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Review article / *Article de synthèse*

A survey on the positive mass theorem for asymptotically flat initial data sets

Enquête sur le théorème de la masse positive pour les ensembles de données initiales asymptotiquement plats

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Abstract. In honor of Yvonne Choquet-Bruhat's 100th birthday, we present this survey on the positive mass theorem. Originating from a conjecture in general relativity regarding the ADM mass, the positive mass theorem has significantly influenced geometry and analysis over the past four decades and continued to inspire new connections. We review seminal contributions as well as recent advances, and then we focus our discussions on the equality case and the counter-examples arising from pp-wave spacetimes.

Résumé. En l'honneur du 100e anniversaire d'Yvonne Choquet-Bruhat, nous présentons cette étude sur le théorème de la masse positive. Issu d'une conjecture en relativité générale concernant la masse de l'ADM, le théorème de la masse positive a influencé de manière significative la géométrie et l'analyse au cours des quatre dernières décennies et continue d'inspirer de nouvelles connexions. Nous passons en revue les contributions fondamentales ainsi que les avancées récentes, puis nous concentrons nos discussions sur le cas de l'égalité et les contre-exemples provenant des espaces-temps d'ondes pp.

Keywords. ADM mass, ADM energy-momentum, Asymptotically flat initial data set, Dominant energy condition, pp-wave.

Mots-clés. Masse ADM, Énergie-momentum ADM, Ensemble de données initiales asymptotiquement plates, Condition d'énergie dominante, Onde pp.

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1. Introduction

I am honored to contribute this survey on the positive mass theorem in celebration of Yvonne Choquet-Bruhat's 100th birthday. I had the chance to meet her in 2008 at the Mittag-Leffler Institute while I was a PhD student participating in the *Geometry, Analysis, and General Relativity* program. She visited briefly to meet her collaborators, including Piotr Chruściel and Jim Isenberg who were key participants of the program. During tea time, we gathered at a large oval table in the sunroom. As I was just starting my research career, her warmth and friendliness left a lasting impression. We are truly fortunate to have a pioneer like Choquet-Bruhat in the general relativity community. Her groundbreaking work and kindness are an amazing example for us all and continue to inspire us today.

Choquet-Bruhat made a fundamental contribution to the Cauchy problem, ensuring that the initial value problem in general relativity is well-posed. For more details, see her survey [1]. Her research laid the groundwork for the $3+1$ formulation of general relativity (ADM formalism), which is crucial for understanding how spacetime evolves and is used in both theoretical and numerical relativity. The ADM approach views spacetime as a series of evolving three-dimensional slices from an initial data set. It simplifies Einstein's field equations into constraint and evolution equations. It also introduces ADM energy and linear momentum, which measure the total mass of an asymptotically flat spacetime. These quantities, defined on the initial data set, are essential for understanding isolated gravitational systems and allow working with just the initial data and constraint equations. We refer the reader to her beautiful book [2].

The positive mass conjecture asserts that the ADM mass is non-negative (or, more precisely, that the ADM energy-momentum is future causal) under suitable energy conditions. Choquet-Bruhat and Marsden proved a very interesting case of the positive mass theorem, for metrics close to the Euclidean metric [3], building on an idea outlined by Brill and Deser [4]. The problem gained wider attention through questions posed by Geroch in his 1975 plenary address at the Joint Mathematical Meeting [5]. This conjecture became one of the early topics in mathematical general relativity and remains central to developments in geometric analysis and general relativity. The search for and affirmation of positivity conditions under various settings has had a significant impact on mathematics, influencing areas like the Yamabe problem in conformal geometry [6].

While there were other partial results toward the positive mass conjecture, the first groundbreaking result was obtained by Schoen and Yau in 1979 [7]. For an earlier history of this problem, we refer to the introduction of [7]. Not only did they completely settle the so-called Riemannian case, but the minimal surface technique they introduced also revolutionized the field. Since then, more general versions, higher-dimensional cases, and new alternative proofs have emerged, making the topic even more dynamic in recent years. Researchers have continued to develop innovative techniques and approaches, expanding the scope and relevance of the positive mass theorem. While the physically interesting case is three spatial dimensions, there are various reasons and motivations for seeking a general statement in higher dimensions. We will provide a more comprehensive list of progress below. For now, we present the statement.

Theorem 1 (Spacetime positive mass theorem). *Let $n \geq 3$ and (M, g, k) be an n-dimensional asymptotically flat initial data set that satisfies the dominant energy condition. Suppose either $3 \leq n \leq 7$ or M is spin. Then*

$$E \geq |P|,$$

where (E, P) is the ADM energy-momentum vector of (M, g, k).

The above theorem says that the ADM energy-momentum vector (E, P) is future-directed and causal. Thus, one can define its Lorentzian norm as the *ADM mass* $m = \sqrt{E^2 - |P|^2}$, which is a spacetime invariant.

The special case of Theorem 1 where $k \equiv 0$ is known as the Riemannian case (or the time-symmetric case). In this case, $P = 0$, and the dominant energy condition implies nonnegative scalar curvature. It is also conventional to denote the ADM energy E by the ADM mass m.

Theorem 2 (Riemannian positive mass theorem). *Let $n \geq 3$ and (M, g) be an n-dimensional asymptotically flat manifold with nonnegative scalar curvature. Suppose either $3 \leq n \leq 7$ or M is spin. Then $m \geq 0$. Furthermore, $m = 0$ if and only if (M, g) is isometric to Euclidean space.*

Schoen and Yau proved this case in dimension three in 1979 and 1981 [7, 8] by introducing minimal surface techniques and extended it to dimensions less than eight [9, 10]. They argued that if the ADM mass $m < 0$, then there exists a codimension 1 area-minimizing minimal surface with the induced metric being asymptotically flat with zero mass. The area-minimizing property allows for a conformal factor that further transforms the induced metric to one with zero scalar curvature and negative mass, contradicting the Riemannian positive mass theorem in one dimension lower and ultimately conflicting with the Gauss–Bonnet Theorem when $n = 3$. Since the above argument does not address the case $m = 0$, Schoen and Yau gave a separate argument to show that if $m = 0$, then the manifold is Ricci flat and thus isometric to Euclidean space.

Lohkamp [11] observed a cut-off technique to simplify the asymptotics of an asymptotically flat manifold with negative ADM mass, showing that it can be compactified to a torus while maintaining positive scalar curvature. Consequently, the Riemannian positive mass theorem is equivalent to the torus rigidity theorem of Schoen and Yau [12]. In more recent years, there have been rapid advances and alternative proofs of the Riemannian positive mass theorem for three dimensions. Huisken and Ilmanen [13] used the inverse mean curvature flow, Li [14] employed the Ricci flow, Bray, Kazaras, Khuri, and Stern [15] considered level sets of harmonic functions and Bochner identity, and Agostiniani, Mazzieri, and Oronzio [16] used Green's function.

For the general case $k \not\equiv 0$ in dimension three, Schoen and Yau proved $E \geq 0$ by introducing Jang's equation [17] and reduce the case to the Riemannian positive mass theorem. Eichmair generalized the Jang equation argument and proved the $E \geq 0$ theorem in dimensions less than eight [18]. These results also show that if $E = 0$, then (M, g, k) can be isometrically embedded in Minkowski spacetime with the second fundamental form k.

For the result $E \geq |P|$ as stated in Theorem 1, Witten provided a novel proof in dimension three [19, 20], which directly generalizes to higher dimensions for spin manifolds [21, 22]. Eichmair, Huang, Lee, and Schoen [23] extended the minimal surface approach in the Riemannian case developed by Schoen and Yau by introducing marginally outer trapped surfaces (MOTS) to prove Theorem 1. They addressed the challenges posed by MOTS, which, unlike minimal surfaces, do not arise from a variation principle of the initial data set. In the same paper, they also provided an alternative approach, showing that the $E \geq 0$ theorem implies $E \geq |P|$ by a new density theorem. Recently, Hirsch, Kazaras, and Khuri [24] provided an alternative proof for the 3-dimensional case using level sets of spacetime harmonic functions. We will briefly review those approaches in Section 3.

