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Review article / *Article de synthèse*

Some aspects of spectral and microlocal methods in Mathematical General Relativity

Quelques aspects des méthodes spectrales et microlocales en relativité générale mathématique

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Abstract. We review some of the important ideas of spectral and microlocal analysis which have been applied to problems in Mathematical General Relativity over the last decades.

Résumé. Nous passons en revue quelques-unes des idées importantes de l'analyse spectrale et microlocale qui ont été appliquées à des problèmes de relativité générale mathématique au cours des dernières décennies.

Keywords. Spectral theory, Microlocal analysis, General Relativity.

Mots-clés. Théorie spectrale, Analyse microlocale, Relativité générale.

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1. Introduction

The aim of this review is to give an overview of important spectral and microlocal methods that have been applied to questions in Mathematical General Relativity over the last decades. We will try to explain some of them using the wave equation on the De Sitter Schwarzschild metric as a leading example.

It is hard to say exactly when spectral and microlocal methods appear first in the context of General Relativity. Their application was certainly pushed by the desire of understanding quantum fields on a curved background starting in the 1970s. Right from the beginning the interaction was very rich, not only existing methods were applied in a new context, but also this new context contributed to developing new methods. For example in their celebrated paper *Fourier Integral Operators II* [1] Duistermaat and Hörmander cite the construction of Feynman propagators for the Klein Gordon equation on a general Lorentzian manifold as a motivation. It later turned out that an important condition on states on curved spacetime, the so called Hadamard condition, is naturally formulated in terms of microlocal analysis, see the work of Radzikowski [2]. Today these states have been constructed by pseudodifferential calculus, see Gérard and Wrochna [3].

In the construction of preferred quantum states on a curved background scattering theory plays an important role and a whole program concerning scattering theory on black hole type backgrounds was initiated by Dimock and Kay in the 1980s, see e.g. [4]. This program was later pushed forward by Bachelot in the 1990s, see e.g. [5] and culminated in his rigorous proof of the Hawking effect in the spherically symmetric setting in 1999 [6]. This picture was later completed by Dappiaggi, Moretti and Pinamonti [7] who showed that the Unruh state which is linked to the Hawking effect fulfills the Hadamard condition. Today our understanding of quantum fields is also very complete for fermions on the Kerr spacetime, see Häfner [8] for the Hawking effect and Gérard, Häfner and Wrochna [9] for the Hadamard property of the corresponding Unruh state. These results rely on classical asymptotic completeness results obtained by Häfner and Nicolas in [10]. For bosons the picture is less complete, but see Klein [11] for the Hadamard property of the Unruh state in the De Sitter Kerr metric. Whereas classical asymptotic completeness results are also known for Klein Gordon fields on the (De Sitter) Kerr spacetime (see Häfner [12], Dafermos, Rodnianski, Shlapentokh-Rothman [13] and Gérard, Georgescu, Häfner [14]), the results of Klein rather use decay estimates as obtained e.g. in Hintz and Vasy [15].

Scattering theory for hyperbolic equations is very much linked to dispersive properties of these equations. The importance of these estimates for non linear problems becomes manifest in the fundamental work on the non linear stability of Minkowski spacetime by Christodoulou and Klainerman [16].

It is very fortunate that the so complicated Einstein equations have explicit solutions like the (De Sitter) Kerr family of solutions describing rotating black holes. However to be sure of the physical importance of these solutions, one has to show that these families are stable as solutions of the Einstein vacuum equations. Again dispersive estimates for the linearized problem play a key role in this context.

Whereas the scattering theory developed in the 1980s and 1990s only showed that the local energy of the underlying equations decays to zero, quantitative decay estimates were now needed. Roughly speaking decay of the solution translates to regularity properties of the underlying resolvent and vice-versa. A particular favorable situation in this context is the case when the resolvent in suitable function spaces has a meromorphic extension in a strip beyond the real axis. The poles of this extension are then called resonances. It turns out that this is the case for the De Sitter Kerr metric.

The mathematical study of resonances for black hole spacetimes was initiated by Bachelot and Motet-Bachlot in 1993 [17]. A bit later Sá Barreto and Zworski [18] succeeded to localize the high frequency resonances of a De Sitter Schwarzschild black hole. Later on Bony and Häfner [19] obtained an expansion of the local propagator of the wave equation on the De Sitter Schwarzschild metric. This expansion in particular gives very precise dispersive estimates. The analysis of Bony and Häfner was extended to include also the horizon by Melrose, Sá Barreto and Vasy [20]. Dyatlov then obtained similar results for the De Sitter Kerr metric [21, 22]. The most general version of this theorem including De Sitter Kerr spacetime for small angular momentum of the black hole is due to Vasy [23]. The work of Vasy is an important breakthrough in this context as it doesn't use the symmetries of Kerr spacetime a part from the time invariance. More generally it develops a robust Fredholm theory for non elliptic problems, that turned out to be crucial in the following. This came out as a kind of surprise since Fredholm theory so far seemed to be reserved for elliptic problems. Not surprisingly, these new methods rapidly also gave interesting results in quantum field theory, see e.g. Vasy and Wrochna [24].

An important obstacle for showing dispersive estimates on black hole spacetimes is trapping that appears in these spacetimes. More precisely to show the resonance expansion theorems one has to control the high energy regime of the resolvent which is directly linked to the trapping. Fortunately the trapping on (De Sitter) Kerr spacetimes is very mild, more precisely it is r-normally hyperbolic. Suitable resolvent estimates in the presence of r-normally hyperbolic trapping have been shown by Wunsch-Zworski in [25] and Dyatlov [26], see also Hintz [27].

The program on dispersive estimates on the De Sitter Kerr family culminated in the proof of the non linear stability of this family for small angular momentum of the black holes by Hintz and Vasy in 2016 [28]. Whereas the restriction to small angular momentum has been removed in the linear framework, see Petersen and Vasy [29], it remains for the non linear framework, because growing modes for the linearized problem can't be excluded so far, but see Hintz [30] for important progress in this direction.

In the context of the Kerr family new difficulties appear. On the linear level, the underlying resolvent can't be extended any more meromorphically beyond the real axis. This is a low frequency problem and very close to what happens when considering evolution equations on an asymptotically euclidean manifold. It turns out that the resolvent has nevertheless certain C^k properties down to the real axis permitting to show polynomial decay for several evolution equations, see e.g. the series of papers by Bony and Häfner [31, 32]. However, for the wave equation the low frequency analysis becomes significantly simpler if one uses a hyperboloidal foliation, see Vasy [33,34]. We also refer to Baskin, Vasy and Wunsch [35,36] for the asymptotically Minkowskian context. Let us also mention the work of Gajic and Warnick [37] and Stucker [38] about the definition of quasinormal modes in this context. The above mentioned low frequency problems are very different from the above mentioned high frequency problems and one of the advantages of spectral and microlocal analysis is that these problems can be very clearly separated and then treated by very different methods.

All this has then been used to show linear stability of the Kerr family for small angular momentum, see Häfner, Hintz and Vasy [39]. Recall that most black holes are rapidly spinning, see e.g. Thorne [40], such that it would be important to remove the restriction to small angular momentum. The very robust Fredholm framework makes that the restriction to small angular momentum only comes from the mode analysis. Indeed all essential features for the analysis like the structure of the horizon or the nature of trapping are the same in the case of large angular momentum (see e.g. Dyatlov [41]). In contrast to the De Sitter Kerr case, mode stability for the Teukolsky equation on the Kerr background is known in the full subextreme range in the Kerr case, see Whiting [42]. This was then used by Andersson, Häfner and Whiting [43] to give a full analysis of the mode solutions for the linearized Einstein equations. As an illustration how the whole analysis can be carried out in the full subextreme case we cite Millet [44] who obtains optimal decay results for the Teukolsky equation on the Kerr background, see also the work of Hintz [45] in this context.

The relatively weak decay for linearized gravity[1] makes the analysis of the non linear problem for the Kerr family more difficult than for the the De Sitter Kerr family. In a series of papers Klainerman–Szeftel [46–48] and Giorgi–Klainerman–Szeftel [49] give a proof of non linear stability of the Kerr family in the case of small angular momentum, see also Shen [50]. We also refer to the work of Dafermos, Holzegel, Rodnianski and Taylor [51] for the non linear stability of the Schwarzschild metric.

Let us also briefly mention the case of negative cosmological constant. If one imposes for example Dirichlet boundary conditions for the Klein Gordon equation on the Anti De Sitter Kerr metric, the trapping becomes more severe. As a consequence the decay for the equation becomes weaker, see e.g. Gannot [52] and Holzegel, Smulevici [53].

Microlocal methods also play an important role in the context of strong cosmic censorship in particular in the case of positive cosmological constant. Considering a (De Sitter) Kerr or (De Sitter) Reissner–Nordström black hole, one can see that there exists a stabilizing effect at the black hole horizon, often called redshift effect and an inverse effect at the Cauchy horizon, called blueshift. It is interesting to see that these redshift and blueshift effects are directly linked

[1] More precisely the solution of the linearized problem will decay to some linearized Kerr plus gauge terms.

to radial point estimates which play an important role in establishing the Fredholm setting for linear equations on the background, see Hintz and Vasy [15]. In the case of zero cosmological constant the asymptotic behavior of the metric at infinity leads to only inverse polynomial decay at the black hole horizon and the blueshift effect at the Cauchy horizon seems to be stronger leading in the end to an instability of the Cauchy horizon, see the work of Dafermos [54] as well as Luk and Oh [55,56]. In the case of positive cosmological constant there is a subtle competition between both effects. For example the question if the solution of the wave equation is H^1 near the cosmological horizon is directly linked to the distance to the real axis of the resonance which is closest to the real axis[2], see Cardoso, Costa, Destounis and Hintz [57]. Unfortunately the exact position of these low frequency resonances is very hard to compute. Nevertheless it is possible that strong cosmic censorship fails for certain black hole parameters. Interestingly, it seems that it is always valid on a quantum level, see the work of Hollands, Wald and Zahn [58]. At first glance it might be surprising that only the resonance position that comes somehow from the analysis outside the black hole determines what is going on on the Cauchy horizon. There is indeed a scattering process between the black hole horizons and the Cauchy horizons and scattering theory for the wave equation has been developed in this context by Kehle and Shlapentokh-Rothman [59] in the spherically symmetric case. Remarkably as already observed by Chandresekhar and Hartle [60] the poles of the corresponding scattering matrix can be computed and they lie too far from the real axis to contribute to the instability although one can give data on the black hole horizons which lead to certain instabilities. Note that asymptotic completeness results inside the black hole can also be obtained by time dependent methods, see the work of Häfner, Nicolas and Mokdad [61] and Mokdad and Provci [62]. However this doesn't give a scattering matrix which is needed to study the behavior in the directions transverse to the Cauchy horizon.

We concentrate in this review on spectral analysis, resolvent analysis and Fourier transform. This however doesn't mean that microlocal analysis cannot be applied in non stationary situations. Indeed in general one will be faced with situations where the operators are not stationary but settle down in some sense to a stationary situation. In this situation the here described analysis is the first of two ingredients for the global analysis of the operator. The second ingredient is a control of regularity of solutions of the non stationary equation and requires in addition other techniques which can however be fully microlocal, see e.g. the recent work of Hintz [63]. We also refer to Hintz and Vasy [64] in this context.

In the above review we have concentrated on the mathematical literature on the subject. There is also a very important physics literature that unfortunately we can't review here. Also it goes without saying that only some of the aspects of microlocal analysis in General Relativity can be treated in this review and inevitable important aspects and important contributions will be missing. We hope that the review gives nevertheless an idea of some of the important ideas that have been developed over the last years.

The review is organized as follows. In Section 2 we review some of the important aspects of the (De Sitter) Kerr family of spacetimes which are used in the microlocal approach. A few of the above mentioned results are presented in some detail in Sections 3, 4 gives some minimal notions of mircolocal and semiclassical analysis. In Section 5 we discuss the wave equation on the De Sitter Schwarzschild metric. Through this relatively simple example we try to explain two of the main ideas for showing decay estimates for hyperbolic equations via microlocal methods. Firstly the Fredholm setting and the underlying estimates (in particular radial point estimates) and secondly the high frequency estimates in the presence of trapping.

[2] There is in fact a zero resonance but which is not included in this discussion.

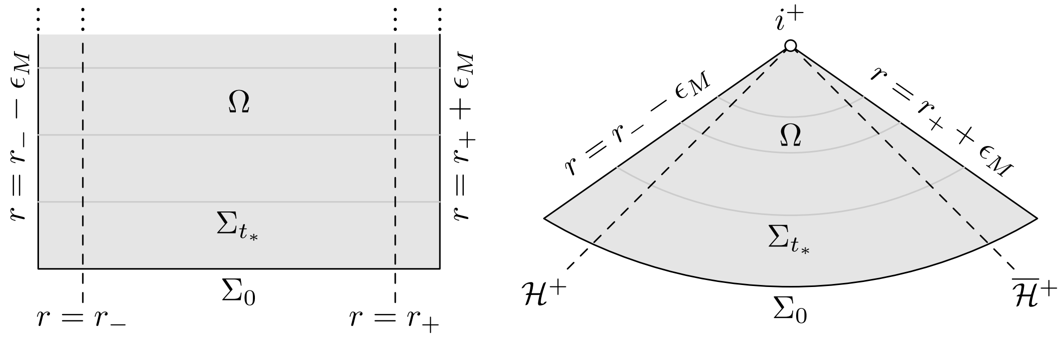

Figure 2.1. Choice of coordinates, source: [28].

