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Review article / *Article de synthèse*

Cosmology, the big bang and the BKL conjecture

Cosmologie, big bang et conjecture BKL

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Abstract. This is a review article of mathematical results in cosmology, written in honor of Yvonne Choquet-Bruhat's 100th birthday. It starts with a brief description of some of the essential questions: strong cosmic censorship; the relation between the future asymptotics and geometrization in the vacuum setting; the cosmic no-hair conjecture; and the BKL-proposal. It then turns to results, starting with ones obtained in situations with symmetry. It continues with a review of future global non-linear stability results, stable big bang formation results, results concerning solutions to linear equations on cosmological backgrounds and numerical results. The article contains a substantial, but very incomplete, list of references to the literature.

Résumé. Il s'agit d'un article de revue sur les résultats mathématiques en cosmologie, écrit en l'honneur du 100ème anniversaire d'Yvonne Choquet-Bruhat. Il commence par une brève description de certaines questions essentielles : la censure cosmique forte, la relation entre l'asymptotique future et la géométrisation dans le cas du vide, la conjecture de l'absence de cheveux dans le cadre cosmique et la proposition BKL. Il aborde ensuite les résultats, en commençant par ceux obtenus dans des situations avec symétrie. Il se poursuit par un examen des résultats de stabilité future non linéaire globale, des résultats de formation stable du big bang, des résultats concernant les solutions d'équations linéaires sur les espaces-temps cosmologiques et des résultats numériques. L'article contient une liste substantielle, mais très incomplète, de références à la littérature.

Keywords. Global analysis, General relativity, Lorentzian geometry, Nonlinear evolution equations, Cosmology, The big bang and the BKL proposal.

Mots-clés. Analyse globale, Relativité générale, Géométrie lorentzienne, Équations d'évolution non linéaires, Cosmologie, Big bang et proposition BKL.

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1. Introduction

From Einstein's attempt to find a static model of the universe; to Friedmann's discovery of expanding solutions; Lemaître's conclusion that the universe is expanding, based on a combination of observations and expanding solutions; and Hubble's observations, the initial history of general relativistic cosmology was long and complicated, and we refer the interested reader to [1] for a detailed discussion. Once it becomes clear that the universe is expanding, it is natural to trace that expansion backward in order to find an origin. However, even if one finds models with a corresponding cosmological, or "big bang", singularity, it is not clear if that is the only explanation, nor is it clear that such a singularity would persist when relaxing the symmetry assumptions associated with the model. Hawking's theorem gives one indication that cosmological singularities are a general feature of solutions to Einstein's equations; see [2, 3] for original references and [4, pp. 431–434] for a textbook presentation. Moreover, the discovery of the cosmic microwave background (CMB) radiation by Penzias and Wilson [5] gives clear observational corroboration to the idea of a big bang. However, in spite of the importance of Hawking's result, it does not say much about the nature of singularities. Does the gravitational field generically blow up in the direction of the singularity? Is the spacetime inextendible? In order to answer these questions, it would seem to be necessary to carry out a more detailed analysis of solutions. Moreover, the high degree of isotropy of the CMB cannot be explained by a thermal equilibrium having been established prior to the surface of last scattering due to the causal structure of the standard models of the universe. For this reason, either more sophisticated models of the universe or more exotic models of matter (inducing, e.g., inflation) would seem to be necessary.

Already at this point, the necessity for a deeper understanding of the space of cosmological solutions to Einstein's equations is apparent. This can be viewed as a mathematical problem, and the purpose of this review article is to give a rough idea of the developments in this subject since the appearance of Yvonne Choquet-Bruhat's seminal paper [6]. Due to the number of results that have been obtained, giving a complete description is not possible. In fact, this review article describes a small subset of the results, and the limited selection is due to the knowledge (or absence thereof) and interests of the author. We begin the discussion in the next section by highlighting some of the questions. In the following sections, we describe results.

2. Cosmology, conjectures

Before describing the objects of study, it is natural to note that there are at least two fundamental problems in the subject of cosmology: we cannot carry out experiments; and we only observe a small part of the universe. Deducing the global properties of the universe on the basis of observations might therefore even in principle be impossible. In cosmology, it is common to compensate for the lack of knowledge by imposing philosophical principles. One example is the *Copernican principle*, a strong version of which states that there are no privileged observers in the universe. This principle does not require the universe to appear the same to all observers. Nevertheless, the currently preferred so called *cosmological principle*, states that the universe can be modeled by a spatially homogeneous and isotropic solution. In such solutions, one cannot distinguish between two points in space, nor between two directions. This quite extreme assumption reduces the possibilities for the geometry to a choice of a 3-dimensional Riemannian manifold of constant curvature and a scalar function of time, describing the expansion and contraction of space. Physicists largely restrict themselves to this situation and are more likely, when problems arise, to introduce exotic matter models than to consider more general geometries. In the mathematical study of cosmology, we do want to consider more general geometries. On the other hand, we are not interested in, e.g., asymptotically flat solutions, in which there are clearly preferred asymptotic regimes. Using the fact that there is a well posed Cauchy problem for Einstein's equations, see e.g. [6–12], it is natural to describe the conditions in terms of initial data. We do not attempt to give a formal definition of the meaning of a cosmological spacetime here (where by a *spacetime*, we mean a time oriented Lorentz manifold). However, we have a spacetime in mind in which the geometry and matter fields in different parts of the initial hypersurface are qualitatively similar. In particular, initial data on a closed manifold such that the mean curvature is negative on some parts and positive on others is not of the type we wish to consider here; this would mean that some observers experience expansion and some contraction. In particular, that the initial manifold is closed is neither a necessary nor a sufficient condition for the corresponding spacetime to be cosmological. In fact, we are mainly interested in initial data sets for which the mean curvature is bounded away from zero by a non-zero constant, in analogy with the conditions of Hawking's theorem.

Next, we describe some of the questions of interest.

2.1. *Strong cosmic censorship, curvature blow up*

As mentioned in the introduction, Hawking's singularity theorem, see e.g. [4, Theorem 55A, p. 431], ensures that singularities appear naturally in general relativity. On the other hand, the only conclusion provided is that there are incomplete timelike geodesics. Moreover, letting N be the timelike future of a point in Minkowski space, letting g denote the Minkowski metric restricted to N, and foliating (N, g) with hyperboloids, the conditions of Hawking's theorem are satisfied (and indicate the occurrence of a singularity), even though (N, g) is simply a subset of Minkowski space. Clearly, it is desirable to obtain more information. At the very least, it would be desirable to prove that if one traces a causal geodesic in an incomplete direction, then a curvature invariant, such as the Kretschmann scalar (i.e., the Riemann curvature tensor contracted with itself), becomes unbounded. Unfortunately, such a statement cannot hold for arbitrary initial data. However, one can conjecture that for generic initial data in an appropriately chosen class, the Kretschmann scalar is unbounded in the incomplete directions of causal geodesics in the corresponding maximal Cauchy development; this is the *curvature blow up conjecture*. It is natural to try to prove the corresponding statement in symmetry classes of solutions. This conjecture is related to the *strong cosmic censorship conjecture*, stating that for generic initial data (in an appropriate class), the corresponding maximal Cauchy development is inextendible. This would mean that, even though Einstein's equations are not deterministic (in the sense that one cannot determine what spacetime one is in on the basis of initial data) they are generically deterministic. In order to obtain mathematical results, one of course has to give a precise definition of the class of data considered and the meaning of the word inextentible.