Challenges in higher dimensions $n \geq 8$ for the positive mass theorem arise because minimal hypersurfaces and MOTS can have singularities in these dimensions, causing the dimensional reduction argument to potentially break down. Lohkamp proposed a program to address these issues in a series of papers [25–28]. For the Riemannian case, Schoen and Yau [29] introduced a different approach using minimal slicings. Additionally, there are perturbation arguments to avoid singularities of minimal hypersurfaces for dimensions $n = 8$ by Smale [30] and $n = 9, 10$ by Chodosh, Mantoulidis, and Schulze [31]. Separately, while we shall only discuss $n \geq 3$ in this survey, there is also a Riemannian positive mass theorem for $n = 2$; see [32].

Neither the spinor proof in [19, 20] nor the MOTS proof in [23] includes a complete proof to characterize the equality case $E = |P|$ of Theorem 1. The natural conjecture states that if $E = |P|$, then $E = |P| = 0$, and thus by $E = 0$ rigidity, (M, g) can be isometrically embedded into Minkowski space with induced second fundamental form k. This would signify that the ADM mass $m = \sqrt{E^2 - |P|^2} = 0$, uniquely characterizes Minkowski spacetime.

The conjecture is shown to be true in dimensions $n = 3, 4$, but surprisingly, in dimensions $n \geq 5$ it has the subtlety depending on asymptotic flat decay rates, to be specified in Section 4 below. For $n \geq 5$, an initial data set with $E = |P|$ that does not exhibit the optimal asymptotic decay rate can have $E = |P| \neq 0$ and exists as a Cauchy hypersurface in a larger class of spacetimes known as pp-waves (short for *plane-fronted waves with parallel rays*). Minkowski spacetime is a special case of pp-waves. Those pp-wave examples also give counter-examples to the Bartnik stationary conjecture, see [33, Section 2].

We summarize the current state of the art below.

Theorem 3 (Equality in the spacetime positive mass theorem). *Let $n \geq 3$ and (M, g, k) be an n-dimensional asymptotically flat initial data set that satisfies the dominant energy condition and has $E = |P|$.*

- *If either $n = 3, 4$ or (M, g, k) satisfies the optimal asymptotic decay rates*[1]*, then $E = |P| = 0$ and (M, g) is isometrically embedded into Minkowski spacetime with the induced second fundamental form k.*
- *If $n \geq 5$ and M is spin, then (M, g, k) is isometrically embedded into a pp-wave spacetime with the induced second fundamental form k.*

We can break down the equality theorem into two separate statements: The first statement is to show that $E = |P|$ implies $E = |P| = 0$, which is the part that requires the optimal asymptotics decay rate. The second statement is to use the fact that $E = 0$ to find an embedding of the initial data set into Minkowski space.

Witten sketched an idea for proving $E = |P|$ rigidity of spin manifolds in his 1981 article [19]. Ashtekar-Horowitz [34] and Yip [35] gave arguments for $n = 3$ under extra spacetime assumptions. A complete, rigorous proof was given in the work of Beig and Chruściel for $n = 3$ by a spinor argument [36] and by Chruściel and Maerten [37] to higher-dimensional spin manifolds under the optimal asymptotic decay rates for dimensions $n \geq 5$. For general initial data sets without the spin assumption, Huang and Lee used the method of Lagrangian Multipliers introduced by Bartnik [38] to prove the first statement (under similar optimal asymptotic decay rates if $n \geq 5$). They also gave an alternative proof of $E = 0$ rigidity [39]. Unlike the other proofs, their proof is self-contained in the sense that it does not depend on how one proves that $E \geq |P|$ theorem or $E \geq 0$ theorem. For $n = 3$, Hirsch and Zhang proved the equality theorem for $n = 3$ from the level sets method [40].

On the other hand, Huang and Lee [33] discovered those pp-wave counterexamples in dimensions $n > 8$. Specifically, they found asymptotically flat spacelike slices in a large class of $(n+1)$-dimensional pp-wave spacetimes that are not isometric to Minkowski spacetime, and those spacelike slices have $E = |P| \neq 0$. Hirsch and Zhang further improved these counterexamples to dimensions $n \geq 5$ and showed that a spin asymptotically flat initial data set which satisfies the dominant energy condition and $E = |P|$ must be embeded into a pp-wave [41].

Given that multiple survey articles already exist on the Riemannian positive mass theorem from different perspectives (e.g., [10, 42, 43]), along with a graduate textbook by Lee [44] on those topics, this survey will primarily focus on the spacetime positive mass theorem, with particular emphasis on the equality case. The outline of the survey paper is as follows: In Section 2, we set the stage and include a variational aspect of the ADM energy-momentum.

In Section 3, we briefly review the different proofs of positivity $E \geq |P|$. In Section 4, we discuss the pp-waves and the equality case $E = |P|$.

2. Definitions, notations, and basic facts

2.1. *Initial data sets, the dominant energy condition, and asymptotic flatness*

Let $n \geq 3$. An *initial data set* is an n-dimensional manifold U equipped with a Riemannian metric g and a symmetric $(0,2)$-tensor k. Loosely speaking, an initial data set consists the information needed to determine the evolution of spacetime by viewing the triple (U, g, k) as a hypersurface

[1] We say that (M, g, k) satisfies the *optimal asymptotic decay rates* if the asymptotics of (g, k) defined by (2) below hold for some $q > 0$ and $0 < \alpha < 1$ satisfying $q + \alpha > n - 2$. This is an additional assumption beyond the usual decay rate condition $q > n/2$ when $n \geq 5$.

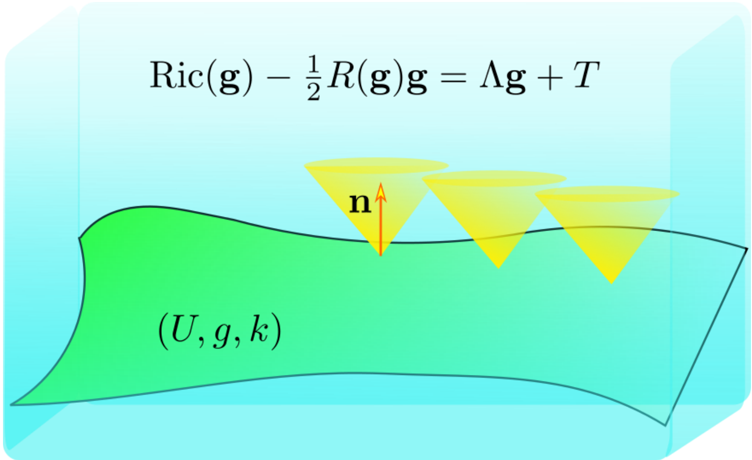

Figure 1. For an initial data set (U, g, k) embedded in a spacetime, the dominant energy condition on (U, g, k) states that $\mu \geq |J|_g$, where $\mu = T(\mathbf{n}, \mathbf{n})$ and $J_i = T(e_i, \mathbf{n})$, with $\mathbf{n}$ being the future-pointing unit normal to U and e_i tangent to U. This condition is equivalent to requiring that $T(\cdot, \mathbf{n})$ is future causal. Note that a spacetime satisfying the dominant energy condition asserts that $T(\mathbf{X}, \mathbf{Y}) \geq 0$ for all future causal vectors $\mathbf{X}$ and $\mathbf{Y}$, which is a stronger condition.

in spacetime $(\mathbf{N}, \mathbf{g})$ with the induced metric g and second fundamental form k. The *Einstein equation* of spacetime is given by

$$\mathrm{Ric}_{\mathbf{g}} - \tfrac{1}{2}R_{\mathbf{g}}\mathbf{g} = \Lambda\mathbf{g} + T,$$

where $\Lambda \in \mathbb{R}$ is the cosmological constant and T is the stress-energy tensor describing a matter field. In this article, we shall only discuss the case $\Lambda = 0$, but note that there are also many progresses on the positive mass theorem for the case $\Lambda < 0$; see, for example, [45–50]. A spacetime is called *vacuum* if T is identically zero. As an example of a non-vacuum matter field, the stress-energy tensor of a perfect fluid is given by $T = p\mathbf{g} + (\rho + p)\mathbf{v} \otimes \mathbf{v}$, where ρ is the energy density, p is the pressure, and $\mathbf{v}$ is the fluid velocity. We do not assume any specific matter fields here; the results apply to any matter field, as long as an energy condition is satisfied, unless otherwise specified.