2. Black hole spacetimes

2.1. *The De Sitter Schwarzschild metric*

We first consider the De Sitter Schwarzschild metric in Boyer Lindquist coordinates. For $\mathfrak{m} > 0$ we put $b_0 = (\mathfrak{m}, 0)$. The manifold and metric (M, g) are then given by

$$M = \mathbb{R}_t \times \tilde{X}, \quad \tilde{X} = \mathscr{I}_{b_0} \times S^2_\omega, \quad \mathscr{I}_{b_0} = (r_-, r_+), \quad g = F_{b_0} \mathrm{d}t^2 - F_{b_0}^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\omega^2, \quad F_{b_0}(r) = 1 - \frac{2\mathfrak{m}}{r} - \frac{\Lambda r^2}{3}.$$

Here $\mathfrak{m} > 0$ is the mass of the black hole and $\Lambda > 0$ is the cosmological constant. We require $9\mathfrak{m}^2\Lambda < 1$ in the following. F_{b_0} then has two positive roots that we call r_-, r_+. The potential $V_{b_0}(r) = F_{b_0}(r)/r^2$ has a unique maximum on (r_-, r_+) given by $r_0 = 3\mathfrak{m}$. The Regge–Wheeler coordinate r_* is defined by the requirement $\mathrm{d}r_*/\mathrm{d}r = F_{b_0}^{-1}$ and that $r_*(3\mathfrak{m}) = 0$. The coordinate $t_*(r_*)$ is defined as a smooth function of r_* such that

$$t_* = \begin{cases} t + r_* & r \le 3\mathfrak{m}, \\ t - r_* & r \ge 4\mathfrak{m}. \end{cases}$$

We find in the coordinates (t_*, r, ω) and for $r \le 3\mathfrak{m}$:

$$g = F_{b_0} \mathrm{d}t_*^2 - 2\,\mathrm{d}t_*\,\mathrm{d}r - r^2\,\mathrm{d}\omega^2$$

as well as a similar formula for $r \ge 4\mathfrak{m}$. This shows that the metric extends for some $\epsilon > 0$ to $\mathscr{M} = \mathbb{R}_{t_*} \times X$, $X = (r_- - \epsilon, r_+ + \epsilon) \times S^2_\omega$. The null hypersurfaces $\mathscr{H}^+ = \mathbb{R}_{t_*} \times \{r_-\} \times S^2_\omega$ and $\overline{\mathscr{H}}^+ = \mathbb{R}_{t_*} \times \{r_+\} \times S^2_\omega$ are called the future event horizon and the future cosmological horizon, see Figure 2.1. A similar construction can be performed in the past, constructing then the past event horizon $\mathscr{H}^-$ and the past cosmological horizon $\overline{\mathscr{H}}-$.

2.2. *The De Sitter Kerr metric*

The De Sitter Kerr metric describes black holes with non zero angular momentum. For $\mathfrak{m} > 0$, $\mathfrak{a} \in \mathbb{R}^3$ we put $b = (\mathfrak{m}, \mathfrak{a})$. Let also $a = |\mathfrak{a}|$. We again start with Boyer Lindquist coordinates. For a suitable interval $\mathscr{I}_b \subset \mathbb{R}$ defined below the metric takes the form

$$g_b = \frac{F_b}{(1+\lambda_b)^2\rho_b^2}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi)^2 - \frac{\kappa_b \sin^2\theta}{(1+\lambda_b)^2\rho_b^2}(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi)^2 - \rho_b^2\left(\frac{\mathrm{d}r^2}{F_b} + \frac{\mathrm{d}\theta^2}{\kappa_b}\right),$$

$$F_b = (r^2+a^2)\left(1 - \frac{\Lambda r^2}{3}\right) - 2\mathfrak{m}r, \quad \rho_b^2 = r^2 + a^2\cos^2\theta, \quad \lambda_b = \frac{\Lambda a^2}{3}, \quad \kappa_b = 1 + \lambda_b \cos^2\theta.$$

Under suitable conditions F_b has two positive roots $r_{b,-}, r_{b,+}$ and $\mathscr{I}_b = (r_{b,-}, r_{b,+})$ in this case. We can construct a similar coordinate t_* as in the De Sitter Schwarzschild case. If one also changes the coordinate ϕ to some coordinate ϕ^*, then the metric g can be extended in the new coordinates (t_*, r, θ, ϕ^*) to the manifold $\mathcal{M} = \mathbb{R}_{t_*} \times X$, $X = (r_{b,-} - \epsilon, r_{b,+} + \epsilon) \times S^2_{\theta,\phi^*}$. Event horizons and cosmological horizons are defined in a similar way to the De Sitter Schwarzschild case.

2.3. *Schwarzschild and Kerr metric*

In Boyer-Lindquist coordinates the Schwarzschild and the Kerr metric are formally obtained by putting $\Lambda = 0$. Note that the interval $\mathcal{I}_{b_0}$ then becomes $\mathcal{I}_{b_0} = (2\mathfrak{m}, \infty)$ in the Schwarzschild case and the interval $\mathcal{I}_b$ becomes $\mathcal{I}_b = (\mathfrak{m} + \sqrt{\mathfrak{m}^2 - a^2}, \infty)$ in the Kerr case. The spacetimes are now asymptotically Minkowskian, no cosmological horizon exists. We then have $\mathcal{M} = \mathbb{R}_{t_*} \times X$, $X = (\mathfrak{m} + \sqrt{\mathfrak{m}^2 - a^2} - \epsilon, \infty)$. Passing to the conformally rescaled metric $\hat{g} = r^{-2} g$, one can construct future and past null infinity in a similar way to the construction of the cosmological horizons. In the Kerr case we will also need the coordinate t which equals t_* near the future event horizon and t near infinity.

2.4. *Some important properties of black hole spacetimes*

2.4.1. *Asymptotic behavior*

The fundamental difference between positive and zero cosmological constant is that the spacetimes are asymptotically De Sitter in the first case and asymptotically Minkowskian in the second case. In the asymptotically Minkowski case we have more precisely:

$$g_{b_0} = g_{(0,0)} + \mathcal{O}(r^{-1}), \quad g_b = g_{b_0} + \mathcal{O}(r^{-2}), \quad r \to \infty.$$

Here $b_0 = (\mathfrak{m}, 0)$, $b = (\mathfrak{m}, \mathfrak{a})$ and $g_{(0,0)}$ is the Minkowski metric. As we will explain in the next section, the different asymptotic behavior has important consequences for the dispersive properties of solutions of the wave equation.

2.4.2. *Superradiance*

It is easy to see that the vector field ∂_t becomes spacelike in a region outside the horizon if the rotation $\mathfrak{a}$ is non zero. This holds for the De Sitter Kerr as well as for the Kerr case. More generally there doesn't exist a global timelike Killing vector field. As a consequence there doesn't exist a positive conserved quantity for the wave equation and the evolution can't be described by a selfadjoint hamiltonian.

2.4.3. *Trapping in the De Sitter Schwarzschild case*

We first consider the De Sitter Schwarzschild case. The associated wave operator is

$$\Box_{g_{b_0}} = F_{b_0}^{-1} \partial_t^2 - \frac{1}{r^2} \partial_r r^2 F_{b_0} \partial_r - r^{-2} \Delta_{\mathbb{S}^2}.$$

Let $P(\sigma) = F_{b_0} \tilde{\Box}_{g_{b_0}}(\sigma)$, where $\tilde{\Box}_{g_{b_0}}(\sigma)$ is the Fourier transform of $\Box_{g_{b_0}}$ with respect to t. Then $P(\sigma)$ has symbol

$$p = \xi^2 + V_{b_0} |\eta|^2 - \sigma^2,$$

where σ, ξ, η are dual to t, r_*, ω and $V_{b_0} = F_{b_0}/r^2$. The projections of the bicharacteristics of p on M are null geodesics, but it is actually more interesting to consider the bicharacteristics themselves. The quantities σ and $|\eta|^2$ are clearly conserved along the evolution and the Hamiltonian equations reduce to

$$\dot{r}_* = 2\xi, \quad \dot{\xi} = -V'_{b_0} |\eta|^2.$$

A solution is clearly given by $(r_*, \xi) = (0,0)$. Linearization around this solution leads to the equation:

$$\dot{r}_* = 2\xi, \quad \dot{\xi} = -V''_{b_0}(0)|\eta|^2 r_*.$$

The matrix

$$\begin{pmatrix} 0 & 2 \\ -V''_{b_0}|\eta|^2 & 0 \end{pmatrix}$$

has eigenvalues $\lambda_\pm = \pm\sqrt{-2V''_{b_0}(0)}|\eta|$. The point (0, 0) is therefore a hyperbolic fixed point of this reduced system.

2.4.4. *r-normally hyperbolic trapping*

In this subsection we describe the trapping arising in black hole spacetimes in a more general framework. We follow closely Dyatlov [26]. Let X be a compact manifold, $p : T^*X \to \mathbb{R}$ a Hamiltonian with associated Hamiltonian vector field H_p. We mimic infinity by an absorption operator Q with semiclassical symbol q, see Section 4.3 for the definition. We define the backward/forward trapped sets $\Gamma_\pm$ and the trapped set K in the following way

(1) $\Gamma_\pm$ are codimension 1 orientable C^∞ submanifolds of T^*X such that $\Gamma_\pm \cap \{p=0\} \cap \{q=0\}$ are compact.
(2) if $(x,\xi) \in \{p=0\} \setminus \Gamma_\pm$, then $e^{\mp tH_p}(x,\xi) \in \{q>0\}$ for some $t \geq 0$;
(3) H_p is tangent to $\Gamma_\pm$.
(4) $\Gamma_\pm$ intersect transversely, $K = \Gamma_+ \cap \Gamma_-$ is called the trapped set, and we assume that $WF'_h(Q) \cap K \cap \{|p| \leq \delta\} = \emptyset$ for $\delta > 0$ small enough, see Section 4.3 for a definition of $WF'_h(Q)$.
(5) K is a symplectic codimension 2 submanifold of T^*X.
(6) if $v \in T_K\Gamma_\pm$, then $de^{\mp tH_p} v$ exponentially approaches $TK \subset T_K\Gamma_\pm$ as $t \to \infty$.

We furthermore define $0 < \nu_{\min} \leq \nu_{\max}$ as the maximal and the minimal numbers such that for each $\epsilon > 0$ there exists a constant C such that for each $v \in T_K\Gamma_\pm$

$$C^{-1}e^{-(\nu_{\max}+\epsilon)t}|\pi(v)| \leq |\pi(de^{\mp tH_p} v)| \leq Ce^{-(\nu_{\min}-\epsilon)t}|\pi(v)|, \quad t \geq 0,$$

where $\pi : T_K\Gamma_\pm \to T_K\Gamma_\pm$ is any fixed smooth linear projection whose kernel is equal to TK. In other words, $\nu_{\min}$ and $\nu_{\max}$ are the minimal and maximal expansion rates in directions transversal to the trapped set. In the above situation we will speak about normally hyperbolic trapping, see Figure 2.2. The notion of r-normally hyperbolic trapping requires in addition that the rate of expansion transversal to K is much larger than the rate of expansion on K. More precisely the trapping is r-normally hyperbolic, if $\nu_{\min} > r\mu_{\max}$, where $\mu_{\max}$ is the maximal expansion rate along K, namely

$$\|de^{tH_p}|_{TK}\| = \mathcal{O}(e^{(\mu_{\max}+\epsilon)t}), \quad \text{as } |t| \to \infty$$

for all $\epsilon > 0$. It can be shown that the trapping on the (De Sitter) Kerr metric is r-normally hyperbolic for all r, see Dyatlov [41]. This is a structurally stable property, see Fenichel [65] as well as Hirsch, Pugh and Shub [66] for details.

3. Stability results obtained via mircolocal methods

In this section we recall some linear and non linear stability results which have been obtained via microlocal methods.

