively. We set its variational formulation (${}^r\mathsf{P}^s$) indexed by r and s at the end of this section. As it will be explained shortly, the index r refers to the geometry of the structure while s is a quadruplet containing the key physical data of the problem.

In Section 3, the problem (${}^r\mathsf{P}^s$) is transformed into a formally equivalent differential inclusion denoted by (${}^r\mathscr{P}^s$) and governed by a multi-valued maximal monotone operator ${}^r\mathscr{A}^s$ defined on a Hilbert space ${}^r\mathscr{H}^s$. Such a deeper and global approach of the dynamics of these structures, considered as Standard Generalized Systems, yields the existence and the uniqueness of a solution ${}^rU^s$ immediately through a simple property of convexity and lower semi-continuity of the global dissipation potentials. Moreover it is efficient and neat with respect to convergence considerations.

Section 4 is devoted to the construction of the framework enabling the study of the asymptotic behavior of ${}^rU^s$. The quadruplet s is then considered as a set of *parameters*. Their relative orders of magnitude are encoded by way of an index denoted I. Through both a deductive and inductive approach, what emerges is a "limit differential inclusion" (${}^r\mathscr{P}^{\mathrm{I}}$) governed by a multi-valued maximal monotone operator ${}^r\mathscr{A}^{\mathrm{I}}$ defined on a Hilbert space ${}^r\mathscr{H}^{\mathrm{I}}$ and whose unique solution is denoted by ${}^rU^{\mathrm{I}}$. It is crucial to observe that our framework lies on an appropriate energetic connection between the elements of spaces ${}^r\mathscr{H}^s$ and ${}^r\mathscr{H}^{\mathrm{I}}$ which are of very different nature. More precisely, through an operator denoted by ${}^r\mathrm{P}^{s\mathrm{I}}$, we may associate to each element U of ${}^r\mathscr{H}^{\mathrm{I}}$ a representative ${}^r\mathrm{P}^{s\mathrm{I}}U$ in ${}^r\mathscr{H}^s$ whose energy converges to the square of the norm of U as s goes to some limit $\bar{s}$ (in particular the thinness ε goes to zero). This means in particular that the problem (${}^r\mathscr{P}^{\mathrm{I}}$) should not be considered as the limit model as such but as a tool to build it: the unique solution ${}^rU^{\mathrm{I}}$ of (${}^r\mathscr{P}^{\mathrm{I}}$) has a representative ${}^r\mathrm{P}^{s\mathrm{I}}\,{}^rU^{\mathrm{I}}$ defined on the genuine physical structure and the relative energetic gap between the unique solution ${}^rU^s$ of the genuine physical problem and ${}^r\mathrm{P}^{s\mathrm{I}}\,{}^rU^{\mathrm{I}}$ goes to zero. In short, we say that ${}^rU^s$ converges toward ${}^rU^{\mathrm{I}}$ in the sense of Trotter. This convergence is here achieved through an elementary adaptation to the case of thin structures of the two-scale convergence developed for homogenization purpose by [9, 10]. This tool is essential to ensure the compactness of the sequence of operators ${}^r\mathscr{A}^s$ in the sense of the resolvent convergence. This is indeed one of the original features of this paper: Kelvin-Voigt viscoelastic behavior is preserved in our limit models, which contrasts with results presented in the literature where most of the time an additional term of delayed memory appears. Of course if we eliminate an additional state variable characterizing the "limit" state we find the term with memory (see (49)). Another topic that should be highlighted is that the strain of the real displacement field on the genuine physical thin plate/slender rod is shown to be actually far from the strain of Kirchhoff–Love/Bernoulli–Navier and even Reissner–Mindlin/Timoshenko displacement fields. The tools and concepts that are used lead to few proofs. However, the reader who is not inclined towards mathematical considerations can be satisfied with the reading of the beginning of the sole subsection 4.1.1 where the "limit" spaces involved by the "limit" problem are defined in (23)-(25).

In Section 5 we present and detail the main properties of our various limit models both in the case of thin plates and of slender beams.

Eventually we present in the Appendix A our adaptation to the case of reduction dimension of the two-scale convergence mentioned earlier.

2. Formulation of the problem

As usual we do not distinguish between $\mathbb{R}^3$ and the Euclidean physical space. Throughout the paper, lower Greek (resp. Latin) indices run from 1 to 2 (resp. 1 to 3). For our purpose it is convenient and capital to introduce

$$ {}^{\alpha}\!(\!(\zeta)\!) := \begin{cases} \widehat{\zeta} := (\zeta_1, \zeta_2) & \text{if } \alpha = 1 \\ \zeta_3 & \text{if } \alpha = 2 \end{cases}, \quad \forall\, \zeta \in \mathbb{R}^3. \tag{1} $$

The thin structure (a thin plate when $r = 1$, a slender beam when $r = 2$, see Fig. 1 below) occupies the closure of ${}^r\Omega^{\varepsilon}$ the images of Ω by the bijections ${}^r\Pi^{\varepsilon}$ defined, respectively, by

$$ \zeta \in \mathbb{R}^3 \mapsto {}^r\Pi^{\varepsilon}\zeta := \left(\varepsilon^{r-1}\widehat{\zeta}, \varepsilon^{2-r}\zeta_3\right) \in \mathbb{R}^3 \tag{2} $$

$$ \Omega := \omega \times (-1, 1) \tag{3} $$

where ω is a bounded domain of $\mathbb{R}^2$ with a Lipschitz-continuous boundary $\partial\omega$. When $r = 2$, without loss of generality, we choose the origin of coordinates in such a way that

$$ \int_{\omega} x_{\alpha}\, d\widehat{x} = \int_{\omega} x_{\alpha} x_{\beta}\, d\widehat{x} = 0, \quad \forall\, \alpha \neq \beta. \tag{4} $$

In the sequel, the space variables x^ε in ${}^r\Omega^\varepsilon$ and x in Ω are systematically connected by

$$x^\varepsilon := {}^r\Pi^\varepsilon x, \quad \forall\, x \in \Omega. \tag{5}$$

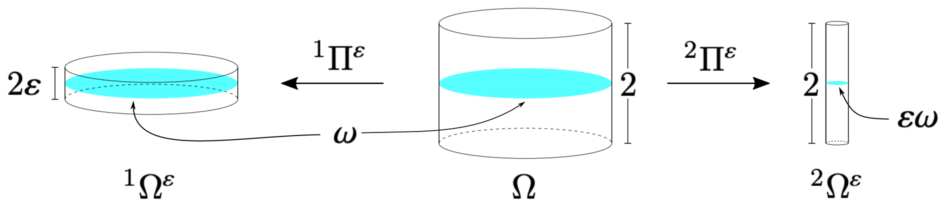

Figure 1. Thin plate and slender beam deriving from a single abstract domain through the bijections ${}^r\Pi^\varepsilon$

Within the context of small strains, we study the dynamic response of a structure made of a nonlinear Kelvin–Voigt viscoelastic material subjected to a given load during the time interval $[0,\mathrm{T}]$. The structure is clamped on a part ${}^r\Gamma^\varepsilon_{\mathrm{D}}$ of the boundary $\partial({}^r\Omega^\varepsilon)$ of ${}^r\Omega^\varepsilon$ such that

$${}^1\Gamma^\varepsilon_{\mathrm{D}} := \gamma_{\mathrm{D}} \times (-\varepsilon,\varepsilon), \quad {}^2\Gamma^\varepsilon_{\mathrm{D}} := \varepsilon\omega \times \{-1\} \tag{6}$$

where γ_{D} is a part of $\partial\omega$ with $\mathscr{H}_1(\gamma_{\mathrm{D}}) > 0$, $\mathscr{H}_n$ being the n-dimensional Hausdorff measure. The structure is also in bilateral contact with a rigid body by Norton or Tresca friction with a “friction” coefficient μ on ${}^r\Gamma^\varepsilon_{\mathrm{C}}$ defined by

$${}^1\Gamma^\varepsilon_{\mathrm{C}} := \gamma_{\mathrm{C}} \times (-\varepsilon,\varepsilon),\ \gamma_{\mathrm{C}} \subset \partial\omega,\ \gamma_{\mathrm{C}} \cap \gamma_{\mathrm{D}} = \varnothing,\ \mathscr{H}_1(\gamma_{\mathrm{C}}) > 0, \quad {}^2\Gamma^\varepsilon_{\mathrm{C}} := \varepsilon\omega \times \{1\} \tag{7}$$

and subjected to a given loading which at each instant t of $[0,\mathrm{T}]$ can be represented by an element ${}^rL^\varepsilon(t)$ of ${}^r\mathscr{U}^{s\,\prime}$ the strong dual of

$${}^r\mathscr{U}^s := \left\{ w \in H^1\left({}^r\Omega^\varepsilon, \mathbb{R}^3\right);\, w = 0 \text{ on } {}^r\Gamma^\varepsilon_{\mathrm{D}},\, w_{N,\varepsilon} = 0 \text{ on } {}^r\Gamma^\varepsilon_{\mathrm{C}} \right\} \tag{8}$$

where $w_{N,\varepsilon} := w \cdot n^\varepsilon$, $w_{T,\varepsilon} := w - w_N\, n^\varepsilon$ are the normal and tangential parts of w on ${}^r\Gamma^\varepsilon_{\mathrm{C}}$, with n^ε the outward unit normal to $\partial({}^r\Omega^\varepsilon)$. Classically ${}^rL^\varepsilon$ stems from body forces of density ${}^rf^\varepsilon$ in $L^2({}^r\Omega^\varepsilon, \mathbb{R}^3)$ and surface forces of density ${}^rg^\varepsilon$ in $L^2({}^r\Gamma^\varepsilon_{\mathrm{N}}, \mathbb{R}^3)$ acting on ${}^r\Gamma^\varepsilon_{\mathrm{N}}$:

$$\begin{cases} {}^1\Gamma^\varepsilon_{\mathrm{N}} := \left(\partial\omega \setminus \left(\gamma_{\mathrm{D}} \cup \gamma_{\mathrm{C}}\right)\right) \times (-\varepsilon,\varepsilon) \cup (\omega \times \{-\varepsilon,\varepsilon\}), \quad {}^2\Gamma^\varepsilon_{\mathrm{N}} := \varepsilon\partial\omega \times (-1,1), \\ {}^rL^\varepsilon(t)(v) := \displaystyle\int_{{}^r\Omega^\varepsilon} {}^rf^\varepsilon\left(x^\varepsilon, t\right) \cdot v\, dx^\varepsilon + \int_{{}^r\Gamma^\varepsilon_{\mathrm{N}}} {}^rg^\varepsilon\left(x^\varepsilon, t\right) \cdot v\, d\mathscr{H}_2,\ \forall\, v \in {}^r\mathscr{U}^s. \end{cases} \tag{9}$$

The density $\rho\delta^\varepsilon$ of the structure, its elasticity tensor a^ε and the density of its viscous pseudo-potential of dissipation $b\,{}^r\mathscr{D}^\varepsilon_{\mathrm{v}}$ satisfy:

$$\begin{cases} \rho > 0,\ b > 0, \\ \delta^\varepsilon \in L^\infty\left({}^r\Omega^\varepsilon\right); \quad \exists\, \alpha_a > 0 \text{ s.t. } \delta^\varepsilon\left(x^\varepsilon\right) \geq \alpha_a \text{ a.e. } x^\varepsilon \in {}^r\Omega^\varepsilon, \\ a^\varepsilon \in L^\infty\left({}^r\Omega^\varepsilon; \mathrm{Lin}\left(\mathbb{S}^3\right)\right); \quad \alpha_a |e|^2 \leq a^\varepsilon\left(x^\varepsilon\right) e \cdot e,\ \forall\, e \in \mathbb{S}^3, \text{ a.e. } x^\varepsilon \in {}^r\Omega^\varepsilon, \\ \exists\, q \in [1,2],\ \exists\, \beta > 0,\ \exists\, \alpha_{\mathrm{v}} > 0; \quad -\alpha_{\mathrm{v}} \leq {}^r\mathscr{D}^\varepsilon_{\mathrm{v}}\left(x^\varepsilon, e\right) \leq \beta\left(1 + |e|^q\right), \quad \forall\, e \in \mathbb{S}^3, \text{ a.e. } x^\varepsilon \in {}^r\Omega^\varepsilon. \end{cases} \tag{H0}$$

Let $s := (\varepsilon, \rho, \mu, b)$ be the key data of the structure and $e^\varepsilon(u)$ the strain tensor associated with the displacement u (the symmetric part of $\nabla^\varepsilon u$, the gradient of u with respect to x^ε-variable). If $\phi_p(\xi) = |\xi|^p/p$ for all ξ in $\mathbb{R}^N, 1 \leq p \leq 2$ and if $\partial J(v)$ denotes the subdifferential at v of any lower semicontinuous convex function J, then the stress tensor ${}^r\sigma^s$, the fields of displacement ${}^ru^s$ and velocity ${}^rv^s$ satisfy:

$$\begin{cases} {}^r\sigma^s \in a^\varepsilon\, e^\varepsilon\left({}^ru^s\right) + b\,\partial\, {}^r\mathscr{D}^\varepsilon_{\mathrm{v}}\big(e^\varepsilon\left({}^rv^s\right)\big) & \text{in } {}^r\Omega^\varepsilon, \\ -\left({}^r\sigma^s\, n^\varepsilon\right)_{T,\varepsilon} \in \partial\phi_p\left({}^rv^s_{T,\varepsilon}\right),\ {}^ru^s_{N,\varepsilon} = 0 & \text{on } {}^r\Gamma^\varepsilon_{\mathrm{C}}, \\ -\mathrm{div}\ {}^r\sigma^s - {}^rf^\varepsilon + \rho\delta^\varepsilon \dfrac{\partial\, {}^rv^s}{\partial t} = 0 & \text{in } {}^r\Omega^\varepsilon, \\ {}^r\sigma^s n^\varepsilon = {}^rg^\varepsilon \text{ on } {}^r\Gamma^\varepsilon_{\mathrm{N}},\ {}^ru^s = 0 & \text{on } {}^r\Gamma^\varepsilon_{\mathrm{D}}. \end{cases} \tag{10}$$

So ${}^rU^s := ({}^ru^s, {}^rv^s)$ has to solve the following problem:

$$({}^r\mathsf{P}^s) \quad \begin{cases} \text{Find } \left({}^ru^s, {}^rv^s = \partial\, {}^ru^s/\partial t\right) \text{ sufficiently smooth in } {}^r\Omega^\varepsilon \times [0,\mathrm{T}] \text{ such that} \\ {}^ru^s = 0 \text{ on } {}^r\Gamma^\varepsilon_{\mathrm{D}} \times [0,\mathrm{T}],\ {}^ru^s_{N,\varepsilon} = 0 \text{ on } {}^r\Gamma^\varepsilon_{\mathrm{C}} \times [0,\mathrm{T}], \\ \left({}^ru^s(\cdot,0), {}^rv^s(\cdot,0)\right) = {}^rU^{s0} := \left({}^ru^{s0}, {}^rv^{s0}\right) \text{ in } {}^r\Omega^\varepsilon, \\ \text{and there exists } \left({}^r\zeta^s, {}^r\xi^s\right) \text{ in } \partial\, {}^r\mathscr{D}^\varepsilon_{\mathrm{v}}\left(x^\varepsilon, e^\varepsilon\left({}^rv^s\right)\right) \times \partial\phi_p\left({}^rv^s_{T,\varepsilon}\right) \text{ satisfying} \\ \displaystyle\int_{{}^r\Omega^\varepsilon} \rho\delta^\varepsilon \frac{\partial^2\, {}^ru^s}{\partial t^2} \cdot w\, dx^\varepsilon + \int_{{}^r\Omega^\varepsilon} a^\varepsilon e^\varepsilon\left({}^ru^s\right) \cdot e^\varepsilon(w)\, dx^\varepsilon + \int_{{}^r\Omega^\varepsilon} b\, {}^r\zeta^s \cdot e^\varepsilon(w)\, dx^\varepsilon \\ \qquad\qquad\qquad\qquad \displaystyle + \int_{{}^r\Gamma^\varepsilon_{\mathrm{C}}} \mu\, {}^r\xi^s \cdot w_{T,\varepsilon}\, d\mathscr{H}_2 = {}^rL^\varepsilon(t)(w), \\ \text{for all } w \text{ sufficiently smooth in } {}^r\Omega^\varepsilon \text{ and such that } w = 0 \text{ on } {}^r\Gamma^\varepsilon_{\mathrm{D}},\ w_{N,\varepsilon} = 0 \text{ on } {}^r\Gamma^\varepsilon_{\mathrm{C}}, \end{cases}$$

Note that on the lateral part ${}^r\Gamma^\varepsilon_{\mathrm{lat}} := \varepsilon^{r-1}\partial\omega \times (-\varepsilon^{2-r}, \varepsilon^{2-r})$, the plate is partly clamped, subjected to surface forces and friction whereas the beam is only subjected to surface forces, while on the basis ${}^r\Gamma^\varepsilon_\pm := \varepsilon^{r-1}\omega \times \{\pm\varepsilon^{2-r}\}$ the plate is subjected to surface forces only whereas the beam is clamped and subjected to friction (to simplify we do not consider the case when surface forces appear on a part of ${}^2\Gamma^\varepsilon_\pm$).