On an initial data set, one can define the *mass density* μ and the *current density* J by

$$\mu = \tfrac{1}{2}\left(R_g - |k|_g^2 + (\mathrm{tr}_g k)^2\right)$$
$$J = \mathrm{div}_g k - d(\mathrm{tr}_g k).$$

The first equation is referred to as the *Hamiltonian constraint* and the second is the *momentum constraint*. We denote the *constraint map* by $\Phi(g, k) := (\mu, J)$.

We say that (U, g, k) satisfies the *dominant energy condition* if $\mu \geq |J|_g$. If (U, g, k) is *embedded in a spacetime* $(\mathbf{N}, \mathbf{g})$, meaning that (U, g) is isometrically embedded in to $\mathbf{N}$ with the induced metric k, then the dominant energy condition is equivalent to requiring $T(\mathbf{X}, \mathbf{n}) \geq 0$ for all future causal vectors $\mathbf{X}$, where $\mathbf{n}$ is the future-directed normal to the hypersurface. The initial data set (U, g, k) is said to be *vacuum* if $\mu = 0, J = 0$. See Figure 1.

When considering the perturbation of the dominant energy condition, it turns out more effective to use the *modified constraint map* $\overline{\Phi}|_{(g,k)}$ *at* (g, k), introduced by Corvino and Huang [51]. The modified operator $\overline{\Phi}|_{(g,k)}$ is defined on initial data (γ, τ) as follows:

$$\overline{\Phi}_{(g,k)}(\gamma, \tau) = \Phi(\gamma, \tau) + \left(0, \tfrac{1}{2}\gamma \cdot J\right), \tag{1}$$

where J is the current density of g and $(\gamma\cdot J)^i := g^{ij}\gamma_{jk}J^k$. A fundamental property of the modified operator is that the dominant energy condition is preserved under perturbation. More precisely, assume (g,k) satisfies the dominant energy condition $\mu \ge |J|_g$ in M. Suppose (γ,τ) is an initial data set with $|\gamma-g|_g < 3$ in M and

$$\overline{\Phi}|_{(g,k)}(\gamma,\tau) = \overline{\Phi}|_{(g,k)}(g,k).$$

Then (γ,τ) also satisfies the dominant energy condition.

The seminal work of Choquet-Bruhat says that if (U,g,k) is a vacuum initial data set, then there exists a unique vacuum spacetime development $(\mathbf{N},\mathbf{g})$ that evolves from (U,g,k). However, note that in general an initial data set satisfying the dominant energy condition may not have a spacetime development satisfying an energy condition, or if it does, the spacetime development may not be unique.

For analysis on asymptotically flat manifolds, it is convenient to use the weighted Hölder spaces, defined below. Let B be a closed ball in $\mathbb{R}^n$ centered at the origin, and consider the exterior region $\mathbb{R}^n\setminus B$. For any $\ell = 0,1,2,\dots$, $\alpha\in(0,1)$, $q\in\mathbb{R}$, we define the *weighted Hölder space* $C^{\ell,\alpha}_{-q}(\mathbb{R}^n\setminus B)$ as the space of functions f on $\mathbb{R}^n\setminus B$ such that

$$\|f\|_{C^{\ell,\alpha}_{-q}(\mathbb{R}^n\setminus B)} := \sum_{|\alpha|\le\ell}\sup_x\left||x|^{|\alpha|+q}\partial^\alpha f(x)\right| + \sum_{|\alpha|=\ell}\left[|x|^{|\alpha|+q}\partial^\alpha f(x)\right]_\alpha < \infty.$$

Now suppose that M is a manifold such that there is a compact set $K\subset M$ and a diffeomorphism $M\setminus K\cong\mathbb{R}^n\setminus B$. Then one can define $C^{\ell,\alpha}_{-q}(M)$ by choosing an atlas for M that consists of the diffeomorphism $M\setminus K\cong\mathbb{R}^n\setminus B$ together with finitely many precompact charts, and then using the $C^{\ell,\alpha}_{-q}(\mathbb{R}^n\setminus B)$ norm on the noncompact chart while using the $C^{\ell,\alpha}$ norm on the other ones. The definition of $C^{\ell,\alpha}_{-q}(M)$ also extends to tensors simply by considering their components with respect to these coordinate charts.

We say that an initial data set (M,g,k) is *asymptotically flat with decay rate q* if M is a complete manifold and there is a compact set $K\subset M$ and a diffeomorphism $M\setminus K\cong\mathbb{R}^n\setminus B$ for some closed ball $B\subset\mathbb{R}^n$ such that

$$(g-\delta,k)\in C^{2,\alpha}_{-q}(M)\times C^{1,\alpha}_{-1-q}(M) \tag{2}$$

where δ is a smooth symmetric $(0,2)$-tensor that coincides with the Euclidean inner product on $M\setminus K\cong\mathbb{R}^n\setminus B$. In addition, we assume $\mu,J\in L^1(M)$, and sometimes it is convenient to make a stronger assumption that $\mu,J\in C^{0,\alpha}_{-n-q_0}(M)$ for some $q_0>0$. Throughout the article, we assume $q>(n-2)/2$.

2.2. *ADM energy and linear momentum, Regge–Teitelboim Hamiltonian, and Killing initial data*

Assume (M,g,k) is asymptotically flat with decay rate $q>(n-2)/2$ in the above sense. We define the *ADM energy E* and the *ADM momentum P* as

$$E(g,k) = \frac{1}{2(n-1)\omega_{n-1}}\lim_{r\to\infty}\int_{|x|=r}\sum_{i,j=1}^n\left(\frac{\partial g_{ij}}{\partial x^i}-\frac{\partial g_{ii}}{\partial x^j}\right)\nu_0^j\,d\sigma_0$$

$$P_i(g,k) = \frac{1}{(n-1)\omega_{n-1}}\lim_{r\to\infty}\int_{|x|=r}\sum_{j=1}^n(k_{ij}-(\operatorname{tr}_g k)g_{ij})\nu_0^j\,d\sigma_0 \qquad i=1,2,\dots,n.$$

Here, the integrals are computed in the coordinate chart $M\setminus K\cong_x\mathbb{R}^n\setminus B$, $\nu_0^j = x^j/|x|$, σ_0 is the $(n-1)$-dimensional Euclidean Hausdorff measure, and ω_{n-1} is the volume of the standard unit sphere in $\mathbb{R}^n$.

Notice that if the decay rate $q>n-2$, then the integrand in the above expressions decays too quickly, so $E=0$ and $P=0$ trivially. On the other hand, a decay rate $q>(n-2)/2$ is sufficient to ensure that the ADM energy-momentum is well-defined in the following sense. While the ADM energy and linear momentum are defined using a specific asymptotically flat coordinate chart, they can be shown to be coordinate invariant, depending only on the structure at infinity, see [21]. Furthermore, although the ADM energy and linear momentum are defined entirely on the initial data set, the Lorentzian norm $-E^2+|P|^2$ is a spacetime invariant. It means that, while the ADM energy-momentum vectors of two asymptotically flat initial data sets embedded in the same spacetime may differ, they have the same Lorentzian norm, see [52].