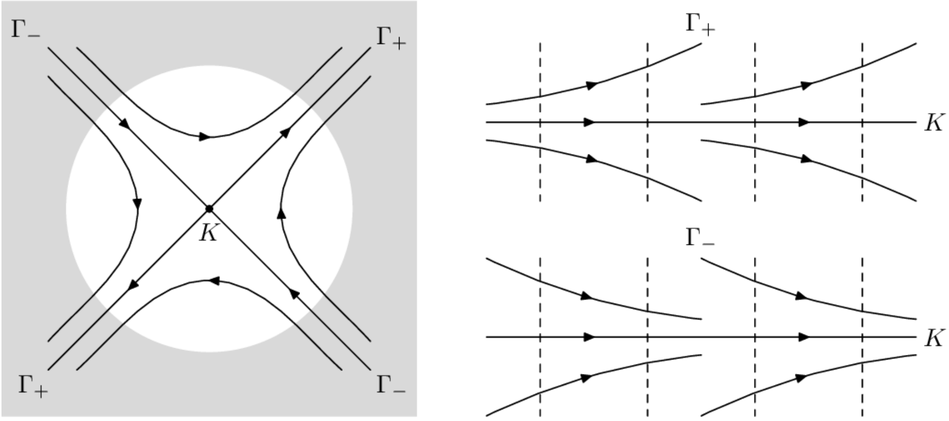

Figure 2.2. Normally hyperbolic trapping, source: [26].

3.1. *The Cauchy problem in General Relativity*

We consider the Einstein vacuum equations

$$\mathrm{Ric}(g) + \Lambda g = 0. \tag{3.1}$$

Initial data for this equation are a triplet (Σ, h, k) consisting of a 3-manifold Σ, a riemannian metric h on Σ and a symmetric 2-tensor k on Σ subject to the *constraint equations*, which are the Gauss–Codazzi equations on Σ implied by (3.1). Let us fix Σ as a submanifold of $\mathscr{M}$. For a Lorentzian metric g on $\mathscr{M}$ we define $\tau(g) = (-g|_\Sigma, \Pi^g_\Sigma)$, where Π^g_Σ is the second fundamental for of Σ within $\mathscr{M}$. A metric g is said to solve the initial value problem with data (Σ, h, k) if

- Σ is spacelike with respect to g;
- $\tau(g) = (h, k)$.

The pioneering work of Choquet-Bruhat [67] shows that the Einstein equations have a local solution for smooth data fulfilling the constraint equations. The non linear stability problem for black hole solutions consists now in slightly perturbing initial data of these solutions and to ask if the initial value problem has a global solution which stays close to one of the members of the family. This is an orbital stability. Note that because of energy radiation each member of the family is not expected to be individually stable.

3.2. *The global nonlinear stability of the De Sitter Kerr family*

The first non linear stability result for black holes was obtained by Hintz and Vasy in 2016 for slowly rotating De Sitter Kerr family. We state it here in its easiest form, see [28] for the full precise formulation of the result. Let us fix a De Sitter Schwarzschild spacetime $(\mathscr{M}, g_{b_0})$. Within this we consider a compact spacelike hypersurface $\Sigma_0 \subset \{t_* = 0\} \subset \mathscr{M}$ extending slightly beyond the event and cosmological horizons. Denote by Σ_{t_*} the translates of Σ_0 by ∂_{t_*} and let $\Omega = \bigcup_{t_* \geq 0} \Sigma_{t_*} \subset \mathscr{M}$ be the spacetime region swept out by these, see Figure 2.1.

Theorem 3.1 (Hintz–Vasy 2016). *Suppose that (h, k) are smooth initial data on Σ_0, satisfying the constraint equations, which are close to the data (h_{b_0}, k_{b_0}) of a De Sitter Schwarzschild spacetime in a high regularity norm. Then there exists a solution g of (3.1) attaining these initial data at Σ_0, and black hole parameters b which are close to b_0, so that*

$$g - g_b = \mathscr{O}(\mathrm{e}^{-\alpha t_*})$$

for a constant $\alpha > 0$ independent of the initial data; that is, g decays exponentially fast to the De Sitter Kerr metric g_b.

Note that data for slowly rotating De Sitter Kerr black holes are included so that the above result is a non linear stability result for the De Sitter Kerr family in case of slow rotation of the black holes.

3.3. *Linear stability of the Kerr family*

As seen above, the decay in the De Sitter Kerr case is very fast. As a consequence passing from the linear to the non linear stability result is not too difficult. This is different in the Kerr case and the full non linear stability of the Kerr family has not been shown so far via spectral and mircolocal methods, but see the series of papers by Klainerman–Szeftel [46], Giorgi–Klainerman–Szeftel [49] for a proof via different methods. However a linear stability result has been obtained by Häfner–Hintz–Vasy via spectral and microlocal methods which we state now in its easiest version. Again we refer to the original version [39] for full details[3]. Consider black hole parameters b close to $b_0 = (\mathfrak{m}_0, 0)$.

Theorem 3.2 (Linear stability of the Kerr family, H-Hintz–Vasy 2019). *Let γ', k' be symmetric 2-tensors on $\Sigma = \mathfrak{t}^{-1}(0)$ satisfying the linearized constraint equations and*

$$|\gamma'| \lesssim r^{-1-\alpha}, \quad |k'| \lesssim r^{-2-\alpha}, 0 < \alpha < 1,$$

(and similar bounds for derivatives ∂_ω, $r\partial_r$ up to order 8). Then there exists a symmetric 2-tensor h on $\mathcal{M}$ such that

$$D_{g_b}\mathrm{Ric}(h) = 0, \quad D_{g_b}\tau(h) = (\gamma', k'),$$

which decays to a linearized Kerr metric,

$$h = \dot{g}_b(\dot{b}) + \tilde{h}, \quad |\tilde{h}| \lesssim t_*^{-1-\alpha+}, \quad \left(\dot{g}_b(\dot{b}) := \frac{\mathrm{d}}{\mathrm{d}s} g_{b+s\dot{b}}\Big|_{s=0}\right).$$

Note that Andersson et al. [69] also obtained a linear stability result for the Kerr family, but their decay assumptions on the data are stronger, which eliminates the linearized Kerr solution. Earlier results by Dafermos, Holzegel and Rodnianski [70] as well as Hung, Keller and Wang [71] establish linear stability of the Schwarzschild solution.

3.4. *Remarks on gauge issues*

To carry out the analysis in the linear or non linear case one has to fix a gauge and one usually then considers a gauge fixed operator. On the linear level one can then consider this gauge fixed operator with general initial data. This is an advantage in an iteration scheme to solve the non linear equation because one doesn't have to worry about the linearized constraint equations at each step. On the other hand the gauge fixed linear operator can now have growing modes which are not physical in the sense that they are not solutions of the linearized Einstein equations without gauge. Unfortunately this kind of scenario is produced when one uses the usual wave map gauge. In the De Sitter Kerr case one even expects exponentially growing modes. In the Kerr case the situation is more favorable and only generalized quadratically growing modes can arise. The handling of these gauge issues is beyond the scope of this review. Let us just say that in the Kerr case one can get rid of these modes by the so called constraint damping, we refer to [28] for the De Sitter Kerr case. In the Kerr case once the constraint damping implemented we obtain a linearized operator L_b and the gauge fixed version of Theorem 3.2 is the following.

[3] Note that a small error in [39] has been corrected in [68], the main theorem remains unchanged.

Theorem 3.3 (H-Hintz–Vasy 2019). *Let $\alpha \in (0,1)$, and let $h_0, h_1 \in \mathscr{C}^\infty(\Sigma; S^2 T^*_\Sigma \mathscr{M})$,*

$$|h_0| \lesssim r^{-1-\alpha}, \quad |h_1| \lesssim r^{-2-\alpha}, 0 < \alpha < 1$$

(and similar bounds for derivatives). Let

$$\begin{cases} L_b h = 0, \\ (h|_\Sigma, \mathscr{L}_{\partial_t} h|_\Sigma) = (h_0, h_1). \end{cases}$$

Then there exist $\dot{b} \in \mathbb{R} \times \mathbb{R}^3$ and vector fields $W, V \in$ on $\mathscr{M}$ such that

$$h = \dot{g}'_b(\dot{b}) + \mathscr{L}_V g_b + \mathscr{L}_W g_b + \tilde{h}, \quad |\tilde{h}| \lesssim t_*^{-1-\alpha+}, \quad |\mathscr{L}_W g_b| \lesssim t_*^{-\alpha+}.$$

Here $\dot{g}'_b(\dot{b})$ is a gauge fixed version of $\dot{g}_b(\dot{b})$ and V lies in a fixed 7-dimensional vector space consisting of asymptotic translations, asymptotic boosts and a Coulomb type vector field.

A part from the Coulomb type solutions, the vector field V lies exactly in the space one would expect. In principle one can read off the change of the black hole parameters and the movement of the black hole in the given gauge. Let us also mention that one can get rid of the gauge solution $\mathscr{L}_W g_b$ by admitting a gauge source term.

3.5. *The large a case*

As will be explained later on, the proof of these theorems relies on a very robust Fredholm framework. The only ingredient which is not stable with respect to perturbations is the mode analysis. This mode analysis for the linearized Einstein equations has been carried out by Andersson, Häfner and Whiting [43] also in the large a case. Precise decay estimates for the Teukolsky equation (including the computation of the leading order term) have been obtained by these techniques by Millet in [44]. We give here a rough version of the theorem without defining precisely the Teukolsky operator T_s. We refer to [72] for a review of the geometric background for the Teukolsky operator.

Theorem 3.4 (Millet 2023). *Let $0 \le a < \mathfrak{m}$. We fix $s \in \{0, \pm 1/2, \pm 1, \pm 3/2, \pm 2\}$. Let u_0, u_1* smooth and compactly supported *on Σ_0. The solution u of the Cauchy problem*

$$\begin{cases} T_s u = 0 \\ u(t=0) = u_0 \\ \dfrac{\partial}{\partial t} u(t=0) = u_1 \end{cases}$$

satisfies:

$$|u(r, t_*, \theta, \phi) - \mathfrak{p}_{u_0,u_1}(r, t_*, \theta, \phi)| \le C r^{-1+} t_*^{-2-|s|+s-\epsilon} \left(\frac{t_*}{r} + 1 \right)^{-1-s-|s|}$$

where $\epsilon > 0$. We have

$$\mathfrak{p} = t_*^{-3-2|s|} \frac{(2|s|+2)\left(\frac{t_*}{r}\right)^{2+|s|+s} + 2(|s|-s+1)\left(\frac{t_*}{r}\right)^{1+|s|+s}}{\left(\frac{t_*}{r}+2\right)^{2+|s|+s}} F_{u_0,u_1},$$

where $F_{u_0,u_1}(r,\theta,\phi)$ can be expressed in terms of hypergeometric functions and spin weighted spherical harmonics. The hypotheses of the above theorem can be relaxed. In particular H^N regularity for N sufficiently large is sufficient. Also the compact support of initial data is not necessary, inverse polynomial decay can be permitted and the exact decay rate then depends on the exact decay rate of the initial data. We refer to [44] for details. Similar results also in the large a case have been obtained by Shlapentokh Rothman and Teixeira Da Costa [73, 74] but the precise decay rates are weaker. For entire spin and small a the above result has been obtained before by Ma–Zhang in [75]. The leading order term has already been computed for the wave equation by Hintz [45] in the large a case.

4. Microlocal and semiclassical analysis

In standard microlocal analysis the asymptotic parameter is given by $|\xi|$, where ξ is the fiber variable. We start our representation with a review of that theory. In the semiclassical theory a small parameter h is added to measure the wave length of oscillations. We are then concerned with asymptotics as $h \to 0, \xi \to \infty$ and to make the link with standard microlocal analysis h can be thought as $h = 1/|\xi|$. A fundamental reference for the subject is Hörmander [76]. We also refer to Zworski [77] for the semiclassical part. Our presentation owes a lot to the lecture notes of Wunsch [78] and Hintz [79] as well as to the thesis of Hintz [80].

4.1. *Microlocal analysis*

In this section we present some of the main objects of microlocal analysis we will need. Let us start with a differential operator on $\mathbb{R}^n$ of order m,

$$A = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha \in \mathrm{Diff}^m(\mathbb{R}^n).$$

We can rewrite this expression using the Fourier transform $\mathscr{F}$:

$$(Au)(x) = \sum_{|\alpha|\le m} a_\alpha(x)\mathscr{F}^{-1}(\xi^\alpha(\mathscr{F}u)(\xi)) = \frac{1}{(2\pi)^n}\int e^{i\langle x,\xi\rangle} \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \hat{u}(\xi)\, d\xi =: (Op(a)u)(x),$$

where $a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ is the symbol of the operator. The above writing suggests that one can permit more general symbols $a(x,\xi)$ and then define the corresponding operator $Op(a)$ by

$$(Op(a)u)(x) = \frac{1}{(2\pi)^n}\int e^{ix\xi} a(x,\xi)\hat{u}(\xi)\, d\xi = \frac{1}{(2\pi)^n}\iint e^{i\langle (x-y),\xi\rangle} a(x,\xi)u(y)\, dy\, d\xi. \tag{4.1}$$

A smooth function a on $T^*\mathbb{R}^n$ is a symbol of order m if the following estimate holds:

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}, \quad \forall \alpha,\beta \in \mathbb{N}^n.$$

Let $S^m(\mathbb{R}^n)$ be the set of symbols of order m. If $a \in S^m$ we define $Op(a)$ by (4.1). If S^m is the set of symbols of order m, we define the set of pseudodifferential operators of order m as

$$\Psi^m(\mathbb{R}^n) = \{Op(a),\, a \in S^m(\mathbb{R}^n)\}.$$

For symbols in S^m one can define other quantizations, for example the Weyl quantization

$$(a^w(x,D)u)(x) = \frac{1}{(2\pi)^n}\iint e^{i\langle (x-y),\xi\rangle} a\left(\frac{x+y}{2},\xi\right) u(y)\, dy\, d\xi$$

is often used. If $A \in \Psi^m(\mathbb{R}^n)$ we define the principal symbol $\sigma_m(A) \in S^m/S^{m-1}$ by

$$\sigma_m(A) =: a(x,\xi) \bmod S^{m-1}.$$

The principal symbol is independent of the choice of coordinates and it doesn't depend on the particular quantization we use. We now collect some important properties of pseudodifferential operators.

