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Review article / *Article de synthèse*

Gluing for the Einstein constraint equations

Recollement pour les équations de contrainte d'Einstein

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Abstract. Initial data for the Einstein equations must satisfy a system of nonlinear partial differential equations, the Einstein constraint equations. Constructing interesting solutions of the constraint equations can then lead to interesting spacetime evolutions. Over the past twenty-five years, various gluing methods to construct solutions of the constraints have shed light on some long-standing questions. In this article we survey some of the methods and results achieved to date.

Résumé. Les données initiales des équations d'Einstein doivent satisfaire un système d'équations aux dérivées partielles non linéaires, les équations de contrainte d'Einstein. La construction de solutions intéressantes des équations de contrainte peut alors conduire à des évolutions intéressantes de l'espace-temps. Au cours des vingt-cinq dernières années, diverses méthodes de recollement permettant de construire des solutions des équations de contrainte ont permis de faire la lumière sur certaines questions de longue date. Dans cet article, nous passons en revue quelques-unes des méthodes et des résultats obtenus à ce jour.

Keywords. Einstein constraint equations, Geometric analysis, Gluing constructions.

Mots-clés. Equations de contrainte d' Einstein, Analyse géométrique, Méthodes de recollement.

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1. Introduction and preliminaries

The foundational work of Yvonne Choquet-Bruhat[1] firmly established that the Einstein field equation can be treated as an evolution problem [5]. The initial data for the vacuum equation are given by a Riemannian manifold (M, g) along with a symmetric $(0,2)$-tensor K, that determine a vacuum spacetime, i.e. a time-orientable Ricci-flat Lorentzian manifold $(\mathscr{S}, \bar{g})$ in which M is a Cauchy surface embedded with first and second fundamental forms g and K, which play the role of the initial metric and its time derivative (in appropriate coordinates). In the non-vacuum case, the initial data will be augmented with data for whatever physical fields are being modeled.

The initial data g and K are not freely prescribed, but must adhere to the Gauss and Codazzi equations relating the ambient geometry to the fundamental forms. The Einstein equation involves the Ricci tensor, and so from appropriate traces of the Gauss and Codazzi equations along with the Einstein equation, one deduces a coupled, nonlinear *underdetermined* elliptic system for g and K, the exploration of which has provided a rich playground for mathematical physicists and geometric analysts, with deep connections to topology and geometry, in part forged by advances in the theory of partial differential equations. Robust methods to construct

[1] For her exposition on this, see her book [1], the classic survey articles [2, 3], as well as [4] for her comments on some of the history of the problem.

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solutions of the constraint equations allow for constructions, theoretical and numerical, of spacetimes. Understanding the solution space to the constraint equations is thus intimately tied to an understanding of the solution space of the Einstein field equation, and a parametrization of the solution space of the constraints would ostensibly isolate the true gravitational degrees of freedom [6], cf. [7].

A fulsome account of the work on the constraint equations, which one can trace back at least eighty years to classical work of Lichnerowicz [8], is well beyond the scope of this article. Rather, we focus on some relatively recent work of the past twenty-five years on constructing solutions of the Einstein constraint equations by *gluing* techniques, discussing some of the methods and applications without giving an exhaustive account. Gluing techniques can be used to address the following natural question: given multiple solutions of the constraint equations, is there another solution (on a connected manifold) which contains chosen regions from the given initial data sets (at least approximately)? There is of course the related question about spacetimes: given regions of multiple spacetimes solving Einstein's field equation, is there a (connected) spacetime satisfying the field equation that contains them?

From the initial data point of view, if one can construct an initial data set which contains regions obtained from several spacetimes, the resulting evolution of the initial data will contain regions isometric to ones in the given spacetimes, by finite speed of propagation. That said, one might naturally try to glue spacetime regions together directly, or at least utilize the Lorentzian structure in the construction. In this direction we mention two different recent series of works. Hintz constructs spacetimes with multiple small black holes by a gluing construction using geometric singular analysis (pseudodifferential b-calculus) techniques (see [9], cf. [10]). In a different direction, Aretakis, Czimek and Rodnianski achieve a suite of gluing results by *characteristic gluing*, which leverages the *null structure* of spacetime, as surveyed with additional references in [11]. While these works are interesting, we do not have the time in the present short survey article to do justice to either, and we will focus on constructing solutions of the constraint equations on a spacelike hypersurface (though we do mention some compelling results obtained by the methods of [11] in Section 4 below).

1.1. *Overview of some spacelike gluing results*

A natural interpretation of the above aim of gluing is that of constructing initial data for a relativistic N-body problem [12], including, say, multi-black hole spacetimes. Such constructions have been achieved in earlier works [13–15], while more recent works have used gluing to construct more exotic such initial data sets, such as the N-body configurations with gravitational shielding from [16], multi-localized initial data sets [17], as well as initial data with small black holes [10]. Naturally, the issue of the two-body and N-body problems in general relativity dates back a century (see [8, 18, 19] and references therein) and various sorts of approaches have been taken to produce initial data to model such spacetimes.

Connected sum construction of initial data was achieved by Isenberg, Mazzeo and Pollack in [20] by the conformal method, and Chruściel and Delay [21] observed how to combine conformal and localized gluing techniques (introduced in [22, 23]) to perform connected sum constructions which leave the data unperturbed outside a connecting neck region (cf. [24, 25] for related results, and [26, 27] for analogous surgery results). Of particular note, [28] established that there is no topological obstruction to admitting an asymptotically Euclidean solution of the vacuum Einstein constraint equations: for a closed manifold Σ and $p \in \Sigma$, there is a vacuum asymptotically flat solution of the Einstein constraints on $\Sigma \setminus \{p\}$. An intriguing implementation of the conformal gluing method was employed by Stavrov in [29] to construct families of initial data to model point particle limits in general relativity (cf. recent work of Hintz [9, 10] along these lines).

Gluing constructions have also shed light on the asymptotics of solutions of the constraints. The original motivation for the construction in [22] was to understand to what extent the asymptotics determine the interior of a vacuum initial data set, in particular how special are Schwarzchild asymptotics. That initial work showed that any compact subset of any time-symmetric asymptotically flat vacuum initial data set is also contained in a vacuum time-symmetric initial data set which is *exactly* Schwarzschild near infinity. While that in itself may seem intriguing, one immediately turns to the spacetime evolution of such data. To be precise, the question open at the time was whether there exist any nontrivial asymptotically simple vacuum spacetimes (sometimes called *purely radiative* spacetimes), a question answered in the affirmative in [30, 31] by constructing suitable families of initial data which are Schwarzschild with small mass in a *uniform* neighborhood of infinity, evolving to an asymptotically hyperboloidal hypersurface, and then invoking the work of [32] for the asymptotics of the evolution.

Another question that has been addressed by gluing methods is that of *gravitational shielding*. A remarkable result was obtained by Carlotto and Schoen [16], who showed that any region of any vacuum initial data set can be localized within a conical region in a vacuum initial data set which is *flat* outside the conical region. While Bartnik's parabolic quasi-spherical construction [33] produced examples of nontrivial asymptotically flat vacuum initial data (time-symmetric, so scalar-flat) containing a flat region, their localized gluing construction produced striking examples with non-compact flat regions. The results in [16] were proven in spirit along the lines of [21–23], with a number additional of technical obstacles to surmount. While the works [21–23] established that any bounded region in an asymptotially flat vacuum solution can be realized as a subdomain in a solution in which each asymptotic end is a slice of a Schwarzschild or Kerr spacetime, the method of proof requires that the end be tuned appropriately to the given data (in terms of the asymptotically conserved quantities: ADM energy-momentum, center-of-mass and angular momentum), whereas the regions in the result of Carlotto and Schoen are glued to flat exteriors outside the conical regions. In any case, all these results illustrate that unique continuation fails spectacularly for the vacuum constraints.

In this article we overview aspects of gluing by the conformal method [20] and compare with the localized approach. While there are many excellent references for the conformal method, localized gluing constructions are relatively newer, and are still being developed, so we will focus on some details of the approach to localized gluing that dates back to the works [22, 23]. In the spirit of "something old and something new", we illustrate the localized gluing technique by applying it to the time-symmetric Einstein–Maxwell constraint equations.

1.2. *Preliminaries*

1.2.1. *The Einstein constraint equations*

The vacuum constraint equations arise from the following. Consider a spacetime splitting $(\mathscr{S} = I \times M, \bar{g})$, with I an interval and (M, g) a Riemannian hypersurface given as a level set of the coordinate x^0 on I (so $\partial/\partial x^0$ is timelike). With $G = \mathrm{Ric}(\bar{g}) - (1/2)R(\bar{g})\bar{g}$ the Einstein tensor, $G_{\alpha\beta}n^\beta$ does not contain any second time derivatives $\bar{g}_{\mu\nu,00}$ of the metric components, where n is a timelike unit normal to M. In fact, using the Gauss and Codazzi equations, one obtains, with $G(n,\cdot)$ a one-form along M,

$$G(n,n) = \tfrac{1}{2}(R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2) \tag{1}$$

$$G(n,\cdot) = \mathrm{div}_g(K - (\mathrm{tr}_g K)g). \tag{2}$$

If we consider a cosmological constant Λ, and let $G_\Lambda = G + \Lambda\bar{g}$, and if T is the stress-energy tensor, the Einstein equation takes the form $G_\Lambda = \kappa T$ for some constant $\kappa > 0$. (In spacetime dimension four, $\kappa = 8\pi G/c^4$.) We observe that $\mu := T(n,n)$ is the energy density as measured by an observer with four-velocity cn, and $c^{-1}J := -c^{-1}T(n,\cdot)$ is corresponding momentum density one-form. The dominant energy condition that the energy-momentum vector of the matter fields is causal future-pointing is given by $\mu \geq |J|_g$. Combining the Einstein equation with (1)–(2) yields the Einstein constraint equations

$$\tfrac{1}{2}(R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2) = \kappa\mu + \Lambda \tag{3}$$

$$\mathrm{div}_g(K - (\mathrm{tr}_g K)g) = \kappa J. \tag{4}$$

In the vacuum ($T = 0$) time-symmetric ($K = 0$) case, the system reduces to the constant scalar curvature (CSC) condition $R(g) = 2\Lambda$. Unless noted, we take $\Lambda = 0$, in which case the time-symmetric vacuum constraints reduce to $R(g) = 0$.

For a non-vacuum example, consider initial data for the time-symmetric Einstein–Maxwell equations, often given as a Riemannian metric g and an electric field E. The constraint (3) becomes $R(g) = 2\kappa|E|_g^2$. In any source-free region, we must also have $\mathrm{div}_g E = 0$. For simplicity we will treat E as a vector field and study the map $\Psi(g,E) = (R(g) - 2\kappa|E|_g^2, \mathrm{div}_g E)$. We note that it could be more natural to specify the field through a differential form η_E, to which a metric g would associate a vector field E_g in such as way that the condition $\mathrm{div}_g E_g = 0$ would follow *for any* g in a source-free region. We will see the echo of this in the gluing method below. For further exploration of specifying the initial data for fields, see [34].

We will focus on the constraints in dimension $n = 3$, though much of what is discussed below extends readily to $n \geq 3$. We employ the Einstein summation convention of summing over repeated upper and lower indices; a semicolon denotes a covariant derivative, and a comma denotes a partial derivative.