To understand heuristically how the decay rate $q>(n-2)/2$ plays a role, we expand the constraint equations in asymptotically flat coordinates:

$$\begin{aligned}\mu &= \frac{1}{2}(R_g-|k|^2+(\mathrm{tr}_g k)^2)=\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial x^j}\left(\frac{\partial g_{ij}}{\partial x^i}-\frac{\partial g_{ii}}{\partial x^j}\right)+O(|x|^{-2-2q})\\ J_i &= (\mathrm{div}_g k)_j-(\mathrm{tr}_g k)_{,j}=\sum_i\frac{\partial}{\partial x^i}(k_{ij}-(\mathrm{tr}_g k)g_{ij})+O(|x|^{-2-2q}).\end{aligned}$$

Observe that the ADM energy-momentum arises from the flux integrals of the divergence terms on the right-hand sides. The decay rate $q>(n-2)/2$ ensures that the exponent of the error terms, $-2-2q<-n$, decays rapidly enough to be integrable on the asymptotically flat end. Together with the assumption that both μ and J are integrable, the flux integrals over the coordinate sphere have a limit as the radius of the sphere going to infinity.

The following density theorem states that initial data sets with the decay rate $q=n-2$ are dense among those with the weaker decay rates $q>(n-2)/2$, and that the ADM energy-momentum varies continuously. This has several applications. For example, to prove the positivity of the ADM mass as stated in Theorem 1, it suffices to verify the inequality for an asymptotically flat initial data set with a decay rate $q=n-2$. For more details about the density theorem, see, for example, [23, Section 6].

Theorem 4 (Density theorem (loosely stated)). *Let (M,g,k) be an n-dimensional asymptotically flat initial data set at the decay rate $q>(n-2)/2$ satisfying the dominant energy condition. For any $\epsilon>0$, there exists an asymptotically flat initial data set $(M,\bar g,\bar k)$ at the decay rate $q=n-2$ such that $(\bar g,\bar k)$ satisfies the dominant energy condition, is ϵ-close to (g,k) in $C^{2,\alpha}_{-q}(M)\times C^{1,\alpha}_{-1-q}(M)$, and $|(\bar E,\bar P)-(E,P)|<\epsilon$.*

Furthermore, we can arrange $(M,\bar g,\bar k)$ to have either the strict dominant energy condition $\bar\mu>|\bar J|_{\bar g}$ everywhere in M or to be vacuum $\bar\mu=|\bar J|_{\bar g}=0$ outside a large compact subset of M.

The ADM energy-momentum has a variational characterization.

Definition 5 (Regge–Teitelboim Hamiltonian). *Let M be an n-dimensional manifold that can carry an asymptotically flat initial data set. Let (f_0,X_0) be a pair of a function and a vector field on M (which we will often call a* lapse-shift pair*) such that (f_0,X_0) is smooth and is equal to a constant $(a,b)\in\mathbb{R}\times\mathbb{R}^n$ in the exterior coordinate chart for $M\setminus K$. We define the* Regge–Teitelboim Hamiltonian *$\mathscr{H}_{\mathrm{RT}}$ corresponding to (f_0,X_0) by, for asymptotically flat initial data set (g,k)*

$$\mathscr{H}_{\mathrm{RT}}(g,k)=(n-1)\omega_{n-1}\left[aE(g,k)+b\cdot P(g,k)\right]-\int_M\langle\Phi(g,k),(f_0,X_0)\rangle_g\,d\mu_g. \tag{3}$$

In the case of vacuum initial data sets, as originally considered by Regge and Teitelboim [53], the Hamiltonian can recover the energy E and the linear momentum by choosing the lapse-shift pair to be asymptotic to the translation vector fields of Minkowski spacetime. Specifically, $(f,X)\to(1,0)$ yields the energy E, while $(f,X)\to(0,\partial/\partial x^i)$ yields P_i, the i-th component of the linear momentum. The task of "minimizing" the ADM energy-momentum (E,P) is transformed into minimizing the functional (3) as follows. Suppose the positivity of mass holds for vacuum (g,k), i.e., $E(g,k) \geq |P(g,k)|$. Then, for any future causal vector (a,b), we have

$$aE(g,k) + b \cdot P(g,k) \geq aE(g,k) - |b||P(g,k)| \geq (a-|b|)|P(g,k)| \geq 0 \tag{4}$$

with equality if and only if $E(g,k) = |P(g,k)|$ and (a,b) is a constant multiple of $(E(g,k), P(g,k))$. This is just a general fact that for non-zero future causal vectors $\mathbf{X}$ and $\mathbf{Y}$, their Lorentzian inner product $-\langle \mathbf{X}, \mathbf{Y}\rangle \geq 0$ with equality if and only if $\mathbf{X}, \mathbf{Y}$ are both null and $\mathbf{X}$ is a constant multiple of $\mathbf{Y}$.

With the minimization task in mind, we compute the first variation of $\mathscr{H}_{\mathrm{RT}}$: for vacuum (g,k), we have

$$D\mathscr{H}\big|_{(g,k)}(h,w) = -\int_M (h,w)\cdot D\Phi|^*_{(g,k)}(f_0, X_0)\,\mathrm{d}\mu_g$$

for all compactly supported (h,w), where $D\Phi|^*_{(g,k)}$ denotes the L^2-formal adjoint equation of the linearization $D\Phi|_{(g,k)}$ at (g,k). The adjoint equation $D\Phi|^*_{(g,k)}$ has geometric significance in the vacuum case, as discovered by Moncrief [54].

Theorem 6. *Let (U,g,k) be a vacuum initial data set and suppose that there exists a nontrivial lapse-shift pair (f,X) on U solving*

$$D\Phi|^*_{(g,k)}(f,X) = 0.$$

Then the vacuum spacetime development of (U,g,k) admits a unique global Killing vector field $\mathbf{Y}$ such that $\mathbf{Y} = f\mathbf{n} + X$ along U, where $\mathbf{n}$ is the future unit normal to U.

Conversely, given a vacuum spacetime equipped with a global Killing vector field $\mathbf{Y}$ and a spacelike hypersurface U with induced initial data (g,k), if we decompose $\mathbf{Y} = f\mathbf{n} + X$ along U, then the lapse-shift pair (f,X) must lie in the kernel of $D\Phi|^*_{(g,k)}$.

In light of the above theorem, a solution (f,X) to $D\Phi|^*_{(g,k)}(f,X) = 0$ is referred to as the *Killing initial data.*

An analogous statement for a non-vacuum case was established by Huang and Lee [33, Theorem 6] using the modified constraint operator. A spacetime $(\mathbf{N},\mathbf{g})$ is said to be a *null perfect fluid* with velocity $\mathbf{v}$ and pressure p if $\mathbf{v}$ is either future null or zero at each point and the Einstein tensor takes the form:

$$\mathrm{Ric}_{\mathbf{g}} - \tfrac{1}{2}R_{\mathbf{g}}\mathbf{g} = p\mathbf{g} + \mathbf{v}\otimes\mathbf{v}. \tag{5}$$

Theorem 7. *Let (U,g,k) be an initial data set satisfying the dominant energy condition. Assume there exists a nontrivial lapse-shift pair (f,X) on U solving the system*

$$D\overline{\Phi}|^*_{(g,k)}(f,X) = 0 \quad \textit{and} \quad fJ + |J|_g X = 0. \tag{6}$$

and assume that f is nonvanishing in U. Then the following holds:

1. *The dominant energy scalar $\sigma(g,k) := \mu - |J|_g$ is constant on U.*
2. *(U,g,k) is embedded inside a null perfect fluid spacetime $(\mathbf{N},\mathbf{g})$ that satisfies the spacetime dominant energy condition and admits a global Killing vector field $\mathbf{Y}$ equal to $f\mathbf{n} + X$ along U, where $\mathbf{n}$ is the future unit normal to U.*

There is also a converse statement.

We remark that the system (6) is closely related to the improvability of the dominant energy condition. See [33].

We refer to a solution (f,X) to $D\overline{\Phi}|^*_{(g,k)}(f,X) = 0$ as the *modified* Killing initial data. These solutions play an important role in the equality theorem of the positive mass theorem; see Section 4. An overall strategy for proving the equality theorem is to show that the initial data set achieving equality has one or more modified Killing initial data.

3. The proofs of positivity $E \geq |P|$

We outline the currently available proofs of the positivity theorem, Theorem 1.