2.2. *Vacuum evolution and geometrization*

In [13], Eardley and Moncrief suggested one way to address the strong cosmic censorship conjecture: by proving global existence results for constant mean curvature (CMC) foliations. Their hope was that the CMC foliations should avoid the singularities and cover the spacetimes outside the singularities. More specifically, and focusing on the expanding direction, there are, due to the work of Fischer and Moncrief on the one hand, see [14–22], and Anderson on the other, see [23, 24], quite detailed conjectures concerning the future asymptotics of solutions to Einstein's vacuum equations (with a vanishing cosmological constant) in the cosmological setting. In these conjectures, the initial manifold is assumed to be closed and to not admit a Riemannian metric of positive scalar curvature. Moreover, the authors focus on CMC foliations. Since the volume of the leaves of the CMC foliation is expected to tend to infinity in the expanding direction, it is natural to carry out an expansion normalization of the metric. In the work of Fischer and Moncrief, the authors rescale the Riemannian metrics on the leaves by multiplying them by the mean curvature squared. Anderson rescales by dividing by the proper time squared (where the proper time is measured to a fixed Cauchy hypersurface). Considering the rescaled metrics as a family of Riemannian metrics on a fixed closed manifold, the idea is that in the limit, this family implements a geometrization of the manifold. In particular, it converges to complete hyperbolic metrics of finite volume on the hyperbolic pieces and collapses on the Seifert fibred pieces. In this sense the evolution leads to an isotropization, on average, in the expanding direction. As will become clear below, there are many results confirming this picture. However, the results are either in symmetric settings or concern perturbations of symmetric settings. This means that the initial manifold must admit a symmetric metric, so that the geometrization must be trivial. In other words, confirming the conjectured picture in a non-trivial setting can be expected to be quite difficult.

2.3. *Cosmic no-hair conjecture*

Turning to cosmological solutions with a positive cosmological constant, the *cosmic no-hair conjecture* states that, in the expanding direction, spacetimes should asymptote to de Sitter space. More specifically, fixing, e.g., a future complete timelike geodesic, say γ, then the behavior in $J^-(\gamma)$ should asymptotically be indistinguishable from de Sitter space. For a more formal definition, see [25, Definition 1.8, pp. 1573–1574] and [25, Conjecture 1.11, p. 1574]. Again, there are many results confirming this proposal in specific situations.

2.4. *The BKL proposal*

This review article is focused on mathematical results. However, the questions that are addressed in the mathematical literature are at least partially inspired by heuristic arguments and proposals arising in the physics literature. One such proposal is due to Belinskiǐ, Khalatnikov and Lifschitz, BKL for short, and dates back to the 60's, 70's and 80's; see, e.g., [26–31]. The rough statement of the BKL proposal is that generic cosmological singularities are spacelike, local and oscillatory (with the exception of matter models such as scalar fields and stiff fluids). More specifically, distinct causal curves typically lose the ability to communicate in the direction of the singularity; each causal curve evolves as its own universe in the direction of the singularity; and the local behavior is modeled by the dynamics of Bianchi type VIII or IX (mixmaster) spactimes, which are expected to exhibit (chaotic) oscillatory behavior in the direction of the singularity. A related idea arising from the BKL proposal is that spatial derivatives should be dominated by time derivatives in the approach to the singularity. From this perspective, it is natural to drop the spatial derivatives in the equations, which leads to an interpretation of Einstein's equations as a family of ODE's (one ODE for each spatial point). Two examples of attempts to make this perspective more precise are given by [32, 33]. See also Section 2.5 below for a more detailed discussion. The ideas of BKL have later been developed further in the physics literature, see, e.g., [34–37]. More mathematical results, deriving conclusions from a priori assumptions, can be found in, e.g., [38–40]. In the next subsection, we try to make the ideas of BKL more precise in the non-oscillatory setting.

2.5. *Initial data on the singularity*

One of the ideas of BKL is that the dynamics of nearby observers decouple and that time derivative terms (or "kinetic terms") asymptotically dominate over spatial derivative terms (or spatial curvature terms) in the direction of the singularity. In [32], the authors discuss ideas of this type in terms of asymptotics of the geometry, and they speak of *velocity-dominated singularities*. A related perspective is developed in [33], in which the authors introduce the terminology that a spacetime can be *asymptotically velocity term dominated (AVTD) near the singularity*. The idea in [33] is to introduce a truncated set of equations, the *velocity term dominated (VTD) system*, and to say that a solution is AVTD if it asymptotes to a solution to the VTD system; see [33] for details. Unfortunately, the VTD system has no covariant formulation. On the other hand, the evolutionary part of the VTD system consists of ODE's for each spatial point. This means that the VTD system is significantly easier to solve than the original equations. Moreover, in [41] the authors prove that, given a real analytic solution to the VTD system in the $3+1$-dimensional Einstein-scalar field or Einstein-stiff fluid setting (satisfying certain conditions), there is a unique corresponding solution to the actual equations. The authors of [41] call the resulting singularities *quiescent*, a terminology that goes back to [42]. In the $3+1$-dimensional vacuum setting, the behavior in the direction of the singularity is expected to be oscillatory (in accordance with the BKL proposal), in which case it is, at this time, not clear how to specify initial data on the singularity. However, in higher dimensions (more specifically, for $n+1$-dimensions with $n \geq 10$), quiescence is expected even in the vacuum setting; see [43]. Accordingly, [41] is generalized in [44] to higher dimensions and more general matter fields.

The solutions constructed in [41] and [44] are expected to be robust. In other words, perturbations of the corresponding initial data are expected to lead to maximal Cauchy developments with similar singularities; see Section 5 below for a partial justification of this expectation. However, in situations with symmetry, e.g., even singularities in $3+1$-dimensional vacuum spacetimes can be quiescent; see, e.g, [45–53]. The reason for this is that the symmetry type can eliminate the mechanism causing the oscillations. One can also impose conditions on data on the singularity that eliminate this mechanism. One corresponding result in the real analytic setting, but in the absence of symmetries, is obtained in [54]. A more general result in the smooth setting is contained in [55].

The VTD solutions in [41, 44] can be interpreted as initial data on the singularity. However, there are many other notions of initial data on the singularity arising when studying symmetry classes of solutions as well as in results such as [54, 55]. For this reason, it is of interest to try to find a unifying perspective. This is the goal of [56]. In particular, [56, Definition 10, p. 5] yields the following definition of initial data on the singularity in the Einstein-scalar field setting with a cosmological constant:

Definition 1. *Let $3 \leq n \in \mathbb{N}$, $\Lambda \in \mathbb{R}$, $(\Sigma, \mathcal{H})$ be a smooth n-dimensional Riemannian manifold, $\mathcal{K}$ be a smooth $(1,1)$-tensor field on Σ and Φ_0 and Φ_1 be smooth functions on Σ. Then $(\Sigma, \mathcal{H}, \mathcal{K}, \Phi_0, \Phi_1)$ are non-degenerate quiescent initial data on the singularity for the Einstein-scalar field equations with a cosmological constant Λ if*

(1) $\mathrm{tr}\mathcal{K} = 1$ *and* $\mathcal{K}$ *is symmetric with respect to* $\mathcal{H}$.
(2) $\mathrm{tr}\mathcal{K}^2 + \Phi_1^2 = 1$ *and* $\mathrm{div}_{\mathcal{H}}\mathcal{K} = \Phi_1 d\Phi_0$.
(3) *The eigenvalues of* $\mathcal{K}$, *say* $\{p_A\}$, *are distinct.*
(4) $\mathcal{O}^A_{BC}$ *vanishes in a neighborhood of* $\bar{x} \in \Sigma$ *if* $1 + p_A(\bar{x}) - p_B(\bar{x}) - p_C(\bar{x}) \leq 0$, *where* $\mathcal{O}^A_{BC} := (\gamma^A_{BC})^2$; γ^A_{BC} *is given by* $[\mathcal{X}_B, \mathcal{X}_C] = \gamma^A_{BC}\mathcal{X}_A$; *and* $\mathcal{X}_A$ *is an eigenvector field corresponding to* p_A *such that* $|\mathcal{X}_A|_{\mathcal{H}} = 1$.

