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Research article / *Article de recherche*

About the critical height of a vertical cut

À propos de la hauteur critique d'une tranchée verticale

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Abstract. The iconic problem of the stability analysis of a vertical cut is revisited after Drucker's celebrated contribution, in order to assess the sensitivity of the analysis to the soil tensile resistance. Within the framework of the exterior approach of the theory of yield design, the same virtual mechanisms as introduced by Drucker are reconsidered and thoroughly implemented. It is observed that, whatever the value of the friction angle, the drastic drop of the non-dimensional stability factor observed by Drucker when the constituent soil does not sustain tension remains confined within the immediate vicinity i.e., a few percent, of zero tensile resistance. Despite the fact that this conclusion cannot be considered as a final one since it is only based upon a full implementation of Drucker's virtual collapse mechanisms, we believe it may bring some relief as to the reliability of classical analyses that do not take any tension cutoff into account, as a response to Drucker's warning about "the consequences of assuming soil unable to take tension".

Résumé. On reprend le problème de l'analyse de stabilité d'une tranchée verticale après la contribution célèbre de Drucker, afin d'évaluer la sensibilité de l'analyse à la résistance en traction du sol. Dans le cadre de l'approche par l'extérieur de la théorie du calcul à la rupture, les mécanismes virtuels introduits par Drucker sont réexaminés pour une mise en œuvre complète. On observe que, quelle que soit la valeur de l'angle de frottement, la chute drastique du facteur adimensionnel de stabilité observée par Drucker lorsque le sol n'offre aucune résistance à la traction reste confinée dans le voisinage immédiat, c'est-à-dire quelques pour cent, de la résistance nulle en traction. Bien que cette conclusion ne puisse être considérée comme définitive, puisqu'elle n'est fondée que sur une mise en œuvre complète des mécanismes virtuels introduits par Drucker, nous pensons qu'elle peut apporter un certain soulagement quant à la fiabilité des analyses classiques en réponse à l'avertissement de Drucker sur « les conséquences qui pourraient résulter d'une hypothèse de sol incapable de résister en traction ».

Keywords. Strength criterion, Maximum resisting rate of work, Tension cutoff, Vertical cut, Critical height.

Mots-clés. Critère de résistance, Puissance résistante maximale, Troncature en traction, Tranchée verticale, Hauteur critique.

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1. Yield design analysis for materials obeying a Coulomb or Tresca strength criterion with tension cutoff

Tresca's and Coulomb's criteria with zero-tension cutoff were considered by Drucker and Prager [1, 2] for the stability analysis of a vertical cut, who, implementing a vanishing virtual velocity field, identified a significant reduction of the critical height of the cut. Salençon and Pecker [3], Chatzigogos, Pecker and Salençon [4] also referred to a Tresca criterion with zero-tension cutoff for the determination of the ultimate bearing capacity of shallow foundations under inclined eccentric loads.

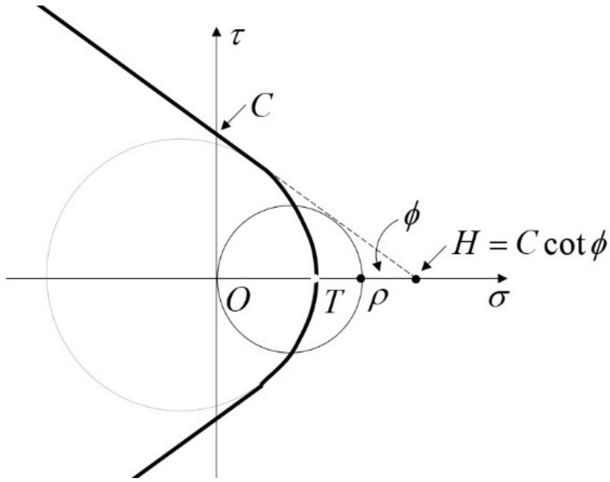

Figure 1. Coulomb's criterion with a tension cutoff set at a value T in the (σ,τ) plane of the Mohr stress representation.

From a general viewpoint, Coulomb or Tresca strength criteria with a zero-tension cutoff, which include cohesionless Coulomb's criterion, illustrate the concept of strength criteria for which the zero-stress state lies on the boundary of the domain of resistance, as considered by Frémond and Friaa [5]. For such criteria, it turns out that, when implementing the exterior approach of the Yield design theory, the minimization process often results in "vanishing" virtual velocity fields, such as those implemented in [2] or [6].