3.1. *Marginally outer trapped hypersurfaces*

We briefly revisit the minimal hypersurface proof of the Riemannian positive mass theorem, originally developed by Schoen and Yau. This proof uses induction on the dimension $3 \leq n \leq 7$ and proceeds by contradiction. Assume there exists an asymptotically flat Riemannian manifold (M, g) with nonnegative scalar curvature and negative mass $m < 0$. A density argument allows us to assume (M, g) has harmonic asymptotics and positive scalar curvature. These conditions imply that the coordinate planes $x^n = \pm\Lambda$ act as barriers for minimal hypersurfaces for sufficiently large Λ.

Consider an $(n-1)$-dimensional vertical cylinder ∂C_ρ with a large radius ρ. For each $h \in [-\Lambda, \Lambda]$, there exists an area-minimizing hypersurface $\Sigma_{\rho,h} \subset C_\rho$ whose boundary is the height h sphere on ∂C_ρ. If $n \leq 7$, this hypersurface is smooth. Each $\Sigma_{\rho,h}$ lies between the hyperplanes $x^n = \pm\Lambda$. The area of $\Sigma_{\rho,h}$ is minimized by some $h_\rho \in (-\Lambda, \Lambda)$. This "height-picking" step is crucial in dimensions $n \geq 4$ because the corresponding surface Σ_{ρ,h_ρ} satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \operatorname{area}(\Phi_t(\Sigma_{\rho,h_\rho})) \geq 0$$

for any variation Φ_t that is a vertical translation along ∂C_ρ. Without the height-picking step, the above inequality only holds for compactly supported variations Φ_t, which is not sufficient. (When $n = 3$, one can use the so-called logarithmic cut-off trick to approximate variations that are vertical translations along ∂C_ρ by compactly supported variations. However, this trick does not work in higher dimensions.) A smooth subsequential limit Σ_∞ of Σ_{ρ,h_ρ} as $\rho \to \infty$ is an $(n-1)$-dimensional asymptotically flat manifold with zero energy and is a stable minimal hypersurface of M.

For $n = 3$, the stability of Σ_∞ and its asymptotics are incompatible with the Gauss–Bonnet theorem. In higher dimensions, one can construct a conformal factor that changes the metric on Σ_∞ to one that is asymptotically flat with zero scalar curvature and negative mass, thus violating the Riemannian positive mass theorem in dimension $n-1$.

Our approach to the spacetime positive mass theorem generalizes the minimal hypersurface proof using marginally outer trapped hypersurfaces (MOTS). A hypersurface Σ is a MOTS if $\theta = 0$, where $\theta = H_\Sigma + \operatorname{tr}_\Sigma k$ is the expansion scalar. In the Riemannian case, θ reduces to the mean curvature.

Let (M, g, k) be an n-dimensional asymptotically flat initial data set satisfying the dominant energy condition $\mu \geq |J|$ and $E < |P|$. By our density theorem, Theorem 4, we may assume (M, g, k) has harmonic asymptotics and satisfies the strict dominant energy condition $\mu > |J|$. Furthermore, we assume P points in the vertical direction $-\partial_n$. The harmonic asymptotics and $E < |P|$ assumptions imply that the planes $x^n = \pm\Lambda$ act as barriers for MOTS for sufficiently large Λ. Consider an $(n-1)$-dimensional vertical cylinder ∂C_r. For each $h \in [-\Lambda, \Lambda]$, the existence theorem of MOTS [55] guarantees the existence of a MOTS $\Sigma_{r,h}$ with boundary equal to the height h sphere on ∂C_r. This MOTS is smooth if $n < 8$ and lies between the planes $x^n = \pm\Lambda$, being stable in the sense of MOTS with boundary. See Figure 2.

A major challenge, distinct from the minimal surface approach, is that MOTS do not arise from a variational principle. Thus, there is no obvious alternative to the height-picking step of

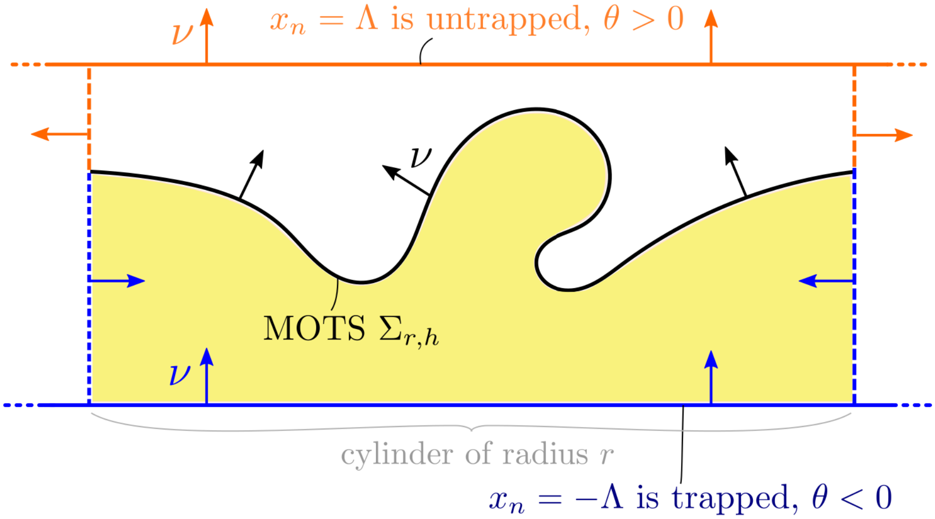

Figure 2. Under the (contradictory) assumption $E < |P|$, two coordinate hyperplanes $x^n = \Lambda$ and $x^n = -\Lambda$, together with the lateral side of a cylinder of large radius, provide barriers for the existence of a MOTS Σ_r with prescribed boundary.

minimizing the area with respect to the height h. To overcome this, we introduce a new functional $\mathscr{F}$ on hypersurfaces with boundary on ∂C_r such that for some $h_r \in (-\Lambda, \Lambda)$, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=h_r} \mathscr{F}(\Sigma_{r,h}) \geq 0.$$

This inequality, along with the MOTS stability, parallels the area inequality in the Riemannian case. Extracting a smooth subsequential limit Σ_∞ of Σ_{r,h_r} as $r \to \infty$, we find that Σ_∞ is an $(n-1)$-dimensional asymptotically flat manifold with zero energy and is a stable MOTS in M. We can construct a conformal factor that changes the metric on Σ_∞ to one with zero scalar curvature and decreases the energy of Σ_∞, thus violating the Riemannian positive mass theorem in dimension $n-1$.

3.2. *Boost argument*

The positive mass theorem $E \geq |P|$ can also be derived from the positive energy theorem $E \geq 0$ using the following reduction argument. Assume that $0 < E < |P|$. Fix $\theta \in (0,1)$ such that $E' := (E - \theta|P|)/(1-\theta^2) < 0$. Using the density theorem, Theorem 4, we may assume that (M, g, k) satisfies $\mu = 0$ and $J = 0$ outside a large compact set. According to [56], there exists a vacuum spacetime development of the asymptotically flat end of (M, g, k) in which the end of (M, g, k) can be deformed to a boosted slice (of slope θ) that has energy E'. The deformed initial data (M', g', k') satisfies the conditions of the $E \geq 0$ theorem, leading to a contradiction. This reduction is well known in the mathematical relativity community when the initial data set is assumed to be a spacelike slice of an asymptotically flat spacetime; see, e.g., [57].