1.2.2. *Some classical solutions*

For later use, we recall a family of time-symmetric solutions to the Einstein–Maxwell constraint equations, each obtained from data on a spacelike hypersurface in a Reissner–Nordström spacetime: $g_{\mathrm{RN}} = (1 - (2m/r) + (Q^2/r^2))^{-1}\,dr^2 + r^2 \mathring{g}_{\mathbb{S}^2}$, with $E_{\mathrm{RN}} = (Q/r^2)\sqrt{1-(2m/r)+(Q^2/r^2)}(\partial/\partial r) =: (Q/r^2)e_r$, where $\mathring{g}_{\mathbb{S}^2}$ is the unit round metric. Since $R(g_{\mathrm{RN}}) = 2Q^2/r^4$, we see the units are such that $\kappa = 2$. We will just employ the data on an asymptotic end $r = |x| \gg 1$, and so we do not pose any relation between m and Q, though that could affect the Reissner–Nordström spacetime structure.

For $Q = 0$, the Reissner–Nordström metric reduces to the Schwarzschild metric, which has vanishing scalar curvature. We write the metric in isotropic form, and we can also re-center the coordinates to obtain $(g_S^{m,c})_{ij}(x) = (1 + m/(2|x-c|))^4\delta_{ij}$. A metric $g_{ij} = u^4\delta_{ij}$ which is conformally Euclidean has vanishing scalar curvature precisely when u is Euclidean harmonic. If we require that u tends to 1 at infinity, to impose asymptotically Euclidean geometry, then u is either identically 1 or it must have singularities, such as for the Schwarzschild metric with $m \neq 0$, and for the following family of solutions.

As recalled earlier, it has been a longstanding question as to how to construct solutions to the Einstein equations to model the interaction of multiple bodies. A classical solution of the vacuum constraints is given by vacuum Brill–Lindquist metrics (see [35,36], cf. [8]): given a finite set $\mathfrak{p} = \{p_1, \ldots, p_N\}$ of N points in $\mathbb{R}^3$ and $\mathfrak{m} = (m_1, \ldots, m_N) \in \mathbb{R}^N$, we consider the metric

$$(g_{\mathrm{BL}}^{\mathfrak{m},\mathfrak{p}})_{ij}(x) = \left(1 + \sum_{k=1}^{N} \frac{m_k}{2|x-p_k|}\right)^4 \delta_{ij} \tag{5}$$

for $x \in \mathbb{R}^3 \setminus \mathfrak{p}$. This metric is conformally Euclidean with harmonic conformal factor, and thus has vanishing scalar curvature, providing a time-symmetric vacuum initial data set. If we take all $m_k > 0$, then the data has $N+1$ asymptotically flat ends. By an ingenious use of the method of images, Misner [37] modified the conformal factor, incorporating an infinite sequence of inversions, to produce a scalar-flat metric with two asymptotically flat ends and N Einstein–Rosen bridges, hence obtaining a type of gluing construction by conformal means. In another direction, one can modify the Brill–Lindquist data by localized gluing techniques to produce data on $\mathbb{R}^3$, even allowing $\mathfrak{p}$ to be infinite; see [17] along with comments in Section 4 below.

1.2.3. *The conformal method*

The vacuum Einstein constraints form an *underdetermined* elliptic system for g and K. The conformal method specifies some parts of g and K, leaving a determined elliptic system. This approach can be traced back to[2] Lichnerowicz [8], Choquet-Bruhat [38], York and Ó Murchadha [6,39,40]. The formulation below is especially well suited to the constant mean curvature (CMC) case with $\mathrm{tr}_g K$ constant[3] (for a variation see [40]). It is based on the transverse-traceless (TT) decomposition of symmetric two-tensors. In this approach, a conformal class $\mathscr{C}$ of metrics is given, say $\mathscr{C} = [\gamma]$. Furthermore a scalar function τ and a $(0,2)$-tensor σ are given on M^n, where σ is TT with respect to γ: $\mathrm{div}_\gamma \sigma = 0$, $\mathrm{tr}_\gamma \sigma = 0$.

From this data we seek a scalar function $u > 0$ and a vector field W such that, with

$$g = u^{\frac{4}{n-2}}\gamma, \quad K = u^{-2}(\sigma + L_\gamma W) + \frac{\tau}{n} u^{\frac{4}{n-2}}\gamma,$$

where $(L_\gamma W)_{ij} = W_{i;j} + W_{j;i} - (2/n)(\mathrm{div}_\gamma W)\gamma_{ij}$ is the conformal Killing operator, (g,K) satisfies the vacuum constraint equations, which take the form

$$\Delta_\gamma u - \frac{n-2}{4(n-1)} R(\gamma) u + \frac{n-2}{4(n-1)} |\sigma + L_\gamma W|^2_\gamma u^{-\frac{3n-2}{n-2}} - \frac{n-2}{4n}\tau^2 u^{\frac{n+2}{n-2}} = 0 \tag{6}$$

$$\mathrm{div}_\gamma (L_\gamma W) - \frac{n-1}{n} u^{\frac{2n}{n-2}}\, d\tau = 0. \tag{7}$$

The system is a semilinear elliptic system for (u, W). In the CMC case (constant τ), the system decouples: the second equation becomes $\mathrm{div}_\gamma(L_\gamma W) = 0$. On a closed manifold, or asymptotically flat manifold with W decaying suitably near infinity, multiplying by W and integrating by parts gives $L_\gamma W = 0$, i.e. W is a *conformal Killing field*. In this case W does not affect K or the equation for u, and we can just take $W = 0$ (which is the only option in the absence of conformal symmetries). Then the conformal method comes down to solving the *Lichnerowicz equation*

$$\Delta_\gamma u - \frac{n-2}{4(n-1)} R(\gamma) u + \frac{n-2}{4(n-1)} |\sigma|^2_\gamma u^{-\frac{3n-2}{n-2}} - \frac{n-2}{4n}\tau^2 u^{\frac{n+2}{n-2}} = 0.$$

As noted earlier, we will focus on $n = 3$, in which case the above becomes

$$\Delta_\gamma u - \tfrac{1}{8} R(\gamma) u + \tfrac{1}{8}|\sigma|^2_\gamma u^{-7} - \tfrac{1}{12}\tau^2 u^5 = 0. \tag{8}$$

1.2.4. *The conformal method and asymptotics*

When gluing initial data with the conformal method (see Section 2), the resulting data can be made arbitrarily close to the given data outside the gluing region. A similar scenario arises when using conformal perturbations to arrange conformally Euclidean, or more generally *harmonic*, asymptotics. Indeed Schoen and Yau [42] showed that a scalar-flat asymptotically flat metric can be approximated by one which is conformally Euclidean outside a compact set, and so that the ADM energy is perturbed by an arbitrarily small amount. The method of proof is straightforward: along a large annular region, patch the original metric to a Euclidean metric near infinity; for large enough annuli, we get approximately scalar-flat data. One then employs a conformal factor to reimpose the vacuum constraint, with appropriate estimates on the size of the perturbation of the metric and the ADM energy. Such an approximation result is useful for proving the Positive

[2] One can find an early analysis of the conformally Euclidean case in [18], which also includes some early study of static spaces and asymptotically Euclidean spaces; see [4] for more comments on the history.

[3] See [41], cf. [7,34], for discussion of the conformal method outside the vacuum CMC regime.

Energy Theorem [43] and the Riemannian Penrose Inequality, since it simplifies the asymptotics with only a small change to the energy, and Bray termed such metrics *harmonically flat* at infinity [44].

An analogous non-time-symmetric version established in [23], *harmonic asymptotics*, has also proved useful in this and other contexts [45, 46]. The formulation is a modification of the conformal method above so that the principal part of the constraints operator in the asymptotic region is a diagonal Laplacian. The asymptotic form of the data in harmonic asymptotics is

$$g_{ij} = u^4\delta_{ij}, \quad K_{ij} = u^2\left(X_{i,j} + X_{j,i} - \tfrac{1}{2}X^k_{,k}\delta_{ij}\right). \tag{9}$$

This form is usually written in terms of the momentum tensor $\pi = K - (\mathrm{tr}_g K)g$, in which case this becomes $\pi_{ij} = u^2(X_{i,j} + X_{j,i} - X^k_{,k}\delta_{ij})$.

The proof for harmonic asymptotic approximation of asymptotically flat initial data sets is similar to the time-symmetric case. Given asymptotically flat data (g,K), smoothly patch the metric to Euclidean while cutting off K to zero, over a large annular region $R < |x| < 2R$, with the resulting data $(\hat{g},\hat{\pi})$ (a family of data depending on R). Let $(\mathscr{L}_g X)_{ij} = X_{i;j} + X_{j;i} - (\mathrm{div}_g X)g_{ij}$. One seeks to reimpose the constraints and obtain harmonic asymptotics by considering data of the form $\bar{g} = u^4\hat{g}$ and $\bar{\pi} = u^2(\hat{\pi} + \mathscr{L}_{\hat{g}}X)$, or $\bar{K} = u^2(\hat{K} + \mathscr{L}_{\hat{g}}X + (1/2)(\mathrm{div}_{\hat{g}}X)\hat{g})$. Observe that it is not the conformal Killing operator $L_{\hat{g}}X$ that appears here, in contrast to the standard formulation of the conformal method. In any case, $(\hat{g},\hat{K})$ is an approximate solution of the vacuum constraints (better for larger R), and one seeks to perturb to an exact solution, for which one considers the map

$$(u,X) \mapsto T(u,X) = \left(R(\bar{g}) - |\bar{K}|^2_{\bar{g}} + (\mathrm{tr}_{\bar{g}}\bar{K})^2, \mathrm{div}_{\bar{g}}(\bar{K} - (\mathrm{tr}_{\bar{g}}K)\bar{g})\right).$$

Outside a compact set, i.e. for $|x| > 2R$, the vacuum constraints $T(u,X) = 0$ can be written in Cartesian components as (omitting metric subscripts for the Euclidean metric)

$$8\Delta u = u\left(-|\mathscr{L}X|^2 + \tfrac{1}{2}(\mathrm{tr}(\mathscr{L}X))^2\right), \quad \Delta X^i = -4u^{-1}u_{,j}(\mathscr{L}X)^j_i + 2u^{-1}u_{,i}\mathrm{tr}(\mathscr{L}X). \tag{10}$$

Solutions in spaces with suitable decay for $u-1$ and X will admit partial expansions of the form $u(x) = 1 + (a/|x|) + O(|x|^{-2})$, and $X^i(x) = (b^i/|x|) + O(|x|^{-2})$. Using (9), one finds the ADM energy is $2a$, and the linear momentum is $P^i = -(1/2)b^i$.

The linearization of $T(u,X)$ at $(1,0)$ is readily found to be Fredholm of index zero between appropriate spaces [23], of which (10) is certainly indicative. The possible existence of a finite-dimensional cokernel is accommodated by finding a complementary finite-dimensional space of *compactly supported* tensors, and showing that one can find (h,k) near $(0,0)$ in this complementing space along with (u,X) close to $(1,0)$ (for large R), such that $(\bar{g}+h, \bar{K}+k)$ solves the vacuum constraints. It can also be shown that the energy and linear momentum can be perturbed an arbitrarily small amount in this construction [23].

In a certain sense, these approximation constructions establishing that a kind of asymptotic behavior is suitably dense can be construed as a type of gluing construction, gluing a given interior to an exterior end in a certain family. Good approximate solutions form a family parametrized by R, which can be corrected with suitable perturbations. In this case the family of exterior data, either the harmonically flat metrics, or more generally data with harmonic asymptotics, is a large family, as compared to a data like the Schwarzschild family, parametrized by m and c. Just as in the conformal gluing we recall below, the given data is perturbed a small (for large R) amount.

Harmonic asymptotics yield a particularly convenient form of the asymptotics for the energy and momenta. For some other applications, one might want to make controlled construction of solutions of the constraints with more general asymptotics. For instance, recent work of Fang, Szeftel and Touati [47, 48] constructs suitable classes of vacuum initial data compatible with the stability results for Minkowski spacetime and for black hole stability. The basic approach is to start with an approximate solution with a desired asymptotic behavior and then perturb to a solution of the constraints, with estimates on the perturbation which ensure the desired decay rate. The perturbation is as above, utilizing a conformal change of metric, together with a perturbation of the momentum tensor of the form $\mathscr{L}_\gamma X$ (for a suitable metric γ), plus a complementing compactly supported perturbation.