These criteria stand as particular cases of intrinsic curve type strength criteria with a tension cutoff set at a value T, which may either be zero, with the meaning that no tension can be sustained by the material, or positive allowing some tensile stresses. As a "tribute" to D. C. Drucker (1918–2001), this contribution is devoted to a more detailed analysis of the minimization process implemented in [2] for the stability analysis of a vertical cut, in the cases of a Tresca or Coulomb strength criterion when a positive tension cutoff T tends to zero, in order to assess the sensitivity of the analysis to the soil tensile resistance.

2. Critical height of a vertical cut

In terms of non-ordered principal stresses, a Coulomb criterion defined by a friction angle ϕ and a cohesion C, with a tension cutoff denoted by T such that $0 \le T < H = C\cot\phi$, can be written as

$$f(\underline{\underline{\sigma}}) = \mathrm{Max}\{\sigma_i(1+\sin\phi) - \sigma_j(1-\sin\phi) - 2C\cos\phi, \sigma_i - T \mid i,j = 1,2,3\} \le 0, \tag{2.1}$$

as shown in Figure 1.

The exterior approach of the yield design theory states that the rate of work by external forces exerted on the structure whose stability is under concern shall not be superior to the maximum resisting rate of work in any virtual velocity field $\underline{U}$. Such virtual velocity fields $\underline{U}$ can be piecewise continuous and continuously differentiable over Ω, with strain rate $\underline{\underline{d}}$ and velocity discontinuities $\underline{V}$ across jump surfaces Σ with normal $\underline{n}$. The maximum resisting rate of work is then defined as

$$\mathrm{P}_{\mathrm{mr}}(\underline{U}) = \int_\Omega \pi(\underline{\underline{d}})\,\mathrm{d}\Omega + \int_\Sigma \pi(\underline{V},\underline{n})\,\mathrm{d}\Sigma \tag{2.2}$$

with

$$\pi(\underline{\underline{d}}) = \mathrm{Sup}\{\underline{\underline{\sigma}} : \underline{\underline{d}} \mid f(\underline{\underline{\sigma}}) \le 0\}, \pi(\underline{V},\underline{n}) = \mathrm{Sup}\{\underline{V}\cdot\underline{\underline{\sigma}}\cdot\underline{n} \mid f(\underline{\underline{\sigma}}) \le 0\}, \tag{2.3}$$

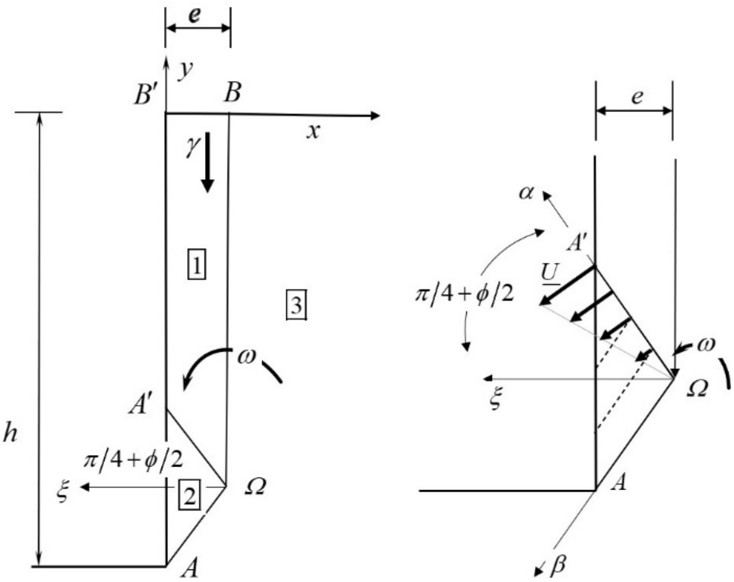

Figure 2. Virtual collapse mechanism considered in this analysis.