3. Existence and uniqueness

To obtain the existence and uniqueness of ${}^rU^s$, we make an assumption on the loading

$${}^rL^\varepsilon \in BV^1\big(0,\mathrm{T}; {}^r\mathscr{U}^{s\,\prime}\big) \tag{H1}$$

where for all Hilbert space H, $BV^1(0,\mathrm{T};H)$ is the space containing all elements of $BV(0,\mathrm{T};H)$ whose distributional time derivative belongs to $BV(0,\mathrm{T};H)$ which itself is the space of all elements of $L^1(0,\mathrm{T};H)$ whose distributional time derivative is an H-valued measure.

The field ${}^rU^s$ is decomposed into two fields through ${}^rU^s = {}^rU^{se} + {}^rU^{sa}$ with ${}^rU^{se}(t) := ({}^ru^{se}(t), 0)$ defined by

$${}^ru^{se}(t) \in {}^r\mathscr{U}^s; \quad {}^r\varphi^s\left({}^ru^{se}(t), u'\right) = {}^rL^\varepsilon(t)(u'), \quad \forall\, u' \in {}^r\mathscr{U}^s,\ \forall\, t \in [0,\mathrm{T}], \tag{11}$$

where

$${}^r\varphi^s(u,u') := \frac{1}{\varepsilon^{2+r}} \int_{{}^r\Omega^\varepsilon} a^\varepsilon e^\varepsilon(u) \cdot e^\varepsilon(u')\, dx^\varepsilon, \quad \forall\, u, u' \in {}^r\mathscr{U}^s. \tag{12}$$

Due to (H0) and (H1), the displacement field ${}^ru^{se}$ is well-defined and lives in $BV^1(0,\mathrm{T}; {}^r\mathscr{U}^s)$.

The other part ${}^rU^{sa}$ of ${}^rU^s$ brings into play an evolution equation set in a Hilbert space ${}^r\mathscr{H}^s$ of possible states with finite total mechanical energy governed by a maximal-monotone operator ${}^r\mathscr{A}^s$. To this end we introduce the bilinear form ${}^rk^s$ associated with the kinetic energy

$${}^rk^s\left(v,v'\right) := \frac{1}{\varepsilon^{2+r}} \int_{{}^r\Omega^\varepsilon} \rho\delta^\varepsilon v \cdot v'\, dx^\varepsilon, \quad \forall\, v, v' \in {}^r\mathscr{V}^s := L^2\left({}^r\Omega^\varepsilon, \mathbb{R}^3\right), \tag{13}$$

and define the space ${}^r\mathscr{H}^s := {}^r\mathscr{U}^s \times {}^r\mathscr{V}^s$ endowed with the following inner product and norm

$$\begin{aligned} {}^r\langle U, U'\rangle^s &:= {}^r\varphi^s\left(u,u'\right) + {}^rk^s\left(v,v'\right), \quad \forall\, U = (u,v),\ U' = \left(u',v'\right) \in {}^r\mathscr{H}^s, \\ {}^r|U|^s &:= \left[{}^r\langle U, U\rangle^s\right]^{1/2}. \end{aligned} \tag{14}$$

The global pseudo-potential of dissipation ${}^r\mathscr{D}^s_{\mathrm{f}}$ involved by friction is

$$ {}^r\mathscr{D}^s_{\mathrm{f}}(v) := \frac{\mu}{\varepsilon^{2+r}} \int_{{}^r\Gamma^\varepsilon_{\mathrm{C}}} \phi_p(v_{T,\varepsilon})\, d\mathscr{H}_2, \quad \forall\, v \in {}^r\mathscr{U}^s, \tag{15} $$

while the global viscous pseudo-potential of dissipation is

$$ {}^r\mathscr{D}^s_{\mathrm{v}}(v) := \frac{b}{\varepsilon^{2+r}} \int_{{}^r\Omega^\varepsilon} {}^r\mathscr{D}^\varepsilon_{\mathrm{v}}\left(x^\varepsilon, e^\varepsilon(v)\right) dx^\varepsilon, \quad \forall\, v \in {}^r\mathscr{U}^s. \tag{16} $$

The situation when $p = 1$ refers to Tresca while $p \in (1,2]$ corresponds to Norton tangential friction with bilateral contact. The purpose of the normalizing factor ε^{2+r} for energies and global pseudo-potentials of dissipation will clearly appear in the next section.

So the multi-valued operator ${}^r\mathscr{A}^s$ defined on ${}^r\mathscr{H}^s$ by

$$ \begin{cases} D\left({}^r\mathscr{A}^s\right) := \Bigg\{ {}^rU = \left({}^ru, {}^rv\right) \in {}^r\mathscr{H}^s; \begin{cases} \mathrm{I)}\ {}^rv \in {}^r\mathscr{U}^s \\ \mathrm{ii)}\ \exists\, {}^rw \in {}^r\mathscr{V}^s \text{ s.t.} \end{cases} \\ \qquad {}^r\varphi^s\left({}^ru, v'\right) + {}^rk^s\left({}^rw, v'\right) + {}^r\mathscr{D}^s_{\mathrm{f}}\left({}^rv + v'\right) - {}^r\mathscr{D}^s_{\mathrm{f}}\left({}^rv\right) \\ \qquad\qquad + {}^r\mathscr{D}^s_{\mathrm{v}}({}^rv + v') - {}^r\mathscr{D}^s_{\mathrm{v}}({}^rv) \geq 0 \quad \forall\, v' \in {}^r\mathscr{U}^s \Bigg\}, \\ -{}^r\mathscr{A}^s\, {}^rU = \left\{ \left({}^rv, {}^rw\right);\ {}^rw \text{ satisfies ii) in the definition of } D\left({}^r\mathscr{A}^s\right) \right\}, \end{cases} $$

obviously satisfies:

Proposition 1. *The operator ${}^r\mathscr{A}^s$ is maximal monotone and for all ${}^r\psi^s = ({}^r\psi^s_u, {}^r\psi^s_v)$ in ${}^r\mathscr{H}^s$*

$$ \begin{cases} {}^r\overline{U}^s = \left({}^r\overline{u}^s, {}^r\overline{v}^s\right) \text{ s.t.} \\ {}^r\overline{U}^s + {}^r\mathscr{A}^s\, {}^r\overline{U}^s \ni {}^r\psi^s \end{cases} \Longleftrightarrow \begin{cases} {}^r\overline{u}^s = {}^r\overline{v}^s + {}^r\psi^s_u, \\ \text{where } {}^r\overline{v}^s \text{ is the unique minimizer on } {}^r\mathscr{U}^s \text{ of } {}^r\mathcal{J}^s; \\ {}^r\mathcal{J}^s\left({}^rv\right) := \frac{1}{2}\left[{}^r\left|\left({}^rv, {}^rv\right)\right|^s\right]^2 + {}^r\left\langle \left({}^r\psi^s_u, -{}^r\psi^s_v\right), \left({}^rv, {}^rv\right)\right\rangle^s \\ \qquad + {}^r\mathscr{D}^s_{\mathrm{f}}\left({}^rv\right) + {}^r\mathscr{D}^s_{\mathrm{v}}\left({}^rv\right) \quad \forall\, {}^rv \in {}^r\mathscr{U}^s. \end{cases} $$

Finally as the very definitions of ${}^r\mathscr{D}^s_{\mathrm{f}}$ and ${}^r\mathscr{D}^s_{\mathrm{v}}$ imply that $({}^r\mathsf{P}^s)$ is formally equivalent to

$$ \left({}^r\mathscr{P}^s\right) \quad \begin{cases} \dfrac{d\,{}^rU^s}{dt} + {}^r\mathscr{A}^s\left({}^rU^s - {}^rU^{se}\right) \ni 0, \\ {}^rU^s(0) = {}^rU^{s0}, \end{cases} $$

a result of [11] yields:

Theorem 2. *Under assumptions* (H0), (H1) *and*

$$ {}^rU^{s0} \in {}^rU^{se}(0) + D\left({}^r\mathscr{A}^s\right) \tag{H2} $$

the problem $({}^r\mathscr{P}^s)$ has a unique solution ${}^rU^s$ belonging to $W^{1,\infty}(0,T;{}^r\mathscr{H}^s)$ and the first line of $({}^r\mathscr{P}^s)$ is satisfied almost everywhere in $(0,\mathrm{T}]$.

4. Asymptotic behavior

Now we consider *s as a quadruplet of parameters taking values in a countable subset $\mathfrak{S}$ of* $(0,+\infty)^4$ with a unique cluster point $\bar{s}$ in $\{0\} \times [0,+\infty) \times [0,+\infty]^2$.

For each value of r we will consider various cases of relative behavior of the elements of s characterized by $\mathrm{I} = (\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)$ in $\{1,2\} \times \{1,2,3\} \times \{1,2\}$. First we let

$$ \rho^{*\mathrm{I}_1} := \begin{cases} \rho\,\varepsilon^{-2} & \mathrm{I}_1 = 1 \\ \rho & \mathrm{I}_1 = 2 \end{cases}, $$

$$^{1}\mu^{*\mathrm{I}_2} := \begin{cases} \mu\varepsilon^{-2} & \mathrm{I}_2 = 1 \\ \mu\varepsilon^{p-2} & \mathrm{I}_2 = 2,3 \end{cases} \quad ; \quad {}^{2}\mu^{*\mathrm{I}_2} := \mu\varepsilon^{-2},\ \mathrm{I}_2 = 1,2,3,$$

$$b^{*\mathrm{I}_3} := b\varepsilon^{q-2},\ \mathrm{I}_3 = 1,2$$

and second make the following assumption to account for the magnitudes of density, thickness and viscosity:

$$\begin{cases} \text{there exists } \left(\bar{\rho}^{\mathrm{I}_1}, {}^{r}\bar{\mu}^{\mathrm{I}_2}, \bar{b}^{\mathrm{I}_3}\right) \text{ in } (0,+\infty)\times[0,+\infty]\times[0,+\infty] \text{ such that} \\ \bar{\rho}^{\mathrm{I}_1} := \lim\limits_{s\to\bar{s}} \rho^{*\mathrm{I}_1}, \\ {}^{1}\bar{\mu}^{\mathrm{I}_2} := \lim\limits_{s\to\bar{s}} {}^{1}\mu^{*\mathrm{I}_2} \text{ with } {}^{1}\bar{\mu}^{1}, {}^{1}\bar{\mu}^{2} \in [0,+\infty) \text{ and } {}^{1}\bar{\mu}^{3} = +\infty, \\ {}^{2}\bar{\mu}^{\mathrm{I}_2} := \lim\limits_{s\to\bar{s}} {}^{2}\mu^{*\mathrm{I}_2} \text{ with } {}^{2}\bar{\mu}^{1} \in [0,+\infty), {}^{2}\bar{\mu}^{\mathrm{I}_2} = +\infty \text{ if } \mathrm{I}_2 = 2,3, \\ \bar{b}^{\mathrm{I}_3} := \lim\limits_{s\to\bar{s}} b^{*\mathrm{I}_3} \text{ with } \bar{b}^{2} = +\infty. \end{cases} \tag{H3}$$

In the sequel, according to the very definition of ${}^{r}\Pi^{\varepsilon}$ in (2), we discard the index ε for the notations of the inverse images of ${}^{r}\Gamma^{\varepsilon}_{\mathrm{D}}$ and ${}^{r}\Gamma^{\varepsilon}_{\mathrm{C}}$. We make the following assumption (H4) on the density, the elasticity tensor of the structure, the viscous pseudo-potential of dissipation and the loading:

$$\begin{cases} \exists\ (\delta, a) \in L^{\infty}\left(\Omega, \mathbb{R}\times \mathrm{Lin}(\mathbb{S}^3)\right) \text{ s.t.} \\ \qquad \alpha_a \le \delta(x),\ \alpha_a |e|^2 \le a(x)\, e\cdot e,\ \forall\ e \in \mathbb{S}^3, \text{ a.e. } x\in\Omega \\ \qquad \delta^{\varepsilon}(x^{\varepsilon}) = \delta(x),\ a^{\varepsilon}(x^{\varepsilon}) = a(x), \text{ a.e. } x\in\Omega \\ \exists\ {}^{r}\mathscr{D}_{\mathrm{V}} \text{ measurable in } \Omega, \text{ convex on } \mathbb{S}^3 \text{ s.t.} \\ \qquad \left.\begin{array}{l} \exists\ q\in[1,2],\ -\alpha_{\mathrm{V}} \le {}^{r}\mathscr{D}_{\mathrm{V}}(x,e) \le \beta(1+|e|^q) \\ {}^{r}\mathscr{D}^{\varepsilon}_{\mathrm{V}}(x^{\varepsilon}, e) = {}^{r}\mathscr{D}_{\mathrm{V}}(x,e) \end{array}\right\} \forall\ e\in\mathbb{S}^3, \text{ a.e. } x\in\Omega \\ \exists\ {}^{r}L_{\varepsilon} \in BV^1\left(0,\mathrm{T}; {}^{r}\mathscr{U}'_{\mathrm{c}}\right) \text{ s.t.} \\ \qquad {}^{r}L^{\varepsilon}(t)(w) = \varepsilon^{2+r}\ {}^{r}L_{\varepsilon}(t)({}^{r}\mathscr{S}_{\varepsilon} w), \forall\ (w,t) \in {}^{r}\mathscr{U}^{s}\times(0,\mathrm{T}) \\ \exists\ {}^{r}L \in BV^1\left(0,\mathrm{T}; {}^{r}\mathscr{U}'_{\mathrm{c}}\right) \text{ s.t.} \\ \qquad {}^{r}L_{\varepsilon} \text{ strongly converges in } BV^1\left(0,\mathrm{T}; {}^{r}\mathscr{U}'_{\mathrm{c}}\right) \text{ toward } {}^{r}L \end{cases} \tag{H4}$$

where

$$\begin{cases} {}^{r}\mathscr{U}_{\mathrm{c}} := \left\{ w \in H^1(\Omega,\mathbb{R}^3);\ w = 0 \text{ on } {}^{r}\Gamma_{\mathrm{D}},\ w_N = 0 \text{ on } {}^{r}\Gamma_{\mathrm{C}} \right\} \\ {}^{r}\mathscr{U}'_{\mathrm{c}} \text{ is the strong dual of } {}^{r}\mathscr{U}_{\mathrm{c}} \\ n \text{ is the outward normal to } \partial\Omega, \quad w_N := w\cdot n, \quad w_T := w - w_N\, n \end{cases} \tag{17}$$

and (see (1))

$${}^{r}(\!({}^{r}\mathscr{S}_{\varepsilon} w)\!)(x) := \frac{1}{\varepsilon}\, {}^{r}(\!(w)\!)(x^{\varepsilon}),\ {}^{3-r}(\!({}^{r}\mathscr{S}_{\varepsilon} w)\!)(x) := {}^{3-r}(\!(w)\!)(x^{\varepsilon}), \quad \forall\ w \in L^2({}^{r}\Omega^{\varepsilon}, \mathbb{R}^3). \tag{18}$$

Remark 3. Equation (18) expresses that the relative orders of magnitude between the in-plane and the out of plane displacements are not the same in the case of plates and in the case of beams. In the classical literature (see [5–7] for example) what is called the "scaled displacement field" (the one which lives on Ω) and very often denoted by $u(\varepsilon)$ is - up to a power of ε - connected to the genuine physical displacement denoted by u^{ε} through formulas of the kind $u^{\varepsilon}_{\alpha}(x^{\varepsilon}) = \varepsilon u_{\alpha}(\varepsilon)(x)$, $u^{\varepsilon}_{3}(x^{\varepsilon}) = u_3(\varepsilon)(x)$ in the case of plates and $u^{\varepsilon}_{\alpha}(x^{\varepsilon}) = u_{\alpha}(\varepsilon)(x)$, $u^{\varepsilon}_{3}(x^{\varepsilon}) = \varepsilon u_3(\varepsilon)(x)$ in the case of beams, which is summarized in (18).