1.2.5. *Remarks on localized deformations*

A fundamental issue for constructing solutions to the constraints by gluing methods is how to obtain controlled perturbations of the constraints map (where $\pi = K - (\mathrm{tr}_g K)g$)

$$(g,\pi) \mapsto \Phi(g,\pi) := (R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2, \mathrm{div}_g \pi). \tag{11}$$

As recalled above, modifying data with a suitable conformal transformation has proven fruitful. Since a conformal factor u solves an elliptic equation, we do not expect to be able to effectively contain the support of $u-1$. With the linearization stability results of Fischer and Marsden [49,50] in mind, one can naturally move outside the conformal regime and consider a general form of perturbations, modifying (g,π) to $(g+\delta g, \pi+\delta\pi)$. The vacuum constraints then give an underdetermined elliptic system for the perturbations $(\delta g, \delta\pi)$. This freedom is what has been employed in [22, 23] and numerous subsequent works to achieve more direct control on the support of the deformation tensors.

Of course, the analysis will focus on the linearized constraints operator, which morally speaking should be locally surjective when the formal adjoint of the linearization has trivial kernel. The presence of nontrivial kernel indicates symmetries in the spacetime evolution of the data, as in [51], so that except in special situations, we expect to be able to effectively perturb the constraints operator. That said, in many gluing constructions, the approximate solution approaches a special configuration which admits symmetries, and this requires some extra care to handle.

To make this precise, we consider the linearization and formal L^2-adjoint for the scalar curvature and vacuum constraints operators. The well-known formulas for the linearization of the scalar curvature operator and its adjoint are given by

$$DR_g(h) = -\Delta_g(\mathrm{tr}_g h) + \mathrm{div}_g \mathrm{div}_g h - h \cdot_g \mathrm{Ric}(g), \quad DR_g^*(f) = -(\Delta_g f)g + \mathrm{Hess}_g f - f\mathrm{Ric}(g). \tag{12}$$

At a Ricci-flat metric, an element of the kernel of DR_g^* is precisely a function f with vanishing Hessian, which in the Euclidean case means f is a constant plus a linear combination of the Cartesian coordinate functions x^1, x^2, x^3.

While the formulas for the linearization of Φ and its formal adjoint are straightforward to compute in the general case, they simplify tremendously at the flat data, about which we will be perturbing. Let $\mathring{g}$ be the Euclidean metric on $\mathbb{R}^3$. If we let $D\mathring{\Phi} = D\Phi_{(\mathring{g},0)}$, we find from (11)

$$D\mathring{\Phi}(h,\omega) = (DR_{\mathring{g}}(h), \mathrm{div}_{\mathring{g}}\omega), \quad D\mathring{\Phi}^*(f,Z) = \left(DR_{\mathring{g}}^*(f), -\tfrac{1}{2}L_Z\mathring{g}\right),$$

where $(L_Z g)_{ij} = Z_{i;j} + Z_{j;i}$ is the Lie derivative, so that at $\mathring{g}$ in Cartesian coordinates, we have $(L_Z\mathring{g})_{ij} = Z_{i,j} + Z_{j,i}$. Now, $L_Z g = 0$ precisely for a Killing vector field Z, which at $\mathring{g}$ must be a linear combination of the generators of translations and rotations. Thus the kernel of $D\mathring{\Phi}^*$ gives a ten-dimensional space of potential obstructions to localized gluing constructions for the vacuum Einstein equations, near Minkowskian initial data. To accommodate for this, we glue to model families of solutions that are governed by ten parameters which can cover this approximate cokernel, such as the Kerr family of asymptotically flat initial data (see, e.g. [21, Appendix F]). We will see in Section 3 below how the mass and center-of-mass parameters can be used to handle the kernel of $DR_{\mathring{g}}^*$, while in the general case, the linear and angular momentum parameters correspond to the translational and rotational Killing fields [21,23]. In general the kernel elements of $D\Phi_{(g,\pi)}^*$ are related to spacetime symmetries [51], and are called *Killing Initial Data*.

2. Gluing by the conformal method

We now discuss the connected sum construction of Isenberg, Mazzeo and Pollack [20]. We start with solutions of the vacuum constraints on manifolds M_1 and M_2 with constant mean curvature τ, given on each as a Riemannian metric γ_j, and a TT tensor μ_j with respect to γ_j, so that the second fundamental form is given $K_j = \mu_j + (\tau/3)\gamma_j$ (recall that we consider dimension three).

To construct a connected sum $M = M_1 \sharp M_2$ of disjoint manifolds M_1 and M_2, one removes a ball around points $p_1 \in M_1$ and $p_2 \in M_2$ and identifies the union of the complements along the boundary spheres (the construction could also be applied where p_1 and p_2 are points on the same manifold, corresponding to adding a handle). We seek to construct a family of solutions on $M_1 \sharp M_2$, such that for any compactly contained domains away from p_1 and p_2, there are solutions in the family arbitrarily close to the original data. The approximate solutions are obtained using conformal blowups of $M_1 \setminus \{p_1\}$ and $M_2 \setminus \{p_2\}$, producing solutions with asymptotically cylindrical ends. The connected sum is then performed by cutting off far down on each cylinder and making appropriate identifications. We assume that K_j is nontrivial, and we also make the nondegeneracy assumption that there are no nontrivial conformal Killing vector fields on M_j that vanish at p_j.

In normal coordinate neighborhoods $U_j \ni p_j$ around each point, the metric takes the form $\gamma_j = \mathrm{d}r_j^2 + r_j^2 h_j(r_j)$, where h_j gives a family of metrics on $\mathbb{S}^2$, with $h_j(0) = \mathring{g}_{\mathbb{S}^2}$ the unit round metric. For $j = 1,2$, let ψ_j be a positive smooth function which is identically one on most of M_j, and which is $r_j^{1/2}$ near each p_j, say for $r_j < \mathring{r}$, and with $\psi_j = 1$ outside of $r_j < 2\mathring{r}$ (take $\log(\mathring{r}^{-2}) > 1$). With the change in radial coordinate $t_j = -\log r_j$, we find $\psi_j^{-4}\gamma_j = \mathrm{d}t_j^2 + \mathring{g}_{\mathbb{S}^2} + O(\mathrm{e}^{-2t_j})$ [25]. One introduces a large parameter $T \gg -2\log \mathring{r} > 1$ to specify how far along each neck we go before transitioning smoothly to an *exactly* cylindrical metric $\mathring{\gamma} = \mathrm{d}s_j^2 + \mathring{g}_{\mathbb{S}^2}$. Indeed with coordinate $s_j = -\log(r_j/\mathring{r}) - (T/2)$, we have $\psi_j^{-4}\gamma_j = \mathrm{d}s_j^2 + \mathring{g}_{\mathbb{S}^2} + \mathrm{e}^{-T}\mathrm{e}^{-2s_j}\mathring{r}^2\hat{h}_j$, and we can smoothly transition in the region $(-1,-1/2)\times\mathbb{S}^2$ to the exactly cylindrical metric on $(-1/2,\infty)\times\mathbb{S}^2$. We appropriately form a quotient along the cylindrical pieces $(-1/2,1/2)\times\mathbb{S}^2$ where the change of coordinates identifies s_1 with $-s_2$; note that $s_j = 0$ where $r_j = \mathring{r}\mathrm{e}^{-T/2}$, while $s_j = -T/2$ where $r_j = \mathring{r}$. With $s = s_1$ for $s \le 0$ and $s = -s_2$ for $s \ge 0$, then on $C_T \cong [-T/2, T/2]\times\mathbb{S}^2$, the metric is $\gamma_T = \mathrm{d}s^2 + \mathring{g}_{\mathbb{S}^2} + \mathrm{e}^{-T}\cosh(2s)\hat{h}_T$, where $\hat{h}_T$ and its covariant derivatives with respect to the cylindrical metric are bounded.

We have now put a metric on the connected sum, patching together the conformal rescalings. Observe that $\psi_j^2\mu_j$ is TT for $\psi_j^{-4}\gamma$, and just like for γ_T, we can patch together the conformally rescaled TT-tensors along the cylinder, to obtain an approximate solution μ_T. We want to conformally rescale appropriately to get a suitable approximate solution to the constraint equations. To do this, we patch the conformal factors $\psi_T := \chi_{1,T}\psi_1 + \chi_{2,T}\psi_2$, in such a way that ψ_T is the sum $\psi_1 + \psi_2$ inside the region corresponding to $[-(T/2)+1, (T/2)-1]\times\mathbb{S}^2$, i.e. most of C_T. In this region, then

$$\psi_T = r_1^{1/2} + r_2^{1/2} = (\mathring{r}\mathrm{e}^{-s-\frac{T}{2}})^{1/2} + (\mathring{r}\mathrm{e}^{s-\frac{T}{2}})^{1/2} = 2\mathring{r}^{1/2}\mathrm{e}^{-\frac{T}{4}}\cosh\left(\frac{s}{2}\right).$$

In the exactly cylindrical piece $-1/2 < s < 1/2$, $\psi_T^4\gamma_T$ is readily seen to have scalar curvature *zero*. By rotational symmetry, then, it is *exactly Schwarzschild* in this region, with mass $2\mathring{r}\mathrm{e}^{-T/2}$. Thus for T large, we have a connected sum with the middle of the neck being a neighborhood of the very small horizon sphere.

From here, one can first modify μ_T to a γ_T-TT tensor $\tilde{\mu}_T = \mu_T - \sigma_T$, with the perturbation σ_T which is globally small, with norm exponentially decaying in T [20, Proposition 5]. This step employs the conformal Killing operator, and this is where the nondegeneracy assumption is used. The proof is interesting and nontrivial, but we focus on the Lichnerowicz operator here for the sake of brevity, noting its role in constant scalar curvature gluing (see, e.g. [25]).

Having sorted out $\tilde{\mu}_T$, one then solves the Lichnerowicz equation

$$\mathscr{N}_T(u) := \Delta_{\gamma_T} u - \tfrac{1}{8}R(\gamma_T)u + \tfrac{1}{8}|\tilde{\mu}_T|^2_{\gamma_T} u^{-7} - \tfrac{1}{12}\tau^2 u^5 = 0.$$

We make some remarks on the proof.

First, ψ_T can be used as an approximate solution, since one can bound $\|\mathscr{N}_T(\psi_T)\|_{C^{k,\alpha}} \le C e^{-T/2}$, where C is independent of T large [20, Proposition 6], cf. [25, Proposition 4.6]. Indeed outside of C_T, $\mathscr{N}_T(\psi_T) = 0$. Now, where $\psi_T = \psi_1 + \psi_2$, we have $\Delta_{\mathring{\gamma}}\psi_T - (1/8)R(\mathring{\gamma})\psi_T = 0$ (we note $R(\mathring{\gamma}) = 2$ for the cylindrical metric). As γ_T is approximately cylindrical in $[-1,1]\times\mathbb{S}^2$, with precise error estimates, and with corresponding estimates on $\tilde{\mu}_T$ and ψ_T, we can estimate $\mathscr{N}_T(\psi_T)$ in this region. The other regions in C_T are handled likewise, using that in $[-T/2,-1]\times\mathbb{S}^2$ we have $\mathscr{N}_T(\psi_1) = 0$, and in $[1, T/2]\times\mathbb{S}^2$ we have $\mathscr{N}_T(\psi_2) = 0$.