Theorem 4.1 (Fundamental properties of pseudodifferential operators). (1) *Algebra property. Ψ^m is a vector space for each $m \in \mathbb{R}$. If $A \in \Psi^m$ and $B \in \Psi^{m'}$, then $AB \in \Psi^{m+m'}$, $A^* \in \Psi^m$. The composition of operators is associative and distributive. The identity operator is in $\Psi^0(X)$.*

(2) *Boundedness. If $A \in \Psi^m$, then $A \in \mathscr{L}(H^s, H^{s-m})$ for all $s \in \mathbb{R}$.*

(3) *Principal symbol homomorphism.*

$$\sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B), \quad \sigma_m(A^*) = \overline{\sigma(A)}.$$

(4) *Symbol of commutator. If* $A \in \Psi^m(X)$, $B \in \Psi^{m'}(X)$, *then* $[A,B] \in \Psi^{m+m'-1}(X)$ *and we have*

$$\sigma_{m+m'-1}([A,B]) = \mathrm{i}\{\sigma_m(A), \sigma_{m'}(B)\} = H_a b, \quad \text{if } a = \sigma_m(A), b = \sigma_m(B).$$

Here $\{.,.\}$ *is the Poisson bracket and* H_a *the hamiltonian vector field associated to* a.

Definition 4.2. (1) *The symbol* $a \in S^m$ *is elliptic at a point* $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ *if there exists* $c, R > 0$ *and a conic (in the* ξ *variable) neighborhood* U *of* (x_0, ξ_0) *in* $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ *such that* $|a(x,\xi)| \geq c\langle\xi\rangle^m$ *for all* $(x,\xi) \in U$, $|\xi| \geq R$. *The set of all points at which* a *is elliptic is called the elliptic set* $\mathrm{Ell}(a)$, *and its complement the characteristic set* $\mathrm{Char}(a)$. *We say that* a *is uniformly elliptic if it is elliptic at every point with constants* R *and* c *uniform in* x.

(2) *The essential support ess supp of* $a \in S^m$ *is the complement of the set of all* $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ *for which* $a \in S^{-\infty}$ *in a conic neighborhood of* (x_0, ξ_0).

(3) *For* $A = Op(a)$ *we define its wavefront set as* $WF'(A) :=$ *ess supp* a.

(4) *Let* $u \in \mathscr{D}'(\mathbb{R}^n)$. *We define the wave front set of* u *as follows.* (x,ξ) *is not contained in* $WF(u)$ *if and only if there exists an operator* $A \in \Psi^0$ *which is elliptic at* (x,ξ) *and a smooth cut-off* $\psi \in C_c^\infty(\mathbb{R}^n)$, $\psi(x) \neq 0$, *such that* $A(\psi u) \in C^\infty(\mathbb{R}^n)$.

We need two additional important results

Theorem 4.3. *Let* $P \in \Psi^m$ *be a uniformly elliptic operator. Then there exists* $Q \in \Psi^{-m}$ *such that*

$$QP - 1 \in \Psi^{-\infty}, \quad PQ - 1 \in \Psi^{-\infty}.$$

Another important result is the sharp Gårding inequality.

Theorem 4.4. *Let* $a \in S^{2m+1}$ *and* $\Re a \geq 0$, *then we have for* $u \in \mathscr{S}$:

$$Re\langle Op(a)u, u\rangle \geq -C\|u\|^2_{H^m}.$$

We now radially compactify each fiber of $T^*\mathbb{R}^n$ to obtain $\overline{T^*}\mathbb{R}^n$ and we will denote by $S^*\mathbb{R}^n = \partial\overline{T^*}\mathbb{R}^n$ the cosphere bundle, which we thus view as a boundary at fiber infinity of $\overline{T^*}\mathbb{R}^n$. We will sometimes also need classical symbols. A classical symbol of order m is then $\langle\xi\rangle^m$ times a smooth function on $\overline{T^*}\mathbb{R}^n$. For a classical symbol a we can write down the Taylor series near $1/|\xi| = 0$ to obtain

$$a(x,\xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x,\hat{\xi})|\xi|^{m-j}, \quad a_{m-j} \in C^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1}),$$

and where the tilde denotes an asymptotic expansion-truncating the expansion at the $|\xi|^{m-N}$ term gives an error that is $\mathscr{O}(|\xi|^{m-N-1})$.

Suppose that $A \in \Psi^m$ has compactly supported Schwartz kernel supported in $U \times U$. Then under a coordinate change $\kappa : U \to V$, the principal symbol of an operator $a = \sigma_m(A)$ transforms as a function on the cotangent bundle and the Hamiltonian vector field is invariantly defined under change of coordinates. This allows to define pseudodifferential calculus on a compact manifold X and the above properties remain valid. In the non compact case, some care has to be taken for example by imposing some control at infinity or by requiring that the operators are properly supported.

4.2. *Propagation of singularities*

Let $P \in \Psi^m(X)$ be a classical operator with real homogeneous principal symbol $p_m = \sigma_m(P) \in S^m_{\mathrm{hom}}(T^*X)$. Fix a boundary defining function ρ of fiber infinity in $\overline{T^*}X$. We rescale the Hamiltonian vector field $\mathbf{H}_{p_m} = \rho^{m-1}H_{p_m} \in \mathcal{V}(S^*X)$ and also rescale the principal symbol $\mathbf{p} = \rho^m p_m \in C^\infty(S^*X)$. Within the characteristic set $\mathrm{Char}(P) = \mathbf{p}^{-1}(0)$, the rescaled vector field induces a flow which is merely a rescaling of the Hamiltonian flow of p_m if we identify the subset $\mathrm{Char}(P) \subset S^*X$ with the corresponding conic set of $T^*X \setminus 0$. We remark that H_{p_m} vanishes at a point $\zeta \in S^*X$ if

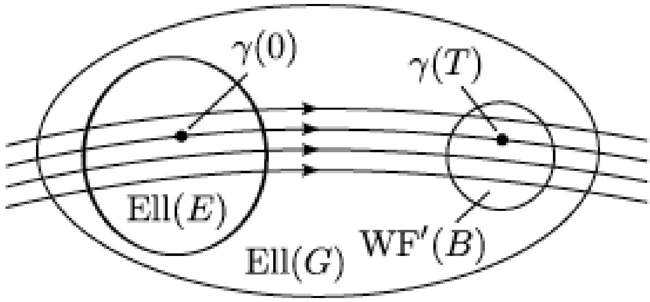

Figure 4.1. Propagation of singularites, source: [80].

and only if H_{p_m} is radial at ζ (i.e. at the ray in $T^* \setminus 0$ associated to ζ), and in this case the integral curve through ζ is constant.

We suppose in the following that $\zeta_0 \in S^*X$ is such that $\mathrm{H}_{p_m}|_{\zeta_0} \neq 0$, hence the hamiltonian flow in Char(P) starting at ζ_0 is non trivial; denote by $\gamma : [0,T] \to S^*X$ a segment of a null bicharacteristic, i.e. an integral curve of $\mathbf{H}_{p_m}$ in Char(P) starting at $\gamma(0) = \zeta_0$. The following theorem was first proven by Duistermaat and Hörmander in [1] (Figure 4.1 illustrates this theorem):

Theorem 4.5. *Let $s, N \in \mathbb{R}$. Let $E, B, G \in \Psi^0$. Assume that E is elliptic at ζ_0, B is elliptic at $\gamma(T)$, and G is elliptic at $\gamma([0,T])$, such that every backward null bicharacteristic starting at a point in $WF'(B)$ reaches* Ell(E) *in finite time. Then*

$$\|Bu\|_{H^s} \le C(\|GPu\|_{H^{s-m+1}} + \|Eu\|_{H^s} + \|u\|_{H^N}).$$

Remark 4.6. (1) *The estimates holds in the strong sense that if $u \in \mathcal{E}'(X)$ is such that the right hand side is finite, then so is the left hand side and the estimate holds. The theorem shows that H^s regularity propagates along null bicharacteristics in the following sense. Assuming that we have microlocal H^s control at $\gamma(0)$, we conclude microlocal H^s control at $\gamma(T)$ provided Pu stays in H^{s-m+1} along the way.*

(2) *The hypothesis excludes radial points, we come back to radial points in the special setting of the De Sitter Schwarzschild metric in Section 5.3.2.*

Sketch of the proof. We sketch the proof in the case when P is selfadjoint, $P = P^*$. We assume further $Pu = 0$. We can suppose $T = 1$. We begin by straightening out the flow of $\mathbf{H}_{p_m}$. We introduce local coordinates $(q_1, q') \in \mathbb{R}\times\mathbb{R}^{2n-2}$ on S^*X near ζ_0 such that $\zeta_0 = (0,0)$ and $\mathbf{H}_{p_m} = \partial_{q_1}$. Let for $\delta > 0$, $U_\delta = \{|q_1|, |q'| < \delta\}$. We start with a formal computation and discuss regularity issues later. For a commutant $A = Op(a) \in \Psi^{2s-m+1}$ such that $A = A^*$ to be chosen later we compute:

$$0 = 2\Im\langle Pu, Au\rangle = \langle \mathrm{i}(PA - AP)u, u\rangle = \langle \mathrm{i}[P,A]u, u\rangle. \tag{4.2}$$

Let $\mathbf{a} = \rho^{2s-m+1}a \in C^\infty(S^*X)$. We claim that we can arrange

$$\mathbf{H}_{p_m}\mathbf{a} = -\mathbf{b}^2 - M\mathbf{a} + \mathbf{e}, \tag{4.3}$$

where $\mathbf{b} \in C^\infty(S^*X)$ is non negative, and positive in $\{q_1 \in (\delta, 1+\delta), |q'| < \delta\}$, while $\mathbf{e} \in C^\infty(S^*X)$ is supported in the neighborhood U_δ and $M > 0$ can be chosen large. We define $\mathbf{a}$ as a product $\mathbf{a} = \mathbf{a}_1(q_1)\mathbf{a}'(q')$, where $\mathbf{a}_1$ and $\mathbf{a}'$ are constructed in the following way. Let

$$\psi \in C_0^\infty(\mathbb{R}^{2n-2}),\ \operatorname{supp}\psi \subset \{|q'| \le 2\delta\},\ \psi(q') = 1 \quad \text{for } |q'| < \delta,$$
$$\chi_1 \in C^\infty(\mathbb{R}),\ \operatorname{supp}\chi_1 \subset (-\delta, \infty),\ \operatorname{supp}(1-\chi_1) \subset (-\infty, \delta) \text{ (turn off function)},$$

$$\chi_0(q_1) = \begin{cases} \mathrm{e}^{-\frac{F}{1+\delta-q_1}} & q_1 < 1+\delta, \\ 0 & q_1 \geq 1+\delta, \end{cases}$$
$$\mathbf{a}_1(q_1) = \chi_0(q_1)\chi_1^2(q_1), \quad \mathbf{a}'(q') = \psi^2(q').$$

It is not difficult to check that (4.3) can be achieved for $M > 0$ given if we choose F sufficiently large. Going back to a we compute

$$H_{p_m} a = \rho^{-m+1}\mathbf{H}_{p_m}(\rho^{-2s+m-1}\mathbf{a}) = \rho^{-2s}(\mathbf{H}_{p_m}\mathbf{a} - (2s-m+1)\mathbf{a}\rho^{-1}\mathbf{H}_{p_m}\rho). \tag{4.4}$$

Noting that $\rho^{-1}\mathbf{H}_{p_m}\rho$ is bounded and choosing M in (4.3) sufficiently large we see that

$$H_{p_m} a = -b^2 + e - M\tilde{a} + r\tilde{a}, \tag{4.5}$$

where $b = \rho^{-s}\mathbf{b}, e = \rho^{-2s}\mathbf{e}, \tilde{a} = \rho^{-m+1}a, r \in S^0$. Let $\tilde{A}$ be a quantization of $\tilde{a}$ and suppose that we arrange $\tilde{A} = C^*C$. Then $-M\langle \tilde{A}u, u\rangle + \langle Op(r\tilde{a})u, u\rangle \leq C\|u\|^2_{H^{s-(1/2)}}$. Let $B \in \Psi^s(X)$ and $E \in \psi^{2s}(X)$ be quantizations of b and e, respectively. We then obtain:

$$\langle B^*Bu, u\rangle \leq \langle Eu, u\rangle + C\|u\|^2_{H^{s-\frac{1}{2}}}. \tag{4.6}$$

It has to be pointed out that the integration by parts above are not justified at the level of regularity supposed in the hypothesis. The above argument therefore needs a regularization procedure and this can be done for $u \in H^{s-1/2}$ microlocally near $\gamma([0,T])$, see [79] for details. Supposing this done, then the estimate (4.6) implies $Bu \in L^2(X)$, therefore $u \in H^s$ microlocally near $\gamma([0,T])$. Thus starting with the a priori knowledge $u \in H^N(X)$, one can iteratively improve the regularity by $1/2$ in each step, until one obtains H^s-regularity as desired. □

4.3. *Semiclassical analysis*

We briefly introduce some elements of semiclassical analysis that we will need in the following. We refer to Zworski [77] for details. We start by introducing the algebra $\Psi^m_h(X)$ of *semiclassical* pseudodifferential operators, depending on a parameter h. The corresponding symbols $a(x,\xi;h)$ (denoted $a \in S^m_h(X)$) satisfy $a(\cdot,\cdot;h) \in S^m(X)$ uniformly in h as $h \to 0$, with $S^m(X)$ defined in Section 4.1. The semiclassical symbol of $A \in \Psi^m_h$, denoted $\sigma_h(A)$, lies in the space $S^m_h(X)/hS^{m-1}_h(X)$.