4.1. *A candidate for the limit behavior*

4.1.1. *The limit framework*

The limit framework may be determined by studying the asymptotic behavior of sequences with bounded total mechanical energy. The "scaling operator" ${}^r\mathscr{S}_\varepsilon$ introduced in (18) plays a key role because one has

$$e^{\varepsilon}(w)(x^{\varepsilon}) = \varepsilon\, {}^r e(\varepsilon, {}^r\mathscr{S}_\varepsilon w) \tag{19}$$

where

$$\left.\begin{aligned} {}^r e_{ij}(\varepsilon, z) \quad &:= \begin{cases} \varepsilon^{2(1-r)} e_{ij}(z) & \text{for } 1 \le i, j \le 2 \\ \varepsilon^{-1} e_{ij}(z) & \text{for } 1 \le i \le 2,\, j = 3 \\ {}^r e_{ji}(\varepsilon, z) & \text{for } 1 \le j < i \le 3 \\ \varepsilon^{2(r-2)} e_{33}(z) & \text{for } i = j = 3, \end{cases} \\ e_{ij}(z) \quad &:= \tfrac{1}{2}\left(\partial_i z_j + \partial_j z_i\right) \end{aligned}\right\} \quad \forall\, z \in H^1\left(\Omega, \mathbb{R}^3\right). \tag{20}$$

and therefore the bilinear forms ${}^r\varphi^s$ and ${}^r k^s$ read as

$${}^r\varphi^s(u, u') = \int_\Omega a\, {}^r e\left(\varepsilon, {}^r\mathscr{S}_\varepsilon u\right) \cdot {}^r e\left(\varepsilon, {}^r\mathscr{S}_\varepsilon u'\right) dx, \quad \forall\, u, u' \in {}^r\mathscr{U}^s, \tag{21}$$

$${}^r k^s(v, v') = \frac{\rho}{\varepsilon^2} \int_\Omega \delta\Big(\varepsilon^2\, {}^r[\![{}^r\mathscr{S}_\varepsilon v]\!] \cdot {}^r[\![{}^r\mathscr{S}_\varepsilon v']\!] + {}^{3-r}[\![{}^r\mathscr{S}_\varepsilon v]\!] \cdot {}^{3-r}[\![{}^r\mathscr{S}_\varepsilon v']\!]\Big)\, dx, \quad \forall\, v, v' \in {}^r\mathscr{V}^s. \tag{22}$$

Remark 4. Similarly to what has been recalled in Remark 3, as soon as we consider ε as a parameter, its influence on the strain tensor is different depending on whether we consider the case of plates or the case of beams. In the case of plates we are led to deal with a strain tensor $e_{ij}(\varepsilon, z)$ of the form

$$\begin{bmatrix} e_{\alpha\beta}(z) & \frac{1}{\varepsilon} e_{\alpha 3}(z) \\ \frac{1}{\varepsilon} e_{\alpha 3}(z) & \frac{1}{\varepsilon^2} e_{33}(z) \end{bmatrix},$$

while the case of beams leads to

$$\begin{bmatrix} \frac{1}{\varepsilon^2} e_{\alpha\beta}(z) & \frac{1}{\varepsilon} e_{\alpha 3}(z) \\ \frac{1}{\varepsilon} e_{\alpha 3}(z) & e_{33}(z) \end{bmatrix}$$

and this is what (20) summarizes.

Thus we let

$$
\begin{cases}
x^R := (-x_2, x_1), \quad \forall\, x \in \mathbb{R}^3, \\
H^1_b(-1,1) := \Big\{ w \in H^1(-1,1); w(-1) = 0 \Big\}, \\
H^1_b\left(\Omega, \mathbb{R}^3\right) := \Big\{ w \in H^1\left(\Omega, \mathbb{R}^3\right); w(\widehat{x}, -1) = 0, \text{ a.e. } \widehat{x} \in \omega \Big\}, \\
H^1_m(\omega) := \Big\{ w \in H^1(\omega); \int_\omega w(\widehat{x})\, d\widehat{x} = 0 \Big\}, \\
H^1_m\big(-1,1; L^2\left(\omega, \mathbb{R}^n\right)\big) := \Big\{ w \in H^1\left(-1,1; L^2\left(\omega, \mathbb{R}^n\right)\right); \int_{-1}^1 w(x)\, dx_3 = 0 \Big\}, \\
V_{\mathrm{KL}} := \Big\{ w \in H^1\left(\Omega, \mathbb{R}^3\right);\ e_{i3}(w) = 0 \Big\}, \quad V_{\mathrm{BN}} := \Big\{ w \in H^1\left(\Omega, \mathbb{R}^3\right);\ e_{\alpha\beta}(w) = e_{\alpha 3}(w) = 0 \Big\}, \\
{}^1\mathcal{U}^0 := V_{\mathrm{KL}} \cap {}^1\mathcal{U}_{\mathrm{c}}, \quad {}^2\mathcal{U}^0 := V_{\mathrm{BN}} \cap {}^2\mathcal{U}_{\mathrm{c}}, \\
{}^1\mathcal{U}^1 := H^1_m\big(-1,1; L^2(\omega, \mathbb{R}^2)\big) \times \{0\}, \\
{}^2\mathcal{U}^1 := \Big\{ w; \exists\, c \in H^1_b(-1,1) \text{ s.t. } \widehat{w}(x) = c(x_3)x^R, w_3 \in L^2\big(-1,1; H^1_m(\omega)\big) \Big\}, \\
{}^1\mathcal{U}^2 := \{0\}^2 \times H^1_m\big(-1,1; L^2(\omega)\big), \\
{}^2\mathcal{U}^2 := \Big\{ \widehat{w} \in L^2\big(-1,1; H^1_m\left(\omega, \mathbb{R}^2\right)\big) \text{ s.t. } \int_\omega x^R \cdot \widehat{w}(x)\, d\widehat{x} = 0 \text{ a.e. } x_3 \in (-1,1) \Big\} \times \{0\}, \\
{}^r\mathcal{U} := {}^r\mathcal{U}^0 \times {}^r\mathcal{U}^1 \times {}^r\mathcal{U}^2.
\end{cases}
\tag{23}
$$

We also introduce ${}^r e_u$ in $L^2(\Omega, \mathbb{S}^3)$ such that

$$
\left({}^r e_u\right)_{ij} := \begin{cases}
e_{ij}\left(u^{2(r-1)}\right) & \text{for } 1 \le i, j \le 2 \\
e_{ij}\left(u^1\right) & \text{for } 1 \le i \le 2, j = 3\,, \\
e_{33}\left(u^{2(2-r)}\right) & \text{for } i = j = 3,
\end{cases}
\quad \forall\, u = \left(u^0, u^1, u^2\right) \in {}^r\mathcal{U},
\tag{24}
$$

and define

$$
\begin{cases}
{}^r\varphi\left(u, u'\right) := \int_\Omega a\, {}^r e_u \cdot {}^r e_{u'}\, dx, \ \forall\, u, u' \in {}^r\mathcal{U} \\
{}^r\mathcal{V}^1 := L^2\left(\Omega, \mathbb{R}^r\right), \quad {}^r\mathcal{V}^2 := \Big\{ v \in L^2\left(\Omega, \mathbb{R}^3\right);\ {}^{3-r}(\!(v)\!) = 0 \Big\}, \\
{}^r\mathcal{H}^{\mathrm{I}} := {}^r\mathcal{U} \times {}^r\mathcal{V}^{\mathrm{I}_1}, \quad \forall\, \mathrm{I} = (\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3) \in \{1,2\} \times \{1,2,3\} \times \{1,2\}, \\
{}^r k^{\mathrm{I}_1}\left(v, v'\right) := \bar{\rho}^{\mathrm{I}_1} \int_\Omega \delta\, v \cdot v'\, dx, \ \forall\, v, v' \in {}^r\mathcal{V}^{\mathrm{I}_1}, \\
{}^r\langle U, U' \rangle^{\mathrm{I}} := {}^r\varphi(u, u') + {}^r k^{\mathrm{I}_1}(v, v'), \ \forall\ U = (u, v),\ U' = \left(u', v'\right) \in {}^r\mathcal{H}^{\mathrm{I}}, \\
{}^r|U|^{\mathrm{I}} := \left[{}^r\langle U, U \rangle^{\mathrm{I}}\right]^{1/2}.
\end{cases}
\tag{25}
$$

The following two propositions suggest that the Hilbert space ${}^r\mathcal{H}^{\mathrm{I}}$ is the required framework to *describe* the asymptotic behavior:

Proposition 5. *For every sequences* ${}^r\mathcal{X}^s = ({}^r\mathcal{X}^s_u, {}^r\mathcal{X}^s_v)$ *in* ${}^r\mathcal{H}^s$ *such that* ${}^r|{}^r\mathcal{X}^s|^s$ *is uniformly bounded, there exist a not relabeled subsequence and* ${}^r\mathcal{X}^{\mathrm{I}} = ({}^r\mathcal{X}^{\mathrm{I}}_u, {}^r\mathcal{X}^{\mathrm{I}}_v)$ *in* ${}^r\mathcal{H}^{\mathrm{I}}$ *such that*

(i) $\left({}^r e_{{}^r\mathcal{X}^{\mathrm{I}}_u}, {}^r\mathcal{X}^{\mathrm{I}}_v\right)$ *is the weak limit in* $L^2\left(\Omega, \mathbb{S}^3 \times \mathbb{R}^{(3-r)\mathrm{I}_1 + (2r-3)}\right)$ *of* $\left({}^r e(\varepsilon, {}^r\mathcal{S}_\varepsilon\, {}^r\mathcal{X}^s_u), {}^{3-r}(\!({}^r\mathcal{S}_\varepsilon\, {}^r\mathcal{X}^s_v)\!)\right)$ *when* $\mathrm{I}_1 = 1$ *or of* $\left({}^r e(\varepsilon, {}^r\mathcal{S}_\varepsilon\, {}^r\mathcal{X}^s_u), {}^r\mathcal{S}_\varepsilon\, {}^r\mathcal{X}^s_v\right)$ *when* $\mathrm{I}_1 = 2$,

(ii) ${}^r\left|{}^r\mathcal{X}^{\mathrm{I}}\right|^{\mathrm{I}} \le \varliminf_{s \to \bar{s}} {}^r\left|{}^r\mathcal{X}^s\right|^s$.

Proposition 6. *For all s in $\mathfrak{S}$ and all $U=(u,v)$ in ${}^r\mathcal{H}^{\mathrm{I}}$, let ${}^r\mathrm{P}^{s\mathrm{I}}U := ({}^r\mathrm{P}^{s\mathrm{I}}_u u, {}^r\mathrm{P}^{s\mathrm{I}}_v v)$ in ${}^r\mathcal{H}^s$ defined by*

$$ {}^r\varphi^s\left({}^r\mathrm{P}^{s\mathrm{I}}_u u, u'\right) := \int_\Omega a\, {}^r e_u \cdot {}^r e\left(\varepsilon, {}^r\mathcal{S}_\varepsilon u'\right) dx, \quad \forall\, u' \in {}^r\mathcal{U}^s, \tag{26} $$

$$ {}^r k^s\left({}^r\mathrm{P}^{s\mathrm{I}}_v v, v'\right) := \begin{cases} \bar\rho^1 \int_\Omega \delta v \cdot {}^{3-r}\!\left(\!\left({}^r\mathcal{S}_\varepsilon v'\right)\!\right) dx, & \text{if } \mathrm{I}_1 = 1 \\ \bar\rho^2 \int_\Omega \delta v \cdot {}^r\mathcal{S}_\varepsilon v'\, dx, & \text{if } \mathrm{I}_1 = 2 \end{cases}, \quad \forall\, v' \in {}^r\mathcal{V}^s. \tag{27} $$

There holds:

(P1) $\exists\, C > 0$ *s.t.* ${}^r|{}^r\mathrm{P}^{s\mathrm{I}}U|^s \le C\, {}^r|U|^{\mathrm{I}}, \quad \forall\, U \in {}^r\mathcal{H}^{\mathrm{I}}, \forall\, s \in \mathfrak{S}$,

(P2) $\lim_{s\to\bar s} {}^r\left|{}^r\mathrm{P}^{s\mathrm{I}}U\right|^s = {}^r|U|^{\mathrm{I}}, \quad \forall\, U \in {}^r\mathcal{H}^{\mathrm{I}}$,

(P3) (i) $\displaystyle\lim_{s\to\bar s} \frac{1}{\varepsilon^{2+r}} \int_{{}^r\Omega^\varepsilon} a^\varepsilon \left[e^\varepsilon\left({}^r\mathrm{P}^{s\mathrm{I}}_u u\right) - {}^r e^\varepsilon_u\right] \cdot \left[e^\varepsilon\left({}^r\mathrm{P}^{s\mathrm{I}}_u u\right) - {}^r e^\varepsilon_u\right] dx^\varepsilon = 0$, *with*

$$ {}^r e^\varepsilon_u\left(x^\varepsilon\right) := \varepsilon\, {}^r e_u(x) \text{ a.e. } x^\varepsilon = {}^r\Pi^\varepsilon x \in {}^r\Omega^\varepsilon, \ \forall\, u \in {}^r\mathcal{U}. \tag{28} $$

$$ \text{(ii)} \quad {}^r\mathrm{P}^{s\mathrm{I}}_v v = {}^r V^{\varepsilon\mathrm{I}}_v := \begin{cases} \dfrac{\bar\rho^{\mathrm{I}_1}}{\rho^{*\mathrm{I}_1}} \left({}^r\mathcal{S}_\varepsilon\right)^{-1} {}^r\check v, & \text{if } \mathrm{I}_1 = 1, \\ \dfrac{\bar\rho^{\mathrm{I}_1}}{\rho^{*\mathrm{I}_1}} \left({}^r\mathcal{S}_\varepsilon\right)^{-1} v & \text{if } \mathrm{I}_1 = 2. \end{cases} $$

where, for all v in $L^2(\Omega,\mathbb{R}^3)$, ${}^r\check v := \begin{cases} (0,0,v_3) & \text{if } r = 1, \\ (\widehat v, 0) & \text{if } r = 2. \end{cases}$

Proof. The proof of Proposition 5 is obvious when $r=1$ (case of plates) but more difficult when $r=2$ (case of beams) and we are aware of two proofs in the literature [12, 13]. Here we propose a simplification of the last part of the proof in [12]. We set $z_\varepsilon := {}^2\mathcal{S}_\varepsilon\, {}^2\mathcal{X}^s_u$, the boundedness of ${}^2e(\varepsilon, z_\varepsilon)$ implies that there exist a not relabeled subsequence and z^0 in ${}^2\mathcal{U}^0$ such that z_ε weakly converges in $H^1(\Omega,\mathbb{R}^3)$ toward z^0. Moreover through a decomposition lemma, it is established in [12] that

$$ \begin{aligned} &\frac{1}{\varepsilon}\widehat z_\varepsilon = \varepsilon\, \widehat w_\varepsilon + c_\varepsilon x^R + d_\varepsilon \\ &w_\varepsilon \in H^1_b\left(\Omega,\mathbb{R}^3\right) \cap {}^2\mathcal{U}^2, \quad |\widehat w_\varepsilon|^2_{L^2(-1,1;H^1_m(\omega,\mathbb{R}^2))} \le \frac{C}{\varepsilon^2} \sum_{\alpha,\beta} \left|e_{\alpha\beta}(z_\varepsilon)\right|^2_{L^2(\Omega)} \\ &c_\varepsilon := \frac{1}{\varepsilon} \frac{\int_\omega x^R \cdot \widehat z_\varepsilon\, d\widehat x}{\int_\omega |\widehat x|^2\, d\widehat x}, \quad d_\varepsilon := \frac{1}{\varepsilon|\omega|} \int_\omega \widehat z_\varepsilon\, d\widehat x \end{aligned} \tag{29} $$

which, first, implies that there exists z^2 in ${}^2\mathcal{U}^2$ such that up to a not relabelled subsequence $\frac{1}{\varepsilon^2} e_{\alpha\beta}(z_\varepsilon) = e_{\alpha\beta}(w_\varepsilon)$ weakly converges in $L^2(\Omega)$ toward $e_{\alpha\beta}(z^2)$ and, next through some handlings, that there exists c in $H^1(-1,1)$ such that c_ε strongly converges in $L^2(-1,1)$ toward c. Lastly to establish that there exists z^1 in ${}^2\mathcal{U}^1$ such that $\frac{1}{\varepsilon} e_{\alpha 3}(z_\varepsilon)$ weakly converges in $L^2(\Omega)$ toward $e_{\alpha 3}(z^1)$, it suffices to observe that

$$ \frac{2}{\varepsilon} e_{\alpha 3}(z_\varepsilon) = \varepsilon \partial_3 w^\varepsilon_\alpha + \frac{dc_\varepsilon}{dx_3} x^R_\alpha + \partial_\alpha\left(\frac{d}{dx_3}\left(d_{\varepsilon_\beta}\right) x_\beta + \frac{z_{\varepsilon_3}}{\varepsilon}\right). \tag{30} $$

As c belongs to $H^1(-1,1)$ one deduces that $\partial_\alpha(\frac{d}{dx_3}(d_{\varepsilon_\beta}) x_\beta + \frac{z_{\varepsilon_3}}{\varepsilon})$ converges in the sense of distributions to an element q_α of $L^2(\Omega)$. Therefore as $\partial_2 q_1 - \partial_1 q_2 = 0$ and ω is a domain, there exists ζ in $L^2(-1,1;H^1(\omega,\mathbb{R}^2))$ such that $q = \widehat\nabla\zeta$ and $z^1 := (cx^R, \zeta)$ meets the required condition.