The next step is to get uniform (in T) bounds on the inverse of the linearization $\mathscr{L}_T = D\mathscr{N}_T|_{\psi_T}$. We observe

$$\mathscr{L}_T = \Delta_{\gamma_T} - \tfrac{1}{8}\left(R(\gamma_T) + 7|\tilde{\mu}_T|^2_{\gamma_T}\psi_T^{-8} + \tfrac{10}{3}\tau^2\psi_T^4\right). \tag{13}$$

Showing that $\mathscr{L}_T$ is invertible for all T large and bounding the inverse are both handled by blowup arguments. Indeed, suppose the existence of a sequence $T_m \nearrow \infty$, with $\sup_M |\eta_m| = 1$ but $\mathscr{L}_{T_m}\eta_m = 0$. By the Schauder estimate and reindexing a subsequence, we can assume that η_m converges to η in C^2. There are two basic scenarios: either we can also arrange the subsequence so that $\sup|\eta_m| \ge c > 0$ on $M_1\setminus\{p_1\}$ or $M_2\setminus\{p_2\}$ (note that $(M_1\sharp M_2, \gamma_T)$ contains more of $(M_1,\gamma_1)\cup(M_2,\gamma_2)$ isometrically as T grows), or η_m converges locally uniformly to zero on these two pieces and the point where $\sup_M|\eta_m| = 1$ is attained travels further down the cylinder. By translation we can center the supremum point, and obtain a nontrivial bounded η on the cylinder $\mathbb{R}\times\mathbb{S}^2$, satisfying $\Delta_{\mathring{\gamma}}\eta - (1/4)\eta = 0$; this follows from (13) by the estimates of $\tilde{\mu}_T$ and ψ_T. As there are no nontrivial such η (by separation of variables or the maximum principle), we have a contradiction.

Thus we must be in the former case, say η is nontrivial on $M_1\setminus\{p_1\}$. In this case, γ_T converges to $\psi_1^{-4}\gamma_1$, $|\tilde{\mu}_T|^2_{\gamma_T}$ converges to $\psi_1^{12}|\mu_1|^2_{\gamma_1}$, so that $\mathscr{L}_T$ (13) converges to

$$\tilde{\mathscr{L}} = \Delta_{\psi_1^{-4}\gamma_1} - \tfrac{1}{8}\left(R(\psi_1^{-4}\gamma_1) + 7|\mu_1|^2_{\gamma_1}\psi_1^4 + \tfrac{10}{3}\tau^2\psi_1^4\right).$$

The conformal Laplacian satisfies a covariance property,

$$\Delta_{\psi_1^{-4}\gamma_1}\eta - \tfrac{1}{8}R(\psi_1^{-4}\gamma_1)\eta = \psi_1^5\left(\Delta_{\gamma_1}(\psi_1^{-1}\eta) - \tfrac{1}{8}R(\gamma_1)\psi_1^{-1}\eta\right),$$

while since $(\gamma_1, K_1 = \mu_1 + (\tau/3)\gamma_1)$ solves the vacuum constraints, we have $R(\gamma_1) = |\mu_1|^2_{\gamma_1} - (2/3)\tau^2$, and so we see that $\tilde{\mathscr{L}}\eta = 0$ can be re-written

$$\left(\Delta_{\gamma_1} - |\mu_1|^2_{\gamma_1} - \tfrac{1}{3}\tau^2\right)(\psi_1^{-1}\eta) = 0.$$

The operator $(\Delta_{\gamma_1} - |\mu_1|^2_{\gamma_1} - (1/3)\tau^2)$ has trivial kernel for nontrivial K_1, which we assume. Thus it admits a positive Green's function G which has a pole or order r_1^{-1} at p_1, with r_1 the distance to p_1. Since $|\eta| \le 1$ and $|\psi^{-1}| \le r_1^{-1/2}$, $\psi^{-1}\eta$ must extend smoothly across p_1, and so we conclude it must vanish, contradicting that η is nontrivial on $M_1\setminus\{p_1\}$.

Similar arguments can be used to control the inverse [20, Proposition 8] (cf. [25, 29]). Writing a Taylor expansion $\mathscr{N}_T(\psi_T + \eta) = \mathscr{N}_T(\psi_T) + \mathscr{L}_T(\eta) + \mathscr{Q}_T(\eta)$, one can then show that for large T, the following map is a contraction on a suitable ball around 0 with small T-dependent radius in a Hölder space:

$$\eta \mapsto -\mathscr{L}_T^{-1}(\mathscr{N}_T(\psi_T) + \mathscr{Q}_T(\eta)) = -\mathscr{L}_T^{-1}(N_T(\psi_T + \eta) - \mathscr{L}_T(\eta)).$$

The keys are that ψ_T is a sufficiently good approximate solution, and there is suitable control on $\mathscr{L}_T^{-1}$ uniform in T large.

The fixed point η_T furnishes the desired conformal factor $\tilde{\psi}_T = \psi_T + \eta_T$, and the solution $(\tilde{\psi}_T^4\gamma_T, \tilde{\psi}_T^{-2}\tilde{\mu}_T + (\tau/3)\tilde{\psi}_T^4\gamma_T)$ of the constraints on $M_1\sharp M_2$. The difference between the solution and the starting data decays exponentially in T on compact subsets of each $M_j \setminus \{p_j\}$.

This result gives us a way to combine solutions to the vacuum constraints (or to add a handle to a solution), with the constituent parts approximately represented in the final solution. There are analogous results in the asymptotically Euclidean and asymptotically hyperboloidal setting, and an interesting implementation [29] of conformal gluing (including a different family of approximate solutions) to construct initial data with some features of point particle solutions.

A natural question to ask is whether and under what conditions one can preserve compact subdomains in the original data. Results in this direction, see e.g. [21, 24, 25], use localized deformations, to which we now turn.

3. Localized gluing

In this section we will outline steps to glue an asymptotically flat initial data set across a large annulus to an initial data set suitably chosen from a given family of initial data. The perturbation will be localized to the annular region.

To bring the ideas across, we will present a proof sketch of the following. For any $R > 0$, let $\mathscr{E}_R = \{x \in \mathbb{R}^3 : |x| > R\}$, and let $\mathscr{E} = \mathscr{E}_1$.

Theorem. *Let (g, E) be asymptoticaly flat, time-symmetric source-free initial data for the Einstein–Maxwell equations on $\mathscr{E}$, with ADM mass $m(g) \neq 0$. For sufficiently large R, there are source-free initial data $(\bar{g}, \bar{E})$ on $\mathscr{E}$ which agree with the given data data on $\mathscr{E} \setminus \mathscr{E}_R$, and on $\mathscr{E}_{2R}$ agree with asymptotically flat data from a spacelike slice in a suitable Reissner–Nordström spacetime, with the same electric charge.*

The statement is for an asymptotic end (which could be outside nontrivial charge distributions), whereas if (M, g, E) comprised a nontrivial complete asymptotically flat solution of the time-symmetric Einstein–Maxwell constraints, then since $R(g) = 2\kappa|E|_g^2 \geq 0$, we would have $m(g) > 0$, by the Positive Mass Theorem.

The corresponding theorem for the non-time symmetric case requires the exterior slice to be chosen suitably in the Kerr–Newman family. For simplicity of exposition, we focus on the time-symmetric case here. Before we begin, we note that non-vacuum gluing results using the conformal method have been explored in [52].

We remark that the argument will show the construction works for *interior* gluing, i.e., the asymptotic exterior could be taken from the given data, while the interior would then be from a suitable member of the model family. For example, in the time-symmetric vacuum case, the interior can be made to be a member of the Schwarzschild family. If this interior solution has positive mass, then the resulting data will be complete with two asymptotically flat ends and will contain the Schwarzschild minimal sphere; as the gluing region will be far out in the asymptotic end, this will be an outermost minimal surface in the end, so the Penrose inequality restricts the mass based on that of the exterior. In any case, we note that interior gluing has been used for long-time existence results for the evolution problem [53, 54].

3.1. *Proof of the Theorem*

The method of proof we follow here is the same as in [22, 23], cf. [21]. We smoothly patch together the given data to data from a slice in the model spacetime across an annular region $\mathscr{A}_R = \{x : R < |x| < 2R\}$ to form an approximate solution. As R increases, the metric approaches $\mathring{g}$, where $\mathring{g}_{ij} = \delta_{ij}$ in the asymptotic coordinates, and E tends to zero (in the general case, K also tends to zero). As such, while the approximation improves with increasing R, it approaches a solution which admits nontrivial kernel of the formal adjoint of linearization of the constraints. This finite-dimensional kernel, arising from symmetries, is precisely the obstruction to a local (and localized) perturbation result, as we now outline.

The first key to achieving localized perturbation in our framework is a weighted elliptic estimate (17), which in the time-symmetric case derives from an absolute estimate (16) without boundary terms. The estimates are used to guarantee that the solution obtained lies in an appropriate weighted space, so that it extends smoothly across $\mathscr{A}_R$. Like the conformal method above, we need to get uniform estimates on a family of linearized operators, but this can only be done transverse to the finite-dimensional approximate kernel of the adjoint of linearized operator (see estimate (18) below). The presence of this approximate cokernel for large R leads us to employ a Lyapunov–Schmidt reduction: we use the weighted linear elliptic estimates and a Picard-type quasi-Newtonian iteration to solve a nonlinear projected problem that puts the constraint data into a finite-dimensional space. We handle the obstruction space by choosing a suitable member of the model family, in this case Reissner–Nordström.

To be precise, we recall the following operator introduced earlier:

$$\Psi(g,E) = (R(g) - 2\kappa|E|_g^2, \mathrm{div}_g E).$$

For a perturbative analysis, we compute its linearization and formal L^2-adjoint, for which one readily finds (and recall (12))

$$D\Psi_{(g,E)}(h,X) = \left(DR_g(h) - 4\kappa g(E,X) - 2\kappa h(E,E), \mathrm{div}_g(X) + \tfrac{1}{2}d(\mathrm{tr}_g h)(E)\right),$$

and hence

$$D\Psi^*_{(g,E)}(f,\psi) = \Big(DR^*_g(f) - 2\kappa f E^\flat \otimes E^\flat - \tfrac{1}{2}\mathrm{div}_g(\psi E)g, -\mathrm{grad}_g\psi - 4\kappa f E\Big). \tag{14}$$

These simplify at the flat data, about which we will be perturbing. Let $\mathring{g}$ be the Euclidean metric on $\mathbb{R}^3$, with connection $\mathring{\nabla}$. If we let $D\mathring{\Psi} = D\Psi_{(\mathring{g},0)}$ with formal L^2-adjoint $D\mathring{\Psi}^*$, then

$$D\mathring{\Psi}(h,X) = (DR_{\mathring{g}}(h), \mathrm{div}_{\mathring{g}}X), \quad D\mathring{\Psi}^*(f,\psi) = (DR^*_{\mathring{g}}(f), -\mathrm{grad}_{\mathring{g}}\psi).$$

The obstruction space is given by the elements (f,ψ) in the kernel of $D\mathring{\Psi}^*$. As observed earlier, then f must be a constant plus a linear combination of the Cartesian coordinate functions x^1, x^2, x^3, while ψ must be a constant. The end of the proof will show how to cover these directions by picking appropriately the parameters (mass, center of mass and charge) of the Reissner–Nordström data to which we glue, as we will detail in Section 3.1.7.

3.1.1. *Weighted spaces*

We suppose $(\overline{\Omega}, \mathring{g})$ is compact, connected with smooth nonempty boundary $\partial\Omega$. It suffices for our purpose here to consider Ω to be an annular region. Given $N > 0$, and $0 < r_1 < r_0$, we let $\tilde{\rho} : \mathbb{R} \to \mathbb{R}$ be a smooth nondecreasing function with $\tilde{\rho}(t) = 0$ for $t \le 0$, $\tilde{\rho}(t) = \mathrm{e}^{-N/t}$ for $0 < t < r_1$, and $\tilde{\rho}(t) = 1$ for $t \ge r_0$. We define ρ on Ω by $\rho(x) = \tilde{\rho}(\mathrm{d}(x,\partial\Omega))$, where $\mathrm{d}(x,\partial\Omega)$ is the $\mathring{g}$-distance of x to $\partial\Omega$; we choose r_0 suitably small, so that in particular ρ is smooth. On an annular region, ρ decays to zero at the boundary spheres, and is identically 1 near the middle of the annulus.