3.3. *Spinor proof*

Witten's proof is notable for its use of spinors, offering a different perspective on the positive mass theorem. A spinor field ψ on a manifold is a section of a spinor bundle, which can be roughly thought of as a field of "spinor-valued" functions. For a 3-dimensional initial data set (M, g, k), M can be given a spin structure. In higher dimensions, this method requires M has a spin structure, which allows us to use spinor fields ψ on M. For convenience, we consider (M, g, k) embedded in a spacetime $(\mathbf{N}, \mathbf{g})$ and choose an orthonormal frame $\{e_0, e_1, \ldots, e_n\}$ where e_0 is the timelike vector. One can also use a framework that is a purely initial data perspective. Define a new connection $\tilde{\nabla}_i = \nabla_i + (1/2)k_{ij}e_j e_0$ and the Dirac operator $\not{D} = e_i \cdot \tilde{\nabla}_i = e_i \cdot \nabla_i + (1/2)k_{ij}e_i e_j e_0$ where ∇_i is the induced connection on M. The Witten equation states

$$\not{D}^2\psi = \tilde{\nabla}^*\tilde{\nabla}\psi + \tfrac{1}{2}(\mu + Je_0)\cdot\psi.$$

For any ψ asymptotic to a constant spinor ψ_0 at infinity, integrating the above Witten's equation and analyzing the boundary term

$$\begin{aligned}\int_M |\tilde{\nabla}\psi|^2 - |\not{D}\psi|^2 + \tfrac{1}{2}\langle\psi, (\mu + Je_0)\cdot\psi\rangle \, \mathrm{d}\mu \\ = \lim_{r\to\infty}\int_{|x|=r}\langle\psi, \tilde{\nabla}_\nu\psi + \nu\cdot\not{D}\psi\rangle \, \mathrm{d}\sigma = \tfrac{1}{2}(n-1)\omega_{n-1}\left(|\psi_0|^2 E + \langle\psi_0, P_j e_j e_0\cdot\psi_0\rangle\right) \qquad (7)\end{aligned}$$

where ν is the outward unit normal. By choosing ψ to solve $\not{D}\psi = 0$ with ψ asymptotic to a certain constant spinor ψ_0 such that $\langle\psi_0, P_j e_j e_0\cdot\psi_0\rangle = -|\psi_0|^2|P|$, one gets

$$\tfrac{1}{2}(n-1)\omega_{n-1}|\psi_0|^2(E - |P|) = \int_M |\tilde{\nabla}\psi|^2 + \tfrac{1}{2}\langle\psi, (\mu + Je_0)\cdot\psi\rangle \, \mathrm{d}\mu \ge 0$$

where we use the dominant energy condition in the last inequality.

The argument also directly leads the $E = 0$ rigidity. Since $E = 0$ implies $|P| = 0$, for any constant spinor ψ_0 at infinity, one can let the harmonic spinor ψ asymptotically approach ψ_0 in (7) to obtain $\tilde{\nabla}\psi = 0$, meaning that ψ is, in fact, a parallel spinor. One then argue that the spacetime metric is flat along M. See [20] for more details.

3.4. *Level set method using spacetime harmonic functions*

We provide an overview of the proof by Hirsch, Kazaras, and Khuri [24]. Additionally, a recent survey by Bray, Hirsch, Kazaras, Khuri, and Zhang [58] discusses the level set method applied to positive mass theorems and other applications.

Let (Ω, g, k) be a 3-dimensional compact initial data set. Consider a scalar-valued function u. For simplicity, we assume that u has no critical points. Let $\Sigma_t = u^{-1}(t)$ be a level set of u, and denote the unit normal to Σ_t by $\eta = \nabla u/|\nabla u|$. By applying the Gauss equation to the level set Σ_t and using $A(e_\alpha, e_\beta) = |\nabla u|^{-1}\nabla^2 u(e_\alpha, e_\beta)$, we obtain

$$|\nabla u|\mathrm{Ric}(\eta,\eta) = \tfrac{1}{2}|\nabla u|(R_g - R_\Sigma) + |\nabla u|^{-1}\left(|\nabla|\nabla u||^2 - \tfrac{1}{2}|\nabla^2 u|^2 + \tfrac{1}{2}(\Delta u)^2 - (\Delta u)(\nabla^2 u)(\eta,\eta)\right). \qquad (8)$$

We have the modified Bochner identity:

$$\Delta|\nabla u| = \frac{1}{|\nabla u|}\left(|\nabla^2 u|^2 + \langle\nabla u, \nabla\Delta u\rangle\right) + |\nabla u|\mathrm{Ric}(\eta,\eta) - |\nabla u|^{-1}|\nabla|\nabla u||^2.$$

Equating the previous two identities gives

$$\Delta|\nabla u| = \frac{1}{|\nabla u|}\langle\nabla u, \nabla\Delta u\rangle + \frac{1}{2}|\nabla u|\left[R_g - R_\Sigma + \frac{|\nabla^2 u|^2 + (\Delta u)^2 - 2|\nabla u|(\Delta u)\nabla^2 u(\eta,\eta)}{|\nabla u|^2}\right]. \qquad (9)$$

So far, the identity holds for any function u. It was observed in [24] that this identity takes a more convenient form when u satisfies the *spacetime harmonic equation* $\Delta u + (\mathrm{tr}_g k)|\nabla u| = 0$, as motivated below. Suppose the initial data set (Ω, g, k) is embedded in a spacetime $(\mathbf{N}, \mathbf{g})$ with the unit normal $\mathbf{n}$. Then the Hessian operator ∇^2 of $\mathbf{g}$ restricted on Ω is given by

$$\nabla^2 u(e_i, e_i) = \nabla^2 u(e_i, e_j) + k_{ij}\mathbf{n}(u).$$

Since $\mathbf{n}(u)$ involves the spacetime vector $\mathbf{n}$ and is not determined on the initial data set (M,g,k), it turns out to be effective to require $\mathbf{n}(u) = |\nabla u|$; namely, the spacetime vector ∇u is null. By taking the trace, it gives the spacetime harmonic equation.

Substituting u, which solves the spacetime harmonic equation, into (9) yields

$$\Delta|\nabla u| = \frac{1}{2}|\nabla u|\left[2\mu - R_\Sigma + \frac{\left|\nabla^2 u + k|\nabla u|\right|^2}{|\nabla u|^2}\right] - \langle \nabla u, \nabla(\mathrm{tr}_g k)\rangle - \langle \nabla^2 u, k\rangle.$$

Consider the identity on an asymptotically flat initial data set (M,g,k) and let Ω be a large coordinate sphere. Consider u with the asymptotics $u(x) \to a\cdot x$ as $|x|\to\infty$ where $a\in\mathbb{R}^3$ and $|a| = 1$. Integrating the previous identity over Ω and applying the coarea formula gives the following integral formula:

$$\int_{\partial\Omega} (\nu(|\nabla u|) + k(\nabla u,\nu))\,\mathrm{d}\sigma = \frac{1}{2}\int_{\underline{u}}^{\overline{u}}\int_{\Sigma_t}\left(2\mu + 2J(\eta) - R_{\Sigma_t} + \frac{\left|\nabla^2 u + k|\nabla u|\right|^2}{|\nabla u|^2}\right)\mathrm{d}\sigma\mathrm{d}t$$

where ν is the outward unit normal to $\partial\Omega$. By carefully analysis, one can show that the integral $\int_{\underline{u}}^{\overline{u}}\int_{\Sigma_t} R_{\Sigma_t}$ is asymptotically nonpositive, and the boundary term is asymptotic to $16\pi(E + a\cdot P)$. To summarize, one obtain

$$E + a\cdot P \ge \frac{1}{16\pi}\int_M\left(\mu + J(\eta) + \frac{\left|\nabla^2 u + k|\nabla u|\right|^2}{|\nabla u|^2}\right)\mathrm{d}\sigma\mathrm{d}t \ge 0.$$

This completes the proof.

4. pp-waves and the equality case $E = |P|$

4.1. *pp-waves*

A pp-wave spacetime (or just a pp-wave for short) is a Lorentzian metric $\mathbf{g}$ defined on $\mathbb{R}^{n+1}$ equipped with a global coordinate chart $\{u, z, x^1, \dots, x^{n-1}\}$ where

$$\mathbf{g} = 2\mathrm{d}u\mathrm{d}z + S\mathrm{d}z^2 + \sum_{a=1}^{n-1}(\mathrm{d}x^a)^2. \tag{10}$$

Here, the scalar-valued function S depends on z and $x^1,\dots,x^{n-1}$, but not on u. The vector $\mathbf{Y} := \partial/\partial u$ is a covariantly constant, nowhere vanishing null vector field. Note that a pp-wave is a special case of a null perfect fluid described by (5) with pressure $p = 0$ and carries a global null Killing vector field $\mathbf{Y}$ with $\mathbf{v} = \eta\mathbf{Y}$ for some scalar function η. The term "pp-waves" stands for *plane-fronted* waves (the wave fronts are constant z slices, whose induced metric is flat) with *parallel rays* ($\mathbf{Y}$ is a parallel null vector). These waves can describe situations in which gravitational waves are idealized as propagating along one direction, uniformly across flat wave fronts perpendicular to this direction.