For a Coulomb criterion with tension cutoff as written in (2.1), the expressions for (2.3) can be found in [7] in the forms

$$\begin{cases} \underset{\phi,C,T}{\pi}(\underline{\underline{d}}) = +\infty \quad \text{if } \operatorname{tr}\underline{\underline{d}} < (|d_1|+|d_2|+|d_3|)\sin\phi \\ \underset{\phi,C,T}{\pi}(\underline{\underline{d}}) = C(|d_1|+|d_2|+|d_3| - \operatorname{tr}\underline{\underline{d}})\tan(\pi/4+\phi/2) \\ \qquad + \dfrac{T}{1-\sin\phi}(\operatorname{tr}\underline{\underline{d}} - (|d_1|+|d_2|+|d_3|)\sin\phi) \quad \text{if } \operatorname{tr}\underline{\underline{d}} \geq (|d_1|+|d_2|+|d_3|)\sin\phi \end{cases} \tag{2.4}$$

and

$$\begin{cases} \underset{\phi,C,T}{\pi}(\underline{V},\underline{n}) = +\infty \quad \text{if } \underline{V}\cdot\underline{n} < |\underline{V}|\sin\phi \\ \underset{\phi,C,T}{\pi}(\underline{V},\underline{n}) = C(|\underline{V}| - \underline{V}\cdot\underline{n})\tan(\pi/4+\phi/2) \\ \qquad + \dfrac{T}{1-\sin\phi}(\underline{V}\cdot\underline{n} - |\underline{V}|\sin\phi) \quad \text{if } \underline{V}\cdot\underline{n} \geq |\underline{V}|\sin\phi. \end{cases} \tag{2.5}$$

Corresponding expressions in the case of a Tresca criterion with tension cutoff are obtained making $\phi = 0$ in these equations.

The same virtual collapse mechanism as first devised by Drucker and Prager is considered in this analysis as a 2-dimensional problem, with notations displayed in Figure 2. Non dimensional factors ε and τ are introduced with the constraints

$$0 < \varepsilon = e/h \leq \tan(\pi/4 - \phi/2)/2, \tag{2.6}$$

$$0 \leq \tau = T/C < \cot\phi. \tag{2.7}$$

It is worth recalling that the material resistance to simple tension ($\sigma_1 = \rho, \sigma_2 = 0$) is $\rho = 2C\tan(\pi/4-\phi/2)$, (Figure 1), so that the reduction factor on the resistance to tension introduced by the tension cutoff is just

$$T/\rho = (\tau/2)\tan(\pi/4+\phi/2). \tag{2.8}$$

The description of the virtual collapse mechanism in Figure 2 goes as follows.

Zone $\boxed{3}$ remaining motionless, the virtual velocity field $\underline{U}$ in zone $\boxed{1}$ defined as $\Omega A'B'B$ consists of an anticlockwise rigid body rotational motion, with angular velocity ω about point Ω. This implies that zone $\boxed{1}$ separates from zone $\boxed{3}$ with a velocity discontinuity $\underline{V}(y)$ along the ξ axis when crossing ΩB, whose magnitude is

$$V(y) = \omega[h(1-\varepsilon\tan(\pi/4+\phi/2)) + y]. \tag{2.9}$$

The velocity field is continuous across $\Omega A'$ and across ΩA. Complying with these boundary conditions, the velocity field in zone $\boxed{2}$, delimited by $\Omega AA'$, is defined as follows: referring to the α and β lines in Figure 2, $\underline{U}$ is constant along any β line and normal to $\Omega A'$ with magnitude $\omega\, x^{\alpha}$, where x^{α} denotes the abscissa of the considered β line along $\Omega\alpha$[1]. The virtual strain rate in zone $\boxed{2}$ is constant, with principal axes along the ξ and y axes and principal values

$$\begin{cases} d_{\xi\xi} = (\omega/2)\tan(\pi/4+\phi/2) \\ d_{yy} = -(\omega/2)\tan(\pi/4-\phi/2), \end{cases} \tag{2.10}$$

so that

$$\operatorname{tr}\underline{\underline{d}} = d_{\xi\xi} + d_{yy} = \omega\tan\phi = (|d_{\xi\xi}| + |d_{yy}|)\sin\phi. \tag{2.11}$$

Implementing the yield design exterior approach in such a virtual velocity field calls for the computation of the rate of work by external forces $\mathrm{P_e}(\omega,\gamma,h,\varepsilon)$, where γ stands for the 2-D specific weight of the soil, and the maximum resisting rate of work $\mathrm{P_{mr}}(\omega,C,\phi,\tau,h,\varepsilon)$, in the whole bulk of the soil. Since zone $\boxed{3}$ remains motionless, it does not contribute to any of these two quantities.