Let $\|u\|^2_{L^2_\rho(\Omega)} = \int_\Omega |u|^2 \rho\, \mathrm{d}\mu_{\mathring{g}}$, while for k a nonnegative integer, let $\|u\|^2_{H^k_\rho(\Omega)} = \sum_{\ell=0}^k \|\mathring{\nabla}^\ell u\|_{L^2_\rho(\Omega)}$. The resulting Hilbert spaces can in fact can be identified as the closure of $C^\infty(\overline{\Omega})$ in the relevant norm [23, Lemma 2.1]. Furthermore by [55, Proposition 2.10], upon choosing N suitably large, there is a constant $C>0$ such that for all $u\in H^k_\rho(\Omega)$,

$$\|u\rho^{1/2}\|_{H^k(\Omega)}\leq C\|u\|_{H^k_\rho(\Omega)}. \tag{15}$$

As such, for a bounded sequence in $H^k_\rho(\Omega)$, there is a $u\in H^k_\rho(\Omega)$ and a subsequence u_i for which $u_i\rho^{1/2}$ converges to $u\rho^{1/2}$ in $H^{k-1}(\Omega)$ and weakly in $H^k(\Omega)$, while u_i converges weakly to u in $H^k_\rho(\Omega)$. We note that for metrics g in a $C^k(\overline{\Omega})$-bounded set, the norms defined as above but using the measures $\mathrm{d}\mu_g$ and connection ∇_g are uniformly equivalent.

The linearized operators enjoy estimates (16)–(17) in these weighted spaces, which are used in solving the linearized equation variationally (see Section 3.1.4). There are also weighted Hölder spaces built with the appropriate scaling to capture the interior Schauder estimates (22)–(23) that follow for the solutions of the linearized equations. These spaces are defined with a natural scaling compatible with the weight function ρ, and in fact there is a suite of weighted spaces compatible with different weights ρ. In [22], we first dealt with power decay, so $\rho=d^N$ near $\partial\Omega$, which has a simple weighted estimate. While the exponential weighting was discussed there, the weighted Hölder spaces used by Chruściel and Delay [21] give a more elegant formulation of the interior estimates, and allow for various kinds of weights all in one framework.

We follow an equivalent implementation as in [55]. We take a smooth function $0<\phi<1$ with $\phi=d^2$ near the boundary, and with $\overline{B_{\phi(x)}(x)}\subset\Omega$ for all $x\in\Omega$. We let φ be of the form $\varphi=\phi^r\rho^s$, with r and s real numbers, and let

$$\|u\|_{C^{k,\alpha}_{\phi,\varphi}(\Omega)}=\sup_{x\in\Omega}\left(\sum_{j=0}^k\varphi(x)\phi^j(x)\|\mathring{\nabla}^j u\|_{C^0(B_{\phi(x)}(x))}+\varphi(x)\phi^{k+\alpha}(x)[\mathring{\nabla}^k u]_{0,\alpha;B_{\phi(x)}(x)}\right).$$

Every derivative is matched with a power of ϕ, corresponding to $\tilde\rho'(t)=Nt^{-2}\tilde\rho(t)$, and by iterating this we have $|\phi^k\rho^{-1}\nabla^k\rho|$ is bounded.

3.1.2. *Integral estimates*

The key to time-symmetric estimates is that DR^*_g is overdetermined-elliptic. In fact

$$\mathrm{Hess}_g f=DR^*_g(f)-\frac{1}{n-1}(\mathrm{tr}_g(DR^*_g(f))+fR(g))g+f\mathrm{Ric}(g),$$

from which we immediately see that there is a constant $C>0$ such that on any open set Ω, $\|f\|_{H^2(\Omega)}\leq C(\|DR^*_g(f,\psi)\|_{L^2(\Omega)}+\|f\|_{H^1(\Omega)})$.

With an application of Rellich's lemma we can replace the right-most norm by $\|f\|_{L^2(\Omega)}$, and together with (14) we obtain

$$\|f\|_{H^2(\Omega)}+\|\psi\|_{H^1(\Omega)}\leq C(\|D\Psi^*_{(g,E)}(f,\psi)\|_{L^2(\Omega)}+\|f\|_{L^2(\Omega)}+\|\psi\|_{L^2(\Omega)}). \tag{16}$$

We can replace the above norms with the weighted Sobolev norms by a method used in [22], namely we can replace Ω with $\Omega_\varepsilon=\{x\in\Omega: d(x,\partial\Omega)>\varepsilon\}$ for small $\varepsilon>0$, multiply the (squared) estimates by $\tilde\rho'(\varepsilon)\geq 0$ and integrate:

$$\int_0^{\varepsilon_0}\tilde\rho'(\varepsilon)(\|f\|^2_{H^2(\Omega_\varepsilon)}+\|\psi\|^2_{H^1(\Omega_\varepsilon)})\,\mathrm{d}\varepsilon\leq C\int_0^{\varepsilon_0}\tilde\rho'(\varepsilon)(\|D\Psi^*_{(g,E)}(f,\psi)\|^2_{L^2(\Omega_\varepsilon)}+\|f\|^2_{L^2(\Omega_\varepsilon)}+\|\psi\|^2_{L^2(\Omega_\varepsilon)})\,\mathrm{d}\varepsilon.$$

Integration by parts together with the coarea formula lead to

$$\|(f,\psi)\|_{H^{2,1}_\rho(\Omega)}:=\|f\|_{H^2_\rho(\Omega)}+\|\psi\|_{H^1_\rho(\Omega)}\leq C(\|D\Psi^*_{(g,E)}(f,\psi)\|_{L^2_\rho(\Omega)}+\|f\|_{L^2_\rho(\Omega)}+\|\psi\|_{L^2_\rho(\Omega)}). \tag{17}$$

We note that (16), and hence (17), holds for *any* data (g,E), with a constant uniform for (g,E) in a bounded $C^2(\overline{\Omega})\times C^1(\overline{\Omega})$ neighborhood.

While we focus on the time-symmetric case, we do note that the estimate

$$\|Z\|_{H^1_\rho(\Omega)} \le C(\|L_Z g\|_{L^2_\rho(\Omega)} + \|Z\|_{L^2_\rho(\Omega)})$$

for Z is more subtle. The proof from [55, Lemma 5.1] is a variation on the argument from [23, Lemma 4.1]; both use the elementary inequality $\Delta_{\mathring{g}}\rho \ge (1/2)N^2 d^{-4}\rho$ near $\partial\Omega$ and an integration-by-parts argument, and the estimate holds for N sufficiently large. The analogue of (17) for the full constraints operator follows.

3.1.3. *Uniform estimate transverse to cokernel*

We can promote (17) to a coercivity estimate, at least in directions transverse to the cokernel $\mathscr{K}$ at flat data. Recall $\mathscr{K} = \mathrm{span}\{1, x^1, x^2, x^3\} \oplus \mathrm{span}\{1\} \subset H^2_\rho(\Omega)\times H^1_\rho(\Omega) =: H^{2,1}_\rho(\Omega)$. Fix a nontrivial smooth bump function ζ supported on Ω. Let S_g be the $L^2(\Omega, \mathrm{d}\mu_g)$-orthogonal complement of $\zeta\mathscr{K}$.

We claim there is a $C > 0$ and a $C^2(\overline{\Omega}) \times C^1(\overline{\Omega})$ neighborhood $\mathscr{U}$ of the flat data $(\mathring{g}, 0)$ such that for all data $(g, E) \in \mathscr{U}$ and all $(f,\psi) \in S_g$,

$$\|(f,\psi)\|_{H^{2,1}_\rho(\Omega)} \le C\|D\Psi^*_{(g,E)}(f,\psi)\|_{L^2_\rho(\Omega)}. \tag{18}$$

The same holds if S_g is replaced by any closed subspace S of $H^{2,1}_\rho(\Omega)$ with $S \cap \mathscr{K} = \{0\}$.

There are a couple ways one can prove this. For instance, one can argue along the lines of the proof of [25, Proposition 3.1], establishing the *unweighted* analogue of (18) on Ω_ε, with a uniform constant for ε small, and proceed as in the proof of (17). Another approach is to use (17) along with a standard compactness argument: from a sequence $(f_i,\psi_i) \in S_{g_i}$, with $(g_i, E_i) \to (\mathring{g}, 0)$, $\|(f_i,\psi_i)\|_{H^{2,1}_\rho(\Omega)} = 1$ and $\|D\Psi^*_{(g,E)}(f_i,\psi_i)\|_{L^2_\rho(\Omega)} \to 0$, one can construct a nontrivial $(f,\psi) \in S_{\mathring{g}} \cap \mathscr{K}$, which furnishes a contradiction. Indeed as noted after (15), there is a $(f,\psi) \in H^{2,1}_\rho(\Omega)$ such that, upon re-indexing a subsequence, $(f_i,\psi_i)\rho^{1/2}$ converges to $(f,\psi)\rho^{1/2}$ in $H^1(\Omega) \times L^2(\Omega)$. In particular, (f_i,ψ_i) converges in $L^2_{\mathrm{loc}}(\Omega)$ to (f,ψ), so that $D\mathring{\Psi}^*(f,\psi) = 0$ weakly. Hence $(f,\psi) \in \mathscr{K}$. Since (f_i,ψ_i) is $L^2(\mathrm{d}\mu_{g_i})$-orthogonal to $\zeta\mathscr{K}$, $(f,\psi) \in S_{\mathring{g}}$. Moreover, since $(f_i,\psi_i) \to (f,\psi)$ in $L^2_\rho(\Omega)$, we see from (17) that (f,ψ) is nontrivial, which is a contradiction.

3.1.4. *The projected problem*

With ζ, $\mathscr{K}$ and S_g as above, let Π_g be the $L^2(\Omega, \mathrm{d}\mu_g)$-orthogonal projection to S_g, and let $\mathring{\Pi} = \Pi_{\mathring{g}}$. For simplicity, we may arrange that ζ is supported where $\rho = 1$. We study the map $(g,E) \mapsto \mathring{\Pi}(\Psi(g,E))$, with linearization $(h,X) \mapsto \mathring{\Pi}(D\Psi_{(g,E)}(h,X))$.

The goal is to solve $\Psi(g,E) = 0$. We start by solving a projected problem:

$$\mathring{\Pi}(\Psi(g_0 + h, E_0 + X)) = 0,$$

where (g_0, E_0) is close to $(\mathring{g}, 0)$. The linearization of this is

$$\mathring{\Pi}(D\Psi_{(g_0,E_0)}(h,X)) = -\mathring{\Pi}(\Psi(g_0,E_0)). \tag{19}$$

To solve this we consider the functional on $H^{2,1}_\rho(\Omega) \cap S_{g_0}$ given by

$$\mathscr{G}_0(f,\psi) = \int_\Omega \left(\frac{1}{2}\rho|D\Psi^*_{(g_0,E_0)}(f,\psi)|^2_{g_0} + \Pi_{g_0}(\Psi(g_0,E_0))\cdot(f,\psi)\right)\mathrm{d}\mu_{g_0}.$$

If $\mathscr{G}_0$ is stationary at (f_0,ψ_0), the Euler–Lagrange equation

$$\Pi_{g_0}(D\Psi_{(g_0,E_0)}\rho D\Psi^*_{(g_0,E_0)}(f_0,\psi_0)) = -\Pi_{g_0}(\Psi(g_0,E_0))$$

is implied by the following, which holds for all (u,v) compactly supported:

$$\int_\Omega (\Pi_{g_0}(D\Psi_{(g_0,E_0)}\rho D\Psi^*_{(g_0,E_0)}(f,\psi)) + \Pi_{g_0}(\Psi(g_0,E_0)))\cdot(u,v)\,\mathrm{d}\mu_{g_0}.$$

This follows trivially for $(u,v)\in\zeta\mathscr{K}$, while for $(u,v)\in S_{g_0}$, the leading projection operator in the above integral equation can be removed, and so it follows by stationarity of the functional at $(f,\psi)\in H^{2,1}_\rho(\Omega)\cap S_{g_0}$.