The reader interested in a justification of why this is a natural notion of initial data on the singularity is referred to [40, 56]. That the solutions to the VTD equations that are used in [41] as a substitute for data on the singularity are equivalent to data on the singularity in the sense of Definition 1 is demonstrated in [56, Section 1.5, pp. 7–9]. That the notion of initial data used in [55] is a special case of Definition 1 is demonstrated in [56, Section 1.5, pp. 7–9]. That the notion of initial data on the singularity used in [45, 47] in the case of $\mathbb{T}^3$-Gowdy is a special case of Definition 1 is demonstrated in [57]. It would be of interest to show that all the notions of initial data on the singularity used in the literature (in the Einstein-scalar field setting) are special cases of Definition 1. In [58], Andrés Franco Grisales demonstrates that data as in Definition 1 give rise to unique developments in the $3+1$-dimensional Einstein-non-linear scalar field setting. In particular, ignoring the condition of non-degeneracy (i.e., the requirement that the eigenvalues be distinct), [58] simultaneously generalizes [41] and [55].

In $3+1$-dimensions, one way to avoid the condition of non-degeneracy in Definition 1 is given in [56, Remark 26, p. 9]. It is also of great interest to generalize Definition 1 to other matter fields.

Partly, this has already been done in [44]; the VTD solutions in [44] can presumably be used to this end. However, there are many matter models of interest that are not covered in [44].

The relation between solutions and data on the singularity are the following. Assume (M,g,ϕ) to be a solution to the Einstein-scalar field equations with a cosmological constant Λ. Assume, moreover, that there is a partial foliation of (M,g,ϕ) by spacelike Cauchy hypersurfaces, say $\{\Sigma_t\}$ for $t\in(0,t_+)$ and assume that the mean curvature of Σ_t, say θ_t is such that θ_t converges uniformly to ∞ as $t\downarrow 0$. Define $\mathcal{K}_t$ to be the *expansion normalized Weingarten map* of Σ_t, i.e., the Weingarten map of Σ_t divided by θ_t. Define $\mathcal{H}_t := \bar{g}_t(\theta_t^{\mathcal{K}_t}\cdot,\theta_t^{\mathcal{K}_t}\cdot)$, where $\bar{g}_t$ is the Riemannian metric induced on Σ_t. Note that $\mathcal{K}_t$ is an operator on the tangent space of Σ_t and that θ_t can be assumed to be strictly positive, so that

$$\theta_t^{\mathcal{K}_t} := \sum_{l=0}^{\infty}\frac{(\ln\theta_t)^l\mathcal{K}_t^l}{l!}.$$

Finally, let ϕ_t be the scalar field induced on Σ_t, $\dot{\phi}_t$ be the future unit normal derivative of ϕ at Σ_t, $\Phi_{1,t} := \dot{\phi}_t/\theta_t$ and $\Phi_{0,t} := \phi_t + \Phi_{1,t}\ln\theta_t$. Then (M,g,ϕ) is said to be a *development of initial data* $(\Sigma,\mathcal{H},\mathcal{K},\Phi_0,\Phi_1)$ *on the singularity* if

$$\mathcal{H}_t\to\mathcal{H},\quad \mathcal{K}_t\to\mathcal{K},\quad \Phi_{0,t}\to\Phi_0,\quad \Phi_{1,t}\to\Phi_1 \tag{1}$$

as $t\downarrow 0$, where the topology defining the convergence should be adapted to the situation.

The notion of initial data on the singularity serves many purposes. First, the definition is covariant. Second, that a solution is a development of initial data on the singularity can be interpreted as saying that it is AVTD, in accordance with the BKL proposal and [33]. Third, if there are initial data on the singularity in a certain symmetry class, then it is natural to conjecture that the behavior is generically quiescent in this symmetry class, and that generic solutions are developments of initial data on the singularity. Conversely, if there are no initial data on the singularity, it is natural to conjecture that the generic behavior is oscillatory. Fourth, in quiescent settings, initial data on the singularity can potentially be used to parameterize solutions. In such settings, they thus fill a double role: they both provide as detailed information concerning the asymptotics as one could wish and parameterize solutions. However, there are at least two caveats to keep in mind. First, the formulation of the limits in (1) introduces a foliation dependence. It is natural to interpret this dependence as a lack of covariance. On the other hand, in many situations there are foliations that are uniquely determined by the spacetime. Assume that (M,g) is a globally hyperbolic spacetime with closed spacelike Cauchy hypersurfaces; satisfies the timelike convergence condition (this holds for the Einstein-scalar field equations with a vanishing cosmological constant); and has a crushing singularity (in the sense that there is a foliation by a family of Cauchy hypersurfaces whose mean curvatures diverge to infinity uniformly). Then (M,g) has a unique CMC crushing foliation in a neighborhood of the singularity. One natural covariant relation between the data on the singularity and the development is then to require that (1) holds for the uniquely determined CMC foliation. The notion of a cosmological time function, see [59], can be used similarly. Nevertheless, it is of interest to ask if the existence of one type of foliation inducing initial data $\mathfrak{I}$ on the singularity leads to the existence of other types of foliations inducing the same data $\mathfrak{I}$.

Second, even though a certain combination of matter model and symmetry class admits initial data on the singularity, it is unlikely that all solutions are developments of initial data on the singularity. In particular, a generalization of the notion of spikes appearing in the $\mathbb{T}^3$-Gowdy symmetric setting, see Section 3.2.1, can be expected to be relevant in the spatially inhomogeneous setting. Nevertheless, one can hope that these exceptional features only occur along hypersurfaces on the singularity; see [57] for an example.

3. Solutions with symmetry

One approach to the study of cosmological solutions to Einstein's equations is to start by studying solutions with a high degree of symmetry and then to gradually relax the symmetry conditions. In view of the fact that the starting point in standard cosmology is the cosmological principle, this is a quite natural way to proceed. The disadvantage of studying classes of symmetric solutions is that they are automatically non-generic. On the other hand, within a symmetry class, it might be possible to obtain a global understanding of the dynamics and to discover unexpected phenomena. Most, though not all, results in the absence of symmetry are stability results. In this case, the advantage is that one obtains results for open sets of initial data. On the other hand, in order to obtain results, one normally has to prove that the perturbed solutions remain close to the background. This means that one is unlikely to discover new phenomena.