The proof of Proposition 6 is then straightforward. First by taking $u' = {}^r\mathcal{X}^s_u := {}^r\mathrm{P}^{s\mathrm{I}}_u u$ in (26), one deduces that there exist a not relabeled subsequence and z in ${}^r\mathcal{U}$ such that ${}^r e(\varepsilon, {}^r\mathcal{S}_\varepsilon\, {}^r\mathcal{X}^s_u)$ weakly converges in $L^2(\Omega,\mathbb{S}^3)$ toward some ${}^r e_z$. Next by choosing $u' = ({}^r\mathcal{S}_\varepsilon)^{-1}(\sum_{i=0}^2 \varepsilon^i y^i)$ with

y in ${}^r\mathscr{U} \cap C^\infty(\overline{\Omega}, \mathbb{R}^3)^3$, we deduce that $z = u$. Lastly by taking $u' = {}^r\mathscr{X}^s_u$ again, one obtains that $|{}^r e(\varepsilon, {}^r\mathscr{S}_\varepsilon\, {}^r\mathscr{X}^s_u)|^2_{L^2(\Omega,\mathbb{S}^3)}$ converges to $|{}^r e_u|^2_{L^2(\Omega,\mathbb{S}^3)}$ which implies the strong convergence of ${}^r e(\varepsilon, {}^r\mathscr{S}_\varepsilon\, {}^r\mathscr{X}^s_u)$ toward ${}^r e_u$ which is also (P3i) because of (19) and the definition of ${}^r e^\varepsilon_u$; in other words $\frac{1}{\varepsilon} e^\varepsilon({}^r\mathscr{X}^s_u)$ "3d$-(3-r)$d converges" toward ${}^r e_u$ (see the Appendix).

The remaining point (P3ii) is obvious. □

Property (P2) states that any element U of ${}^r\mathscr{H}^{\mathrm{I}}$ has a representative ${}^r\mathrm{P}^{s\mathrm{I}}U$ in ${}^r\mathscr{H}^s$ whose rescaled energy ${}^r\langle {}^r\mathrm{P}^{s\mathrm{I}}U, {}^r\mathrm{P}^{s\mathrm{I}}U\rangle^s$ is arbitrarily close to the square of the norm of U in ${}^r\mathscr{H}^{\mathrm{I}}$, ensuring that ${}^r\mathscr{H}^{\mathrm{I}}$ is appropriate to describe the asymptotic behavior. Note that through (25) the "abstract velocities" living in the space ${}^r\mathcal{V}^{\mathrm{I}_1}$ involve only the component ${}^{3-r}\!(\!(\cdot)\!)$!

4.1.2. *The limit operator ${}^r\mathscr{A}^{\mathrm{I}}$*

According to the theory of Trotter of approximation of semi-groups of operators acting on variable spaces [1, 2, 8], we examine the asymptotic behavior of the resolvent $(\mathbb{I} - {}^r\mathscr{A}^s)^{-1}$ of ${}^r\mathscr{A}^s$ in order to guess the limit operator ${}^r\mathscr{A}^{\mathrm{I}}$. By due account of Proposition 1, we consider sequences ${}^r z^s$ with uniformly bounded global friction and viscous pseudo-potentials of dissipation ${}^r\mathscr{D}^s_{\mathrm{f}}({}^r z^s)$, ${}^r\mathscr{D}^s_{\mathrm{v}}({}^r z^s)$ and "total energy functional" $[{}^r|({}^r z^s, {}^r z^s)|^s]^2$, which will permit to define the space ${}^r\mathcal{Z}^{\mathrm{I}}$ of "virtual limit admissible generalized velocities" and the limit global potentials of dissipations ${}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{f}}$ and ${}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{v}}$. Let

$$
{}^r\mathcal{Z}^{\mathrm{I}} := \left\{ \begin{array}{l} z \in {}^r\mathscr{U};\ {}^{3-r}\!(\!(z^0)\!) = 0 \text{ if } \mathrm{I}_1 = 2, \\ \qquad {}^{3-r}\!(\!(z^0)\!) = 0 \text{ on } {}^r\Gamma_{\mathrm{C}} \text{ if } \mathrm{I}_2 = 2, z^0 = 0 \text{ on } {}^r\Gamma_{\mathrm{C}} \text{ if } \mathrm{I}_2 = 3, z = 0 \text{ if } \mathrm{I}_3 = 2 \end{array} \right\}, \tag{31}
$$

$$
{}^1\mathscr{D}^{\mathrm{I}}_{\mathrm{f}}(z) := \begin{cases} 2\,{}^1\bar{\mu}^1 \int_{\gamma_{\mathrm{C}}} \phi_p\left(z^0_3\right) d\mathscr{H}_1 & \text{if } \mathrm{I}_2 = 1 \\ 2\,{}^1\bar{\mu}^2 \int_{\gamma_{\mathrm{C}}} \phi_p\left(\widehat{z^0_T}\right) d\mathscr{H}_1 & \text{if } \mathrm{I}_2 = 2 \\ 0 & \text{if } \mathrm{I}_2 = 3 \end{cases}, \qquad \forall\, z \in {}^1\mathcal{Z}^{\mathrm{I}}, \tag{32}
$$

$$
{}^2\mathscr{D}^{\mathrm{I}}_{\mathrm{f}}(z) := \begin{cases} {}^2\bar{\mu}^1 \int_\omega \phi_p\left(\widehat{z^0}(\hat{x}, 1)\right) d\hat{x} & \text{if } \mathrm{I}_2 = 1 \\ 0 & \text{if } \mathrm{I}_2 = 2, 3 \end{cases}, \qquad \forall\, z \in {}^2\mathcal{Z}^{\mathrm{I}}, \tag{33}
$$

$$
{}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{v}}(z) := \begin{cases} \bar{b}^1 \int_\Omega {}^r\mathscr{D}_{\mathrm{v}}({}^r e_z)\, dx & \text{if } \mathrm{I}_3 = 1 \\ 0 & \text{if } \mathrm{I}_3 = 2 \end{cases}, \qquad \forall\, z \in {}^r\mathcal{Z}^{\mathrm{I}}. \tag{34}
$$

A simple argument of lower semi-continuity yields:

Proposition 7. *For all sequence ${}^r z^s$ in ${}^r\mathscr{U}^s$ such that $[{}^r|({}^r z^s, {}^r z^s)|^s]^2 + {}^r\mathscr{D}^s_{\mathrm{f}}({}^r z^s) + {}^r\mathscr{D}^s_{\mathrm{v}}({}^r z^s) \leq C$, there exist a not relabeled subsequence and ${}^r z$ in ${}^r\mathcal{Z}^{\mathrm{I}}$ such that ${}^r e(\varepsilon, {}^r\mathscr{S}_\varepsilon\, {}^r z^s)$ weakly converges in $L^2(\Omega, \mathbb{S}^3)$ toward ${}^r e_{{}^r z}$ and*

$$
\left[{}^r\left|\left({}^r z, {}^r(\mathring{z})^{\mathrm{I}}\right)\right|^{\mathrm{I}}\right]^2 + {}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{f}}\left({}^r z\right) + {}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{v}}\left({}^r z\right) \leq \varliminf_{s \to \bar{s}} \left(\left[{}^r\left|\left({}^r z^s, {}^r z^s\right)\right|^s\right]^2 + {}^r\mathscr{D}^s_{\mathrm{f}}\left({}^r z^s\right) + {}^r\mathscr{D}^s_{\mathrm{v}}\left({}^r z^s\right)\right)
$$

with

$$
{}^r\!\left(\mathring{z}\right)^{\mathrm{I}} := \begin{cases} {}^{3-r}\!(\!(z^0)\!) & \text{if } \mathrm{I}_1 = 1 \\ z^0 & \text{if } \mathrm{I}_1 = 2 \end{cases}, \ \forall\, z = \left(z^0, z^1, z^2\right) \in {}^r\mathscr{U}. \tag{35}
$$

Note that therefore friction appears only when $\mathrm{I}_2 = 1$ in the case of beams.

Thus taking advantage of the concept of multi-valued operators, we introduce the following operator ${}^r\mathscr{A}^{\mathrm{I}}$:

- When $\mathrm{I}_3 = 1$:

$$\begin{cases} D\left({}^r\!\mathscr{A}^{\mathrm{I}}\right) := \\ \left\{ U=(u,v)\in {}^r\mathscr{H}^{\mathrm{I}}\,; \quad \begin{cases} \text{i) } \exists\ \tilde{v}\in {}^r\mathscr{Z}^{\mathrm{I}} \text{ s.t. } {}^r\left(\mathring{\tilde{v}}\right)^{\mathrm{I}} = v, \\ \text{ii) } \exists\ w\in {}^r\mathscr{V}^{\mathrm{I}_1} \text{ s.t. } {}^r\left\langle (u,w), \left(z, {}^r\!\left(\mathring{z}\right)^{\mathrm{I}}\right)\right\rangle^{\mathrm{I}} + {}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{f}}(z+\tilde{v}) - {}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{f}}(\tilde{v}) \\ \qquad + {}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{v}}(z+\tilde{v}) - {}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{v}}(\tilde{v}) \geq 0,\ \forall\ z\in {}^r\mathscr{Z}^{\mathrm{I}} \end{cases} \right\}, \\ -{}^r\!\mathscr{A}^{\mathrm{I}}\, U = \{(\tilde{v}, w) \text{ satisfying i) and ii)}\}. \end{cases}$$

- When $\mathrm{I}_3 = 2$:

$$\begin{cases} D\left({}^r\!\mathscr{A}^{\mathrm{I}}\right) := {}^r\mathscr{U}\times\{0\}, \\ -{}^r\!\mathscr{A}^{\mathrm{I}}\, U = \{0\}\times {}^r\mathscr{V}^{\mathrm{I}_1}, \end{cases}$$

and the very definition of ${}^r\!\mathscr{A}^{\mathrm{I}}$ implies:

Proposition 8. *The operator ${}^r\!\mathscr{A}^{\mathrm{I}}$ is maximal monotone and for all ${}^r\psi = ({}^r\psi_u, {}^r\psi_v)$ in ${}^r\mathscr{H}^{\mathrm{I}}$, when* $\mathrm{I}_3 = 1$*:*

$$\begin{cases} {}^r\bar{U}^{\mathrm{I}} = \left({}^r\bar{u}^{\mathrm{I}}, {}^r\bar{v}^{\mathrm{I}}\right) \text{ s.t.} \\ {}^r\bar{U}^{\mathrm{I}} + {}^r\!\mathscr{A}^{\mathrm{I}}\, {}^r\bar{U}^{\mathrm{I}} \ni {}^r\psi \end{cases} \iff \begin{cases} \left({}^r\bar{u}^{\mathrm{I}}, {}^r\bar{v}^{\mathrm{I}}\right) = \left({}^r\bar{z} + {}^r\psi_u, {}^r\!\left(\mathring{\bar{z}}\right)^{\mathrm{I}}\right), \\ \text{where } {}^r\bar{z} \text{ is the unique minimizer on } {}^r\mathscr{Z}^{\mathrm{I}} \text{ of } {}^r\mathscr{J}^{\mathrm{I}}; \\ {}^r\mathscr{J}^{\mathrm{I}}(z) := \dfrac{1}{2}\left[{}^r\left|\left(z, {}^r\!\left(\mathring{z}\right)^{\mathrm{I}}\right)\right|^{\mathrm{I}}\right]^2 + {}^r\left\langle \left({}^r\psi_u, -{}^r\psi_v\right), \left(z, {}^r\!\left(\mathring{z}\right)^{\mathrm{I}}\right)\right\rangle^{\mathrm{I}} \\ \qquad + {}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{f}}(z) + {}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{v}}(z),\ \forall\ z\in {}^r\mathscr{Z}^{\mathrm{I}} \end{cases}$$

while when $\mathrm{I}_3 = 2$*:* ${}^r\bar{U}^{\mathrm{I}} + {}^r\!\mathscr{A}^{\mathrm{I}}\, {}^r\bar{U}^{\mathrm{I}} \ni {}^r\psi \iff ({}^r\bar{u}^{\mathrm{I}}, {}^r\bar{v}^{\mathrm{I}}) = ({}^r\psi_u, 0)$.

Hence if ${}^rU^{\mathrm{I}e}(t) = ({}^ru^{\mathrm{I}e}(t), 0)$ is defined by

$$ {}^ru^{\mathrm{I}e}(t) \in {}^r\mathscr{U};\ {}^r\varphi\left({}^ru^{\mathrm{I}e}(t), w\right) = {}^rL(t)(w), \quad \forall\ w\in {}^r\mathscr{U}, \quad \forall\ t\in[0,\mathrm{T}] \tag{36}$$

we have:

Theorem 9. *Under assumptions* (H1) *to* (H4) *and*

$$ {}^rU^{\mathrm{I}0} \in {}^rU^{\mathrm{I}e}(0) + D\left({}^r\!\mathscr{A}^{\mathrm{I}}\right) \tag{H5}$$

the differential inclusion

$$({}^r\mathscr{P}^{\mathrm{I}}) \qquad \begin{cases} \dfrac{d\,{}^rU^{\mathrm{I}}}{dt} + {}^r\!\mathscr{A}^{\mathrm{I}}\left({}^rU^{\mathrm{I}} - {}^rU^{\mathrm{I}e}\right) \ni 0 \\ {}^rU^{\mathrm{I}}(0) = {}^rU^{\mathrm{I}0} \end{cases}$$

has a unique solution ${}^rU^{\mathrm{I}}$ belonging to $W^{1,\infty}(0,\mathrm{T};{}^r\mathscr{H}^{\mathrm{I}})$ and the first line of $({}^r\mathscr{P}^{\mathrm{I}})$ is satisfied almost everywhere in $(0,\mathrm{T}]$.