We conclude that $\Psi(g_0,E_0)+D\Psi_{(g_0,E_0)}\rho D\Psi^*_{(g_0,E_0)}(f_0,\psi_0)\in\zeta\mathscr{K}$, and so we see that (19) is solved with $(h,X)=\rho D\Psi^*_{(g_0,E_0)}(f_0,\psi_0)$. Moreover, combining (18) with $\mathscr{G}_0(f_0,\psi_0)\le 0$ we find

$$\|(f_0,\psi_0)\|_{H^{2,1}_\rho(\Omega)}\le C\|\Pi_{g_0}(\Psi(g_0,E_0))\|_{L^2_{\rho^{-1}}(\Omega)}\le C'\|\mathring{\Pi}(\Psi(g_0,E_0))\|_{L^2_{\rho^{-1}}(\Omega)}. \tag{20}$$

3.1.5. *Pointwise estimates*

The above shows we can solve the desired equation at the linear level, with an integral estimate. To iterate linear corrections to solve the nonlinear problem, we require a pointwise estimate of the variational solution, in part to establish the convergence of a quasi-Newtonian iteration scheme.

The operator $(f,\psi)\mapsto D\Psi_{(g,E)}D\Psi^*_{(g,E)}(f,\psi)$, and in fact for our purposes, the weighted version

$$(f,\psi)\mapsto\rho^{-1}D\Psi_{(g,E)}\rho D\Psi^*_{(g,E)}(f,\psi)=:(L_{11}f+L_{12}\psi,L_{21}f+L_{22}\psi), \tag{21}$$

is an elliptic operator of mixed order that admits Douglis–Nirenberg weights [56]: L_{ij} is order s_i+t_j, where $s_1=0$, $s_2=-1$, $t_1=4$, $t_2=3$, i.e. L_{11} is order four, L_{12} and L_{21} are order three, and L_{22} is order two. For example, at the flat data $(\mathring{g},0)$, $D\mathring{\Psi}D\mathring{\Psi}^*(f,\psi)=(2\Delta^2_{\mathring{g}}f,-\Delta_{\mathring{g}}\psi)$. The diagonal terms of (21) have the analogous principal parts, but there are also off-diagonal terms (though they are small for E small), and terms from the weight, which can in principle blow up on approach to the boundary at a rate which is a power of d^{-1}. This behavior can be accommodated in the interior elliptic estimates by a scaling argument, which is efficiently coded by the weighted Hölder norms introduced earlier. Indeed if we write L_{ij} in local coordinates as $L_{ij}=\sum_{|\beta|\le s_i+t_j}b^\beta_{ij}\partial^\beta$, then it is not hard to see that for $0<\alpha<1$, there is a $C>0$ such that $\|b^\beta_{ij}\|_{C^{-s_i,\alpha}_{\phi,\phi^{s_i+t_j-|\beta|}}(\Omega)}\le C$. We record the following interior Schauder estimate (see [55, Theorem 5.8], e.g.), with $\varphi_j=\phi^{(3/2)+4-t_j}\rho^{1/2}$, i.e. $\varphi_1=\phi^{3/2}\rho^{1/2}$ and $\varphi_2=\phi^{5/2}\rho^{1/2}$:

$$\begin{aligned}&\|f\|_{C^{4,\alpha}_{\phi,\varphi_1}(\Omega)}+\|\psi\|_{C^{3,\alpha}_{\phi,\varphi_2}(\Omega)}\\&\le C(\|L_{11}f+L_{12}\psi\|_{C^{0,\alpha}_{\phi,\phi^4\varphi_1}(\Omega)}+\|L_{21}f+L_{22}\psi\|_{C^{1,\alpha}_{\phi,\phi^2\varphi_2}(\Omega)}+\|f\|_{L^2_\rho(\Omega)}+\|\psi\|_{L^2_{\phi^2\rho}(\Omega)}).\end{aligned} \tag{22}$$

We apply this to get a pointwise estimate of (f_0,ψ_0), where $(h,X)=\rho D\Psi^*_{(g_0,E_0)}(f_0,\psi_0)$ is the variational solution to (19). The integral terms can be estimated by (20). We can also replace terms in (22) involving the operator by the their projected values: since $\rho=1$ on the support of ζ,

$$\rho^{-1}D\Psi_{(g_0,E_0)}\rho D\Psi^*_{(g_0,E_0)}(f_0,\psi_0)-\rho^{-1}\mathring{\Pi}(D\Psi_{(g_0,E_0)}\rho D\Psi^*_{(g_0,E_0)}(f_0,\psi_0))\in\zeta\mathscr{K}.$$

Using the fact that $\zeta\mathscr{K}$ is finite-dimensional (so all norms on it are equivalent) and an elementary compactness argument, one can derive (cf. the proof of [55, Proposition 6.4])

$$\begin{aligned}&\|f_0\|_{C^{4,\alpha}_{\phi,\varphi_1}(\Omega)}+\|\psi_0\|_{C^{3,\alpha}_{\phi,\varphi_2}(\Omega)}+\|(f_0,\psi_0)\|_{H^{2,1}_\rho(\Omega)}\\&\le C(\|\rho^{-1}\mathring{\Pi}(\Psi(g_0,E_0))\|_{C^{0,\alpha}_{\phi,\phi^4\varphi_1}(\Omega)\times C^{1,\alpha}_{\phi,\phi^2\varphi_2}(\Omega)}+\|f_0\|_{L^2_\rho(\Omega)}+\|\psi_0\|_{L^2_{\phi^2\rho}(\Omega)})\\&\le C(\|\mathring{\Pi}(\Psi(g_0,E_0))\|_{C^{0,\alpha}_{\phi,\phi^4\varphi_1\rho^{-1}}(\Omega)\times C^{1,\alpha}_{\phi,\phi^2\varphi_2\rho^{-1}}(\Omega)}+\|\mathring{\Pi}(\Psi(g_0,E_0))\|_{L^2_{\rho^{-1}}(\Omega)})\end{aligned} \tag{23}$$

where we used the variational estimate (20). We define norms such that we can abbreviate the above result as

$$\|(f_0,\psi_0)\|_{\mathscr{B}^{4,3}}\le C\|\mathring{\Pi}(\Psi(g_0,E_0))\|_{\mathscr{B}^{0,1}}. \tag{24}$$

3.1.6. *Iteration*

If we let $(h_0, X_0) = \rho D\Psi^*_{(g_0,E_0)}(f_0, \psi_0)$ and define

$$\|(h,X)\|_{\mathscr{B}^{2,2}} = \|(h,X)\|_{C^{2,\alpha}_{\phi,\phi^{\frac{7}{2}}\rho^{-\frac{1}{2}}}(\Omega)} + \|(h,X)\|_{L^2_{\rho^{-1}}(\Omega)},$$

it follows from (24) that

$$\|(h_0,X_0)\|_{\mathscr{B}^{2,2}} \le C\|(f_0,\psi_0)\|_{\mathscr{B}^{4,3}} \le C'\|\mathring{\Pi}(\Psi(g_0,E_0))\|_{\mathscr{B}^{0,1}}.$$

The point is that by design, $\mathring{\Pi}(\Psi(g_0+h_0, E_0+X_0))$ is quadratic in (h_0, X_0), i.e. in $\mathring{\Pi}(\Psi(g_0,E_0))$. We set up a recursion by a quasi-Newtonian scheme to determine a sequence $(h_m, X_m) = \rho D\Psi^*_{(g_0,E_0)}(f_m,\psi_m)$, satisfying

$$\mathring{\Pi}(D\Psi_{(g_0,E_0)}\rho D\Psi^*_{(g_0,E_0)}(f_{m+1},\psi_{m+1})) = -\mathring{\Pi}(\Psi(g_m,E_m)),$$

where $g_m = g_0 + \sum_{k=0}^{m-1} h_k$, and $E_m = E_0 + \sum_{k=0}^{m-1} X_k$. Note that the linearization is computed at the starting data, which we take to be smooth.

For sufficiently small $\Psi(g_0,E_0)$, the resulting convergence is not quadratic, but it is super-linear; for more details, see [55, Section 6.3 and Appendix D], cf. [22,23], and [21, Appendix G]. We obtain $g_m \to g = g_0 + h$ and $E_m \to E = E_0 + X$, with $\mathring{\Pi}(\Psi(g,E)) = 0$, in other words, $\Psi(g,E) \in \zeta\mathscr{K}$. Here $(h,X) = \rho D\Psi^*_{(g_0,E_0)}(f,\psi) \in \mathscr{B}^{2,2}$, where $f = \sum_{k=0}^{\infty} f_k$ and $\psi = \sum_{k=0}^{\infty}\psi_k$, with $(f,\psi) \in \mathscr{B}^{4,3}\cap S_{g_0}$, and $\|(h,X)\|_{\mathscr{B}^{2,2}} \le C\|\mathring{\Pi}(\Psi(g_0,E_0))\|_{\mathscr{B}^{0,1}}$. By elliptic bootstrapping on the quaslinear equation $\Psi(g,E) \in \zeta\mathscr{K}$, one can show that (h,X) extends smoothly by zero across $\partial\Omega$.

3.1.7. *Handling the cokernel*

Thus far we have not glued anything, we have just referenced data near $(\mathring{g}, 0)$ on a domain Ω. We will now restrict to $\Omega = \mathscr{A}_1$, the unit annulus. Given asymptotically flat initial data (g,E) satisfying $\Psi(g,E) = 0$, we use a cutoff function to smoothly patch (g,E) to $(g_{\mathrm{RN}}, E_{\mathrm{RN}})$ in $\mathscr{A}_R$, obtaining $(\hat{g}_R, \hat{E}_R)$. We may assume for exposition that $g_{ij}(x) - \delta_{ij} = O(|x|^{-1})$ and $E^i(x) = O(|x|^{-2})$ (and derivatives used in the proof attain additional decay), though we could formulate somewhat more generally. For instance, we could pose the starting solution of the constraints is asymptotically flat of order q, i.e. in asymptotic coordinates $|x| > 1$, $\partial_x^\alpha(g_{ij}(x) - \delta_{ij}) = O(|x|^{-q-|\alpha|})$ and $\partial_x^\beta E^i = O_1(|x|^{-1-q-|\beta|})$ for $q > 1/2$, $|\alpha| \le 2$ and $|\beta| \le 1$. Notice that $R(g) = 2\kappa|E|_g^2 = O(|x|^{-2-2q})$ which is integrable, and hence the ADM mass exists. In any case, we consider $q = 1$ for exposition, and the interested reader can readily generalize.