3.1. *Spatially homogeneous solutions*

For a solution to be spatially homogeneous, it has to arise from homogeneous initial data. This means that the isometry group of the initial data has to be transitive. If the initial manifold Σ is 3-dimensional, this means that the universal covering manifold of Σ is either a 3-dimensional Lie group, or it is $\mathbb{S}^2 \times \mathbb{R}$. In the latter case, which is referred to as *Kantowski–Sachs*, the isometry group of simply connected initial data is the isometry group of the standard Riemannian metric on $\mathbb{S}^2 \times \mathbb{R}$ (note that this isometry group does not have a 3-dimensional transitive subgroup). If G is a 3-dimensional Lie group and $\mathfrak{g}$ is the corresponding Lie algebra, let, for $X \in \mathfrak{g}$, $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ be defined by $\mathrm{ad}_X(Y) = [X, Y]$. Define $\eta_G \in \mathfrak{g}^*$ by $\eta_G(X) := \mathrm{tr}\, \mathrm{ad}_X$. If $\eta_G = 0$, then the Lie group is said to be *unimodular* (or of *Bianchi class A* in the cosmology literature). If $\eta_G \neq 0$, the Lie group is said to be *non-unimodular* (or of *Bianchi class B* in the cosmology literature). *Bianchi class A/B initial data* are left invariant initial data on a Bianchi class A/B Lie group. *Bianchi class A/B solutions* are solutions arising from Bianchi class A/B initial data. In what follows, we do not discuss the Kantowski–Sachs setting but rather focus on the Bianchi solutions. Imposing spatial homogeneity typically means that the combined Einstein-matter equations reduce to a system of ODE's. However, in the case of kinetic theory (the Einstein–Vlasov and Einstein–Boltzmann equations, e.g.), the equations are still PDE's.

As far as formulations of the equations are concerned, an important step in obtaining a unified framework for discussing Bianchi spacetimes was taken in [60], see also [61]. Many subsequent formulations, such as [62, 63], used to prove several of the results mentioned below, are based on the ideas of [60].

Since the literature in the spatially homogeneous setting is vast, we can here only mention an extremely limited selection of the results. Nevertheless, we refer to [60, 61, 64, 65] and the references cited therein for a description of the state of the art prior to the mid 90's; here our main focus will be on results obtained thereafter.

3.1.1. *The direction of the singularity*

Once the occurrence of a big bang and the high degree of isotropy of the CMB was established, a fundamental question immediately presented itself:

> ... if the 3 °K background radiation was last scattered at a redshift $z = 7$, then the radiation coming to us from two directions in the sky separated by more than about 30° was last scattered by regions of plasma whose prior histories had no causal relationship. [...] Robertson–Walker models therefore give no insight into why the observed microwave radiation from widely different angles in the sky has very precisely (0.2%) the same temperature;

see [66]. In order to address this problem, Misner suggested considering Bianchi type IX spacetimes. These are special examples of Bianchi class A spacetimes arising from left invariant initial data on a Lie group with a Lie algebra given by that of SU(2). Misner hoped that such spacetimes would have a different causal structure, allowing the establishment of a thermal equilibrium prior to the surface of last scattering. Beyond having the maximal number of degrees of freedom in the spatially homogeneous setting, and beyond playing a central role in the BKL proposal, Misner's expectation was thus that the Bianchi type IX spacetimes should play a central role in resolving the problem with high degree of isotropy of the CMB. In addition, there are vacuum Bianchi type IX solutions (namely the locally rotationally symmetric ones) such that the maximal Cauchy development can be extended through Cauchy horizons. In fact, there are two inequivalent maximal extensions, indicating that the initial data do not determine the spacetime; see [67, 68] for the original articles and [69, Theorem 25.4, p. 262] for a textbook presentation. Proving that inextendibility holds for initial data that are not locally rotationally symmetric is thus a natural goal and would correspond to proving strong cosmic censorship in the corresponding symmetry class.

The oscillatory setting. In the BKL proposal, the vacuum Bianchi type VIII and IX solutions are models for the expected oscillatory behavior (Bianchi type VIII solutions arise from left invariant initial data on Lie groups with a Lie algebra equal to that of $\mathrm{Sl}(2,\mathbb{R})$). That oscillations are to be expected in the vacuum Bianchi type VIII and IX settings can also be seen directly from Definition 1; there are no vacuum Bianchi type VIII or IX initial data on the singularity in the sense of Definition 1, since the Lie algebras in these cases are such that the p_D occurring in Definition 1 have to satisfy $2p_A = 1 + p_A - p_B - p_C > 0$ for all distinct A, B and C, which is not consistent with the requirements that the sum of the p_D equals one and the sum of the p_D^2 equals one. In spite of extensive discussions in the physics literature, the first mathematical result concerning the singularity in Bianchi type IX spacetimes was obtained in [70] (to the best of our knowledge). In particular, [70] contains a proof of the fact that, with the exception of the known locally rotationally solutions that give rise to Cauchy horizons through which the spacetimes can be extended, all Bianchi type VIII and IX vacuum initial data give rise to developments such that the Kretschmann scalar is unbounded in the incomplete directions of causal geodesics. This verifies both the curvature blow up conjecture and the C^2-version of the strong cosmic censorship conjecture in this setting. Moreover, the expansion normalized eigenvalues of the Weingarten map do not converge. In this sense, the solutions exhibit the oscillatory behavior conjectured by BKL. On the other hand, the result does not resolve the issue of the causal structure. Moreover, it does not yield the conclusion that solutions asymptote to an orbit of the BKL map.

Next, consider Einstein's equations coupled to an orthogonal perfect fluid with a linear equation of state $p = (\gamma - 1)\rho$, where ρ denotes the *energy density*, p the *pressure* and γ is a constant (here assumed to satisfy $\gamma \in (2/3, 2)$); the orthogonality assumption corresponds to the condition that fluid vector field be perpendicular to the hypersurfaces of spatial homogeneity. In this setting, generic Bianchi type IX solutions converge to an attractor on which the dynamics are determined by the BKL map; see [71] as well as [72, 73] for an overview of the state of the art in 2009. This is an indication that the BKL map is a good model for the dynamics. Unfortunately, the methods of [71] do not seem to be well suited to proving a similar result for Bianchi type VIII solutions. Nevertheless, in [74], the author proves that generic Bianchi type VIII and IX vacuum solutions converge to an attractor. Moreover, he proves that there is a set of initial data of full measure such that the corresponding solutions have particle horizons. On the other hand, he conjectures that for a Baire generic set of initial data, the corresponding solutions do not have particle horizons. The issue of the causality is thus quite subtle.

Turning to the question of whether solutions asymptote to an orbit of the Kasner map, the first results were obtained in [75, 76]; see also [77]. In these results, one fixes an orbit and then proves that there are solutions converging to the orbit. In fact, in the first two results, the authors prove that there is a co-dimension one Lipschitz submanifold of the state space such that initial data in the submanifold yield solutions converging to the given orbit. Similar results were later obtained in the magnetic Bianchi type VI_0 setting in [78]. These results are certainly very interesting. However, they do not imply that there is a set of initial data of positive measure such that the corresponding solutions asymptote to an orbit of the Kasner map. However, in [79], the authors prove that there is a subset of the Kasner circle, say S, of full measure such that for each orbit originating with a point in S, there is a co-dimension one Lipschitz submanifold of the state space such that initial data in the submanifold yield solutions converging to the given orbit. Moreover, they prove that the union of all these Lipschitz submanifolds yields a subset of the state space with positive measure. The argument is long and technical, but in the end, one concludes that the set of initial data such that the corresponding solutions asymptote to an orbit of the Kasner map has positive measure. Clearly, this is an important step forward. However, it is still unclear if a similar statement holds for a subset of the state space of full measure.