Note that for the singular case (i.e. when $\mathrm{I}_3 = 2$) the problem $({}^r\mathscr{P}^{\mathrm{I}})$ reduces to

$$ {}^rU^{\mathrm{I}}(t) = {}^rU^{\mathrm{I}0}, \quad {}^rU^{\mathrm{I}0} = \left({}^ru^{\mathrm{I}0}, 0\right).$$

4.2. *Convergence*

To prove the "convergence" of the solution ${}^rU^s$ to $({}^r\mathscr{P}^s)$ toward the solution ${}^rU^{\mathrm{I}}$ to $({}^r\mathscr{P}^{\mathrm{I}})$, as ${}^rU^s$ and ${}^rU^{\mathrm{I}}$ *do not inhabit in the same space* and by due account of Propositions 5 and 6, we use the framework of the Theory of Trotter of approximation of semi-groups of linear operators acting on variable spaces [1, 2]. We state:

$$ {}^r\mathscr{X}^s \text{ in } {}^r\mathscr{H}^s \text{ converges in the sense of Trotter toward } {}^r\mathscr{X}^{\mathrm{I}} \text{ in } {}^r\mathscr{H}^{\mathrm{I}} \text{ if } \lim_{s\to\bar{s}} {}^r\left|{}^r\mathscr{X}^s - {}^r\mathrm{P}^{s\mathrm{I}}\, {}^r\mathscr{X}^{\mathrm{I}}\right|^s = 0. \tag{37}$$

Propositions 5 and 6 immediately imply:

Proposition 10. *The sequence* ${}^r\mathscr{X}^s = ({}^r\mathscr{X}_u^s, {}^r\mathscr{X}_v^s)$ *in* ${}^r\mathscr{H}^s$ *converges in the sense of Trotter toward* ${}^r\mathscr{X}^{\mathrm{I}} = ({}^r\mathscr{X}_u^{\mathrm{I}}, {}^r\mathscr{X}_v^{\mathrm{I}})$ *in* ${}^r\mathscr{H}^{\mathrm{I}}$ *if and only if both limits are satisfied:*

(i) $\lim_{s\to\bar{s}} \frac{1}{\varepsilon^{2+r}} \int_{{}^r\Omega^\varepsilon} a^\varepsilon \left(e^\varepsilon\left({}^r\mathscr{X}_u^s\right) - {}^re^\varepsilon_{{}^r\mathscr{X}_u^{\mathrm{I}}}\right)\cdot\left(e^\varepsilon\left({}^r\mathscr{X}_u^s\right) - {}^re^\varepsilon_{{}^r\mathscr{X}_u^{\mathrm{I}}}\right) dx^\varepsilon = 0,$

(ii) $\lim_{s\to\bar{s}} {}^rk^s\left({}^r\mathscr{X}_v^s - {}^rV^{\varepsilon\mathrm{I}}_{{}^r\mathscr{X}_v^{\mathrm{I}}}, {}^r\mathscr{X}_v^s - {}^rV^{\varepsilon\mathrm{I}}_{{}^r\mathscr{X}_v^{\mathrm{I}}}\right) = 0.$

As $\int_{{}^r\Omega^\varepsilon} a^\varepsilon\, {}^re^\varepsilon_{{}^r\mathscr{X}_u^{\mathrm{I}}}\cdot {}^re^\varepsilon_{{}^r\mathscr{X}_u^{\mathrm{I}}}\, dx^\varepsilon = \varepsilon^{2+r}\int_\Omega a\, {}^re_{{}^r\mathscr{X}_u^{\mathrm{I}}}\cdot {}^re_{{}^r\mathscr{X}_u^{\mathrm{I}}}\, dx$*, this notion of convergence is the appropriate one from the mechanical point of view: a convergence result of* relative energetic gaps measured on the physical structure *(the only one which has a meaning because the total mechanical energies are going to zero!) between the state* ${}^r\mathscr{X}^s$ *and the image on the genuine physical configuration* ${}^r\Omega^\varepsilon$ *of the limit state* ${}^r\mathscr{X}^{\mathrm{I}}$*.*

Thus according to a non-linear extension of Trotter theory [8], our key result of convergence:

Theorem 11. *Under assumptions* (H1) *to* (H5) *and*

$$\lim_{s\to\bar{s}} {}^r\left|{}^r\mathrm{P}^{s\mathrm{I}}\, {}^rU^{\mathrm{I}0} - {}^rU^{s0}\right|^s = 0 \tag{H6}$$

the solution ${}^rU^s$ *to* $({}^r\mathscr{P}^s)$ *converges to the solution* ${}^rU^{\mathrm{I}}$ *to* $({}^r\mathscr{P}^{\mathrm{I}})$ *in the sense that* $\lim_{s\to\bar{s}} {}^r|{}^r\mathrm{P}^{s\mathrm{I}}\, {}^rU^{\mathrm{I}}(t) - {}^rU^s(t)|^s = 0$ *uniformly on* $[0,T]$*. In addition,* $\lim_{s\to\bar{s}} {}^r|{}^rU^s(t)|^s = {}^r|{}^rU^{\mathrm{I}}(t)|^{\mathrm{I}}$ *uniformly on* $[0,T]$*.*

stems from the definitions (11) and (36) of ${}^ru^{se}$ and ${}^ru^{\mathrm{I}e}$, their time regularities and the following result:

Proposition 12. *There hold:*

(i) $\lim_{s\to\bar{s}} {}^r\left|{}^r\mathrm{P}^{s\mathrm{I}}\left(\mathbb{I}+{}^r\mathscr{A}^{\mathrm{I}}\right)^{-1}\psi-(\mathbb{I}+{}^r\mathscr{A}^s)^{-1}\,{}^r\mathrm{P}^{s\mathrm{I}}\psi\right|^s=0, \quad \forall\,\psi=(\psi_u,\psi_v)\in{}^r\mathscr{H}^{\mathrm{I}}$,

(ii) $\lim_{s\to\bar{s}} {}^r\left|{}^r\mathrm{P}^{s\mathrm{I}}\,{}^rU^{\mathrm{I}e}(t)-{}^rU^{se}(t)\right|^s=0, \quad \forall\,t\in[0,\mathrm{T}]$.

Proof.
(i) Proposition 1 implies that ${}^r\overline{U}^s=({}^r\bar{u}^s,{}^r\bar{v}^s):=(\mathbb{I}+{}^r\mathscr{A}^s)^{-1}\,{}^r\mathrm{P}^{s\mathrm{I}}\psi$ is such that ${}^r\bar{u}^s={}^r\bar{v}^s+{}^r\mathrm{P}_u^{s\mathrm{I}}\psi_u$ and ${}^r\bar{v}^s$ is the unique minimizer on ${}^r\mathscr{U}^s$ of ${}^r\widetilde{\mathcal{J}}^s$ defined by

$${}^r\widetilde{\mathcal{J}}^s(v)=\frac{1}{2}\left[{}^r|(v,v)|^s\right]^2+\int_\Omega a\,{}^re_{\psi_u}\cdot{}^re\left(\varepsilon,{}^r\mathscr{S}_\varepsilon\,v\right)dx+{}^rk^{\mathrm{I}_1}\left(-\psi_v,{}^r\mathscr{S}_\varepsilon\,v\right)+{}^r\mathscr{D}_{\mathrm{f}}^s(v)+{}^r\mathscr{D}_{\mathrm{v}}^s(v),\ \forall\,v\in{}^r\mathscr{U}^s.$$

Hence ${}^r\bar{v}^s$ is bounded in ${}^r\mathscr{U}^s$ and ${}^r\mathscr{V}^s$. According to Propositions 7 and 8, there exist ${}^rv^*$ in ${}^r\mathscr{Z}^{\mathrm{I}}$ and a not relabeled subsequence such that ${}^re(\varepsilon,{}^r\mathscr{S}_\varepsilon\,{}^r\bar{v}^s)$ weakly converges in $L^2(\Omega,\mathbb{S}^3)$ toward ${}^re_{{}^rv^*}$ and

$${}^r\mathcal{J}^{\mathrm{I}}\left({}^rv^*\right)\le\varliminf_{s\to\bar{s}}{}^r\widetilde{\mathcal{J}}^s\left({}^r\bar{v}^s\right).$$

To prove that the entire sequence converges toward ${}^re_{{}^r\bar{z}}$ with ${}^r\bar{z}$ the unique minimizer of ${}^r\mathcal{J}^{\mathrm{I}}$ on ${}^r\mathscr{Z}^{\mathrm{I}}$ and

$${}^r\mathcal{J}^{\mathrm{I}}\left({}^r\bar{z}\right)=\lim_{s\to\bar{s}}{}^r\widetilde{\mathcal{J}}^s\left({}^r\bar{v}^s\right),\quad {}^r\left|\left({}^r\bar{z},{}^r(\mathring{\bar{z}})^{\mathrm{I}}\right)\right|^{\mathrm{I}}=\lim_{s\to\bar{s}}{}^r\left|\left({}^r\bar{v}^s,{}^r\bar{v}^s\right)\right|^s, \tag{38}$$

it remains to show that for all $z=(z^0,z^1,z^2)$ in ${}^r\mathscr{Z}^{\mathrm{I}}$ such that z^i, $i=1,2$, belongs to ${}^r\mathscr{U}_{\mathrm{c}}$ there exists ${}^rz^s$ in ${}^r\mathscr{U}_{\mathrm{c}}$ such that ${}^re(\varepsilon,{}^r\mathscr{S}_\varepsilon\,{}^rz^s)$ weakly converges in $L^2(\Omega,\mathbb{S}^3)$ toward ${}^re_z$ with

$$\begin{cases}\varlimsup_{s\to\bar{s}}{}^r\left|\left({}^rz^s,{}^rz^s\right)\right|^s\le{}^r\left|\left(z,{}^r(\mathring{z})^{\mathrm{I}}\right)\right|^{\mathrm{I}},\\ \varlimsup_{s\to\bar{s}}{}^r\mathscr{D}_{\mathrm{f}}^s\left({}^rz^s\right)\le{}^r\mathscr{D}_{\mathrm{f}}^{\mathrm{I}}(z),\\ \varlimsup_{s\to\bar{s}}{}^r\mathscr{D}_{\mathrm{v}}^s\left({}^rz^s\right)\le{}^r\mathscr{D}_{\mathrm{v}}^{\mathrm{I}}(z),\\ \varlimsup_{s\to\bar{s}}{}^r\widetilde{\mathcal{J}}^s\left({}^rz^s\right)\le{}^r\mathcal{J}^{\mathrm{I}}(z).\end{cases} \tag{39}$$

which is achieved by

$${}^rz^s:=\left({}^r\mathscr{S}_\varepsilon\right)^{-1}\left(z^0+\sum_{i=1}^{2}\varepsilon^iz^i\right).$$

(ii) As ${}^rU^{se}(t)$ and ${}^rU^{\mathrm{I}e}(t)$ are the unique minimizers of

$$\frac{1}{2}\left[{}^r\left|(\cdot,\cdot)\right|^s\right]^2-{}^rL^\varepsilon(t)\quad\text{and}\quad\frac{1}{2}\left[{}^r\left|(\cdot,\cdot)\right|^{\mathrm{I}}\right]^2-{}^rL(t),$$

respectively, it suffices to use the preceding result i) by simply replacing the linear forms $\int_\Omega a\,{}^re_{\psi_u}\cdot{}^re(\varepsilon,{}^r\mathscr{S}_\varepsilon\cdot)\,dx$, $\varphi(\psi_u,\cdot)$ by ${}^rL^\varepsilon(t)$, ${}^rL(t)$, respectively, and make $\rho=\mu=0$, $\psi_v=0$. □

5. Conclusive remarks and proposal of an "asymptotic model"

According to each value of I in $\{1,2\}\times\{1,2,3\}\times\{1,2\}$ we give a more explicit way of writing $({}^r\mathscr{P}^{\mathrm{I}})$ in the form of variational equations $({}^r\mathrm{P}^{\mathrm{I}})$ posed over the abstract domain Ω. Their "mechanical interpretation" is given right after. We recall that the space ${}^r\mathscr{Z}^{\mathrm{I}}$ of virtual limit admissible generalized velocities and the limit global potentials of dissipations ${}^r\mathscr{D}_{\mathrm{f}}^{\mathrm{I}}$ and ${}^r\mathscr{D}_{\mathrm{v}}^{\mathrm{I}}$ are defined in (31)-(34) while the "limit loading" ${}^rL(t)$ satisfies:

$${}^rL(t)(v):=\int_{{}^r\Omega}{}^rf(x,t)\cdot v^0(x,t)\,dx+\int_{{}^r\Gamma_{\mathrm{N}}}{}^rg(x,t)\cdot v^0(x,t)\,d\mathscr{H}_2,\ \forall\,v=\left(v^0,v^1,v^2\right)\in{}^r\mathscr{Z}^{\mathrm{I}},$$
$$\left({}^rf(\cdot,t),{}^rg(\cdot,t)\right)\in L^2\left(\Omega,\mathbb{R}^3\right)\times L^2\left({}^r\Gamma_{\mathrm{N}},\mathbb{R}^3\right). \tag{40}$$

Let ${}^1\langle\delta\rangle := \int_{-1}^{1} \delta(\widehat{x},x_3)\,dx_3$ and ${}^2\langle\delta\rangle := \int_\omega \delta(\widehat{x},x_3)\,d\widehat{x}$. We denote the time derivative by an upper dot and consider the following initial conditions:

$$
{}^r u^{\mathrm{I}}(0) = {}^r \mathrm{u}^{\mathrm{I}0} = \left({}^r\mathrm{u}^{\mathrm{I}0,0}, {}^r\mathrm{u}^{\mathrm{I}0,1}, {}^r\mathrm{u}^{\mathrm{I}0,2}\right), \quad {}^r v^{\mathrm{I}}(0) = {}^r\mathrm{v}^{\mathrm{I}0}, \quad \forall\,\mathrm{I}. \tag{41}
$$

In the case of plates we use the equivalent characterization of V_{KL}:

$$
V_{\mathrm{KL}} := \Big\{ w \in H^1\left(\Omega,\mathbb{R}^3\right); \exists\left(w^{\mathrm{M}}, w^{\mathrm{F}}\right) \in H^1\left(\omega;\mathbb{R}^2\right)\times H^2(\omega) \text{ s.t. } \widehat{w}(x) = w^{\mathrm{M}}(\widehat{x}) - x_3 \widehat{\nabla} w^{\mathrm{F}}(\widehat{x}), w_3(x) = w^{\mathrm{F}}(\widehat{x}) \Big\}, \tag{42}
$$

from which we infer

$$
{}^1u^{\mathrm{I}0\mathrm{M}} = 0,\ {}^1u^{\mathrm{I}0\mathrm{F}} = 0 \text{ and } \partial_\nu\, {}^1u^{\mathrm{I}0\mathrm{F}} = 0 \text{ on } \gamma_{\mathrm{D}},\ {}^1u^{\mathrm{I}0\mathrm{M}}\cdot\nu = 0 \text{ and } \partial_\nu\, {}^1u^{\mathrm{I}0\mathrm{F}} = 0 \text{ on } \gamma_{\mathrm{C}}, \tag{43}
$$

where ν is the unit outer normal vector along $\partial\omega$ and $\partial\nu := \nu\cdot\widehat{\nabla}$ is the normal derivative. These relations are also valid for all z^0 such that $z=(z^0,z^1,z^2)$ belongs to ${}^1\mathcal{Z}^{\mathrm{I}}$ with moreover

$$
z^{0\mathrm{F}} = 0 \text{ in } \omega \text{ if } \mathrm{I}_1 = 2,\ z^{0\mathrm{F}} = 0 \text{ if } \mathrm{I}_2 = 2,3,\ z^{0\mathrm{M}} = 0 \text{ on } \Gamma_{\mathrm{C}} \text{ if } \mathrm{I}_2 = 3, \tag{44}
$$