We will be concerned in particular with the class $\Psi^0_h(X)$. We moreover require that symbols of the elements of $\Psi^0_h(X)$ are classical in the sense that they have an asymptotic expansion in non negative powers of h. More precisely we require that $a \in \Psi^0_h$ writes

$$a \sim \sum_{j=0}^{\infty} h^j a_j, \text{ meaning } \left|\partial_x^\alpha \partial_\xi^\beta \left(a - \sum_{j=0}^{N-1} h^j a_j\right)\right| \leq C_{\alpha,\beta} h^N \langle\xi\rangle^{-N-|\beta|}$$

for all multiindices α, β and $N \in \mathbb{N}$. For $A, A' \in \Psi^0_h(X)$ with principal symbols a, a' we have

$$\sigma_h(A \circ A') = aa', \quad \sigma_h(A^*) = \bar{a}, \quad \sigma_h(\mathrm{i}h^{-1}[A, A']) = \{a, a'\} = H_a a'.$$

In the context of semiclassical analysis we will mostly use the Weyl quantization:

$$a^w(x, hD)u(x) = \frac{1}{(2\pi h)^n}\iint a\left(\frac{x+y}{2}, \xi\right) \mathrm{e}^{\frac{\mathrm{i}}{h}\langle x-y,\xi\rangle} u(y)\,\mathrm{d}y\,\mathrm{d}\xi.$$

Moreover each $A = a^w(x, hD) \in \Psi^0_h(X)$ acts $L^2(X) \to L^2(X)$ with norm uniformly bounded in h. In fact we have the bound [77, Theorem 13.13]

$$\limsup_{h\to 0} \|A\|_{L^2(X)\to L^2(X)} \leq \sup_{T^*X} |\sigma_h(A)|. \tag{4.7}$$

Each $A \in \Psi^k_h(X)$ has a semiclassical wave front set $WF'_h(A)$, a closed (and not necessarily conic) subset of the fiber-radially compactified cotangent bundle $\overline{T^*}X$; a point $(x,\xi) \in \overline{T^*}X$ does not lie in $WF'_h(A)$ if and only if the full symbol of A satisfies $a(x',\xi') = \mathcal{O}(h^\infty\langle\xi'\rangle^{-\infty})$ for h small enough and $(x',\xi')\in T^*X$ in a neighborhood of (x,ξ) in $\overline{T^*X}$. We will mostly use the subalgebra of *compactly microlocalized* pseudodifferential operators $\Psi_h^{\mathrm{comp}}(X)\subset\Psi_h^0(X)$. For $A\in\Psi_h^{\mathrm{comp}}(X)$ the wave front set is a compact subset of T^*X. We say that $A=B+\mathcal{O}(h^\infty)$ microlocally in some set $U\subset T^*X$ if $WF_h'(A-B)\cap U=\emptyset$. We will often consider sequences $u(h_j)\in L^2(X)$, where $h_j\to 0$ is a sequence of positive numbers; we typically suppress the dependence on j and write $u=u(h)$. For such a sequence, we say that $u=\mathcal{O}(h^\delta)$ microlocally in some set $U\subset T^*X$, if $\|Au\|_{L^2}=\mathcal{O}(h^\delta)$ for each $A\in\Psi_h^{\mathrm{comp}}(X)$ such that $WF_h'(A)\subset U$. The notion $u=o(h^\delta)$ is defined similarly.

4.4. *Semiclassical defect measures*

Definition 4.7. *Assume that $h_j\to 0$ and $u=u(h_j)\in L^2(X)$ is uniformly bounded as $j\to\infty$. We say that u converges to a non-negative Radon measure μ on T^*X if for each $A=A(h)\in\Psi_h^{\mathrm{comp}}(X)$, we have*

$$\langle A(h_j)u(h_j),u(h_j)\rangle\to\int_{T^*X}\sigma_h(A)\,\mathrm{d}\mu\quad\text{as } j\to\infty. \tag{4.8}$$

See [77, Chapter 5] for an introduction to defect measures, in particular,

(1) *for each sequence $u(h_j)$ uniformly bounded in L^2, there exists a subsequence h_{j_k} such that $u(h_{j_k})$ converges to some μ [77, Theorem 5.2];*
(2) *we have $\mu(U)=0$ for some open set $U\subset T^*X$ if and only if $u=o(1)$ microlocally on U.*

The following basic properties of microlocal measures are easy to prove, see e.g. [26].

Lemma 4.8 (Ellipticity). *Take $P\in\Psi_h^0(X)$ and denote $p=\sigma_h(P)$. Assume that $u=u(h_j)$ converges to some measure μ and $Pu=o(1)$ microlocally in some open set $U\subset T^*X$. Then $\mu(U\cap\{p\neq 0\})=0$.*

Lemma 4.9 (Propagation). *Take $P,W\in\Psi_h^0$, denote $p=\sigma_h(P)$, $w=\sigma_h(W)$, and assume $P^*=P$. Assume that $u=u(h_j)$ converges to some measure μ and denote $f:=(P-\mathrm{i}hW)u$. Then for each $a\in C_0^\infty(T^*X)$ and for each $Y\in\Psi_h^{\mathrm{comp}}(X)$ such that $Y=1+\mathcal{O}(h^\infty)$ microlocally in a neighborhood of* $\operatorname{supp}a$ *we have,*

$$\left|\int_{T^*X}(H_p-2\Re w)a\,\mathrm{d}\mu\right|\le 2\|a\|_\infty\limsup_{h\to 0}(h^{-1}\|Yf\|_{L^2}\|Yu\|_{L^2}). \tag{4.9}$$

In particular, if $f=o(h)$ microlocally in a neighborhood of $\operatorname{supp}a$, *then*

$$\int_{T^*X}(H_p-2\Re wa)\,\mathrm{d}\mu=0.$$

5. The wave equation on the De Sitter Schwarzschild metric

In this section we explain the general strategy for showing decay estimates via microlocal methods for a simple example: the wave equation on the De Sitter Schwarzschild metric. Let $\mathcal{M}=\mathbb{R}_{t_*}\times X$, $X=(r_--\epsilon,r_++\epsilon)\times\mathbb{S}^2_\omega$. Near $r=r_-$ resp. $r=r_+$ the metric is given by

$$g=F\,\mathrm{d}t_*^2-2\,\mathrm{d}t_*\,\mathrm{d}r-r^2\,\mathrm{d}\omega^2\quad\text{resp.}\quad g=F\,\mathrm{d}t_*^2+2\,\mathrm{d}t_*\,\mathrm{d}r-r^2\,\mathrm{d}\omega^2,$$

where $F(r)=1-(2\mathfrak{m}/r)-(1/3)\Lambda r^2$. Recall that the surface gravities are given by $\kappa_\pm=(1/2)F'(r_\pm)$. The wave operator is given near $r=r_-$ resp. $r=r_+$ by

$$\Box_g=2D_{t_*}D_r+\frac{1}{r^2}D_rr^2FD_r-r^{-2}\Delta_{\mathbb{S}^2}\quad\text{resp.}\quad\Box_g=-2D_{t_*}D_r+\frac{1}{r^2}D_rr^2FD_r-r^{-2}\Delta_{\mathbb{S}^2}.$$

Fourier transforming this with respect to t_*, i.e. putting

$$\hat{\Box}_g(\sigma)=\mathrm{e}^{\mathrm{i}t_*\sigma}\Box_g\mathrm{e}^{-\mathrm{i}t_*\sigma}.$$

then gives

$$\hat{\Box}_g(\sigma)=-2\sigma D_r+\frac{1}{r^2}D_rr^2FD_r-r^{-2}\Delta_{\mathbb{S}^2}\quad\text{resp.}\quad\hat{\Box}_g(\sigma)=2\sigma D_r+\frac{1}{r^2}D_rr^2FD_r-r^{-2}\Delta_{\mathbb{S}^2}.$$

We will consider in the following the forward problem

$$\Box_g u = f, \tag{5.1}$$

where $f \in C_0^\infty(\mathcal{M})$ and we require that $\operatorname{supp} u \subset \{t_* \geq -T\}$ for some $T > 0$. Let $\bar{H}^s(X)$ resp. $\dot{H}^s(X)$ be the Sobolev spaces based on extendible resp. supported distributions. Initial value problems can easily be transformed into forward problems and vice versa, see e.g. [44]. The existence of a forward solution is assured by the global hyperbolicity of the spacetime.

5.1. *A key theorem*

A first important result is that the Fourier transformed wave operators form an analytic family of Fredholm operators. Let $\gamma = \max\{(1/\kappa_-), (1/\kappa_+)\}$. We have

Theorem 5.1. *Let $s \geq 1/2$, $\mathcal{X}^s = \{u \in \bar{H}^s(X); \hat{\Box}_g(0) u \in \bar{H}^{s-1}(X)\}$, $\mathcal{Y}^{s-1} = \bar{H}^{s-1}(X)$. The operators*

$$\hat{\Box}_g(\sigma) : \mathcal{X}^s \to \mathcal{Y}^{s-1}$$

form an analytic family of Fredholm operators of index 0 for all $\sigma \in \mathcal{C}$ such that

$$\Im\sigma > \frac{1-2s}{2\gamma}. \tag{5.2}$$

Moreover $\hat{\Box}_g(\sigma)$ is invertible if $\Im\sigma > 0$.

Theorem 5.2. *Let $t_0 \in \mathbb{R}$. There are $C, \nu > 0$ such that for $0 < \delta < C$ and $s > (1/2) + \gamma\delta$ any solution to (5.1) with $f \in e^{-\delta t_*}\bar{H}^{s-1+\nu}(M)$ and with $\operatorname{supp}(u) \cup \operatorname{supp}(f) \subset \{t_* > t_0\}$ has an asymptotic expansion*

$$u(t_*, r, \omega) - \sum_{j=1}^{N} \sum_{k=0}^{k_j} t_*^k e^{-i\sigma_j t_*} a_{j,k}(r,\omega) \in e^{-\delta t_*}\bar{H}^s(M), \tag{5.3}$$

where $\sigma_1, \dots, \sigma_N$ are the (finitely many) quasinormal mode frequencies with

$$\Im\sigma_j > -\delta$$

and k_j their multiplicities, and where $e^{-i\sigma_j t_*} a_{k,j}$ are the C^∞ generalized quasinormal modes with frequency σ_j.

Remark 5.3. (1) Quasinormal mode frequencies are also called resonances, they correspond to those σ such that $\hat{\Box}_g(\sigma)$ has a non trivial kernel. By analytic Fredholm theory one obtains a meromorphic family of operators $\hat{\Box}_g^{-1}(\sigma) : \mathcal{Y}^{s-1} \to \mathcal{X}^s$. Resonances or quasinormal mode frequencies are then the poles of this family of operators. These poles may be of higher order which gives rise to the t_*^k coefficients.

(2) Note that close to the black hole horizon, $\hat{\Box}_g(\sigma) = e^{i\sigma r_*}\tilde{\Box}_g(\sigma)e^{-i\sigma r_*}$, where $\tilde{\Box}_g(\sigma)$ is the Fourier transformed operator with respect to t. Therefore v is mode solution for $\hat{\Box}_g(\sigma)$ ($\hat{\Box}_g(\sigma)v = 0$) if and only if $u = e^{-i\sigma r_*}v$ is mode solution for $\tilde{\Box}_g(\sigma)$ and the requirement that v is smooth at the horizon translates to the usual outgoing condition $u \sim e^{-i\sigma r_*}$, $r_* \to -\infty$ for u.