The exterior we attach depends on the parameters m, Q, as well as $c \in \mathbb{R}^3$ about which we center $(g_{\mathrm{RN}}, E_{\mathrm{RN}})$. We then pullback to the unit annulus, as follows: for $x \in \mathscr{A}_1$, let $(\tilde{g}_R)_{ij}(x) = (\hat{g}_R)_{ij}(Rx)$, and $\tilde{E}^i_R(x) = R\hat{E}^i_R(Rx)$. Observe that $(\tilde{g}_R, \tilde{E}_R)$ differs from $(\mathring{g}, 0)$ in $\mathscr{A}_1$ by $O(R^{-1})$, while $\Psi(\tilde{g}_R, \tilde{E}_R)(x) = R^2\Psi(\hat{g}_R, \hat{E}_R)(Rx) = O(R^{-1})$. We see the scaling was chosen so that we can solve the vacuum constraints on $\mathscr{A}_1$ and simply scale back. Moreover, for all large R, we can apply the previous analysis to find (h_R, X_R) that extend smoothly by zero outside $\mathscr{A}_1$, such that with $(\overline{g}_R, \overline{E}_R) = (\tilde{g}_R + h_R, \tilde{E}_R + X_R)$, we have $\mathring{\Pi}(\Psi(\overline{g}_R, \overline{E}_R)) = 0$.

The final step is to arrange $\Psi(\overline{g}_R, \overline{E}_R) = 0$, for which we recall $\mathscr{K} = \mathrm{span}\{1, x^1, x^2, x^3\} \oplus \mathrm{span}\{1\}$. Projecting $\Psi(\overline{g}_R, \overline{E}_R)$ onto the second summand yields (by scaling)

$$R\int_{\mathscr{A}_1} \mathrm{div}_{\overline{g}_R}\overline{E}_R \,\mathrm{d}\mu_{\overline{g}_R} = R\int_{\partial\mathscr{A}_1} \tilde{E}_R\cdot\nu_{\tilde{g}_R}\,\mathrm{d}\sigma_{\tilde{g}_R} = \int_{\{|x|=2R\}} E_{\mathrm{RN}}\cdot\nu_{g_{\mathrm{RN}}}\,\mathrm{d}\sigma_{g_{\mathrm{RN}}} - \int_{\{|x|=R\}} E\cdot\nu_g\,\mathrm{d}\sigma_g.$$

Since $\mathrm{div}_g E = 0$, the integral $\int_{\{|x|=R\}} E\cdot\nu_g\,\mathrm{d}\sigma_g$ is independent of $R > 1$, and the value is $4\pi Q$. Thus if we patch the solution to Reissner–Nordström data with charge Q, then we conclude $\int_{\mathscr{A}_1}\mathrm{div}_{\overline{g}_R}\overline{E}_R\,\mathrm{d}\mu_{\overline{g}_R} = 0$. Since $\mathrm{div}_{\overline{g}_R}\overline{E}_R = b\zeta$ for a constant b, we conclude $b = 0$, and hence $\mathrm{div}_{\overline{g}_R}\overline{E}_R = 0$. The reader might compare this with our earlier remark in Section 1.2.1 about how to prescribe the electric field data.

For the rest of the section, we extend the summation convention to all pairs of repeated indices (even if both are down). Since in Cartesian coordinates, $DR_{\mathring{g}}(h) = h_{ij,ij} - h_{ii,jj}$, we see that by simple expansion of the scalar curvature term

$$\begin{aligned}\int_{\mathscr{A}_1} (R(\overline{g}_R) - 2\kappa|\overline{E}_R|^2_{\overline{g}_R})\,\mathrm{d}x &= \int_{\partial\mathscr{A}_1} ((\tilde{g}_R)_{ij,i} - (\tilde{g}_R)_{ii,j})\,\mathring{\nu}^j\,\mathrm{d}\mathring{\sigma} + O(R^{-2})\\ &= R^{-1}\int_{\partial\mathscr{A}_R} ((\hat{g}_R)_{ij,i} - (\hat{g}_R)_{ii,j})\,\mathring{\nu}^j\,\mathrm{d}\mathring{\sigma} + O(R^{-2})\\ &= R^{-1}16\pi(m - m(g)) + O(R^{-2}).\end{aligned}$$

So the projection is to leading order controlled by the mass m of the Reissner–Nordström exterior.

The projection onto the linear elements requires a bit more attention. For the metric g, we find that for any $r > r_0 > 1$,

$$\int_{\{r_0\le|x|\le r\}} x^k(g_{ij,ij} - g_{ii,jj})\,\mathrm{d}x = \left(\int_{\{|x|=r\}} - \int_{\{|x|=r_0\}}\right)(x^k(g_{ij,i} - g_{ii,j}) - (\delta^k_i g_{ij} - \delta^k_j g_{ii}))\mathring{\nu}^j\,\mathrm{d}\mathring{\sigma}.$$

Note that in the last terms of the integral on the right, we could replace the terms of the form $g_{\ell m}$ with $g_{\ell m} - \delta_{\ell m}$. Moreover, *using the constraint equations*, the integrand $(g_{ij,ij} - g_{ii,jj}) = O(|x|^{-4})$, and so the left hand side is $O(\log r)$.

If instead the above is evaluated at the Reissner–Nordström data, by using the asymptotic parity and the fact that the center of mass does not appear in the $|x|^{-1}$ term in the expansion, we can show that for $|c| \le r/2$ and for m bounded, $\int_{\{|x|\ge r\}} x^k((g_{\mathrm{RN}})_{ij,ij} - (g_{\mathrm{RN}})_{ii,jj})\,\mathrm{d}x = O(r^{-1})$, on the one hand, while on the other

$$\lim_{r\to\infty}\int_{\{|x|=r\}} (x^k((g_{\mathrm{RN}})_{ij,i} - (g_{\mathrm{RN}})_{ii,j}) - (\delta^k_i(g_{\mathrm{RN}})_{ij} - \delta^k_j(g_{\mathrm{RN}})_{ii}))\mathring{\nu}^j\,\mathrm{d}\mathring{\sigma} = 16\pi m c^k.$$

Assembling all of this, noting that the rescaled exterior is a Reissner–Nordström with center $\tilde{c} = cR^{-1}$, we obtain

$$\begin{aligned}&\int_{\mathscr{A}_1} x^k(R(\overline{g}_R) - 2\kappa|\overline{E}_R|^2_{\overline{g}_R})\,\mathrm{d}x\\ &= \int_{\partial\mathscr{A}_1} x^k((\tilde{g}_R)_{ij,i} - (\tilde{g}_R)_{ii,j})\,\mathring{\nu}^j\,\mathrm{d}\mathring{\sigma} + O(R^{-2})\\ &= R^{-2}\int_{\{|x|=2R\}} (x^k((g_{\mathrm{RN}})_{ij,i} - (g_{\mathrm{RN}})_{ii,j}) - (\delta^k_i(g_{\mathrm{RN}})_{ij} - \delta^k_j(g_{\mathrm{RN}})_{ii}))\mathring{\nu}^j\,\mathrm{d}\mathring{\sigma}\\ &\quad - R^{-2}\int_{\{|x|=R\}} (x^k(g_{ij,i} - g_{ii,j}) - (\delta^k_i g_{ij} - \delta^k_j g_{ii}))\mathring{\nu}^j\,\mathrm{d}\mathring{\sigma} + O(R^{-2})\\ &= R^{-1}\left(16\pi m\tilde{c}^k + O\left(\frac{\log R}{R}\right)\right) + O(R^{-2}).\end{aligned} \tag{25}$$

Thus we get a map

$$(m,\tilde{c}) \mapsto \frac{R}{16\pi}\int_{\mathscr{A}_1} (1,x)(R(\overline{g}_R) - 2\kappa|\overline{E}_R|^2_{\overline{g}_R})\,\mathrm{d}x = (m - m(g) + \xi(m,\tilde{c}), m\tilde{c} + \Xi(m,\tilde{c})),$$

where $\xi(m,\tilde{c}) = O(R^{-1})$ and $\Xi(m,\tilde{c}) = O(\log R/R)$. Moreover, the construction is *continuous* with respect to m and $\tilde{c}$, and so ξ and Ξ are continuous; this is fairly straightforward but cumbersome to prove (see, e.g. [17, Section A.7]). A fixed-point or degree argument can finish the proof from here. For example, the function

$$F(m,\tilde{c}) = \left(m(g) - \xi(m,\tilde{c}), -\frac{\Xi(m,\tilde{c})}{m(g) - \xi(m,\tilde{c})}\right),$$

is readily seen to map $\{(m,\tilde{c}) : |m - m(g)| \le |m(g)|/2, |\tilde{c}| \le 1/2\}$ to itself, and the Brouwer fixed point theorem can then be applied. The fixed point gives parameters that solve the problem.

We remark that we did not assume the original metric g enjoyed enough parity symmetry to admit a well-defined center of mass. If this were the case, we could argue that the resulting center $c = R\tilde{c}$ is close to that of the original, cf. [22, 23].

3.2. *Further applications of deformation*

Suppose the starting data, say (g_0, E_0), is such that $D\Psi^*_{(g_0,E_0)}$ admits only trivial kernel in Ω. The same perturbative analysis above can be applied, the only difference is that since there is no kernel to deal with, we do not need the projected operator, and so the technique yields a localized deformation result: given a compactly contained subdomain of Ω and sufficiently small compactly supported S and σ, say, there are h and X which smoothly extend by zero across $\partial\Omega$ with $\Psi(g_0 + h, E_0 + X) = \Psi(g_0, E_0) + (S, \sigma)$. Similar comments apply to scalar curvature deformation [22], as well as for the general constraints case [23], though in the latter case maintaining the dominant energy condition at the borderline is somewhat subtle [55].

As mentioned earlier, one can combine conformal gluing and localized gluing to obtain connected sum constructions leaving the original data intact outside the gluing neck [21, 24, 25]. The additional assumption needed, beyond any needed for the conformal gluing, is that the corresponding operator for the localized gluing problem has trivial kernel. If you have a sequence of solutions of the nonlinear problem (vacuum constraints, or a scalar curvature equation, say) which converges to a solution for which the linearized adjoint has trivial kernel in Ω, then you can *glue back in* the original data inside some appropriate subdomain, and use the localized perturbations (unobstructed in this case) to reimpose the nonlinear equation.

For an interesting application combining localized deformations of constraint data with conformal deformations, consider an asymptotically flat metric g with nonnegative scalar curvature $R(g)$, which is not *static* outside of some bounded region $\Omega \subset \mathbb{R}^3$ (in the sense that the adjoint DR_g^* of the linearization of the scalar curvature has trivial kernel in the exterior $\mathbb{R}^3 \setminus \overline{\Omega}$). It is possible using localized deformations to find a compactly contained domain $V \subset R^3 \setminus \overline{\Omega}$ such that for S compactly supported in V, and for $\varepsilon > 0$ sufficiently small, there is a metric $\bar{g}_\varepsilon$ with $R(\bar{g}_\varepsilon) = R(g) + \varepsilon S$, and $\bar{g}_\varepsilon - g = O(\epsilon)$ is supported in V. If $S \geq 0$ and is nontrivial, one can then use an elementary conformal transformation to *bring the ADM mass down* while keeping the scalar curvature nonnegative, with conformal factor $u_\varepsilon > 0$ which is one in a neighborhood of $\overline{\Omega}$ [57]. In terms of asymptotically flat extensions of a given region, then, this shows that unless the data outside the region were static in the above sense, there would be extensions with nonnegative scalar curvature and lower ADM mass, and hence in particular the original configuration does not minimize the Bartnik mass. To summarize, a localized deformation is used to bump the scalar curvature up (adding energy density) in a compact set outside the region $\overline{\Omega}$, which does not change the ADM mass, and then a conformal deformation removes some of this scalar curvature and in so doing brings down the ADM mass, leaving the original geometry of $\overline{\Omega}$ intact.

Similar considerations hold for the general (non-time-symmetric) case, though maintaining the dominant energy condition seems to require a modification of the constraints operator to have trivial kernel [55, 58]. An interpretation of the kernel elements for the modified operator, akin to that in [51], and has been established in the recent work [59] on constraint deformations and the dominant energy condition, with striking applications to the Bartnik mass.