The limit problem for thin plates reads as:

$$
({}^1\mathrm{P}^{\mathrm{I}})\begin{cases}
\mathrm{I}=(1,1,1):\quad \exists\,\zeta\in\partial\,{}^1\mathcal{D}_{\mathrm{V}}\left({}^1e_{1_{\dot u^{\mathrm{I}}}}\right) \text{ and } \exists\,\xi\in\partial\phi_p\left({}^1\dot u^{\mathrm{I}0\mathrm{F}}\right) \text{ s.t.}\\
\overline{\rho}^1\int_\omega {}^1\langle\delta\rangle\,{}^1\ddot u^{\mathrm{I}0\mathrm{F}} z^{0\mathrm{F}}\,d\widehat{x} + \int_\Omega\left[a\,{}^1e_{1_{u^{\mathrm{I}}}} + \overline{b}^1\zeta\right]\cdot{}^1e_z\,dx + 2\,{}^1\overline{\mu}^1\int_{\gamma_{\mathrm{C}}}\xi z^{0\mathrm{F}}\,d\mathcal{H}_1 = {}^1L(t)(z^0),\\
\qquad\qquad \forall\, z=\left(z^0,z^1,z^2\right)\in{}^1\mathcal{Z}^{\mathrm{I}},\\[1em]
\mathrm{I}=(1,2,1):\exists\,\zeta\in\partial\,{}^1\mathcal{D}_{\mathrm{V}}\left({}^1e_{1_{\dot u^{\mathrm{I}}}}\right) \text{ and } \exists\,\xi\in\partial\phi_p\left({}^1\dot u_T^{\mathrm{I}0\mathrm{M}}\right) \text{ s.t.}\\
\overline{\rho}^1\int_\omega {}^1\langle\delta\rangle\,{}^1\ddot u^{\mathrm{I}0\mathrm{F}} z^{0\mathrm{F}}\,d\widehat{x} + \int_\Omega\left[a\,{}^1e_{1_{u^{\mathrm{I}}}} + \overline{b}^1\zeta\right]\cdot{}^1e_z\,dx + 2\,{}^1\overline{\mu}^2\int_{\gamma_{\mathrm{C}}}\xi\cdot z^{0\mathrm{M}}\,d\mathcal{H}_1 = {}^1L(t)(z^0),\ \forall\, z\in{}^1\mathcal{Z}^{\mathrm{I}},\\
{}^1u^{\mathrm{I}0\mathrm{F}}(t) = {}^1\mathrm{u}^{\mathrm{I}0,0\mathrm{F}} \text{ on } \gamma_{\mathrm{C}},\ \forall\, t\in[0,\mathrm{T}],\\[1em]
\mathrm{I}=(1,3,1):\quad \exists\,\zeta\in\partial\,{}^1\mathcal{D}_{\mathrm{V}}\left({}^1e_{1_{\dot u^{\mathrm{I}}}}\right) \text{ s.t.}\\
\overline{\rho}^1\int_\omega {}^1\langle\delta\rangle\,{}^1\ddot u^{\mathrm{I}0\mathrm{F}} z^{0\mathrm{F}}\,d\widehat{x} + \int_\Omega\left[a\,{}^1e_{1_{u^{\mathrm{I}}}} + \overline{b}^1\zeta\right]\cdot{}^1e_z\,dx = {}^1L(t)(z^0),\ \forall\, z\in{}^1\mathcal{Z}^{\mathrm{I}},\\
{}^1u^{\mathrm{I}0}(t) = {}^1\mathrm{u}^{\mathrm{I}0,0} \text{ on } {}^1\Gamma_{\mathrm{C}},\ \forall\, t\in[0,\mathrm{T}],\\[1em]
\mathrm{I}=(2,1,1):\quad \exists\,\zeta\in\partial\,{}^1\mathcal{D}_{\mathrm{V}}\left({}^1e_{1_{\dot u^{\mathrm{I}}}}\right) \text{ s.t.}\\
\overline{\rho}^2\int_\omega {}^1\langle\delta\rangle\,{}^1\ddot u^{\mathrm{I}0\mathrm{M}}\cdot z^{0\mathrm{M}}\,d\widehat{x} + \int_\Omega\left[a\,{}^1e_{1_{u^{\mathrm{I}}}} + \overline{b}^1\zeta\right]\cdot{}^1e_z\,dx = {}^1L(t)(z^0),\ \forall\, z\in{}^1\mathcal{Z}^{\mathrm{I}},\\
{}^1u^{\mathrm{I}0\mathrm{F}}(t) = {}^1\mathrm{u}^{\mathrm{I}0,0\mathrm{F}},\ \forall\, t\in[0,\mathrm{T}],\\[1em]
\mathrm{I}=(2,2,1):\quad \exists\,\zeta\in\partial\,{}^1\mathcal{D}_{\mathrm{V}}\left({}^1e_{1_{\dot u^{\mathrm{I}}}}\right) \text{ and } \exists\,\xi\in\partial\phi_p\left({}^1\dot u_T^{\mathrm{I}0\mathrm{M}}\right) \text{ s.t.}\\
\overline{\rho}^2\int_\omega {}^1\langle\delta\rangle\,{}^1\ddot u^{\mathrm{I}0\mathrm{M}}\cdot z^{0\mathrm{M}}\,d\widehat{x} + \int_\Omega\left[a\,{}^1e_{1_{u^{\mathrm{I}}}} + \overline{b}^1\zeta\right]\cdot{}^1e_z\,dx + 2\,{}^1\overline{\mu}^2\int_{\gamma_{\mathrm{C}}}\xi\cdot z^{0\mathrm{M}}\,d\mathcal{H}_1 = {}^1L(t)(z^0),\ \forall\, z\in{}^1\mathcal{Z}^{\mathrm{I}},\\
{}^1u^{\mathrm{I}0\mathrm{F}}(t) = {}^1\mathrm{u}^{\mathrm{I}0,0\mathrm{F}},\ \forall\, t\in[0,\mathrm{T}],\\[1em]
\mathrm{I}=(2,3,1):\quad \exists\,\zeta\in\partial\,{}^1\mathcal{D}_{\mathrm{V}}\left({}^1e_{1_{\dot u^{\mathrm{I}}}}\right) \text{ s.t.}\\
\overline{\rho}^2\int_\omega {}^1\langle\delta\rangle\,{}^1\ddot u^{\mathrm{I}0\mathrm{M}}\cdot z^{0\mathrm{M}}\,d\widehat{x} + \int_\Omega\left[a\,{}^1e_{1_{u^{\mathrm{I}}}} + \overline{b}^1\zeta\right]\cdot{}^1e_z\,dx = {}^1L(t)(z^0),\ \forall\, z\in{}^1\mathcal{Z}^{\mathrm{I}},\\
{}^1u^{\mathrm{I}0\mathrm{F}}(t) = {}^1\mathrm{u}^{\mathrm{I}0,0\mathrm{F}},\ \forall\, t\in[0,\mathrm{T}],\\
{}^1u^{\mathrm{I}0\mathrm{M}}(t) = {}^1\mathrm{u}^{\mathrm{I}0,0\mathrm{M}} \text{ on } \gamma_{\mathrm{C}},\ \forall\, t\in[0,\mathrm{T}],\\[1em]
\mathrm{I}_3 = 2:\quad {}^1u^{\mathrm{I}}(t) = {}^1\mathrm{u}^{\mathrm{I}0},\ {}^1v^{\mathrm{I}}(t) = 0.
\end{cases}
$$

As for the beams, it is convenient to use the equivalent characterization of V_{BN}:

$$V_{\mathrm{BN}} := \Big\{ w \in H^1\left(\Omega, \mathbb{R}^3\right); \exists\left(c_w, w^{\mathrm{T}}, w^{\mathrm{L}}\right) \in \mathbb{R} \times H^2\left(-1,1;\mathbb{R}^2\right) \times H^1(-1,1) \text{ s.t. } \widehat{w}(x) = w^{\mathrm{T}}(x_3) + c_w x^R,\ w_3(x) = w^{\mathrm{L}}(x_3) - \widehat{x} \cdot \frac{dw^{\mathrm{T}}}{dx_3}(x_3) \Big\} \tag{45}$$

which implies

$$\begin{cases} {}^2u^{\mathrm{I0L}}(-1) = 0,\ {}^2u^{\mathrm{I0T}}(-1) = 0,\ \dfrac{d}{dx_3}{}^2u^{\mathrm{I0T}}(-1) = 0,\ c_{{}^2u^{\mathrm{I0}}} = 0, \\ {}^2u^{\mathrm{I0L}}(1) = 0,\ \dfrac{d}{dx_3}{}^2u^{\mathrm{I0T}}(1) = 0. \end{cases} \tag{46}$$

These relations are also valid for all z^0 such that $z = (z^0, z^1, z^2)$ belongs to ${}^2\mathcal{Z}^{\mathrm{I}}$ with moreover:

$$z^{0\mathrm{T}} = 0 \ \text{ if } \mathrm{I}_1 = 2, \quad z^{0\mathrm{T}}(1) = 0 \ \text{ if } \mathrm{I}_2 = 2,3. \tag{47}$$

The limit problem for slender beams then reads as:

$$({}^2\mathsf{P}^{\mathrm{I}}) \begin{cases} \mathrm{I} = (1,1,1): \quad \exists\, \zeta \in \partial\, {}^2\mathcal{D}_{\mathrm{v}}\left({}^2e_{{}^2\dot{u}^{\mathrm{I}}}\right) \text{ and } \exists\, \xi \in \partial\phi_p\left({}^2\dot{u}^{\mathrm{I0T}}(1)\right) \text{ s.t.} \\ \bar{\rho}^1 \int_{-1}^1 {}^2\langle\delta\rangle\, {}^2\ddot{u}^{\mathrm{I0T}} \cdot z^{0\mathrm{T}}\, dx_3 + \int_\Omega \left[a\, {}^2e_{{}^2u^{\mathrm{I}}} + \bar{b}^1\zeta\right] \cdot {}^2e_z\, dx \\ \qquad\qquad + {}^2\bar{\mu}^1 |\omega|\, \xi \cdot z^{0\mathrm{T}}(1) = {}^2L(t)(z^0),\ \forall\, z \in {}^2\mathcal{Z}^{\mathrm{I}}, \\ \\ \mathrm{I} = (1,2,1) \text{ or } (1,3,1): \exists\, \zeta \in \partial\, {}^2\mathcal{D}_{\mathrm{v}}\left({}^2e_{{}^2\dot{u}^{\mathrm{I}}}\right) \text{ s.t.} \\ \bar{\rho}^1 \int_{-1}^1 {}^2\langle\delta\rangle\, {}^2\ddot{u}^{\mathrm{I0T}} \cdot z^{0\mathrm{T}}\, dx_3 + \int_\Omega \left[a\, {}^2e_{{}^2u^{\mathrm{I}}} + \bar{b}^1\zeta\right] \cdot {}^2e_z\, dx = {}^2L(t)(z^0),\ \forall\, z \in {}^2\mathcal{Z}^{\mathrm{I}}, \\ \\ \mathrm{I} = (2,1,1) \text{ or } (2,2,1): \quad \exists\, \zeta \in \partial\, {}^2\mathcal{D}_{\mathrm{v}}\left({}^2e_{{}^2\dot{u}^{\mathrm{I}}}\right) \text{ s.t.} \\ \bar{\rho}^2 \int_{-1}^1 {}^2\langle\delta\rangle\, {}^2\ddot{u}^{\mathrm{I0L}} z^{0\mathrm{L}}\, dx_3 + \int_\Omega \left[a\, {}^2e_{{}^2u^{\mathrm{I}}} + \bar{b}^1\zeta\right] \cdot {}^2e_z\, dx = {}^2L(t)(z^0),\ \forall\, z \in {}^2\mathcal{Z}^{\mathrm{I}}, \\ {}^2u^{\mathrm{I0T}}(t) = {}^2\mathrm{u}^{\mathrm{I0,0T}},\ \forall\, t \in [0,\mathrm{T}], \\ \\ \mathrm{I} = (2,3,1): \quad \exists\, \zeta \in \partial\, {}^2\mathcal{D}_{\mathrm{v}}\left({}^2e_{{}^2\dot{u}^{\mathrm{I}}}\right) \text{ s.t.} \\ \bar{\rho}^2 \int_{-1}^1 {}^2\langle\delta\rangle\, {}^2\ddot{u}^{\mathrm{I0L}} z^{0\mathrm{L}}\, dx_3 + \int_\Omega \left[a\, {}^2e_{{}^2u^{\mathrm{I}}} + \bar{b}^1\zeta\right] \cdot {}^2e_z\, dx = {}^2L(t)(z^0),\ \forall\, z \in {}^2\mathcal{Z}^{\mathrm{I}}, \\ {}^2u^{\mathrm{I0T}}(t) = {}^2\mathrm{u}^{\mathrm{I0,0T}},\ \forall\, t \in [0,\mathrm{T}], \\ {}^2u^{\mathrm{I0L}}(t)(1) = {}^2\mathrm{u}^{\mathrm{I0,0L}}(1),\ \forall\, t \in [0,\mathrm{T}], \\ \\ \mathrm{I}_3 = 2: \quad {}^2u^{\mathrm{I}}(t) = {}^2\mathrm{u}^{\mathrm{I0}},\ {}^2v^{\mathrm{I}}(t) = 0. \end{cases}$$

Even if $({}^r\mathsf{P}^{\mathrm{I}})$ involves *abstract* fields defined in an "abstract plate" or an "abstract beam" occupying Ω, we will use the language of Mechanics to comment it. For almost all t in $[0,\mathrm{T}]$ there exists an element ${}^r\sigma^{\mathrm{I}}$ in $L^2(\Omega, \mathbb{S}^3)$ - a kind of "stress field" - such that

$${}^r\sigma^{\mathrm{I}} \in a\, e_{{}^ru^{\mathrm{I}}} + \bar{b}^{\mathrm{I}_3}\, \partial\, {}^r\mathcal{D}_{\mathrm{v}}^{\mathrm{I}}\left(e_{{}^r\dot{u}^{\mathrm{I}}}\right), \tag{48}$$

which makes it possible to express just below the "equations of motion" for both plates and beams (of course only if $\mathrm{I}_3 = 1$) subjected to smooth enough loading such that ${}^1g_3$ vanishes on $\partial\gamma_{\mathrm{N}} \times (-1,1)$ with $\gamma_{\mathrm{N}} = \partial\omega \setminus (\gamma_{\mathrm{D}} \cup \gamma_{\mathrm{C}})$. In the case of plates, we follow [5] and denote by ${}^1N^{\mathrm{I}}$ and ${}^1M^{\mathrm{I}}$ the stress resultant and the bending moment, respectively. In the case of beams, ${}^2N^{\mathrm{I}}$, ${}^2M^{\mathrm{I}}$ and ${}^2T^{\mathrm{I}}$ respectively stand for the axial normal force, the bending moment and the shear force (see [7]). They satisfy the equations below in which their dynamic ('dyn') and friction ('f') components are subscripted.