(3) The analogue of Theorem 5.1 is false in the Schwarzschild case. Nevertheless the resolvent has good regularity properties down to the real axis, see Vasy [33, 34]. This can then be used to show polynomial decay estimates, see Häfner, Hintz, Vasy [39], Hintz [45] and Millet [44].

(4) There exists a generalized version for the De Sitter Kerr metric (see Vasy [23], Petersen-Vasy [29]), but a priori nothing is said about the existence of resonances with strictly positive imaginary part. The non existence of mode solutions with positive imaginary part of σ has been shown by Whiting in the whole subextreme case for the Kerr solution.

The analogue question for the De Sitter Kerr case remains open, but see Hintz [30] for results in this direction.

(5) High frequency resonances (meaning for large real part of σ) are well approximated by so called pseudopoles which can be computed and which are given by

$$\frac{\sqrt{1-9\Lambda\mathfrak{m}^2}}{3^{\frac{3}{2}}\mathfrak{m}}(\pm(\ell+1/2)-\mathrm{i}(1/2+n)), \quad \ell\in\mathbb{N}, n\in\mathbb{N}_0.$$

These resonances are very closely linked to the trapping of the hamiltonian flow.

(6) As $\Box_g(0)1=0$, 0 is a resonance, in particular the solution *does not* decay. However it can be shown (see [19]) that this is the only real resonance. Together with the high frequency analysis of Section 5.4 this gives the existence of a strip beyond the real axis in which the only resonance is zero. As a corollary one obtains the existence of a constant c with

$$u(t_*,r,\omega)-c\in \mathrm{e}^{-\delta t_*}\bar{H}^s(M).$$

The first version of Theorem 5.2 was shown by Bony and Häfner in [19]. This was formulated in the original Boyer Lindquist coordinates and excluded the horizons. A generalization of this including the horizons was subsequently obtained by Melrose, Sá-Barreto and Vasy in [20]. Later on the theorem was generalized by Dyatlov to the De Sitter Kerr case in [21,22]. The most general version of this was obtained by Vasy in [23] first in the small a case and later generalized to the large a case in collaboration with Petersen [29].

In the remaining of this section we will explain some of the principal ideas in the proof of Theorems 5.1 and 5.2. Since $\mathscr{M}$ is globally hyperbolic there exists a unique forward solution of (5.1). Standard energy estimates give a first control of the solution of the form

$$|u(t_*,r,\omega)|\lesssim \mathrm{e}^{Ct_*}.$$

It is then easy to see that the Fourier transform of u

$$\hat{u}(\sigma)=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\sigma t_*}u(t_*)\,\mathrm{d}t_*$$

is holomorphic in $\Im\sigma>C$. We now take Fourier transform of Equation (5.1) to obtain:

$$\hat{\Box}_g(\sigma)\hat{u}(\sigma)=\hat{f}(\sigma).$$

We then can write

$$u(t_*,x)=\frac{1}{2\pi}\int_{\Im\sigma=C+1}\mathrm{e}^{-\mathrm{i}\sigma t_*}\hat{\Box}_g^{-1}(\sigma)\hat{f}(\sigma)\,\mathrm{d}\sigma.$$

This doesn't give so far any better behavior of u, but $\hat{\Box}_g^{-1}(\sigma)$ is a meromorphic family of bounded operators and we can shift the contour to $\Im\sigma=-\delta$. The residue theorem then gives the expansion in (5.3). The most delicate part is the estimate of the error, which requires in particular high frequency estimates of the resolvent $\hat{\Box}_g^{-1}(\sigma)$.

Remark 5.4. By choosing s large enough the strip where the operators form an analytic Fredholm family can be made arbitrary large. For $\Im\sigma=-C$ and C large enough this line lies below an infinite number of resonances. The sum in (5.3) is then infinite, but still converges in some appropriate sense, see [19].

5.2. *Fredholm operators*

Let us recall that a Fredholm operator $T:X\to Y$ is a bounded operator with closed rank such that kernel and cokernel coker $=Y/\mathrm{range}(T)$ are finite dimensional. The index is then defined as $\mathrm{ind}T=\dim\ker T-\dim\mathrm{coker}T$. The situation is particularly interesting when we have an analytic family $D\ni z\mapsto T(z)$ of Fredholm operators. The index is then a continuous function of z and the Fredholm alternative says that either $T(z)$ is invertible for no $z\in D$ or $F^{-1}(z)$ form a meromorphic family of operators. Typically elliptic operators are Fredholm between suitable Sobolev spaces and vice versa the Fredholm property between these Sobolev spaces requires ellipticity. We refer to [76, Section 19.2] for details.

The operators $\hat{P}(\sigma)$ we are interested in are not elliptic. An analytic Fredholm theory can nevertheless be developed as observed by Vasy [23]. A central observation is that the Fredholm theory follows from suitable estimates. A general setting is the following, see e.g. [44, Lemma 6.23]:

Lemma 5.5. *Let $X_0 \subset X_1 \subset X_2$ and $Y_0 \subset Y_1 \subset Y_2$ be Banach spaces (with continuous dense inclusions). Let $P : X_1 \to Y_2$ be a bounded operator such that $P|_{X_0}$ is bounded from X_0 to Y_1. We assume that both inclusions $X_1 \subset X_2$ and $Y_0 \subset Y_1$ are compact and that there exists $C > 0$ such that for all $u \in X_1$ and all $v \in Y_1^*$:*

$$\begin{aligned} \|u\|_{X_1} &\le C(\|Pu\|_{Y_1} + \|u\|_{X_2}), \\ \|v\|_{Y_1^*} &\le C(\|P^* v\|_{X_1^*} + \|v\|_{Y_0^*}). \end{aligned}$$

Under these assumptions, P is Fredholm as an operator between the Banach space $\mathscr{X} := \{u \in X_1; Pu \in Y_1\}$ (endowed with the norm $\|u\|_{\mathscr{X}}^2 = \|u\|_{X_1}^2 + \|Pu\|_{Y_1}^2$) and Y_1.

Let $\beta = \min\{(1/\kappa_-), (1/\kappa_+)\}$ if $\Im\sigma \ge 0$, $\beta = \max\{(1/\kappa_-), (1/\kappa_+)\}$ if $\Im\sigma \le 0$. We will obtain the following global estimates.

(1) Let $s > s_0 > (1/2) - \beta\Im\sigma$. Then we have

$$\|u\|_{\bar{H}^s(X)} \le C(\|\hat{\Box}_g(\sigma)u\|_{\bar{H}^{s-1}(X)} + \|u\|_{\bar{H}^{s_0}(X)}). \tag{5.4}$$

(2) Let $s_0 < s' < (1/2) + \beta\Im\sigma$. Then we have:

$$\|u\|_{\dot{H}^{s'}(X)} \le C(\|\hat{\Box}_g^*(\sigma)u\|_{\dot{H}^{s-1}(X)} + \|u\|_{\dot{H}^{s_0}(X)}). \tag{5.5}$$

The compact Sobolev embedding $\bar{H}^s(X) \hookrightarrow \bar{H}^{s_0}(X)$ and Lemma 5.5 then imply the Fredholm property. In the following we will only show (5.4), the arguments for (5.5) are quite similar.

5.3. *Bounded frequency analysis*

Let us recall that we have near $r = r_\pm$:

$$\hat{\Box}_g(\sigma) = r^{-2}D_r r^2 F D_r \pm 2\sigma r^{-2} D_r r^2 - r^{-2}\Delta_{\mathbb{S}^2}.$$

We will apply the unitary transform $U : L^2((r_-, r_+)\times\mathbb{S}^2, r^2\,\mathrm{d}r\,\mathrm{d}\omega) \to L^2((r_-, r_+)\times\mathbb{S}^2, \mathrm{d}r\,\mathrm{d}\omega), u \mapsto ru$ and then work with the operator

$$r\hat{\Box}_g(\sigma)r^{-1} = r^{-1}D_r r^2 F D_r r^{-1} \pm 2\sigma r^{-1} D_r r - r^{-2}\Delta_{\mathbb{S}^2},$$

which we call again $\hat{\Box}_g(\sigma)$ in the following. The above operator has principal classical symbol (ξ is now dual to r):

$$\sigma_2(\hat{\Box}_g(\sigma)) = F\xi^2 + r^{-2}|\eta|^2 =: p.$$

The operator is therefore elliptic over all points (r,ω) of the base manifold with $r_- < r < r_+$. We will distinguish the elliptic and non elliptic regions.

5.3.1. *Elliptic type estimates*

Proposition 5.6. *Let $\chi \in C_0^\infty(r_- + \delta, r_+ - \delta)$ for some $\delta > 0$. Then we have*

$$\|\chi u\|_{\bar{H}^s} \lesssim \|\hat{\Box}_g(\sigma)u\|_{\bar{H}^{s-2}} + \|u\|_{\bar{H}^{s-1}}.$$

Proof. Let $\phi \in C_0^\infty((-1,1))$ with $\phi = 1$ in a neighborhood of 0. We write

$$\chi(x) = \chi(x)\phi(p) + \chi(x)(1-\phi(p)).$$

Note that $\chi(r)\phi(p) \in S^{-\infty}$, $\chi(x)(1-\phi(p)) \in S^0$. By pseudodifferential calculus we only have to consider the second term. Now,

$$Op(\chi(x)(1-\phi(p))) = Op\left(\chi(x)\frac{1-\phi(p)}{p}\right)\hat{\Box}_g(\sigma) + R,\, R \in \Psi^{-1}.$$

The estimate for $s = 0$ then follows by pseudodifferential calculus. Higher order estimates are obtained by applying $Op(\xi^2 + |\eta|^2)$. □

5.3.2. *Radial point estimates*

The characteristic manifold is

$$\Sigma = \{(r,\omega,\xi,\eta) \in T^*X \setminus 0; p(r,\omega,\xi,\eta) = 0\}$$

and it splits into positive and negative energy shells:

$$\Sigma = \Sigma_+ \cup \Sigma_-, \Sigma_\pm = \Sigma \cap \{\pm\xi > 0\}.$$

As $F(r_-) = 0$ there is a lack of ellipticity at $r = r_-$. Note also that we have

$$H_p = (-F'\xi^2 + 2r^{-3}|\eta|^2)\partial_\xi + 2\xi F\partial_r + r^{-2}H_{|\eta|^2}.$$

The

$$\Lambda_\pm^\pm = \left\{(r_\pm,\omega,\xi,0); \xi \in \mathbb{R}^*, \omega \in \mathbb{S}^2\right\} \cap \Sigma_\pm$$

are the radial points of H_p. The lower index indicates the point on the base manifold, the upper index the energy level. It is not difficult to see that Λ_-^+ and Λ_+^- are sources of the hamiltonian vector field whereas Λ_-^- and Λ_+^+ are sinks.

Proposition 5.7. *Suppose that* $u \in \bar{H}^s$ *with* $s > (1/2) - (1/\kappa_\pm)\Im\sigma$. *Let* $\chi_\pm \in C_0^\infty((r_\pm - \delta, r_\pm + \delta))$. *Then for* $\delta > 0$ *sufficiently small we have:*

$$\|\chi_\pm u\|_{\bar{H}^s} \lesssim \|\hat{\Box}_g(\sigma)u\|_{\bar{H}^{s-1}} + \|u\|_{\bar{H}^{s-\frac{1}{2}}}.$$

Remark 5.8.
(1) The condition $s > (1/2) - (1/\kappa_\pm)\Im\sigma$ gives the condition (5.2) in Theorem 5.1.
(2) Let us try to understand where the condition $s > (1/2) - \beta\Im\sigma$ comes from. As a toy model we consider
$$P_\sigma = D_x x D_x - \sigma D_x, \quad x \in (-1,1).$$
In this case $\beta = 1$. We have
$$P_\sigma (x \pm \mathrm{i}0)^{\mathrm{i}\sigma} = 0.$$
We have $(x \pm \mathrm{i}0)^{\mathrm{i}\sigma} \in H^{(1/2-\Im\sigma)-}$ but not in H^s for $s > (1/2) - \Im\sigma$. Imposing Sobolev regularity $s > (1/2) - \Im\sigma$ thus excludes these solutions.
(3) The estimate is a very simplified version of the radial point estimate obtained by Vasy, see [23, Proposition 2.3]. This kind of estimates goes back to Melrose, see [81]. They are central in Vasy's method concerning the Fredholm theory for non elliptic problems. We also refer to Zworski [82] for a review of this method.

Main ideas of the proof. We will prove the proposition only for a model problem. We will consider the operator which is given near $r = r_\pm$ by

$$\hat{P}(\sigma) = r^{-1}D_r r^2 F D_r r^{-1} \pm 2\sigma r^{-1} D_r r$$

and $\hat{P}(\sigma)$ acts on $L^2((r_- - \epsilon, r_+ + \epsilon); dr)$.