Another application is the gluing of initial data to *interpolate* their scalar curvatures [60], or more generally interpolating (μ, J) from the constraints operator [55]. As an interesting example, one can glue initial data for a Kerr solution (including Schwarzschild) to a different Kerr solution across an annular region where the dominant energy condition can be maintained (see Remark after Theorem 1.4 in [55]).

Finally, we discussed earlier the use of the conformal method to achieve solutions with certain prescribed asymptotic behavior, with stability for the Einstein evolution in mind. Motivated by questions about the asymptotics posed in [16], the work [61] proceeds to seed the desired asymptotics, and then correct to solutions of the constraints, but in a different manner than that of [47, 48], in part employing variational methods of the kind used in [16, 23].

3.3. *Alternate approach to solving the projected problem*

We review the steps in the procedure shown above for gluing to an asymptotically flat model family. We start by patching initial data to a member of the model family, with the following parameters at our disposal: R (measuring the scale), the exterior mass m and center of mass c; for the Einstein–Maxwell constraints, we match the electric charge Q. For large enough R the patched data is sufficiently close to flat, and we solve the constraints up to a finite-dimensional obstruction space. To accomplish this, we iterate linear corrections, in spaces with suitable decay at the boundary of the annulus. The procedure is controlled using basic elliptic estimates, both integral and pointwise, and with suitable weights. The integral estimates for the time-symmetric case are relatively straightforward, while the weighted Schauder estimates follow from the standard interior estimates and scaling. From here, a topological argument using the finite-dimensional space of parameters allows us to cover the cokernel and solve the constraints. Similar comments apply in the non-time-symmetric case.

The main first step, then, is how to solve the relevant linear equations with compact support. For $g = \mathring{g} + h$ with h small, the scalar curvature to leading order is governed by a divergence:

$$R(\mathring{g} + h) = DR_{\mathring{g}}(h) + \mathscr{Q}(h) = \mathrm{div}_{\mathring{g}}\mathrm{div}_{\mathring{g}}(h - (\mathrm{tr}_{\mathring{g}} h)\mathring{g}) + \mathscr{Q}(h),$$

where $\mathscr{Q}(h)$ is a combination with bounded coefficients of terms of components of $h \otimes \partial^2 h$ and $\partial h \otimes \partial h$. A similar expansion holds for the constraints operator. At the linear level, then, solving for a compactly supported perturbation for the constraints map near $(\mathring{g}, 0)$ amounts to being able to solve suitable divergence equations with compact support.

Recent works [62, 63] have indeed developed this approach near the flat data (with an extension to data near hyperbolic data obtained in [64]). For instance, to solve a double divergence equation, the authors use a Bogovskiĭ-type operator as follows. Let $\Omega \subset \mathbb{R}^3$ be an open set, which for simplicity we take to be convex, and containing an open ball B, and let $\eta \in C_c^\infty(B)$ with $\int_{\mathbb{R}^3} \eta(x)\,\mathrm{d}x = 1$. Then for any $f \in C_c^\infty(\mathbb{R}^n)$ we define the tensor Sf to have Cartesian components

$$(Sf)^{ij}(x) = \int_{\mathbb{R}^3} \left(\int_{|x-y|}^\infty \eta\left(r\frac{x-y}{|x-y|} + y \right) r^2\,\mathrm{d}r \right) \frac{(x-y)^i (x-y)^j}{|x-y|^3} f(y)\,\mathrm{d}y.$$

If $f \in C_c^\infty(\Omega)$, then Sf is supported in Ω as well, and moreover, if $\int_\Omega f(x)\,\mathrm{d}x = 0 = \int_\Omega x^k f(x)\,\mathrm{d}x$, for $k \in \{1, 2, 3\}$, then $\mathrm{div}_{\mathring{g}}\mathrm{div}_{\mathring{g}}(Sf) = f$. The integral conditions are of course readily seen to be the orthogonality of f to the cokernel of the double divergence. One can furthermore use a straightforward finite patching argument to handle more general domains such as the annuli used in gluing constructions.

As a remark related to this and to the Einstein–Maxwell equations, we recall the well-known procedure for solving the divergence equation $\mathrm{div}_{\mathring{g}} X = f$, with f and the solution X compactly supported (see, e.g. [65, Section 1.3]). Of course we assume that $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = 0$. In one dimension this is straightforward, as $X(x) = \int_{-\infty}^x f(t)\,\mathrm{d}t$ satisfies the required conditions. We can proceed inductively. Suppose f is compactly supported in $D = D_n \times I \subset \mathbb{R}^{n+1}$, where D_n is an open rectangle in $\mathbb{R}^n$ and $I \subset \mathbb{R}$ is an open interval. For $x \in \mathbb{R}^n$, we let $F(x) = \int_{\mathbb{R}} f(x,t)\,\mathrm{d}t$. By assumption, then $\int_{\mathbb{R}^n} F(x)\,\mathrm{d}x = 0$, so by induction, there are $Y^1, \dots, Y^n$ supported in D_n with

$\partial_i Y^i(x) = F(x)$. If θ is supported in I with $\int_{\mathbb{R}} \theta(t)\,\mathrm{d}t = 1$, we let $X^i(x, x^{n+1}) = Y^i(x)\theta(x^{n+1})$ for $i \in \{1, \ldots, n\}$ and

$$X^{n+1}(x, x^{n+1}) = \int_{-\infty}^{x^{n+1}} (f(x,t) - \theta(t)F(x))\,\mathrm{d}t.$$

The vector field X with components $X^1, \ldots, X^{n+1}$ is supported in D and has divergence f.

Returning to the constraints, one would still need to solve the nonlinear projected problem, which is essentially done by an iteration scheme. That said, one major benefit of the Bogovskiĭ-type operator is that regularity is maintained when the data is in appropriate Sobolev spaces, a question which arises when considering the natural regularity for initial data for the evolution problem. Moreover the method has been extended beyond the asymptotically flat regime to give an asymptotically hyperboloidal construction [64], and this is certainly intriguing since the asymptotically hyperboloidal case proves to be more challenging (for instance the conformal infinity is not a point as it is for the asymptotically Euclidean case), see for example [66–68]. The Bogovskiĭ-type operator method has also been used for other localization results such as Carlotto–Schoen type gravitational shielding constructions [62]. Finally, one often wants to connect the solutions of the Einstein constraints to the spacetimes they determine, whether in theory or numerically. Numerical implementation of gluing constructions seems quite challenging, and though there have been a handful of works in this direction [69–72], alternative methods to do gluing constructions could prove useful if they are more readily implemented numerically. It is natural to wonder if the Bogovskiĭ-type operator approach of [63] might be helpful in this direction, and likewise for the characteristic gluing constructions of [11] involving transport-elliptic differential equations, which are of a different character than the spacelike gluing.

4. Further applications to spacetimes

As mentioned earlier, one of the early goals for the gluing construction was to create vacuum initial data which would evolve to a vacuum spacetime that is asymptotically simply in the sense of Penrose conformal compactification, a *purely radiative* spacetime. Since work of Friedrich [32] the following was known: if one could construct a family of solutions to the vacuum constraints with small mass tending to zero, each of which is Schwarzschild in a *uniform* neighborhood of spacelike infinity, then for members of the family with sufficiently small mass, the spacetime evolution would be an asymptotically simple vacuum spacetime. The result in [22] on its own did not quite give such a construction. The main issue, as one can see from (25), is that as the mass m tends to zero, the projection onto the linear directions is on the order of mc, and so for small mass, the center of mass may need to grow larger to accommodate. To control the center, Chruściel and Delay [30] restrict to a family of parity-symmetric metrics, for which the center of mass in the construction can always be taken to vanish. In [31], we construct a larger family of solutions to the constraints with small masses and controlled centers, by using the second variation of the mass at the Euclidean data [73, 74] to control the mass from below in terms of the center. With either approach, the gluing to Schwarzschild at infinity then finishes the initial data construction.

In a completely different direction, Li and Yu [75] construct vacuum initial data, free of trapped surfaces, whose evolution will contain a trapped surface (cf. [76]). Of note, the solution contains an interior region of Minkowskian data, while the exterior is an exterior of a space-time slice of a Kerr spacetime. The construction builds on the monumental work of Christodoulou on formation of trapped surfaces *in vacuum* [77], from data on null hypersurfaces; in particular, the construction in [75] uses data given at past null infinity.

While we do not have the space and time to discuss any details here, there is one feature of the proof that we want to note. The construction uses gluing to Kerr exterior as in [23], but in a regime where the limiting data is not Minkowksian, but Kerrian. As such, the cokernel has *smaller* dimension than at the flat data, so the gluing is not quite as obstructed. This has been observed in other places, such as in initial data constructions with positive cosmological constant [78] and in more recent work on *multi-localized* initial data [17]. In the latter work, the relevant approximate kernel is that of a Schwarzschild, which is one-dimensional (for non-zero mass). The motivation for the multi-localized initial data constructed in [17] comes from the nonlinear stability result of Anderson and Pasqualotto [79]. The template for the construction is the Brill–Lindquist metric (5), and we glue ends on around each of the punctures in the set $\mathfrak{p}$ to make initial data on $\mathbb{R}^3$. Of note is that we are able to handle certain configurations where $\mathfrak{p}$ is infinite, and in particular, we construct smooth metrics on $\mathbb{R}^3$ with zero scalar curvature and infinitely many minimal spheres, each of them exterior to all the others.

As remarked earlier, in recent years *characteristic gluing* has been developed in a series of papers by Aretakis, Czimek and Rodnianski; for an overview and further references, see [11]. For the characteristic initial value problem, one specifies certain free data on what will be two transversely intersecting null hypersurfaces in the vacuum spacetime, after which one can appeal to local well-posedness results for the characteristic initial value problem for the Einstein vacuum equation [80, 81]. For the characteristic gluing problem, one seeks to glue data on two spheres along a null hypersurface. As with the spacelike case, the relevant equations that must be satisfied arise from the vacuum Einstein equation together with the fundamental compatibility equations for an embedding, in this case the compatibility coming from a double null foliation. The resulting equations for metric and curvature components are the *null structure equations*, of transport-elliptic character along either the ingoing or outgoing null hypersurface. As is noted in [11], there are obstructions to gluing data from two spheres, such as a monotonicity deriving from the Raychaudhuri equation.

There are a number of noteworthy applications of their technique, and we just mention a few. Their methods can recover the asymptotic gluing to Kerr as in [23], as well as localized gluing from [16], and without loss of decay in the transition region. In [82], the method is employed to achieve *obstruction-free* gluing. The asymptotic gluing construction along the lines of [22, 23] glues initial data with certain energy–momenta to Kerr data with energy–momenta which is close to the starting one, as we say above, tuned to cover the linear obstruction from the approximate cokernel. A natural question is whether one can glue pieces from spacetimes whose parameters are not necessarily close, and results in this direction have been obtained in [82].

The characteristic gluing construction has been employed by Kehle and Unger [83] to perform a specific *event horizon gluing* in the context of the Einstein–Maxwell charged scalar field system, in spherical symmetry. As an upshot, they produce a counterexample to the third law of black hole thermodynamics.

5. Conclusion

Gluing methods for constructing solutions of the Einstein constraint equations have garnered a number of noteworthy achievements. While more than twenty years have passed since the gluing results of [22] and [20], new techniques and applications are still being developed and continue to highlight the interplay between initial data results and the evolution problem.

Dedication

This article is dedicated to Mme. Yvonne Choquet-Bruhat, for giving so much to so many of us for so many years. The memory of Yvonne that is emblazoned upon me comes from over twenty years ago at a workshop in Cargèse, Corsica, marking the fiftieth anniversary of her landmark paper on the initial-value problem. Walking back to the seminar room with a colleague, we encountered Yvonne taking a bit of respite from the sunshine under a tree; while it was not an apple tree, what sprung to my imagination was the universe (or the spirit of Sir Isaac Newton) willing one to fall and land in the palm of her hand.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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