Even though much attention has been devoted to Bianchi types VIII and IX, Bianchi type $\mathrm{VI}_{-1/9}$ is an equally general class of spatially homogeneous spacetimes. In fact, it can be argued that it is a more relevant model for oscillatory behavior in lower degrees of symmetry. However, due to the difficulty of analyzing the asymptotics of solutions to the relevant systems of ODE's, so far there are, to the best of our knowledge, no results. See, however, [80] for a formulation of the equations and a heuristic discussion of the asymptotics.

Oscillatory behavior can arise for two reasons; either because there are enough degrees of freedom (as in the case of Bianchi types $\mathrm{VI}_{-1/9}$, VIII and IX) or because of the presence of appropriate matter models. For examples of the latter, the reader is referred to, e.g., [78, 81, 82].

The convergent (or quiescent) setting. The behavior in the direction of the singularity can be convergent/quiescent mainly for two reasons. One possibility is that the matter fields are such that solutions naturally become convergent; this happens, e.g., in the presence of a scalar field or a stiff fluid. Another possibility is that the geometry is such that the mechanism causing oscillations is absent; this corresponds to condition (4) in Definition 1. In the case of orthogonal perfect fluids and Bianchi class A types different from VIII and IX, the situation was quite well understood already in [64, 83]. On the other hand, [63, 84] gives a good understanding of Bianchi class B orthogonal perfect fluid solutions which are not of type $\mathrm{VI}_{-1/9}$. In particular, [64, 70, 71, 84, 85] yield the C^2-version of strong cosmic censorship for non-stiff Bianchi class A and B spacetimes different from $\mathrm{VI}_{-1/9}$. The stiff fluid case is addressed in [71, 86, 87]. It is reasonable to expect convergent/quiescent solutions to arise from initial data on the singularity; see [57] for a justification of this statement in the Bianchi class A orthogonal stiff fluid setting.

Recollapse. Under certain circumstances, spacetimes are expected to recollapse, meaning that there is a singularity both to the future and to the past. More specifically, if the spatial topology of the spacetime is $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{S}^1$, then the universe is in [88] conjectured to recollapse. This expectation goes under the name of the *closed universe recollapse conjecture*. In [89], the authors prove the closed universe recollapse conjecture in the case of diagonal Bianchi type IX universes satisfying the dominant energy condition and with non-negative average principal pressures. See also [24] for a discussion of this conjecture in more general settings.

3.1.2. *The expanding direction*

Turning to the expanding direction, it is natural to distinguish between non-accelerated expansion and accelerated expansion.

Non-accelerated expansion. For simple matter models with non-negative pressure and satisfying the dominant energy condition, one expects recollapse in the case of Kantowski–Sachs and Bianchi type IX. What remains to be considered is thus Bianchi class A types different from IX as well as Bianchi class B types. The Bianchi class A vacuum setting is discussed in [90–92]. Note that [91, 92] contains a discussion of the relation between the asymptotics and the conjectures formulated in Section 2.2. Since the relevant manifolds in the Bianchi class A setting are Seifert fibred, one expects collapse of the fibres after appropriate expansion normalization, and this is what happens.

It is also of interest to study the future asymptotics of solutions in the presence of matter fields. There is a very extensive literature on this subject. We here give a few examples. A fundamental question in cosmology is if one can deduce isotropy of the universe from the isotropy of the CMB. There is a result of this nature, due to Ehlers, Geren and Sachs, and referred to as the EGS theorem, see [93]. However, the theorem states that if the universe has a perfectly isotropic CMB, then the universe is spatially homogeneous and isotropic. It would be more interesting to know that if the CMB is almost isotropic, then the universe is almost isotropic. Such a result would be referred to as an almost-EGS theorem. However, that such a result cannot hold follows from [94–96] in which the authors study Bianchi type VII_0 solutions; see also [97]. In the above discussions, we have focused on orthogonal perfect fluids. However, it is also of interest to consider tilted perfect fluids; see [98–105] for some examples of results in this setting. For results in the presence of kinetic matter, see, e.g., [82, 106–110].

Accelerated expansion. The fundamental paper [111] on the asymptotics of Bianchi solutions to Einstein's equations with a positive cosmological constant establishes that, with the exception of Bianchi type IX, solutions asymptote to de Sitter space. Concerning the matter, it is only necessary to assume that the stress energy tensor obey the strong and dominant energy conditions. In the case of Bianchi type IX, a similar conclusion holds, assuming the cosmological constant is initially large enough relative to the spatial scalar curvature. This would seem to establish the cosmic no-hair conjecture in the spatially homogeneous setting. However, such a conclusion presupposes that the solution exists globally to the future. In the case of a specific matter model, this has to be established; see, e.g., [112]. After the appearance of observational support for accelerated expansion in the late universe, interest in mechanisms yielding accelerated expansion increased. In the mathematical literature, some corresponding examples of studies are given in [113–115].

3.2. *Spacetimes with a* 2*-dimensional isometry group*

A natural next step is to consider spacetimes with a 2-dimensional group of isometries. In this setting, even the issue of global existence of solutions is non-trivial, and the corresponding problem is a natural starting point. Once global existence has been established, it is then natural to try to analyze the asymptotics, prove strong cosmic censorship in the corresponding symmetry class etc. In what follows, when we speak of strong cosmic censorship, we always have the C^2-version in mind.

3.2.1. *Global existence, strong cosmic censorship*

One of the first examples of global existence for arbitrary initial data (in a symmetry class of the type of interest here), is the one concerning Gowdy spacetimes obtained in [13]. However, it was followed by a large collection of results, proving global existence of geometrically natural foliations for several different matter fields; see, e.g., [116–127]. The symmetry classes studied are Gowdy symmetry, $\mathbb{T}^2$-symmetry and surface symmetry. In the first two cases, the basic symmetry assumption is that there is a compact and connected 2-dimensional Lie group of isometries acting effectively on the set of initial data on a closed 3-manifold. Due to [128, 129], this means that the group has to be $\mathbb{T}^2$. Moreover, the initial hypersurface must be $\mathbb{T}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{S}^3$ or one of the Lens spaces (which means that a covering space of it is $\mathbb{S}^3$, so that we ignore Lens spaces from now on); see [117]. The 2-torus action on the initial hypersurface gives rise to two Killing vector fields of the corresponding development, say X_i, $i = 1, 2$. Then $\mathcal{T}_i := \epsilon_{\alpha\beta\gamma\delta} X_1^\alpha X_2^\beta \nabla^\gamma X_i^\delta$ are the associated *twist constants*, where $\epsilon_{\alpha\beta\gamma\delta}$ are the components of the volume form and ∇ is the Levi-Civita connection of the development. In the vacuum setting, the twist constants are constants, but for general matter models they are functions. The Gowdy setting corresponds to demanding that the twist constants vanish (this happens automatically in the case of $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathbb{S}^3$ spatial topology); see [130] for the original article by Gowdy. The possible topologies (ignoring the Lens spaces) are thus $\mathbb{T}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathbb{S}^3$ in the Gowdy setting. The general case that the twist constants do not vanish is referred to as the $\mathbb{T}^2$-symmetric setting. In this case, the relevant spatial topology is $\mathbb{T}^3$. One can take a slightly more general perspective, insisting on only local Killing vector fields; see [123, 131]. In the surface symmetric setting, the topology of the spacetime is $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{F}$, where $\mathbb{F}$ is a closed 2-manifold. Letting $\mathbb{G}$ denote the universal cover of $\mathbb{F}$, the isometry group of one of the standard metrics on 2-dimensional Euclidean, spherical or hyperbolic space should act on the last factor of $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{G}$ by isometries of the lifted spacetime metric. The surface symmetric setting is only interesting outside of vacuum. Even though it is of course interesting to know that there is a global constant mean curvature foliation, for example, the corresponding results should be understood as a first step to analyzing the asymptotics in general and proving strong cosmic censorship in particular.