Also to avoid technical issues we will suppose $u \in C_0^\infty((r_- - \epsilon, r_+ + \epsilon))$. We prove the proposition at $r = r_-$, the proof at $r = r_+$ being essentially the same. We also put $x = r - r_-$. Let χ_- be as in the hypotheses with $x\chi'_-(x)\le 0$. We drop the index—in the following. Let $\psi\in C^\infty(\mathbb{R})$, $\psi(x)=0$ for $x\le 0$ and $\psi(x)=1$ for $x\ge 1$.

$$A=C^*C,\quad C=Op(c),\quad c:=\xi^{s-\frac12}\psi(\xi)\chi(x).$$

Let $a:=\sigma_{2s-1}(A)$. Now,

$$\begin{aligned}2\Im\langle\hat P(\sigma)u,Au\rangle &= \langle \mathrm{i}(\hat P^*(\sigma)A-A\hat P(\sigma))u,u\rangle\\ &= \langle \mathrm{i}[\hat P(\sigma),A]u,u\rangle+\langle \mathrm{i}(\hat P^*(\sigma)-\hat P(\sigma))Au,u\rangle\\ &= \langle \mathrm{i}[\hat P(\sigma),A]u,u\rangle-4\Im\sigma\langle r^{-1}D_xrAu,u\rangle.\end{aligned}\tag{5.6}$$

The symbolic calculation gives

$$H_pa=-(2s-1)\xi^{2s}\psi^2(\xi)F'\chi^2(x)-2F'\psi(\xi)\psi'(\xi)\xi^{2s+1}\chi^2(x)+4\xi^{2s}\psi^2(\xi)F\chi(x)\chi'(x).\tag{5.7}$$

Note that the second term is in $S^{-\infty}$ and that the third term is negative.

We have

$$\langle \mathrm{i}[\hat P(\sigma),A]u,u\rangle=\langle Op(H_pa)u,u\rangle+\langle Ru,u\rangle,\quad R\in\Psi^{2s-1}.$$

Applying the sharp Gårding inequality, Theorem 4.4, we find

$$\langle \mathrm{i}[\hat P(\sigma),A]u,u\rangle-4\Im\sigma\langle r^{-1}D_xrAu,u\rangle\le(-(2s-1)2\kappa_-+\tilde\epsilon(\delta)-4\Im\sigma)\|Op(\psi(\xi)\chi(x))u\|^2_{\bar H^s}+\|u\|^2_{\bar H^{s-\frac12}},$$

where $\tilde\epsilon(\delta)\to 0$, $\delta\to 0$. Here and in the following $\bar H^s=\bar H^s((r_--\epsilon,r_-+\epsilon))$.

We also have

$$|2\Im\sigma\langle\hat P(\sigma)u,Au\rangle|\le C_{\hat\epsilon}\|\hat P(\sigma)u\|^2_{\bar H^{s-1}}+\hat\epsilon\|Op(\psi(\xi)\chi(x))u\|^2_{\bar H^s}+C\|u\|^2_{\bar H^{s-1}}.$$

Putting everything together and choosing $\delta,\hat\epsilon$ small enough gives:

$$\|Op(\psi(\xi)\chi(x))u\|^2_{\bar H^s}\le C\left(\|\hat P(\sigma)u\|^2_{\bar H^s}+\|u\|^2_{\bar H^{s-\frac12}}\right).\tag{5.8}$$

We now want to obtain the same estimate with $Op(\psi(\xi)\chi(x))$ replaced by $\chi(x)$. To show the estimate for $Op(\psi(-\xi)\chi(x))$ we can replace $a(x,\xi)$ by $a(x,-\xi)$, then the above proof works for $-\hat P(\sigma)$ instead of $\hat P(\sigma)$. Writing $1=\psi(\xi)+\psi(-\xi)+\psi_0(\xi)$ and using that $\psi_0(\xi)\in S^{-\infty}$ we see that we can indeed replace $Op(\psi(\xi)\chi(x))$ by $\chi(x)$. □

Remark 5.9. *Using a regularization procedure one can show that it is sufficient to suppose $Cu\in L^2$. What is usually called a radial point estimate is rather (5.8), the important property of $B=Op(\psi(\xi)\chi(x))$ is that backward bicharacteristics starting in $WF'(B)$ tend to Λ^+_-, where C has to be elliptic. In this very simplified model this is used in the proof via the sign of $4\xi^{2s}\psi^2(\xi)F\chi(x)\chi'(x)$ in (5.7). Also $\hat P(\sigma)$ can be replaced by $G\hat P(\sigma)$ with $G\in\Psi^0$ elliptic at Λ^+_- as long as the closure of all backward bicharacteristics starting in $WF'(B)$ is contained in the elliptic set of G. We refer to [23] for details.*

5.3.3. *Proof of Theorem 5.1*

Putting the estimates in Sections 5.3.1 and 5.3.2 together we obtain the following global estimates:

Proposition 5.10. *Let $u\in\bar H^s$ and $s>1/2-\beta\Im\sigma$. Then we have*

$$\|u\|_{\bar H^s(X)}\le C\left(\|\hat\Box_g(\sigma)u\|_{\bar H^{s-1}(X)}+\|u\|_{\bar H^{s-\frac12}(X)}\right).$$

Proof. Let us first choose $\chi_\pm$ as in Proposition 5.7 and $\chi_\pm=1$ on $(r_\pm-(\delta/2),r_\pm+(\delta/2))$. We then choose $\hat\chi_\pm$ with $\mathrm{supp}\hat\chi_-\subset[r_--\epsilon,r_--(\delta/2))$, $\mathrm{supp}\hat\chi_+\subset(r_++(\delta/2),r_++\epsilon]$ and $\chi_0\in C_0^\infty(r_-+(\delta/2),r_+-(\delta/2))$ with $\hat\chi_-+\chi_-+\chi_0+\chi_++\hat\chi_+=1$. We then use Proposition 5.6 on $\mathrm{supp}\chi_0$ and Proposition 5.7 on $\sup\chi_\pm$. Concerning $\mathrm{supp}\hat\chi_\pm$ let us observe that the symbol $p=F\xi^2+r^{-2}|\eta|^2$ is hyperbolic there, because F is negative on this interval. The estimate follows then from classical hyperbolic estimates, we refer to [44] for details. □

Proof of Theorem 5.1. We obtain in a similar way a suitable estimate also for the adjoint operator. The two estimates then give the Fredholm property. To show that the operators are invertible for $\Im\sigma > 0$ we have to show that kernel and cokernel are empty. We only consider the kernel, the proof for the cokernel is similar. Let $\Im\sigma > 0$, $u \in \bar{H}^s$ and $\hat{\Box}_g(\sigma)u = 0$. Then

$$F\tilde{\Box}_g(\sigma)e^{-i\sigma h(r)r_*(r)}u(r,\omega) = 0,$$

where h is some smooth function with $h(r) = 1$ for $r \le 3\mathfrak{m}$ and $h(r) = -1$ for $r \ge 4\mathfrak{m}$ and $\tilde{\Box}_g(\sigma)$ denotes the Fourier transform with respect to t:

$$F\tilde{\Box}_g(\sigma) = r^{-2}FD_r(r^2F)D_r - r^{-2}F\Delta_{\mathbb{S}^2} - \sigma^2.$$

$F\tilde{\Box}_g(0)$ is a selfadjoint positive operator on $L^2((r_-, r_+)\times\mathbb{S}^2; F^{-1}r^2\,dr\,d\omega^2)$. But for $\Im\sigma > 0$ we have $e^{-i\sigma h(r)r_*(r)}u(r,\omega) \in L^2((r_-, r_+)\times\mathbb{S}^2; F^{-1}r^2\,dr\,d\omega^2)$, which is a contradiction. Thus $u|_{(r_-,r_+)\times\mathbb{S}^2} = 0$. To see that u really is zero on the whole manifold $(r_- - \epsilon, r_+ + \epsilon)\times\mathbb{S}^2$ we can follow [82, Lemma 1], see [43, Proof of Theorem 4.5] for details. The index 0 property then follows from the continuous dependence of the index on σ. □

5.4. *High frequency analysis: trapping*

Whereas the bounded frequency analysis is sufficient to prove Theorem 5.1, the proof of Theorem 5.2 requires a precise control of the estimates when $\Re\sigma \to \infty$. This requires a precise analysis of the semiclassical flow whose projection on the base manifold gives the geodesic flow. The analysis then becomes more complicated, but the general structure of elliptic, propagation and radial point estimates remains similar. However there is a new aspect which we want to discuss in this subsection which is trapping. To analyze the trapping it is easier to go back to the initial Boyer Lindquist coordinates. We will present a toy model, the proof for the existence of a resonance free strip follows very closely the proof of Dyatlov [26] for the general situation.

5.4.1. *Resonance free strip*

Fourier transforming with respect to t and fixing the angular momentum ℓ leads to considering the operator:

$$\tilde{P}_\ell - \sigma^2 = D_{r_*}^2 + V\ell(\ell+1) - \sigma^2, \quad V = \frac{F}{r^2}. \tag{5.9}$$

We introduce the semiclassical parameter $h^2 = 1/\ell(\ell+1)$. As explained in Section 2.4.3 trapping is concentrated at $r_* = 0$. We introduce a Taylor development of V at $r_* = 0$. We will put $x = r_*$ in the following. This leads to considering the local model

$$\tilde{P} - \lambda = h^2D_x^2 - x^2 - \lambda, \quad \lambda = \mathcal{O}(h), \tag{5.10}$$

which we will use as a toy model in the following. Note that in the model (5.9) the critical energies are $\sigma^2 \sim V(3\mathfrak{m})\ell(\ell+1)$, after rescaling this becomes $\lambda \sim \sqrt{V(3\mathfrak{m})}$. In the toy model however the critical energy is zero, we therefore consider $\lambda = \mathcal{O}(h)$. We obtain semiclassical estimates that can be translated into high frequency estimates in the original scaling.

The symbol writes $\xi^2 - x^2 = (\xi - x)(\xi + x)$ which suggests that the operator is unitary equivalent to $P = h(xD_x + D_xx)$. We can indeed introduce the transform

$$Uu(x) = \frac{1}{\sqrt{\pi}h}e^{-i\frac{x^2}{2h}}\int e^{i\frac{\sqrt{2}xy}{h}}e^{-i\frac{y^2}{2h}}u(y)\,dy.$$

It can be checked that $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isometry and that $\tilde{P}U = UP$. We will in the following consider the operator $P = (h/2)(xD_x + D_xx)$. Note that

$$\sigma_h(P) = x\xi =: p, \quad H_p = -\xi\partial_\xi + x\partial_x.$$

Thus the Hamiltonian equations can be explicitly solved, $\xi(s) = \mathrm{e}^{-t}\xi_0$, $x(s) = \mathrm{e}^{t}x_0$, where (x_0,ξ_0) are the initial data. In the abstract setting of Section 2.4.4 this means that we have:

$$\Gamma_+ = \{\xi = 0\}, \quad \Gamma_- = \{x = 0\}, \quad K = \{x = \xi = 0\}.$$

To analyze the situation near the trapped set, we will consider the situation on a compact manifold and add an absorption operator. Concretely, we put $X = \mathbb{S}^1 = \mathbb{R}/(6\mathbb{Z})$; we view X as the interval $[-3,3]$ with the endpoints glued together. We add an absorption operator $Q = Op(q)$, $q \in C^\infty(T^*X)$ such that $q(x,\xi) = 0$ for all $|(x,\xi)| \le 1$ and $q(x,\xi) = 1$ for all $|(x,\xi)| \ge 2$. We will now consider the operator $P - \mathrm{i}Q - \lambda$ on X. Let $R(\lambda) = (P - \mathrm{i}Q - \lambda)^{-1}$.