The rate of work by external forces $\mathrm{P_e}(\omega,\gamma,h,\varepsilon)$ results from the contribution of zones $\boxed{1}$ and $\boxed{2}$. With

$$\mathrm{P_e^1}(\omega,\gamma,h,\varepsilon) = (\gamma\omega\, h^3\varepsilon^2/2)(1-(5/3)\varepsilon\tan(\pi/4+\phi/2)) \tag{2.12}$$

$$\mathrm{P_e^2}(\omega,\gamma,h,\varepsilon) = (\gamma\omega\, h^3\varepsilon^3/3)\tan(\pi/4+\phi/2), \tag{2.13}$$

$\mathrm{P_e}(\omega,\gamma,h,\varepsilon)$ results in

$$\mathrm{P_e}(\omega,\gamma,h,\varepsilon) = (\gamma\,\omega\, h^3\varepsilon^2/2)(1-\varepsilon\tan(\pi/4+\phi/2)).^2 \tag{2.14}$$

Since the virtual velocity field is continuous across ΩA and $\Omega A'$, the maximum resisting rate of work $\mathrm{P_{mr}}(\omega,C,\phi,\tau,h,\varepsilon)$ results from the addition of the maximum resisting rate of work along ΩB, which can be calculated from (2.5), and the maximum resisting rate of work developed in zone $\boxed{2}$, to be computed from (2.4) with (2.11):

$$\mathrm{P_{mr}}(\omega,C,\phi,\tau,h,\varepsilon) = \mathrm{P_{mr}^{\Omega B}}(\omega,C,\phi,\tau,h,\varepsilon) + \mathrm{P_{mr}^2}(\omega,C,\phi,\tau,h,\varepsilon). \tag{2.15}$$

We get

$$\mathrm{P_{mr}^{\Omega B}}(\omega,C,\phi,\tau,h,\varepsilon) = (\omega h^2/2)C\tau(1-\varepsilon\tan(\pi/4+\phi/2))^2 \tag{2.16}$$

$$\mathrm{P_{mr}^2}(\omega,C,\phi,\tau,h,\varepsilon) = \omega\, h^2\varepsilon^2 C\tan(\pi/4+\phi/2).^3 \tag{2.17}$$

Finally, implementing the yield design exterior approach results in the fundamental equation that can be written as

$$\frac{\gamma h}{C} \le \frac{2\tan(\pi/4+\phi/2)}{1-\varepsilon\tan(\pi/4+\phi/2)} + \tau\frac{1-\varepsilon\tan(\pi/4+\phi/2)}{\varepsilon^2}, \tag{2.18}$$

to be minimized with respect to ε, with τ as a parameter, within constraints (2.6) and (2.7). The first term on the right hand side of this inequation, which results from $\mathrm{P_{mr}^2}(\omega,C,\phi,\tau,h,\varepsilon)$ and does not depend on τ, decreases with ε, while the second term, which only depends on τ, increases when ε decreases. Thus, for any value of τ complying with (2.7), the minimization process will

[1] Cf. [2, 7].

[2] Previous works did not require a complete computation of this quantity.

[3] As in [2, 8].

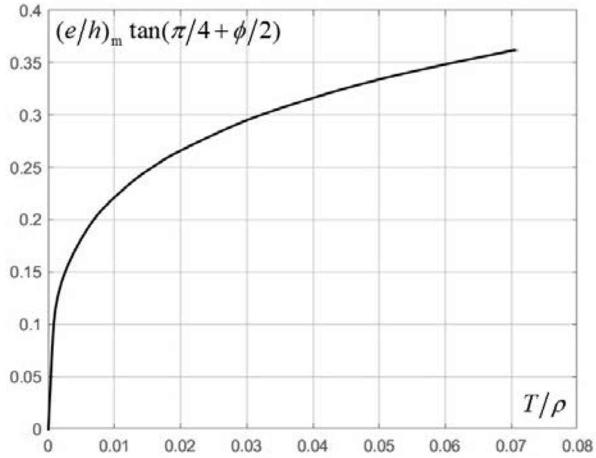

Figure 3. $(e/h)_{\rm m}\tan(\pi/4+\phi/2)$ as a function of T/ρ.

adjust the geometrical variable ε to the value $\varepsilon_{\rm m}(\tau)$ that yields the best balance between these two contributions, resulting in the best upper bound for $\gamma h/C$. Referring to $T/\rho=\tau\tan(\pi/4+\phi/2)/2$ instead of τ, Equation (2.18) can be written as

$$\frac{\gamma h}{C}\tan(\pi/4-\phi/2)\leq\frac{2}{1-\varepsilon\tan(\pi/4+\phi/2)}+2\frac{T}{\rho}\frac{1-\varepsilon\tan(\pi/4+\phi/2)}{\varepsilon^2\tan^2(\pi/4+\phi/2)},\tag{2.19}$$

or, introducing

$$E(\phi)=\varepsilon\tan(\pi/4+\phi/2),\tag{2.20}$$

$$\frac{\gamma h}{C}\tan(\pi/4-\phi/2)\leq\frac{2}{1-E(\phi)}+2\frac{T}{\rho}\frac{1-E(\phi)}{[E(\phi)]^2}\tag{2.21}$$