Strong cosmic censorship in the Gowdy setting. In [132] (see also [33]), the authors prove strong cosmic censorship for polarized Gowdy spacetimes. The Gowdy spacetimes have a two dimensional group of isometries, but the additional requirement of polarization means that the essential equations that need to be solved are linear. This significantly simplifies the analysis. Nevertheless, there are infinite dimensional sets of initial data leading to maximal Cauchy developments that are extendible. Proving that there is an open and dense subset of initial data whose maximal Cauchy developments are inextendible is therefore non-trivial.

A proof of strong cosmic censorship in the general $\mathbb{T}^3$-Gowdy symmetric vacuum setting can be found in [133–137] (these results can also be used to deduce strong cosmic censorship in the polarized $\mathbb{T}^3$-symmetric Einstein–Maxwell setting, see [138]). In this case, the essential equations can be thought of as wave map equations with hyperbolic space as a target. In particular, the equations are non-linear, and this causes substantial complications and rich dynamics. In particular, spikes appear. These features were first discovered numerically, see below, and later understood analytically, see [139] and [140]. Moreover, they are expected to be important more generally. Nevertheless, it is possible to prove that there is an open and dense set of initial data such that the corresponding maximal Cauchy developments are inextendible. In fact, quite detailed information can be derived concerning the asymptotics; see also [57] for an interpretation of the asymptotics in the direction of the singularity in terms of data on the singularity. Note, however, that strong cosmic censorship has neither been demonstrated in the $\mathbb{S}^2 \times \mathbb{S}^1$ nor in the $\mathbb{S}^3$ vacuum Gowdy-settings.

Strong cosmic censorship in the $\mathbb{T}^2$- and surface symmetric settings. In the vacuum $\mathbb{T}^2$-symmetric setting there are no results concerning strong cosmic censorship. This is due to the expectation that the behavior in the direction of the singularity should be oscillatory and therefore much more complicated; see the discussion of numerical work below for details. However, in the expanding direction there are some partial results as well as numerical simulations, see below. On the other hand, including matter sometimes improves the situation. In order to understand why, it is useful to recall that in the case of Bianchi type IX vacuum spacetimes, e.g., the locally rotationally symmetric solutions are such that the maximal Cauchy development can be extended in inequivalent ways. They are therefore counterexamples to determinism. Due to the existence of these examples, proving strong cosmic censorship in the vacuum Bianchi type IX setting is subtle. Considering, on the other hand, Bianchi type IX solutions with non-trivial orthogonal perfect fluid matter, it can easily be demonstrated that the Ricci tensor contracted with itself blows up in the direction of the singularity for all solutions. Proving strong cosmic censorship in this setting is therefore trivial. In this sense, including matter can simplify the situation. Similarly, the reason proving strong cosmic censorship in the $\mathbb{T}^3$-Gowdy symmetric vacuum setting is non-trivial is that there are counterexamples to extendibility in that setting as well. Results concerning strong cosmic censorship in the surface symmetric setting with collisionless matter can be found in [141], see also [142, 143]. See also [144] for a future inextendibility result which covers, e.g., the Einstein–Vlasov equations in the $\mathbb{T}^2$-symmetric setting, including vacuum. A proof of strong cosmic censorship in the $\mathbb{T}^2$-symmetric Einstein–Vlasov setting with non-trivial Vlasov matter can be found in [145]. Finally, [146] extends this result to the case of a positive cosmological constant. Note, however, that [146] actually also covers the vacuum setting, assuming the area element of the group action does not vanish in the limit (in the direction of the singularity).

One bounce in the $\mathbb{T}^3$-Gowdy symmetric vacuum setting. To the best of our knowledge, there are no results in the oscillatory and spatially inhomogeneous setting. However, Warren Li recently proved a result describing one BKL bounce in the $\mathbb{T}^3$-Gowdy symmetric vacuum setting; see [147].

3.2.2. *The expanding direction*

Beyond proving strong cosmic censorship, it is of interest to derive detailed asymptotics in the expanding direction; in particular, it is of interest to verify the conjectures formulated in Section 2.2. In the case of polarized $\mathbb{T}^3$-Gowdy symmetric vacuum spacetimes, such an analysis can be found in [148, 149]. In fact, the asymptotics derived in [148] are detailed enough that they uniquely determine the solution, see [149]. The reason the analysis is so complete in this setting is that the essential equation is linear in this case. Note also that the conclusions follow as a very special case of the theory developed in [150]. In the case of general vacuum $\mathbb{T}^3$-Gowdy spacetimes, the future asymptotics are analyzed in [133, 151] (see also [138] for corresponding conclusion in the polarized $\mathbb{T}^3$-symmetric Einstein–Maxwell setting). In the $\mathbb{T}^2$-symmetric vacuum setting, there are general results that do, however, not provide very detailed information, see [152]. In the polarized $\mathbb{T}^2$-symmetric vacuum setting, there are results providing more detailed information, assuming one starts out close enough to an asymptotic regime, see [153]. There are also numerical studies of this situation, see below.

Including matter, there are results in the case of the Einstein–Vlasov equations with hyperbolic symmetry, see [126, 154]. In the case of the Einstein–Vlasov equations with surface symmetry and a positive cosmological constant, there are results in [155, 156]. In [25], the authors derive detailed information concerning the future asymptotics in the $\mathbb{T}^3$-Gowdy symmetric Einstein–Vlasov setting. In particular, the authors prove the cosmic no-hair conjecture in this setting.

3.2.3. *Specifying data on the singularity*

There is an extensive literature on specifying data on the singularity for classes of spacetimes with a 2-dimensional group of isometries, see, e.g., [45–51]. It would be interesting to relate these results to Definition 1. This has been done in [57] in the case of $\mathbb{T}^3$-symmetric Gowdy spacetimes. However, it remains to be done for the other cases. Related results concerning isotropic singularities can be found in [157–160].

3.3. *The* U(1)*-symmetric setting*

Finally, it is of interest to discuss the U(1)-symmetric setting; see [161] for a general review.

3.3.1. *Existence results and future asymptotics*

For general ideas concerning the reduction of the equations in the U(1)-symmetric setting, the reader is referred to [162]. The question of local existence, using variables that can be expected to be useful to treat global issues, is discussed in [163, 164]. Global existence is then discussed in [165]. In [166] (see also [167]), the authors prove a small data future global existence result which includes a proof of future causal geodesic completeness; the existence of a CMC foliation covering the causal future of the initial hypersurface; and the conclusion that the geometry asymptotes to that of a model spacetime. This result treats the so-called polarized case. An extension to the non-polarized case is given in [168, 169].

