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Historical Commentary / *Commentaire historique*

On Yvonne

À propos d'Yvonne

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Abstract. It is a pleasure and honor to present here a personal perspective on Yvonne Choquet-Bruhat's scientific work.

Résumé. C'est un plaisir et un honneur de présenter ici une perspective personnelle sur le travail scientifique d'Yvonne Choquet-Bruhat.

Keywords. Mathematical general relativity, Einstein equations, General relativistic Cauchy problems.

Mots-clés. Relativité générale mathématique, Équations d'Einstein, Problèmes de Cauchy en relativité générale.

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To make this note less subjective, let me start by mentioning the citation list of Yvonne Choquet-Bruhat on MathSciNet, which is opened by her 2009 book

"General relativity and the Einstein equations"

published by Oxford University Press. (I had the pleasure to accompany Yvonne to the OUP offices when she was preparing her book for publication ⌣̈.) In this list the book is followed by the milestone papers

"Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires", Acta Math. 88 (1952), 141–225,

and

"Global aspects of the Cauchy problem in general relativity" with Robert Geroch, Commun. Math. Phys. 14 (1969), 329–335.

These last two papers are to mathematical relativity as what Banach's "Théorie des opérations linéaires" is to functional analysis.

Indeed, her brilliant proof, that Einstein equations possess a well posed Cauchy problem, is the foundation on which mathematical general relativity is built. Nowadays we take it for granted that our universe is described by evolution equations, with initial data prescribed on an initial data surface. The fact that this is the case for general relativity was not known until Yvonne's "Théorême d'existence…" [1]. There are several difficulties here. The first is the diffeomorphism invariance of the Einstein equations, which implies that solutions of Einstein equations can only be unique up to diffeomorphisms. The first discovery in [1] was, that this can be cured by introducing harmonic coordinates. The next difficulty is to prove that the resulting "harmonically-reduced" equations admit local solutions. Her paper provides the first proof of this for a large class of equations with the relevant structure. Last but not least, one needs to show that a solution of the reduced equations provide a solution of the equations of real interest, namely the Einstein equations. It is shown in [1] how the general relativistic constraint equations can be used to solve this problem. A self-contained presentation of these constructions using modern techniques can be found in [2].

The local constructions of [1] immediately raise the question of existence and properties of global solutions of Einstein equations. The second milestone on which mathematical general relativity rests is the already-mentioned paper with Robert Geroch [3]. The paper unravels the fundamental role of the notion of global hyperbolicity in this context, and presents the proof that every set of, say vacuum, general relativistic initial data evolves to a unique spacetime which is maximal in the set of globally hyperbolic spacetimes inducing the given initial data. A nice conceptual simplification of the proof can be found in [4].

It turns out that this is not the end of the story: there exist well behaved initial data sets, such as initial data for the Kerr black holes or for the Bianchi IX cosmological models, for which the maximal globally hyperbolic developments can be extended beyond a Cauchy horizon in a non-unique way. The contents of the *cosmic censorship conjectures* is that this happens only in very special circumstances; more on this can be found in [5, 6]. To resolve this one needs to understand the global structure of generic solutions of the general relativistic Cauchy problem, which is not a minor task. Among the most significant achievements on this topic in the class of spatially compact spacetimes with one Killing vector are the results presented by Yvonne and Vince Moncrief in [7]. In this context it should be added that Yvonne Choquet-Bruhat's analysis of Fuchsian partial differential equations, reviewed in her already mentioned monograph [5], remains one of the best tools to explore the dynamics of cosmological spacetimes near singularities.

Needless to say, these are not the only contributions of Yvonne: MathSciNet lists 239 publications, including 11 books. One finds there seminal papers on the constraint equations in general relativity, including the development, with Demetrios Christodoulou, of the framework for analysing these equations on asymptotically flat manifolds [8]. These constraint equations constitute a set of nonlinear equations that has to be satisfied by the initial data, and one of the key goals of mathematical general relativity is to construct exhaustive families of their solutions, and to understand the properties of these. For many years a standard tool for this has been the "conformal method", some aspects of which have been intensively studied in the mathematics literature in the context of the Yamabe problem [9–11] mainly on compact manifolds. The paper [8] provides key tools to understand elliptic systems on asymptotically flat manifolds, including the general relativistic constraint equations.

One of the fundamental properties of asymptotically flat solutions of the constraint equations is positivity of mass [12, 13]. In a pioneering work with Marsden, Choquet-Bruhat proves positivity for near-Minkowskian initial data [14]. The arguments used there are used nowadays in state-of-the-art gluing theorems for the constraint equations [15, 16].

The general relativistic constraint equations on compact manifolds become quite a headache when matter fields transform in an inconvenient way under conformal transformations. An example is provided by the scalar field, which has been addressed by Yvonne and collaborators in [17]. More on the general relativistic constraint equations can be found in [18, 19], see also [20–24] and references therein.

Choquet-Bruhat's pioneering work with Daniel Bancel, "Existence, uniqueness, and local stability for the Einstein–Maxwell–Boltzman system", Comm. Math. Phys. 33 (1973), 83–96, provides the first rigorous treatment of the Boltzmann equation within Einsteinian gravity. Her prophetic "Ondes asymptotiques et approchées pour des systèmes d'équations aux dérivées partielles non linéaires" J. Math. Pures Appl. (9) 48 (1969), 117–158, has been given new life in recent papers [25,26], which come close to settling Burnett's conjecture [27] concerning a class of high-frequency limits of solutions of Einstein's equations.

I had the pleasure and honor to coauthor a series of papers with Yvonne on the characteristic Cauchy problem in general relativity. There we present a variation of an approach of Rendall, perhaps making clearer the constraints character of some of Einstein equations. In [28] we observe that all the constraints can be reduced to linear equations, the only obstruction to global existence being positivity of a conformal factor. This should be compared to the much more intricate structure of the already-mentioned spacelike constraints. Natural extensions of this work can be found in [29, 30]. It should be mentioned that the role of the characteristic Cauchy problem in general relativity has been steadily growing, see [31–34].

Amongst the various honors received, Yvonne was the first female member of the French Academy of Sciences. She has been included in the list of the 85 original and important female contributors to the physics of the 20th Century on the web site URL https://web.archive.org/web/20161007090528/http://cwp.library.ucla.edu/exp.html. Together with James York, from Cornell University, she was awarded the 2003 Dannie Heineman Prize for Mathematical Physics,"for their separate as well as joint work in proving the existence and uniqueness of solutions to Einstein's gravitational field equations so as to improve numerical solution procedures with relevance to realistic physical solutions". The prize is presented in recognition of outstanding publications in the field of mathematical physics. In 2006 Yvonne Choquet-Bruhat delivered the Emmy Noether Lecture at the International Congress of Mathematicians in Madrid on the topic of "Mathematical problems in General Relativity". Her biography has been included on the web site 'Biographies of women mathematicians", URL https://mathwomen.agnesscott.org/women/bruhat.htm, next to Hypathia, Sofia Kovalevskaya, Emmy Noether, and Miryam Mirzakhani. A volume in her honor can be found on URL https://celebratio.org/ChoquetBruhat_Y/article/1108/.

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