Example 8. The Minkowski metric $\mathbf{g} = -\mathrm{d}t^2 + \sum_{a=1}^n (\mathrm{d}y^a)^2$ is a special case of the pp-wave metric. To see this, we can introduce the null coordinates $U = y^n - t$ and $Z = \frac{1}{2}(y^n + t)$ and re-express the Minkowski metric as $\mathbf{g} = 2\mathrm{d}U\mathrm{d}Z + \sum_{a=1}^{n-1}(\mathrm{d}y^a)^2$. Thus, we recover the pp-wave metric by setting $S = 0$ in (10). Another choice of coordinates is to set

$$U = u - (x^1 + \dots + x^{n-1}), \quad Z = z, \quad y^a = x^a + z,$$

and express the Minkowski metric as

$$\mathbf{g} = 2\mathrm{d}u\mathrm{d}z + \mathrm{d}z^2 + \sum_{a=1}^{n-1}(\mathrm{d}x^a)^2. \tag{11}$$

Then, we recover the pp-wave metric (10) by setting $S = 1$.

The second expression (11) of the Minkowski metric is more relevant to what we will discuss below. Notice that the induced metric from (11) on the constant u slices is the Euclidean metric with the induced second fundamental form $k \equiv 0$.

We consider the constant u-slices in (10), assuming that $S > 0$ everywhere. Additionally, we use x^n to denote the z coordinate in (10).

Lemma 9. *We assume $S > 0$. Let g and k be respectively the induced metric and second fundamental form on the constant u-slices in (10). Then $(\mathbb{R}^n, g, k)$ is an initial data set where g and k are given by*

$$\begin{aligned} g &= S(\mathrm{d}x^n)^2 + \sum_{a=1}^{n-1} (\mathrm{d}x^a)^2 \\ k_{ab} &= 0 \quad \text{for } a, b = 1, \dots, n-1 \\ k_{jn} &= \tfrac{1}{2} S^{-\frac{1}{2}} S_{,j} \quad \text{for } j = 1, \dots, n. \end{aligned}$$

We also have

$$\begin{aligned} \mu &= -\tfrac{1}{2} S^{-1} \Delta' S \\ J &= \frac{1}{2} S^{-\frac{3}{2}} (\Delta' S) \frac{\partial}{\partial x^n}, \end{aligned}$$

and in particular, $\mu = |J|_g$.

Proof. Along a constant u-slice we have $\mathbf{Y} = \partial/\partial u = S^{-\frac{1}{2}} \mathbf{n} + S^{-1} \partial/\partial x^n$ where $\mathbf{n} = \nabla u / |\nabla u|$. To compute the second fundamental form, we use that $\mathcal{L}_{\partial/\partial u} \mathbf{g} = 0$ because $\partial/\partial u$ is Killing and thus

$$\begin{aligned} k_{ij} &= \tfrac{1}{2} \mathcal{L}_{\mathbf{n}} \mathbf{g}(\partial_i, \partial_j) = -\tfrac{1}{2} f^{-1} \mathcal{L}_X \mathbf{g}(\partial_i, \partial_j) = -\tfrac{1}{2} S^{\frac{1}{2}} \mathcal{L}_X g(\partial_i, \partial_j) \\ &= -\tfrac{1}{2} S^{\frac{1}{2}} \left(X_{i,j} + X_{j,i} - 2\Gamma^n_{ij} X_n \right) = S^{\frac{1}{2}} \Gamma^n_{ij}. \end{aligned}$$

Computing the Christoffel symbols gives the desired formula for k. The computations for μ, J are more involved, and we refer to Lemma C.2 of [33]. □

It easily follows that the dominant energy condition holds if and only if $\Delta' S \le 0$. Note that such a function S cannot exist for $n = 3$ because Liouville's theorem says that any superharmonic function on $\mathbb{R}^2$ that is bounded below must be constant. In the next lemma, we summarize the properties that S have such that $(\mathbb{R}^n, g, k)$ is asymptotically flat with the decay rate q.

Lemma 10. *Let $n > 3$. There exist nonconstant smooth functions S on $\mathbb{R}^n$ with the following properties:*

(1) *$\Delta' S \le 0$ everywhere, strictly negative somewhere, and $\Delta' S$ is integrable on $\mathbb{R}^n$.*
(2) *$S \equiv 1$ in $\{|x^n| \ge C\}$ for some constant $C > 0$.*
(3) *$\lim_{\rho \to \infty} \int_{|x'| = \rho} -\sum_{a=1}^{n-1} \partial S / \partial x^a x^a / |x'| \, \mathrm{d}\sigma$ exists and is positive.*
(4) *For each nonnegative integer ℓ and each $\alpha \in (0,1)$, we have $S - 1 \in C^{\ell,\alpha}_{-q}(\mathbb{R}^n)$ with $q = n - 3 - (\ell + \alpha)$.*

Proof. Let F be a smooth nonnegative function on $\mathbb{R}^{n-1}$ with coordinates $x' = (x^1, \dots, x^{n-1})$, such that $F = O(|x'|^{-s})$ for some $s > n - 1$. We can solve $\Delta' \psi = -F$ on $\mathbb{R}^{n-1}$ via convolution with the fundamental solution of the Laplacian on Euclidean $\mathbb{R}^{n-1}$. As long as F is not identically zero, $\psi(x')$ will be a positive, globally superharmonic function on $\mathbb{R}^{n-1}$. For $n > 3$, it must have the expansion

$$\psi(x') = A |x'|^{3-n} + (\text{lower order terms}),$$

and since ψ is positive, the constant A must also be positive. Now define

$$S(x', x^n) = 1 + \phi(x^n) \psi(x'),$$

where ϕ is chosen to be any nontrivial, compactly supported, smooth, nonnegative function on $\mathbb{R}$. Note that $\Delta' S = -\phi(x^n)F(x') \le 0$ and is strictly negative somewhere. It is straightforward to verify that S satisfies Items (1), (2), and (3). Since the derivatives of S in the x^n direction do not decay any faster than $|x'|^{3-n}$, we can only conclude that $S-1$ and its derivatives of any order are $O(|x|^{3-n})$. Thus, $S-1 \in C^{\ell,\alpha}_{-q}(\mathbb{R}^n)$ with $q = n-3-\ell-\alpha$ by the definition of weighted Hölder spaces. □

Theorem 11. *For each $n \ge 5$, there exist complete, asymptotically flat initial data sets $(\mathbb{R}^n, g, k)$ that satisfy $\mu = |J|$ and $E = |P| > 0$. These examples have asymptotic decay rate $q > (n-2)/2$.*

Proof. We first describe the case $n > 8$ by Huang and Lee [33]. Choose any S as in Lemma 10 and use this choice in Lemma 9 to construct initial data $(\mathbb{R}^n, g, k)$. We claim that for $n > 8$, this is the desired example. By construction (g, k) is clearly complete. The main task is to show that (g, k) is asymptotically flat at the decay rate $q > (n-2)/2$.