3.3.2. *Data on the singularity*

In the direction of the singularity, it is in the generic vacuum setting natural to expect oscillations, see the below discussion of numerical work. However, in the polarized setting, it is natural to expect quiescent asymptotics. In [53], the authors construct real analytic solutions satisfying these expectations using Fuchsian techniques. See also [52].

4. Future global non-linear stability in the expanding direction

In the cosmological setting, spacelike Cauchy hypersurfaces with mean curvature bounded away from zero appear naturally (in the absence of an asymptotically flat regime, hypersurfaces with vanishing mean curvature are, for many matter models, expected to be surrounded by hypersurfaces with mean curvature bounded away from zero, and hypersurfaces with significant positive mean curvature on some parts and significant negative mean curvature on other parts are unnatural in cosmology for the reasons given in the introduction). For a hypersurface satisfying this condition, the volume form is increasing (or decreasing) everywhere on the hypersurface. Moreover, due to Hawking's singularity theorem, the corresponding spacetime is either past or future timelike geodesically incomplete (assuming appropriate matter content). For these reasons, it is, when studying stability, natural to consider future expanding and future causally geodesically complete spacetimes and past contracting and past causally geodesically incomplete spacetimes in cosmology. When we speak of stability in the expanding direction, we typically assume that the background solution is future causally geodesically complete and has a foliation of the causal future of a Cauchy hypersurface such that each leaf of the foliation has strictly positive mean curvature. It is natural to divide the results that have been obtained according to the rate of the expansion, in particular according to whether the expansion is accelerated or not.

4.1. *Non-accelerated expansion*

In the case of non-accelerated expansion, it is, in accordance with the ideas described in Section 2.2, to be expected that the most favorable situation is when the spatial geometry is close to hyperbolic. Accordingly, the first future global non-linear stability result is the stability of the Milne model in the vacuum setting, see [170]; the metric of the Milne model is $-dt^2 + t^2\gamma$ on $(0,\infty) \times \Sigma$, where (Σ, γ) is a closed hyperbolic 3-manifold. The argument is based on considering energies arising from the Bel-Robinson tensor. This result was later generalized to higher dimensions in [171] using rougher energy estimates, and to different matter models, see, e.g., [172–174]. It is also worth noting that while asymptotic stability is obtained in [170], only stability is obtained in [171], since there in higher dimensions is a non-trivial moduli space of negative Einstein manifolds; this adds an additional complication to the problem.

4.2. *Accelerated expansion*

The first result on future global non-linear stability in the cosmological setting is due to Helmut Friedrich, see [175]. In [175], the author proves, among other things, the global non-linear stability of de Sitter space. The method of proof is to derive a set of conformal field equations with respect to which the global stability problem becomes a local problem. However, this method is based on conformal invariance of the equations. The stability result [175] is extended to the Einstein–Maxwell–Yang–Mills setting in [176]. Similar methods have been used to obtain, e.g., the results [177–179].

The methods developed in [175] apply to specific matter models and specific spacetime dimensions. However, future global non-linear stability is expected to hold more generally. In [180], future global non-linear stability is demonstrated in the Einstein-non-linear scalar field setting. The particular situation considered in [180] is that of a potential with a strictly positive non-degenerate lower bound. However, the methods apply in any spacetime dimension ≥ 4. Moreover, they are based on coordinates similar to those used by Choquet-Bruhat to prove local existence in [6]. However, modifications are necessary; the contracted Christoffel symbols are replaced by appropriately chosen so-called *gauge source functions* and additional correction terms are added to the equations. Using the resulting equations, stability can be demonstrated using a bootstrap argument. Moreover, due to the causal structure of solutions, it is possible to prove stability results that are future global in time given only local assumptions in space. Consequently, it is possible to prove future global non-linear stability results for all spatially locally homogeneous solutions that exhibit de Sitter like expansion at the same time. The results in [180] were later generalized in [181] to the case of an exponential potential using similar methods. After [180] appeared, related results using similar methods (with additional complications due to the matter fields) appeared in, e.g., [25, 182–187]. In most of these references, the authors prove future global non-linear stability of specific spatially locally homogeneous solutions. However, in [25] the authors prove future global non-linear stability of arbitrary $\mathbb{T}^3$-Gowdy symmetric solutions to the Einstein–Vlasov system with a positive cosmological constant. In other words, the background solutions in this case do not only exhibit substantial anisotropies initially, they also exhibit substantial spatial inhomogeneities. Next, [186] contains, beyond stability results, a proof of the fact that for any choice of closed manifold for the spatial topology and any given choice of currently preferred standard model, there is a solution to the Einstein–Vlasov equations with a positive cosmological constant and the prescribed spatial topology such that observers in the corresponding spacetime cannot distinguish between their universe and the prescribed standard model; see [186] for details. In other words, given the currently preferred standard models of the universe, it is not possible to determine the global shape of the universe on the basis of observations.

Later, different methods were developed to prove future global non-linear stability in the presence of a positive cosmological constant, see [188]. In [188], the arguments rely on a conformal rescaling of the metric, but are still based on a particular choice of wave gauge, in analogy with [180]. The advantage of the methods developed in [188] is that the proof becomes substantially shorter. Moreover, the methods apply to perfect fluids with a linear equation of state of the form $p = K\rho$, where $0 < K \leq 1/3$. In other words, as opposed to earlier results concerning perfect fluids, the methods immediately include the radiation case. The methods developed in [188] have then been applied in, e.g., [189–191].

5. Big bang formation in the absence of symmetries

Prior to the work of Rodnianski and Speck, results concerning big bang singularities in the absence of symmetries were restricted to the ones obtained by specifying initial data on the singularity, see Section 2.5. The first stability result was obtained in [192], based on a study of the linearized equations in [193]. In [192], the authors prove that the occurrence of big bang singularities in spatially flat and isotropic solutions to the Einstein-scalar field equations (and to the Einstein-stiff fluid equations) is a past globally non-linearly stable phenomenon. The gauge used in the proof is determined by setting the shift vector field to zero and demanding that the time coordinate t equal $1/\theta$, where θ is the mean curvature of the constant-t hypersurface (we here use the opposite sign convention to that of [192] concerning the mean curvature). However, the geometric objects are expressed with respect to coordinate vector fields. The result [192] was later extended to prove past (future) global non-linear stability of the big bang (big crunch) in the case of spatially homogeneous and isotropic solutions with spherical spatial geometry and scalar field matter, see [194]. In [195], stable big bang formation is obtained in the case of hyperbolic spatial geometry. Moreover, the authors in this case also prove future global non-linear stability. In [196], the authors extend the results of [192] to cover the case of moderate anisotropies. In particular, [196] yields stable big bang formation for solutions to Einstein's vacuum equations with $\mathbb{T}^D$ spatial topology, assuming $D \geq 38$. In [192, 194, 196], the arguments are based on using a CMC foliation. This is very useful since it synchronizes the singularity. On the other hand, the CMC condition leads to an elliptic equation for the lapse function. In this sense, the gauge is non-local. On the other hand, the causal structure is such that particle horizons appear. In other words, two causal curves going into the singularity typically asymptotically lose the ability to communicate. This means that, from a causal point of view, the behavior localizes. It would therefore be of interest to prove a past global non-linear stability result which localizes in the spatial directions. This is done in [197]. In this paper, the authors use a different gauge which makes it possible to localize the analysis. The foliation the authors use in this case is determined by the condition that the scalar field be constant along the leaves of the foliation. In particular, the result requires the presence of a scalar field and therefore does not apply in the vacuum setting. In [198], the authors generalize this result to the presence of a perfect fluid with a linear equation of state $p = c_s^2\rho$, where $1/n < c_s^2 < 1$ and n is the spatial dimension (note that for $n = 3$, the case $0 \leq c_s^2 \leq 1/n$ is expected to be unstable in the direction of the singularity due to [199]). From these results, it is clear that the borderline case of a radiation fluid is of particular importance. It is therefore of interest to note that [200] yields stable big bang formation of the FLRW solutions to the Einstein–Vlasov-scalar field equations (the result simultaneously covers the spatially flat, hyperbolic and spherical settings); note that Vlasov matter behaves as a radiation fluid in the direction of the singularity.