Plates ($r = 1$):

- ${}^1\sigma^{\mathrm{I}}_{i3} = {}^1\sigma^{\mathrm{I}}_{3i} = 0$,
- ${}^1N^{\mathrm{I}}_{\alpha\beta} := \int_{-1}^{+1} {}^1\sigma^{\mathrm{I}}_{\alpha\beta}\, dx_3$,

  ${}^1N^{\mathrm{I}}_{\mathrm{dyn}} := \bar{\rho}^2\, {}^1\langle\delta\rangle\, {}^1\ddot{u}^{\mathrm{I0M}}$ if $\mathrm{I} \in \{2\} \times \{1,2,3\} \times \{1\}$, ${}^1N^{\mathrm{I}}_{\mathrm{dyn}} = 0$ if not,

  ${}^1N^{\mathrm{I}}_{\mathrm{f}} := 2\, {}^1\bar{\mu}^{\mathrm{I}}\xi,\ \xi \in \partial\phi_p\big({}^1\dot{u}^{\mathrm{I0M}}_T\big)$ if $\mathrm{I} \in \{1,2\} \times \{2\} \times \{1\}$, ${}^1N^{\mathrm{I}}_{\mathrm{f}} = 0$ if not,
- ${}^1M^{\mathrm{I}}_{\alpha\beta} := \int_{-1}^{+1} x_3\, {}^1\sigma^{\mathrm{I}}_{\alpha\beta}\, dx_3$,

  ${}^1M^{\mathrm{I}}_{\mathrm{dyn}} := \bar{\rho}^1\, {}^1\langle\delta\rangle\, {}^1\ddot{u}^{\mathrm{I0F}}$ if $\mathrm{I} \in \{1\} \times \{1,2,3\} \times \{1\}$, ${}^1M^{\mathrm{I}}_{\mathrm{dyn}} = 0$ if not,

  ${}^1M^{\mathrm{I}}_{\mathrm{f}} := 2\, {}^1\bar{\mu}^1\xi,\ \xi \in \partial\phi_p\big({}^1\dot{u}^{\mathrm{I0F}}\big)$ if $\mathrm{I} = (1,1,1)$, ${}^1M^{\mathrm{I}}_{\mathrm{f}} = 0$ if not,
- $-\partial_\beta\, {}^1N^{\mathrm{I}}_{\alpha\beta} = \int_{-1}^{+1} {}^1\!f_\alpha\, dx_3 + {}^1\!g_\alpha(\hat{x},1) + {}^1\!g_\alpha(\hat{x},-1) - {}^1N^{\mathrm{I}}_{\mathrm{dyn}}$ in ω,

  ${}^1N^{\mathrm{I}}_{\alpha\beta}\cdot\nu_\beta = \int_{-1}^{+1} {}^1\!g_\alpha\, dx_3$ on γ_{N}, ${}^1N^{\mathrm{I}}_{\alpha\beta}\nu_\beta\tau_\alpha = -\,{}^1N^{\mathrm{I}}_{\mathrm{f}}$ on γ_{C},

  ${}^1u^{\mathrm{I0M}}\cdot\nu = 0$ on γ_{C}, ${}^1u^{\mathrm{I0M}} = 0$ on γ_{D},
- $$-\partial_{\alpha\beta}\, {}^1M^{\mathrm{I}}_{\alpha\beta} = \int_{-1}^{+1} {}^1\!f_3\, dx_3 + {}^1\!g_3(\hat{x},1) + {}^1\!g_3(\hat{x},-1) + \int_{-1}^{+1} x_3\partial_\alpha\, {}^1\!f_\alpha\, dx_3 + \partial_\alpha\, {}^1\!g_\alpha(\hat{x},1) - \partial_\alpha\, {}^1\!g_\alpha(\hat{x},-1) - {}^1M^{\mathrm{I}}_{\mathrm{dyn}} \quad \text{in } \omega,$$

  $${}^1M^{\mathrm{I}}_{\alpha\beta}\nu_\alpha\nu_\beta = \int_{-1}^{+1} x_3\, {}^1\!g_\alpha\nu_\alpha\, dx_3 \quad \text{on } \gamma_{\mathrm{N}},$$

  $$\big(\partial_\alpha\, {}^1M^{\mathrm{I}}_{\alpha\beta}\big)\nu_\beta + \partial_\tau\big({}^1M^{\mathrm{I}}_{\alpha\beta}\nu_\beta\tau_\beta\big) = \begin{cases} -\int_{-1}^{+1} x_3\, {}^1\!f_\alpha\nu_\alpha\, dx_3 + \big({}^1\!g_\alpha(\hat{x},1) - {}^1\!g_\alpha(\hat{x},-1)\big)\nu_\alpha \\ \qquad + \int_{-1}^{+1} {}^1\!g_3\, dx_3 + \int_{-1}^{+1} x_3\partial_\tau\, {}^1\!g_\alpha\tau_\alpha\, dx_3 \quad \text{on } \gamma_{\mathrm{N}} \\ -{}^1M^{\mathrm{I}}_{\mathrm{f}} \text{ and } \partial_\nu\, {}^1u^{\mathrm{I0F}} = 0 \quad \text{on } \gamma_{\mathrm{C}}, \end{cases}$$

  ${}^1u^{\mathrm{I0F}} = \partial_\nu\, {}^1u^{\mathrm{I0F}} = 0$ on γ_{D},

where it is recalled that ν is the unit outer normal vector along $\partial\omega$, $\partial\nu$ the normal derivative $\nu\cdot\widehat{\nabla}$, $\tau = (-\nu_2, \nu_1)$ and ∂_τ the tangential derivative $\tau\cdot\widehat{\nabla}$.

Beams ($r = 2$):

- For almost all x_3 in $(-1,1)$ there exists (c_1, c_2, c_3) in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ such that $\partial_\alpha\, {}^2\sigma^{\mathrm{I}}_{\alpha\beta} = (c_1)_\beta + c_2(x^R)_\beta$, $\partial_\alpha\, {}^2\sigma^{\mathrm{I}}_{3\alpha} = c_3$ in ω, ${}^2\sigma^{\mathrm{I}}_{i\beta}\nu_\beta = 0$ on $\partial\omega$,
- ${}^2N^{\mathrm{I}} := \int_\omega {}^2\sigma^{\mathrm{I}}_{33}\, d\hat{x}$,

  ${}^2N^{\mathrm{I}}_{\mathrm{dyn}} := \bar{\rho}^2\, {}^2\langle\delta\rangle\, {}^2\ddot{u}^{\mathrm{I0L}}$ if $\mathrm{I} \in \{2\} \times \{1,2,3\} \times \{1\}$, ${}^2N^{\mathrm{I}}_{\mathrm{dyn}} = 0$ if not,
- ${}^2M^{\mathrm{I}} := \int_\omega \hat{x}\, {}^2\sigma^{\mathrm{I}}_{33}\, d\hat{x}$,

  ${}^2T^{\mathrm{I}} := \dfrac{d}{dx_3}\, {}^2M^{\mathrm{I}}$ if $\mathrm{I} = (1,1,1)$,

  ${}^2M^{\mathrm{I}}_{\mathrm{dyn}} := \bar{\rho}^1\, {}^2\langle\delta\rangle\, {}^2\ddot{u}^{\mathrm{I0T}}$ if $\mathrm{I} \in \{1\} \times \{1,2,3\} \times \{1\}$, ${}^2M^{\mathrm{I}}_{\mathrm{dyn}} = 0$ if not,

  ${}^2T^{\mathrm{I}}_{\mathrm{f}} := {}^2\bar{\mu}^1|\omega|\xi,\ \xi \in \partial\phi_p\big({}^2\dot{u}^{\mathrm{I0T}}(1)\big)$ if $\mathrm{I} = (1,1,1)$,

- $-\dfrac{d}{dx_3}\,{}^2N^{\mathrm{I}} = \displaystyle\int_\omega {}^2f_3\,d\hat{x} + \int_{\partial\omega} {}^2g_3\,d\mathcal{H}_1 - {}^2N^{\mathrm{I}}_{\mathrm{dyn}} \quad \text{in } (-1,1),$
  ${}^2u^{\mathrm{I0L}}(-1) = {}^2u^{\mathrm{I0L}}(-1) = 0,$
- $-\dfrac{d^2}{dx_3^2}\,{}^2M^{\mathrm{I}} = \displaystyle\int_\omega \left(\widehat{{}^2f} + \hat{x}\frac{\partial^2 f_3}{\partial x_3}\right) d\hat{x} + \int_{\partial\omega} \left(\widehat{{}^2g} + \hat{x}\frac{\partial^2 g_3}{\partial x_3}\right) d\mathcal{H}_1 - {}^2M^{\mathrm{I}}_{\mathrm{dyn}} \quad \text{in } (-1,1),$
  ${}^2T^{\mathrm{I}}(1) = -\displaystyle\int_\omega \hat{x}\,{}^2f_3(\hat{x},1)\,d\hat{x} - \int_{\partial\omega} \hat{x}\,{}^2g_3(\hat{x},1)\,d\mathcal{H}_1 - {}^2T^{\mathrm{I}}_{\mathrm{f}},$
  ${}^2u^{\mathrm{I0T}}(-1) = \dfrac{d}{dx_3}\,{}^2u^{\mathrm{I0T}}(-1) = \dfrac{d}{dx_3}\,{}^2u^{\mathrm{I0T}}(1) = 0,$
  ${}^2u^{\mathrm{I0T}}(1) = 0 \text{ if } \mathrm{I}_1 = 2 \text{ or } \mathrm{I}_2 = 2,3.$

We therefore clearly see how and when friction and dynamical effects do occur and notice the remarkable correspondance between reduction elements of the efforts for plates and beams of which we spared the reader a unifying writing.

So, by introducing the additional fields of displacements $({}^r u^{\mathrm{I}_1}, {}^r u^{\mathrm{I}_2})$ and introducing a generalized strain $e_{{}^r u^{\mathrm{I}}}$, the behavior of the abstract structure is viscoelastic of the same non-linear Kelvin-Voigt type as that of the material the real structure is made of. Of course when $\mathcal{D}_{\mathrm{V}}$ is a quadratic form $\frac{1}{2} b\,\dot{e}\cdot\dot{e}$ it is possible to eliminate ${}^1u^{\mathrm{I}_1}$ and ${}^1u^{\mathrm{I}_2}$ (see [14]) and to obtain a viscoelastic behavior which is no longer of Kelvin-Voigt type but rather with fading memory:

$$ {}^1\sigma^{\mathrm{I}} = a^{\mathrm{KL}} e\big({}^1u^{\mathrm{I0}}(t)\big) + b^{\mathrm{KL}} e\big({}^1\dot{u}^{\mathrm{I0}}(t)\big) + \int_0^t \kappa(t-\tau) e\big({}^1\dot{u}^{\mathrm{I0}}(\tau)\big)\,d\tau, \tag{49} $$

$$ \begin{cases} a^{\mathrm{KL}} := a_{\wedge\wedge} - a_{\wedge\perp}(a_{\perp\perp})^{-1} a_{\perp\wedge}, \\ b^{\mathrm{KL}} := b_{\wedge\wedge} - b_{\wedge\perp}(b_{\perp\perp})^{-1} b_{\perp\wedge}, \end{cases} \tag{50} $$

$a_{\wedge\wedge}$, $a_{\wedge\perp}$, $a_{\perp\wedge}$ and $a_{\perp\perp}$ stemming from the decomposition $\mathbb{S}^3 = \mathbb{S}^\wedge \oplus \mathbb{S}^\perp$:

$$ \mathbb{S}^\wedge := \{\sigma \in \mathbb{S}^3; \sigma_{i3} = 0\}, \quad \mathbb{S}^\perp := \{\sigma \in \mathbb{S}^3; \sigma_{\alpha\beta} = 0\}, \tag{51} $$

the same being done for b and one has:

$$ \begin{cases} \kappa(t)e = a_{\wedge\perp} w^e(t) + b_{\wedge\perp} \dot{w}^e(t), \\ w^e \in \mathbb{S}^\perp\,;\; a_{\perp\perp} w^e + b_{\perp\perp} \dot{w}^e = 0, \quad w^e(0) = (u^b)^\perp - (u^a)^\perp, \\ (u^a)^\perp := (a_{\perp\perp})^{-1} a_{\perp\wedge} \hat{e}, \quad (u^b)^\perp := (b_{\perp\perp})^{-1} b_{\perp\wedge} \hat{e}, \\ (\hat{e})_{\alpha\beta} := e_{\alpha\beta},\ \forall\, e \in \mathbb{S}^3. \end{cases} \tag{52} $$

The feature of the evolution is therefore the same in both cases: a juxtaposition of a dynamic evolution for a part of the displacement and of a quasi-static one (possibly static) for the other part, depending on the relative magnitude of the density and the thinness ε. However, in the case of a very high viscosity (i.e. when $\bar{b}^2 = +\infty$) the motion is frozen in the initial state. Dynamic evolution concerns the transverse component of the displacement for ρ of order ε^2 and in the case of beams (resp. plates) the longitudinal (resp. in-plane) one for ρ of order 1. For plates, as in the elastic case, the friction involves the in-plane or transverse component of the tangential velocity according to the relative magnitudes of the viscosity coefficient μ and the thickness. On the contrary, in the case of beams, friction appears only when ρ is of order ε^2 *and* when the viscosity coefficient is of order ε^2. Of course, Kelvin-Voigt viscosity reinforces the presence of additional state variables of displacement (u^1, u^2) as already mentioned in [15] in the purely (anisotropic) elastic beams case. These additional state variables also allow to maintain the short memory viscosity character.

Next we propose our simplified but accurate enough modeling not by considering $({}^r\mathscr{S}_\varepsilon)^{-1}$ ${}^ru^{\mathrm{I}0}(t)$ but by taking into account our convergence result (see Theorem 11) and the crucial Proposition 10 which leads to

$$\lim_{s\to\bar{s}} \frac{1}{\varepsilon^{2+r}} \int_{{}^r\Omega^\varepsilon} a^\varepsilon \left(e^\varepsilon\left({}^ru^s\right) - {}^re^\varepsilon_{{}^ru^{\mathrm{I}}}\right)\cdot\left(e^\varepsilon\left({}^ru^s\right) - {}^re^\varepsilon_{{}^ru^{\mathrm{I}}}\right) dx^\varepsilon = 0. \tag{53}$$

Hence as denoted in [2, 13], ${}^re^\varepsilon_{{}^ru^{\mathrm{I}}}$ is a good approximation of the strain tensor of ${}^ru^s$ in the sense that the *relative error* made by replacing $e^\varepsilon({}^ru^s)$ by ${}^re^\varepsilon_{{}^ru^{\mathrm{I}}}$ tends to zero.
As ${}^re^\varepsilon_{{}^ru^{\mathrm{I}}}$ is not necessarily the strain tensor of *a field of* ${}^r\mathscr{U}^s$, we are led to consider

$${}^ru^{\mathrm{I}s} := \left({}^r\mathscr{S}_\varepsilon\right)^{-1}\left({}^ru^{\mathrm{I}0}_\varepsilon + \sum_{i=1}^{2}\varepsilon^i\, {}^ru^{\mathrm{I}i}_\varepsilon\right),$$

where ${}^ru^{\mathrm{I}i}_\varepsilon$ is a smooth approximation in ${}^r\mathscr{U}_{\mathrm{c}}$ of ${}^ru^{\mathrm{I}i}$, $i=1,2$, which leads to

$$\lim_{s\to\bar{s}} \frac{1}{\varepsilon^{2+r}} \int_{{}^r\Omega^\varepsilon} a^\varepsilon \left(e^\varepsilon\left({}^ru^{\mathrm{I}s}\right) - {}^re^\varepsilon_{{}^ru^{\mathrm{I}}}\right)\cdot\left(e^\varepsilon\left({}^ru^{\mathrm{I}s}\right) - {}^re^\varepsilon_{{}^ru^{\mathrm{I}}}\right) dx^\varepsilon = 0. \tag{54}$$

Thus ${}^ru^{\mathrm{I}s}$ is our proposal of approximation for ${}^ru^s$. It is obtained by first solving $({}^r\mathrm{P}^{\mathrm{I}})$ which actually corresponds to a three-dimensional problem yet set on a "reasonable" fixed domain Ω and involving fields with simplified kinematics and second by means of ${}^ru^{\mathrm{I}s}$ which also involves the fixed domain Ω. It is therefore easy to implement a numerical method of approximation.

For almost all t in $[0,T]$, ${}^r\sigma_s$ defined by ${}^r\sigma_s(x) := {}^r\sigma^s({}^r\Pi^\varepsilon x)$ converges weakly in $L^2(\Omega,\mathbb{S}^3)$ toward some ${}^r\Sigma^{\mathrm{I}}$. So if ${}^r\Sigma^{\mathrm{I}\varepsilon}({}^r\Pi^\varepsilon x) := \varepsilon\, {}^r\Sigma^{\mathrm{I}}(x)$ one has:

$$\lim_{s\to\bar{s}} \frac{1}{|{}^r\Omega^\varepsilon|}\int_{{}^r\Omega^\varepsilon}\left({}^r\sigma^s - {}^r\Sigma^{\mathrm{I}\varepsilon}\right)(x^\varepsilon)\cdot q(x^\varepsilon)\,dx^\varepsilon = \lim_{s\to\bar{s}}\int_\Omega\left({}^r\sigma_s - {}^r\Sigma^{\mathrm{I}}\right)(x)\cdot q(x)\,dx = 0,\ \forall\, q\in L^2\left(\Omega,\mathbb{S}^3\right), \tag{55}$$

that is to say $\frac{1}{\varepsilon}\,{}^r\sigma^s$ 3d-(3-r)d converges toward ${}^r\Sigma^{\mathrm{I}}$, which satisfies ${}^1\Sigma^{\mathrm{I}}_{i3} = 0$ (see the Appendix).