Lemma 5.11. *For each $\epsilon > 0$ and h small enough depending on ϵ we have*

$$\|R(\lambda)\|_{L^2\to L^2} = o(h^{-2}) \quad \textit{if } |\lambda| = \mathscr{O}(h), \quad \Im\lambda > -(\tfrac{1}{2} - \epsilon)h. \tag{5.11}$$

Remark 5.12. *Let us note that the same bound on the resolvent holds in general situations of normally hyperbolic trapping as described in Section 2.4.4, see [26]. We will give here the same proof for the model operator, which has the advantage of avoiding the technical issues while keeping the main ideas of the proof. Further gaps can be obtained for r-normally hyperbolic trapping, see also [26].*

To prove the lemma we will proceed by contradiction. Suppose that (5.11) does not hold. Then for some $\epsilon > 0$ small enough, there exist sequences $h_j, h_j \to 0$, $\lambda(h_j)$, $u = u(h_j) \in L^2$ with the following properties

$$\|u\|_{L^2} = 1, \quad |\lambda| = \mathscr{O}(h), \quad \Im\lambda > -(\tfrac{1}{2} - \epsilon)h \tag{5.12}$$

and

$$(P - \mathrm{i}Q - \lambda)u = \mathscr{O}(h^2). \tag{5.13}$$

The interesting properties are obtained when commuting $(P-\lambda)u$ with the defining functions of $\Gamma_\pm$. Indeed we have

$$hD_x(P-\lambda)u = (P - \mathrm{i}h - \lambda)hD_x u.$$

Now suppose that $\Im\lambda \ge -((1/2) - \epsilon)h$ for some $\epsilon > 0$. Then $h + \Im\lambda \ge ((1/2) + \epsilon)h$ and we can write

$$hD_x u = (P - \mathrm{i}h - \lambda)^{-1} hD_x(P-\lambda)u. \tag{5.14}$$

A simple application of the spectral theorem gives

$$\|hD_x u\| \lesssim h^{-1}\|hD_x(P-\lambda)u\|. \tag{5.15}$$

Note that the corresponding commutation relation with x

$$x(P-\lambda) = (P + \mathrm{i}h - \lambda)xu.$$

doesn't give an analogous estimate because of the different sign.

The estimate (5.15) can be microlocalized. Let for $\delta \ll 1$

$$U_\delta = \{|\xi| < \delta, |x| < \delta\}.$$

Note that

$$(P-\lambda)u = \mathscr{O}(h^2) \text{ microlocally on } U_{2\delta}. \tag{5.16}$$

Let $\phi \in C_0^\infty((-4\delta,4\delta))$, $\phi = 1$ on $[-3\delta,3\delta]$, $\theta_+ = \xi\phi(x)\phi(\xi)$, $\Theta_+ = Op_h(\theta_+)$. Applying Θ_+ to (5.16) and using that $[\Theta_+, P] = \mathrm{i}hOp_h(H_p\theta_+) + \mathscr{O}(h^2) = -\mathrm{i}h\Theta_+ + \mathscr{O}(h^2)$ microlocally on $U_{2\delta}$ together with (5.16) gives an appropriate microlocalized version of (5.15):

$$\Theta_+ u = \mathscr{O}(h), \quad \text{microlocally on } U_{2\delta}. \tag{5.17}$$

A central lemma is then the following:

Lemma 5.13. *There exists a constant C such that for each* $\delta_0 > 0$,

$$\mu(U_\delta \cap \{|x| < \delta_0\}) \le C\delta_0. \tag{5.18}$$

Proof. We first note that for $a \in C_0^\infty(\Gamma_+ \cap U_\delta)$ we have:

$$\left|\int_{\Gamma_+\cap U_\delta} H_{\theta_+} a\, \mathrm{d}\mu\right| \le C\|a\|_\infty. \tag{5.19}$$

This follows indeed from Lemma 4.9 and (5.17). Now choose $a = \tilde{a}(x)\chi(\xi)$, with $\chi \in C_0^\infty((-\delta,\delta))$, $\chi(0) = 1$ and

$$\operatorname{supp}\tilde{a} \subset (-2\delta_0, \delta_1), \quad \|\tilde{a}\|_\infty \le 1, \quad \partial_x \tilde{a} \ge -\frac{2}{\delta_1}, \quad \partial_x \tilde{a} \ge \frac{1}{3\delta_0} \quad \text{for } |x| \le \delta_0$$

for some fixed $\delta_1 \in (0,\delta)$ independent of δ_0. Noting that $\theta_+ = \xi$ on U_δ we find

$$\int_{\Gamma_+\cap U_\delta} H_{\theta_+} a\, \mathrm{d}\mu \ge \frac{1}{3\delta_0}\mu(\{|x| \le \delta_0\}) - \frac{2}{\delta_1}\mu(\{\delta_0 \le |x| \le \delta_1\}).$$

The LHS is bounded by a δ_0-independent constant by (5.19); so is the second term on the RHS (since μ is a finite measure). Multiplying both sides by $3\delta_0$ we obtain (5.18). □

Lemma 5.14.

$$\mu(U_\delta \setminus \Gamma_+) = 0, \quad \mu(U_\delta) > 0.$$

Proof. The lemma follows from two type of estimates

(1) Take $A \in \Psi_h^0(\mathbb{R})$ such that $WF_h'(A) \cap \{p = 0\} \cap \Gamma_+ = \emptyset$. Then

$$\|Au\|_{L^2} \le Ch^{-1}\|(P - \mathrm{i}Q - \lambda)u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

This follows from the elliptic estimate [83, Proposition 2.4] and propagation of singularities with a complex absorbing operator [83, Proposition 2.4], since for each $(x,\xi) \in WF_h'(A)$, there exists $t \ge 0$ such that $\mathrm{e}^{-tH_p}(x,\xi) \in \{p - \mathrm{i}q \ne 0\}$.

(2) Let $B \in \Psi_h^{\mathrm{comp}}(X)$ such that $\sigma_h(B) \ne 0$ on $\{p = 0\} \cap K$:

$$\|u\|_{L^2} \le Ch^{-1}\|(P - \mathrm{i}Q - \lambda)u\|_{L^2} + \|Bu\|_{L^2}.$$

To prove this second estimate we use the elliptic estimate and propagation of singularities, noting that for each $(x,\xi) \in T^*\mathbb{R}$, there exists $t \ge 0$ such that $\mathrm{e}^{-tH_p}(x,\xi) \in \{p - \mathrm{i}q \ne 0\} \cup \{\sigma_h(B) \ne 0\}$, see [41, Lemma 4.1].

□

Proof of Lemma 5.11. Starting with (5.13) and passing to a subsequence, we may assume that

$$u \to \mu, \quad h^{-1}\Im\lambda \to \omega \in \mathscr{C}.$$

By Lemma 5.14 we know that

$$\mu(U_\delta) > 0, \quad \mu(U_\delta \setminus \Gamma_+) = 0.$$

We can now apply Lemma 4.9 to see that for each $a \in C_0^\infty(U_\delta)$ we have

$$\int_{\Gamma_+\cap U_\delta} H_p a\, \mathrm{d}\mu = 2\Im\omega \int_{\Gamma_+\cap U_\delta} a\, \mathrm{d}\mu.$$

For $t \ge 0$, since $\mathrm{e}^{-tH_p}(\Gamma_+ \cap U_\delta) \subset \Gamma_+ \cap U_\delta$, we have

$$\mu(\mathrm{e}^{-tH_p}(U_\delta)) = \mathrm{e}^{2t\Im\omega}\mu(U_\delta).$$

By Lemma 5.13, as $t \to \infty$

$$\mu(\mathrm{e}^{-tH_p}(U_\delta)) \le \mu(U_\delta \cap \{|x| < \delta \mathrm{e}^{-t}\}) \le C\mathrm{e}^{-t}.$$

However, since $\Im\omega \ge -(1/2) + \epsilon$, we see that $\mathrm{e}^{2t\Im\omega}$ decays exponentially slower than e^{-t}, as $t \to \infty$. Since $\mu(U_\delta) > 0$ we arrive at a contradiction, finishing the proof of the Proposition. □

5.4.2. *Approximation of resonances generated by the trapping*

As we have just seen the relatively mild trapping leads to a resonance free strip. Nevertheless the trapping creates resonances that we now want to compute in a high frequency limit. The resonances can be computed as the eigenvalues of a scaled operator which is no longer selfadjoint. The operator P coming from the De Sitter Schwarzschild metric is an operator with holomorphic coefficients

$$P(w, hD_w; h) = (hD_w)^2 + V(w,h); \quad V(w,h) = Fr^{-2}(1 + h^2 r\partial_r F). \tag{5.20}$$

Here w replaces the Regge–Wheeler coordinate r_*. We put $\Gamma_\theta = \mathrm{e}^{\mathrm{i}\theta}\mathbb{R}$ and we parametrize Γ_θ by x, $w = \mathrm{e}^{\mathrm{i}\theta}x$. The scaled operator is then defined by $P_\theta(x, hD_x; h) = P(w, hD_w; h)|_{\Gamma_\theta}$. Note that the RHS of this equation is well defined because all the coefficients are holomorphic. The important proposition is the following (see [18]):

Proposition 5.15. *For $-2\theta < \arg z < 2\pi - 2\theta$ the operator*

$$P_\theta(x, hD_x; h) - z : H^2(\Gamma_\theta) \to L^2(\Gamma_\theta),$$

is a Fredholm operator of index 0. The eigenvalues of $P_\theta(x, hD_x; h)$ in $-2\theta < \arg z < 2\pi - 2\theta$ coincide with the poles of the meromorphic continuation of $(P_\theta(x, hD_x; h) - z)^{-1}$ there.

To figure out what is a good choice for θ we come back to our toy model $\tilde{P} = h^2 D_x^2 - x^2$ in the previous section. We then see that a good choice is $\theta = \pi/4$. Indeed we then find $P_\theta = -\mathrm{i}(h^2 D_x^2 + x^2)$, which is just $-\mathrm{i}$ times the (semiclassical) harmonic oscillator and its eigenvalues are of course known. Putting all this together leads to the following characterization of the high frequency resonances in the De Sitter Schwarzschild case (see [18]):

Proposition 5.16. *There exists $K > 0$ and $\theta > 0$ such that for any $C > 0$ there exists an injective map, $\tilde{b}$, from the sets of pseudo-poles*

$$\frac{(1 - 9\Lambda\mathfrak{m}^2)^{1/2}}{3^{\frac{3}{2}}\mathfrak{m}}\left(\pm\mathbb{N} \pm \frac{1}{2} - \mathrm{i}\frac{1}{2}\left(\mathbb{N}_0 + \frac{1}{2}\right)\right)$$

into the set of poles of the meromorphic continuation of $(P - \sigma^2)^{-1}$ such that the poles in

$$\Omega_C = \{\sigma : \Im\sigma > -C, |\sigma| > K, \Im\sigma > -\theta|\Re\sigma|\}$$

are in the image of $\tilde{b}$ and for $\tilde{b}(\nu) \in \Omega_C$,

$$\tilde{b}(\nu) - \nu \to 0, \quad \textit{when } |\nu| \to \infty.$$

If $\Re\nu = ((1 - 9\Lambda\mathfrak{m}^2)^{1/2}/3^{3/2}\mathfrak{m})(\pm\ell \pm (1/2))$, $\ell = 1, 2, \ldots$, then the corresponding pole, $\tilde{b}(\nu)$ has multiplicity $2\ell + 1$.

5.5. *More general situations*

5.5.1. *De Sitter Kerr metric*

The proof for the equivalent of Theorems 5.1 and 5.2 for the De Sitter Kerr metric follows the same lines as in the De Sitter Schwarzschild case. The radial point estimates are now formulated in terms of propagation estimates which permit to propagate regularity out of these points. As already mentioned the trapping keeps the same structure and the proof for the high energy resolvent estimates follows the same lines. At first glance the new aspect seems to be superradiance. Nevertheless thanks to the very robust Fredholm setting this point is now reduced to the analysis of mode solutions.

5.5.2. *Kerr metric*

The Kerr case is asymptotically Minkowskian and one has to deal with this aspect. As an immediate consequence, the underlying resolvent doesn't possess a meromorphic extension in a strip beyond the real axis. It has nevertheless good C^k properties down to the real axis, see [33,34] and [39]. Another important aspect is that we find a radial point structure of the Hamiltonian flow also at infinity.

5.5.3. *Other fields*

As long as the fields are massless and not charged, most of the here presented aspects remain valid also for other hyperbolic field equations than the wave equation. Note that the presence of mass or charge of the field can also create growing modes, we refer to Shlapentokh-Rothmann for the Klein Gordon equation on the Kerr metric [84] and to Besset–Häfner [85] for the charged Klein Gordon equation on the De Sitter Reissner–Nordström or De Sitter Kerr Newman black hole. In the case of linearized gravity, one has of course to deal with the gauge. The presented methods are well adapted to generalized wave map gauges. In these generalized wave map gauges the Fourier linearized Einstein operator $\widehat{L}_g(\sigma)$ will have a kernel for $\sigma = 0$ consisting of linearized (De Sitter) Kerr solutions and some special gauge solutions. The existence of this kernel makes the analysis of $\widehat{L}_g^{-1}(\sigma)$ more delicate close to $\sigma = 0$.

5.5.4. *Perturbations of the (De Sitter) Kerr metric*

As long as the metric remains stationary and the important structure of radial points, trapping etc. is not changed the analysis remains more or less the same on the linear level. Note that in contrast to the trapping the radial point structure is a priori not stable with respect to perturbations; however the estimates are, see [23, Remark 2.5] for a discussion of this point. A delicate part is the analysis of the mode solutions which most often is analyzed on the exact spacetimes by separation of variables and ode techniques. This can however often be analyzed in a perturbative regime.

If the spacetime itself is not stationary but say decays to a (De Sitter) Kerr solution, the here presented analysis has to be completed by a second step. Note however that this second step can be purely microlocal, see [63]. These aspects go beyond the scope of this review.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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