In spite of the interest of the results mentioned above, they do not cover the regime one would naturally expect from, e.g., Definition 1; i.e., that a sufficient condition should be that the eigenvalues, say ℓ_A, of the expansion normalized Weingarten map satisfy $1 + \ell_A - \ell_B - \ell_C > 0$ for $B \neq C$. However, this is remedied in [201]. In this article, the authors prove stable big bang formation for the full range of spatially homogeneous and spatially flat solutions to the Einstein-scalar field equations for which one expects stability (in the polarized $U(1)$-symmetric setting, the conditions can even be relaxed, in accordance with condition (4) in Definition 1). Again, the authors use a CMC gauge with vanishing shift vector field. However, in [201] the authors use a Fermi–Walker propagated frame as opposed to considering the components of the geometry with respect to coordinate vector fields. Moreover, the equations are for the components of the second fundamental form with respect to the Fermi–Walker propagated frame, the connection coefficients of the frame, the components of the frame (and co-frame) with respect to coordinate vector fields, the scalar field and the lapse function. The unknowns and the equations are therefore quite different from the ones in [192]. The advantage of the formulation in [201] is that the variables are such that one can formulate (and improve) quite mild bootstrap assumptions. Since the bootstrap assumptions are not very strong, it is easier to improve them. On the other hand, the conclusion that the bootstrap assumptions are satisfied does not yield that much information. Nevertheless, the information is sufficient to conclude that the eigenvalues of the expansion normalized Weingarten map converge, that the expansion normalized quantities associated with the scalar field converge and that the curvature blows up.

All the results mentioned above are stability results. In other words, the starting point is a given background solution (or family of background solutions). The conclusions of the results are then that perturbing initial data corresponding to the relevant background solution yields solutions that also have big bang singularities. In analogy with Hakwing's theorem, it would be desirable to prove that there is a general condition, which does not refer to a specific background solution, ensuring big bang formation with curvature blow up. Such a result is obtained in [202]. The result is in the Einstein-non-linear scalar field setting and the assumptions involve: bounds on the potential; the expected bounds on the eigenvalues of the expansion normalized Weingarten map; and, for a fixed bound on suitable Sobolev norms on the expansion normalized quantities (introduced in connection with (1)), a corresponding lower bound on the mean curvature (the conditions are such that, in particular, they are satisfied for all solutions with a CMC foliation that induce data on the singularity). The conclusions include past causal geodesic incompleteness and curvature blow up; see [202] for details. Since the results of [202] hold in the case of the Einstein-non-linear scalar field equations, they can be combined with the future global non-linear stability results in this setting, described in Section 4.2, in order to conclude past and future global non-linear stability of a large class of spatially locally homogeneous solutions. In particular, [202] applies in all the Einstein-scalar field settings in which stable big bang results were previously obtained. Many of the methods used in [202] are inspired by those developed in [201]. However, there are also significant differences. In particular, in [201] there is a background solution to prove stability of, and in the main theorem of [202] there is not. One important step in [202] is therefore to construct a substitute for a background solution.

6. Linear equations on cosmological backgrounds

There is a growing body of work concerning linear systems of wave equations on cosmological backgrounds. There is of course a vast physics literature on this subject, but we here focus on mathematical results. Some very early work is contained [203]. Since the central equation in the case of polarized Gowdy is linear, [132, 148, 149] are, simultaneously, studies of linear equations on cosmological backgrounds are directly relevant for the behavior of solutions to Einstein's equations. In [204], the authors study scalar perturbations of the Einstein–Euler equations. Further studies, mainly of the Klein–Gordon equation on spatially homogeneous backgrounds, are contained in, e.g., [85, 205–214]. Most of the results concern the equations on background geometries with convergent asymptotics. However, it is worth noting that [211] contains results in the case that the background geometry has chaotic dynamics. In [150], a general theory for linear systems of wave equations on cosmological backgrounds is developed. However, the geometries have to be such that the resulting equations are separable. The corresponding results cover many, though not all, of the previously mentioned results. For example, geometries which are similar to those of Kasner spacetimes are included (and solutions to systems of wave equations on such backgrounds are analyzed, both as far as past and future asymptotics are concerned). The main goal in [150] is to derive optimal energy estimates. This is done both with and without derivative losses. Moreover, in some situations it is demonstrated that there is a homeomorphism between the regular initial data and asymptotic data. In [215], the case of spatially inhomogeneous backgrounds is considered. In this case, a priori assumptions concerning the geometry in the direction of a crushing singularity are made. Systems of linear wave equations on the corresponding backgrounds are then studied. In particular, optimal energy estimates and expansions of the solutions in the direction of the singularity are derived. However, the assumptions are general enough that the limits of the coefficients of the equations need not be continuous. This means that one has to localize along the causal future of a causal curve going into the singularity in order to derive detailed asymptotics.

In [216], conformal scattering of the Maxwell-scalar field system on de Sitter space is studied. Some aspects of [150] are improved in [217] and then applied to Maxwell's equations on a Kasner background (in the direction of the singularity). As already mentioned, the linearized Einstein-scalar field equations are studied in [193]. A scattering theory for the linearized Einstein-scalar field equations on a generalized Kasner background is developed in [218]; see also [219] for a result on scattering for the wave equation on de Sitter backgrounds in even spacetime dimensions.

7. Numerical work

There is a substantial literature of numerical results concerning cosmological solutions to Einstein's equations. It is not realistic to do it justice here. However, we wish to give some references that can be used as a starting point for those who wish to acquaint themselves with the literature.

7.1. *Results in symmetric settings*

Some of the early numerical work was carried out in [220], concerning Bianchi type IX solutions, and [221], concerning solutions with a 1- or 2-dimensional symmetry group. In fact, the numerical studies of solutions with symmetries ran in parallel with the mathematical studies described in Section 3.2 and 3.3. Moreover, the mathematical studies benefited from the numerical studies and vice versa. In [222], the algorithms in the Bianchi IX setting are improved and in [223] the authors probe the BKL proposal with numerical techniques. More detailed studies of quiescent and oscillatory behavior for cosmological singularities can be found in, e.g., [224–229]. Some studies concerning the future asymptotics in the $\mathbb{T}^2$-symmetric setting are to be found in [230, 231].

The idea that spikes, appearing, e.g., in $\mathbb{T}^3$-symmetric Gowdy spacetimes, occur naturally in the direction of the singularity arose due to numerical studies (the importance of spikes was later confirmed in mathematical results; cf., e.g., [137, 139]); see [140, 225, 232–237] for some references concerning spikes.

7.2. *Results in the absence of symmetries*

By now there is also a substantial literature in the absence of symmetries. Some of the early examples can be found in [238–241]. See [242, 243] for two more recent examples.

Declaration of interests

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