Recall that our asymptotic flatness condition requires $g_{ij} - \delta_{ij} \in C^{2,\alpha}_{-q}(\mathbb{R}^n)$ and $k_{ij} \in C^{1,\alpha}_{-q}(\mathbb{R}^n)$ for some $(n-2)/2 < q < n-2$, and $(\mu, J) \in L^1(\mathbb{R}^n)$. For our (g, k) from Lemma 9, this is equivalent to requiring that $S - 1 \in C^{2,\alpha}_{-q}$ for some $(n-2)/2 < q < n-2$ and that $\Delta' S$ is integrable. By Item (4) of Lemma 10, this imposes the condition on n:

$$(n-3) - (2+\alpha) > \frac{n-2}{2} \text{ for some } \alpha \in (0,1), \quad \text{or equivalently,} \quad n > 8.$$

To see that $E = |P| > 0$, we evaluate the ADM energy-momentum by integrating over large capped cylinders. The caps do not contribute, and we can see that

$$E = -P_n = \frac{1}{2(n-1)\omega_{n-1}} \lim_{\rho\to\infty} \int_{|x'|=\rho} -\sum_{a=1}^{n-1} \frac{\partial S}{\partial x^a} \frac{x^a}{|x'|} \, \mathrm{d}\mu > 0,$$

and $P_1 = \cdots = P_{n-1} = 0$.

To extend the example down to dimensions $n \ge 5$, Hirsch and Zhang observed that instead of choosing the constant u-slices in (10) as above, one can choose other spacelike slices, such as graphs over the u-slices. By choosing appropriate graphing functions, they can improve the derivatives in the x^n direction and thus construct examples for $n \ge 5$. See [41]. □

4.2. *Proof that $E = |P|$ implies $E = |P| = 0$*

We review the proof by Huang and Lee [59] for the first statement of the equality theorem.

Theorem 12. *Let (M, g, k) be an n-dimensional asymptotically flat initial data set with decay rate q and assume that the positive mass theorem is true near (g, k). Furthermore, we make the stronger asymptotic assumption that with this decay rate q,*

$$q + \alpha > n - 2,$$

where α is the Hölder exponent in the definition of asymptotic flatness. If (g, k) satisfies the dominant energy condition, then for each asymptotically flat end, $E = |P|$ implies that $E = |P| = 0$.

We set up a variational setting to prove the above theorem. Let (M, g, k) be an asymptotically flat initial data set satisfying the dominant energy condition, as well as the assumption $E = |P|$. Given a scalar function f_0 and a vector field X_0 so that $(f_0, X_0) \to (a, b)$, we introduce the *modified Regge–Teitelboim Hamiltonian* $\mathscr{H}$ corresponding to (g, k) and (f_0, X_0) by

$$\mathscr{H}(\gamma, \tau) = (n-1)\omega_{n-1} \left[aE(\gamma, \tau) + b \cdot P(\gamma, \tau) \right] - \int_M \overline{\Phi}|_{(g,k)}(\gamma, \tau) \cdot (f_0, X_0) \, \mathrm{d}\mu_g$$

where the volume measure $\mathrm{d}\mu_g$ and the inner product in the integral are both with respect to g. The functional is obtained from the classical Regge–Teitelboim Hamiltonian in Definition 5 by replacing the usual constraint operator with the modified constraint operator $\overline{\Phi}|_{(g,k)}$. We also choose the volume form with respect to g.

Define the constraint set $\mathscr{C}_{(g,k)}$ to be the set of asymptotically flat initial data sets (γ,τ) such that $\overline{\Phi}_{(g,k)}(\gamma,k) = \overline{\Phi}|_{(g,k)}(g,k)$. A key property of the modified constraint operator is that if (g,k) satisfies the dominant energy condition, then the dominant energy condition holds for all $(\gamma,\tau)\in\mathscr{C}_{(g,k)}$ close to (g,k). Choosing the pair (f_0,X_0) asymptoting to $(a,b):=(E,-P)$, we apply the positive mass inequality to conclude that (g,k) locally minimizes the functional $\mathscr{H}$ on $\mathscr{C}_{(g,k)}$ as below: For $(\gamma,\tau)\in\mathscr{C}_{(g,k)}$, by the same computation as in (4), we have

$$\mathscr{H}(\gamma,\tau)-\mathscr{H}(g,k) = (n-1)\omega_{n-1}\left[aE(\gamma,\tau)+b\cdot P(\gamma,\tau)\right]-(n-1)\omega_{n-1}\left[aE+b\cdot P\right] \tag{12}$$

$$\geq (n-1)\omega_{n-1}\left[EE(\gamma,\tau)-P\cdot P(\gamma,\tau)\right]\geq 0, \tag{13}$$

where the inequality $E(\gamma,\tau)\geq|P(\gamma,\tau)|$ is used. The equality holds when $(\gamma,\tau)=(g,k)$.

Using the method of Lagrange multipliers, there exists a Lagrange multiplier (f_1,X_1) asymptotic to zero such that

$$D\mathscr{H}|_{(g,k)}(h,w) = \int_M (f_1,X_1)\cdot D\overline{\Phi}|_{(g,k)}(h,w)\,\mathrm{d}\mu_g = \int_M D\overline{\Phi}|^*_{(g,k)}(f_1,X_1)\cdot(h,w)\,\mathrm{d}\mu_g$$

Since the left hand side equals $-\int_M (h,w)\cdot D\overline{\Phi}|^*_{(g,k)}(f_0,X_0)\,\mathrm{d}\mu_g$, just as the first variation of the classical Regge–Teitelboim Hamiltonian (3), we construct a pair $(f,X)=(f_0,X_0)+(f_1,X_1)$ that solves $D\overline{\Phi}|^*_{(g,k)}(f,X)=0$ and is asymptotic to $(E,-P)$. To summarize, the above construction yields a modified Killing initial data (f,X) with the prescribed asymptotics $(f,X)\to(E,-P)$.

Note that in the non-vacuum case, there need not be a corresponding spacetime Killing vector field. However, intriguingly, the spinor proof by Beig and Chruściel uses the harmonic spinor to construct (f,X) that satisfies a similar over-determined equation with the asymptotic to $(E,-P)$. Their key observation is that for asymptotically flat initial data sets with the optimal decay rate, this implies $E=|P|=0$. It completes the proof.

4.3. *Proof that* $E=|P|=0$ *implies embedding in Minkowski spacetime*

Suppose $E=0$. We extend the above variational setting to show that the initial data set can be embedded into Minkowski space [39]. Unlike other proofs [17–20, 40], this proof does not rely on the specific method used to establish the $E\geq 0$ theorem. Instead, it characterizes the minimizer of the modified Regge–Teitelboim Hamiltonian by producing additional modified Killing initial data, under the assumption that the positivity of mass holds.

Theorem 13. *Let (M,g,k) be a asymptotically flat initial data set with decay rate $q>(n-2)/2$ and assume that the positive mass theorem is true near (g,k). If (g,k) satisfies the dominant energy condition and $E=0$, then (M,g,k) is embedded into the Minkowski spacetime.*

As in the proof of Theorem 12, we use the fact that (g,k) minimizes a modified Regge–Teitelboim Hamiltonian $\mathscr{H}$ on the constraint set $\mathscr{C}_{(g,k)}$. Then we invoke the method of Lagrange multiplier to construct a lapse-shift pair (f,X) that solves $D\overline{\Phi}|^*_{(g,k)}(f,X)=0$ with the prescribed asymptotics $(f,X)\to(a,b)$, where $(a,b)=(E,-P)$.

When $E=|P|=0$, it turns out that (g,k) minimizes the modified Regge–Teitelboim Hamiltonian for many choices of (a,b), because the last term of (12) is alway zero. By similar computations as in (13), we can choose any a and b with $a=|b|$. This allows us to construct an entire $(n+1)$-dimensional space of lapse-shift pairs that can be thought of as being asymptotic to the $(n+1)$-dimensional space of translational Killing fields on Minkowski space as we approach spatial infinity. Moreover, having so many solutions implies that (g,k) is vacuum, by [33, Corollary 6.6].

Once we know that (g,k) is vacuum, it follows that each of these lapse-shift pairs is actually vacuum Killing initial data for (g,k) by Moncrief's theorem, and we can extend them to become actual Killing fields on the vacuum development of (g,k). The final step is to show that having such a space of Killing fields on an asymptotically flat Lorentzian manifold that are asymptotic to the $(n+1)$ translational directions implies that the Lorentzian manifold must be Minkowski space.

Declaration of interests

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