Remark 13. Note that, as mentioned in part ii) of the proof of Proposition 12, this paper encompasses the full treatment of purely linearly elastic beams and plates in the static case (see the Appendix from (66) onward). Hence we have a good approximation of the real strain tensor in the sense of (53) and also of the real stress in the structure as obviously

$$\begin{aligned}\lim_{s\to\bar{s}} \frac{1}{\varepsilon^{2+r}} &\int_{{}^r\Omega^\varepsilon} (a^\varepsilon)^{-1}\left({}^r\sigma^s - {}^r\Sigma^{\mathrm{I}\varepsilon}\right)\cdot\left({}^r\sigma^s - {}^r\Sigma^{\mathrm{I}\varepsilon}\right)dx^\varepsilon \\ &= \lim_{s\to\bar{s}} \frac{1}{\varepsilon^{2+r}} \int_{{}^r\Omega^\varepsilon} a^\varepsilon\left(e^\varepsilon({}^ru^s) - {}^re^\varepsilon_{{}^ru^{\mathrm{I}}}\right)\cdot\left(e^\varepsilon({}^ru^s) - {}^re^\varepsilon_{{}^ru^{\mathrm{I}}}\right)dx^\varepsilon,\end{aligned} \tag{56}$$

with ${}^r\Sigma^{\mathrm{I}\varepsilon}({}^r\Pi^\varepsilon x) := \varepsilon\,{}^r\Sigma^{\mathrm{I}}(x)$ and ${}^r\Sigma^{\mathrm{I}}(x) = a(x)\,{}^re_{{}^ru^{\mathrm{I}}}$ (see also (19)-(20)), ${}^re_{{}^ru^{\mathrm{I}}}$ being the strong limit in $L^2(\Omega,\mathbb{S}^3)$ of ${}^re\big(\varepsilon, {}^r\mathscr{S}_\varepsilon({}^ru^\varepsilon)\big) = \frac{1}{\varepsilon}e^\varepsilon({}^ru^\varepsilon)$, where ${}^ru^\varepsilon$ is the solution of the elasticity problem set in ${}^r\Omega^\varepsilon$ with a loading ${}^rL^\varepsilon$ such that ${}^rL^\varepsilon(w) = \varepsilon^{2+r}\,{}^rL({}^r\mathscr{S}_\varepsilon w)$. We recall that in the case of plates ${}^1\Sigma^\varepsilon_{i3} = 0$!

Remark 14. It is worthwhile to observe that in problems concerning thin linearly elastic or nonlinear Kelvin-Voigt viscoelastic plates, the field of displacement in the real plate which occupies ${}^1\Omega^\varepsilon$ is far from a Kirchhoff–Love field and even from a Reissner–Mindlin one because $({}^1e^\varepsilon_{{}^1u^{\mathrm{I}}})_{i3}$ depends on x_3^ε even in the case of an homogeneous plate. It is the *abstract* field ${}^1u^{\mathrm{I}0}$ which does satisfy $e_{i3}({}^1u^{\mathrm{I}0}) = 0$ in Ω. We will pay particular attention to this type of properties in a forthcoming paper dedicated to the case of thin viscoelastic plates.

Similarly, in the case of beams, the field of displacement in ${}^2\Omega^\varepsilon$ is far from a Bernoulli–Navier field and even from a Timoshenko one because $({}^2e^\varepsilon_{{}^2u^{\mathrm{I}}})_{i3}$ depends on x_3^ε even in the case of an homogeneous beam. It is the *abstract* field ${}^2u^{\mathrm{I}0}$ whose components $\alpha\beta$ and $\alpha 3$ of its symmetrized gradient do vanish in Ω!

Remark 15. It is also possible to deal with the not too much realistic case $2 < p, q \leq +\infty$ by the same method, the variant being that ${}^r\mathscr{D}^s_{\mathrm{f}}$, ${}^r\mathscr{D}^s_{\mathrm{v}}$, ${}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{f}}$ and ${}^r\mathscr{D}^{\mathrm{I}}_{\mathrm{v}}$ are only lower semicontinuous functions and some trivial approximation processes are in order.

Remark 16. A more practical approach is when two other physical data concerning the magnitudes of the stiffness and of the loading are taken into account and we refer the reader to the [4, Remark 3] for its mathematical treatment.

Appendix A. 3d-(3-r)d convergence and asymptotic modeling of thin structures

We present here the tool used in the proof of Proposition 6. It is an adaptation to the case of dimension reduction problems of two-scale convergence introduced in [9, 10] for periodic homogenization.

Let H be a finite dimensional space. We recall that r takes its value in $\{1,2\}$.

Definition 17. *A sequence of functions u^ε in $L^2({}^r\Omega^\varepsilon, H)$ is said to 3d-(3-r)d converge to a limit u_0 belonging to $L^2(\Omega, H)$ if, for any ψ in $L^2(\Omega, H)$, we have:*

$$\lim_{\varepsilon\to 0} \frac{1}{|{}^r\Omega^\varepsilon|}\int_{{}^r\Omega^\varepsilon} u^\varepsilon(x^\varepsilon)\cdot\psi\big(\big({}^r\Pi^\varepsilon\big)^{-1}x^\varepsilon\big)\,dx^\varepsilon = \frac{1}{|\Omega|}\int_\Omega u_0(x)\cdot\psi(x)\,dx \tag{57}$$

where we recall $x^\varepsilon = {}^r\Pi^\varepsilon x$, a.e. $x\in\Omega$ (see (2)).

Proposition 18. *From each sequence u^ε in $L^2({}^r\Omega^\varepsilon, H)$ such that $\frac{1}{|{}^r\Omega^\varepsilon|}|u^\varepsilon|^2_{L^2({}^r\Omega^\varepsilon,H)}$ is bounded we can extract a subsequence, and there exists a limit u_0 in $L^2(\Omega,H)$ such that this subsequence 3d-(3-r)d converges to u_0 and*

$$\frac{1}{|\Omega|}|u_0|^2_{L^2(\Omega,H)} \leq \varliminf_{\varepsilon\to 0}\frac{1}{|{}^r\Omega^\varepsilon|}|u^\varepsilon|^2_{L^2({}^r\Omega^\varepsilon,H)}. \tag{58}$$

Proof. As u_ε defined by

$$u_\varepsilon(x) = u^\varepsilon\left(x^\varepsilon\right) \text{ a.e. } x\in\Omega \tag{59}$$

satisfies

$$\frac{1}{|{}^r\Omega^\varepsilon|}\int_{{}^r\Omega^\varepsilon}\left|u^\varepsilon\left(x^\varepsilon\right)\right|^2 dx^\varepsilon = \frac{1}{|\Omega|}\int_\Omega |u_\varepsilon(x)|^2\,dx \tag{60}$$

there exists a not relabeled subsequence such that u_ε weakly converges toward some u_0 in $L^2(\Omega, H)$ with:

$$\lim_{\varepsilon\to 0}\frac{1}{|{}^r\Omega^\varepsilon|}\int_{{}^r\Omega^\varepsilon} u^\varepsilon\left(x^\varepsilon\right)\cdot\psi\left(x^\varepsilon\right)dx^\varepsilon = \lim_{\varepsilon\to 0}\frac{1}{|\Omega|}\int_\Omega u_\varepsilon(x)\cdot\psi(x)\,dx = \frac{1}{|\Omega|}\int_\Omega u_0(x)\cdot\psi(x)\,dx \tag{61}$$

and consequently (58). □

This 3d-(3-r)d limit u_0 may give accurate information on the behavior of u^ε:

Proposition 19. *Let u^ε be a sequence of functions in $L^2({}^r\Omega^\varepsilon, H)$ that 3d-(3-r)d converges to a limit u_0 belonging to $L^2(\Omega,H)$. Assume that*

$$\lim_{\varepsilon\to 0}\frac{1}{|{}^r\Omega^\varepsilon|}|u^\varepsilon|^2_{L^2({}^r\Omega^\varepsilon,H)} = \frac{1}{|\Omega|}|u_0|^2_{L^2(\Omega,H)}. \tag{62}$$

Then, for any sequence v^ε which 3d-(3-r)d converges to a limit v_0 belonging to $L^2(\Omega,H)$ we have:

$$\lim_{\varepsilon\to 0}\frac{1}{|{}^r\Omega^\varepsilon|}\int_{{}^r\Omega^\varepsilon} u^\varepsilon\left(x^\varepsilon\right)\cdot v^\varepsilon\left(x^\varepsilon\right)dx^\varepsilon = \frac{1}{|\Omega|}\int_\Omega u_0(x)\cdot v_0(x)\,dx \tag{63}$$

$$\lim_{\varepsilon\to 0}\frac{1}{|{}^r\Omega^\varepsilon|}\int_{{}^r\Omega^\varepsilon}|u^\varepsilon\left(x^\varepsilon\right) - u_0^\varepsilon\left(x^\varepsilon\right)|^2\,dx^\varepsilon = 0 \tag{64}$$

with $u_0^\varepsilon(x^\varepsilon) := u_0(x)$ a.e. $x^\varepsilon\in{}^r\Omega^\varepsilon$.

Proof. It is an obvious consequence of (59), which implies that u_ε converges strongly in $L^2(\Omega, H)$, v_ε weakly, toward u_0, v_0, respectively. □

The relation (64) expresses that u_0^ε is a rather good approximation of u^ε in the sense that the *relative* gap in $L^2({}^r\Omega^\varepsilon, H)$ tends to zero:

$$\lim_{\varepsilon\to 0} \frac{\left|u^\varepsilon - u_0^\varepsilon\right|_{L^2({}^r\Omega^\varepsilon,H)}}{\left|u_0^\varepsilon\right|_{L^2({}^r\Omega^\varepsilon,H)}} = 0. \tag{65}$$

Application:

A standard problem of equilibrium of a linearly elastic thin plate ($r = 1$) or slender beam ($r = 2$) occupying ${}^r\Omega^\varepsilon$ with elasticity tensor a^ε, clamped on ${}^r\Gamma_{\mathrm{D}}^\varepsilon$ defined in (6) and submitted to a given loading represented by a continuous linear form ${}^rL^\varepsilon$ on $H^1_{{}^r\Gamma_{\mathrm{D}}^\varepsilon}({}^r\Omega^\varepsilon,\mathbb{R}^3) := \{v \in H^1({}^r\Omega^\varepsilon,\mathbb{R}^3);\ v = 0 \text{ on } {}^r\Gamma_{\mathrm{D}}^\varepsilon\}$ can be formulated as:

$$\begin{cases} \text{Find } {}^ru^\varepsilon \text{ in } H^1_{{}^r\Gamma_{\mathrm{D}}^\varepsilon}\left({}^r\Omega^\varepsilon,\mathbb{R}^3\right) \text{ such that} \\ \displaystyle\int_{{}^r\Omega^\varepsilon} a^\varepsilon\left(x^\varepsilon\right) e^\varepsilon\left({}^ru^\varepsilon\right)\cdot e^\varepsilon(v)\, dx^\varepsilon = {}^rL^\varepsilon(v),\ \forall\ v \in H^1_{{}^r\Gamma_{\mathrm{D}}^\varepsilon}\left({}^r\Omega^\varepsilon,\mathbb{R}^3\right). \end{cases} \tag{66}$$

Observe that the scaling operator ${}^r\mathscr{S}_\varepsilon$ introduced in (18) implies (see (19) and (20)):

$$e^\varepsilon(w)(x^\varepsilon) = \varepsilon\ {}^re\left(\varepsilon, {}^r\mathscr{S}_\varepsilon w\right)(x). \tag{67}$$

Denoting ${}^r\Gamma_{\mathrm{D}} := \left({}^r\Pi^\varepsilon\right)^{-1}\,{}^r\Gamma_{\mathrm{D}}^\varepsilon$ and assuming

$$\begin{cases} \exists\ \alpha > 0, \exists\ a \in L^\infty\left(\Omega, \mathrm{Lin}\left(\mathbb{S}^3\right)\right) \text{ s.t. } \alpha|e|^2 \le a(x)\, e\cdot e,\ a^\varepsilon(x^\varepsilon) = a(x), \text{ a.e. } x \in \Omega,\ \forall\ e \in \mathbb{S}^3 \\ \exists\ {}^rL_\varepsilon \in H^1_{{}^r\Gamma_{\mathrm{D}}}(\Omega,\mathbb{R}^3)' \text{ the strong dual of } H^1_{{}^r\Gamma_{\mathrm{D}}}\left(\Omega,\mathbb{R}^3\right) := \left\{w \in H^1(\Omega,\mathbb{R}^3);\ w = 0 \text{ on } {}^r\Gamma_{\mathrm{D}}\right\} \text{ such that} \\ \text{i)}\ \ {}^rL^\varepsilon(v) = \varepsilon^{2+r}\ {}^rL_\varepsilon({}^r\mathscr{S}_\varepsilon v),\ \forall\ v \in H^1_{{}^r\Gamma_{\mathrm{D}}^\varepsilon}\left({}^r\Omega^\varepsilon,\mathbb{R}^3\right) \\ \text{ii)}\ {}^rL_\varepsilon \text{ strongly converges in } H^1_{{}^r\Gamma_{\mathrm{D}}}\left(\Omega,\mathbb{R}^3\right)' \text{ toward } {}^rL \end{cases} \tag{68}$$

(which corresponds to assumption (H4) for both a^ε and the loading $(f^\varepsilon, g^\varepsilon)$), the field

$${}^ru_\varepsilon := {}^r\mathscr{S}_\varepsilon\left({}^ru^\varepsilon\right) \tag{69}$$

does satisfy:

$$\begin{cases} \text{Find } {}^ru_\varepsilon \text{ in } H^1_{{}^r\Gamma_{\mathrm{D}}}(\Omega,\mathbb{R}^3) \text{ such that} \\ \displaystyle\int_\Omega a^\varepsilon(x)\,{}^re\left(\varepsilon, {}^ru_\varepsilon\right)\cdot {}^re(\varepsilon, v)\, dx = {}^rL_\varepsilon(v), \quad \forall\ v \in H^1_{{}^r\Gamma_{\mathrm{D}}}\left(\Omega,\mathbb{R}^3\right) \end{cases} \tag{70}$$

We can replicate the proof of Proposition 6 to show that there exists some ${}^ru = ({}^ru^0, {}^ru^1, {}^ru^2)$ in ${}^r\mathscr{U}$ (see (23)) such that ${}^re(\varepsilon, {}^ru_\varepsilon)$ strongly converges toward ${}^re_{{}^ru}$ defined in (24), with

$$\text{i)}\ \begin{cases} {}^ru \in {}^r\mathscr{U}; \\ \displaystyle\int_\Omega a\ {}^re_{{}^ru}\cdot {}^re_v\, dx = {}^rL(v),\ \forall\ v \in {}^r\mathscr{U} \end{cases}$$

$$\text{ii)}\ \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{2+r}} \int_{{}^r\Omega^\varepsilon} a^\varepsilon\left({}^re^\varepsilon({}^ru^\varepsilon) - {}^re^\varepsilon_{{}^ru}\right)\cdot\left({}^re^\varepsilon({}^ru^\varepsilon) - {}^re^\varepsilon_{{}^ru}\right) dx^\varepsilon = 0,$$

$$\text{where } {}^re^\varepsilon_{{}^ru}(x^\varepsilon) := \varepsilon\,{}^re_u(x) \text{ a.e. } x^\varepsilon = {}^r\Pi^\varepsilon x \in {}^r\Omega^\varepsilon, \forall\ u \in {}^r\mathscr{U}$$

that is to say $\dfrac{1}{\varepsilon}\,{}^re^\varepsilon({}^ru^\varepsilon)$ 3d$-(3-r)$d converges toward ${}^re_{{}^ru}$ (see (P3) and (28) in Proposition 6) and the strain of the real field ${}^ru^\varepsilon$ in the domain ${}^r\Omega^\varepsilon$ is close to ${}^re^\varepsilon_{{}^ru}$ in the sense that the *relative* energetic gap tends to zero which is the only significant notion as the energy of u^ε tends to zero!

Moreover we have the same result for the stress field ${}^r\sigma^\varepsilon$, as ii) above is equivalent to:

$$\text{iii)}\ \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{2+r}} \int_{{}^r\Omega^\varepsilon} \left(a^\varepsilon\right)^{-1}\left({}^r\sigma^\varepsilon - {}^r\Sigma^\varepsilon\right)\cdot\left({}^r\sigma^\varepsilon - {}^r\Sigma^\varepsilon\right) dx^\varepsilon = 0, \quad \text{with } {}^r\Sigma^\varepsilon := a^\varepsilon\ {}^re^\varepsilon_{{}^ru}.$$

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