It follows that, for any given value of ϕ, with T/ρ as a parameter, the minimum of the right hand of (2.21) with respect to ε is obtained for the same value of $E(\phi)=\varepsilon\tan(\pi/4+\phi/2)$ denoted by $E_{\rm m}(T/\rho)$, where $E_{\rm m}=(e/h)_{\rm m}\tan(\pi/4+\phi/2)$, (Figure 3).

The corresponding value of the right hand member of (2.21) is the best upper bound for the left hand member of this equation. It is capped by the best available upper bounds in the absence of a tension cutoff, i.e., 3.83 whatever ϕ, (3.77649 for $\phi=0$) as usually agreed upon [9–12].

The results of the minimization process are displayed in Figure 4, where $(\gamma h/C)\tan(\pi/4-\phi/2)$ is plotted as a function of $\tau=T/\rho=\tan(\pi/4+\phi/2)T/2C$.

It comes out, through this normalization process, that all the results fit on a parabolic like curve with a horizontal axis and a summit on the vertical axis at $\gamma h\tan(\pi/4-\phi/2)/C=2$, which is the critical value obtained by Drucker and has also been established as a lower bound estimate through a simple three-zone relevant stress field.

It also appears that the mechanisms in Figure 2 only prevail over the classical log-spiral one for small values of T/ρ, namely when $T/\rho\leq 0.07$, while the drastic drop to the critical and exact value $\gamma h/C=2\tan(\pi/4+\phi/2)$ for $\tau=0$ remains confined within the immediate vicinity of $T/\rho=0$. From a practical viewpoint, if a 20% drop is considered as acceptable, T/ρ must be superior to 3%, which means that the tensile resistance of the soil shall not be less than $0.06C\tan(\pi/4-\phi/2)$. It is worth insisting on the fact that these results are valid whatever the value of ϕ.

Obviously, strictly speaking, the conclusions enounced here above cannot be considered as final since they only result from a full implementation of Drucker's virtual collapse mechanisms. Nevertheless, it can be observed that, from a physical viewpoint, these mechanisms, when

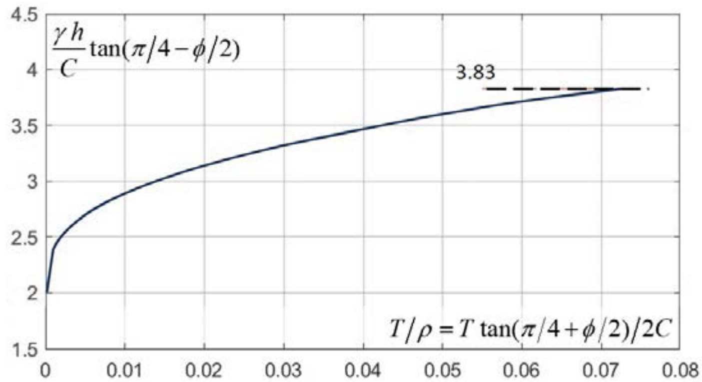

Figure 4. Normalized results of the minimization process on Equation (2.21).

$T/\rho \searrow 0$, with $E_\mathrm{m}(T/\rho) \searrow 0$, are perfectly suited to matching the zero normal stress constraint, which is imposed as a boundary condition along the stress-free vertical and horizontal edges of the cut, with the vanishing tensile resistance of the material. Hence, as a response to Drucker's warning about "*the consequences of assuming soil unable to take tension*" [2], we believe that the analysis that has been developed here may bring some relief as to the reliability of classical analyses that do not take any tension cutoff into account.

In the particular case of a purely cohesive frictionless material ($\phi = 0$), numerical static lower bound approaches have been developed, assuming full tensile resistance of the material ($\rho = 2C$), which culminated with $\gamma h/C \geq 3.77522$ as a lower bound estimate in [11]. Checking the sensitivity of such approaches to a reduction of the tensile resistance would, obviously, help to comfort the preceding conclusion.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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