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Review article / *Article de synthèse*

On the regularity problems of Einstein equations

Sur les problèmes de régularité pour les équations d'Einstein

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Abstract. In this survey, we provide a review of recent progresses in the local well-posedness problem of Einstein equations in (3+1)-D with low regularity and its applications.

Résumé. Dans cette enquête, nous rendons compte des progrès récents dans le problème de la bonne pose locale des équations d'Einstein en (3+1)-D avec une faible régularité et ses applications.

Keywords. Cauchy problem, Einstein equations, Local well-posedness.

Mots-clés. Problème de Cauchy, Équations d'Einstein, Bien-posé local.

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1. Introduction

Many fundamental theories of physics are modelled by nonlinear hyperbolic equations, such as general relativity, gas and fluid mechanics, elastodynamics etc. The main theories of these PDEs are on the well-posedness of the solution of the equation with certain initial or boundary conditions, which are usually classified into local-in-time or global-in-time well-posedness. Pioneered by Schauder [1], Choquet-Bruhat [2] and Leray [3], the local-in-time well-posedness of classical solutions for nonlinear hyperbolic equations of multiple spatial dimension were studied by physical methods such as energy methods, pointwise estimates based on Kirchhoff formulas.

In [2], without assuming analyticity for the Cauchy data, Choquet-Bruhat proved local existence result for the following second order hyperbolic system[1]

$$\begin{cases} A^{\mu\nu}(\Phi,\boldsymbol{\partial}\Phi)\partial_\mu\partial_\nu\Phi = F(\Phi,\boldsymbol{\partial}\Phi), \\ \Phi|_{t=0} = \Phi_0, \quad \partial_t\Phi|_{t=0} = \Phi_1 \end{cases} \tag{1.1}$$

where Φ is a vector-valued function, the principal coefficients $A^{\mu\nu}$ and F are smooth functions of their variables. In particular she provided an constructive proof for local existence and uniqueness of solutions for Einstein vacuum equations under wave coordinates. Together with

[1] By default $\partial_0 = \partial_t$ unless specified otherwise. Under the Einstein summation convention, the lowercase Latin indices such as i, j, k start from 1, while the Greek indices such as μ, ν begin from 0. Here, $\boldsymbol{\partial}$ represents both the spatial and time derivatives, while ∂ without a subscript is reserved solely for spatial derivatives.

Geroch, in [4], she also proved that for each initial data set, there is a unique maximal future development.

Consider Einstein vacuum equations

$$\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0 \tag{1.2}$$

where $\mathbf{R}_{\alpha\beta}$ denotes Ricci curvature tensor of the Lorentzian spacetime $(\mathcal{M}, \mathbf{g})$. An initial data set for (1.2) consists of a three dimensional surface Σ_0 together with a Riemmannian metric g and a symmetric 2-tensor k satisfying the constraint equations

$$\begin{cases} \nabla^j k_{ij} - \nabla_j \mathrm{tr} k = 0 \\ R_s - |k|^2 + (\mathrm{Tr} k)^2 = 0 \end{cases} \tag{1.3}$$

where ∇ is the Levi-Civita connection of the metric g on Σ_0, and R_s is the scalar curvature of (Σ, g). For a given initial data set (g, k, Σ_0), solving the Cauchy problem is to find a metric $\mathbf{g}$ satisfying (1.2) and an embedding of Σ_0 in $\mathcal{M}$ such that the metric induced by $\mathbf{g}$ on Σ_0 is g and the 2-tensor k is the second fundamental form of the hypersurface $\Sigma_0 \subset \mathcal{M}$.

With the wave coordinates x^α, the metric takes

$$\mathbf{g} = -n^2 \mathrm{d}t^2 + g_{ij}(\mathrm{d}x^i + v^i \mathrm{d}t)(\mathrm{d}x^j + v^j \mathrm{d}t),$$

where n is the lapse function of $x^0 = t$ and the vector-valued function v^i is the shift of the metric, g_{ij} is the induced Riemannian metric on Σ_t, the level set of t. Since $\mathbf{g}$ is Lorentzian, there is a constant $c > 0$ such that

$$c^{-1}|\xi|^2 < g_{ij}\xi^i\xi^j \le c|\xi|^2, \quad n^2 - |v|_g^2 \ge c.$$

Schematically, with $\Phi = (\mathbf{g}_{\mu\nu})$, under the wave coordinates, the reduced Einstein equation system takes the form

$$\mathbf{g}^{\mu\nu}(\Phi)\partial_\mu\partial_\nu\Phi = \mathcal{N}(\Phi, \boldsymbol{\partial}\Phi). \tag{1.4}$$

$\mathbf{g}^{\mu\nu}$ is the inverse metric of the Lorentz metric $\mathbf{g}$; the function $\mathcal{N} = (\mathcal{N}_{\mu\nu})$ on the right-hand side is smooth on its variables, and $\mathcal{N}(\Phi, \boldsymbol{\partial}\Phi)$ is quadratic in $\boldsymbol{\partial}\Phi$. The method Choquet-Bruhat adopted is based on pointwise estimates and a generalized Kirchhoff formula, which later inspired important works such as the breakdown criterion for solutions of Einstein equations given by Klainerman–Rodnianski in [5–7]. We will go back to this point in Section 5. Meanwhile, the energy method adopted in [1, 3] has become the classical method in studying hyperbolic PDEs.

A key motivation of the work [2] was to extend the Cauchy–Kowalevski theorem to non-analytic Cauchy data, as the assumption of analyticity is meaningless in a physical theory where coordinate changes are only restricted to be sufficiently differentiable. Furthermore, in [2], Choquet-Bruhat noted: “It seemed to me that, for the problems considered by the theory of relativity, it would be interesting to obtain, under the minimal possible amount of assumptions, an existence theorem easy to use, enabling [one] to find properties of the solutions that can be compared with the classical properties of light waves and gravitational potentials, and to have formulas which can be an efficient method of calculating gravitational fields, at least approximately, that correspond to given initial conditions.”

Since the seminal work [2] of Choquet-Bruhat, extensive work has been done on the well-posedness of the quasi-linear wave equation (1.1) in $\mathbb{R}^{n+1}$ for $n \ge 2$, including applications to Einstein-vacuum equations. The assumptions regarding the regularity of the Cauchy data have been significantly relaxed. The classical approach for proving local well-posedness relies on energy methods, Sobolev embeddings, and the classical iteration argument. Constructing solutions with lower regularity data may involve using Strichartz estimates or bilinear estimates (when the equations satisfy the null form condition). These approaches may go beyond physical methods, requiring constructing and controlling a parametrix and employing Fourier analysis, which are challenging to extend to quasilinear equations.

Next, we provide a detailed review of the local well-posedness results for Cauchy problems concerning Einstein equations and similar types of quasilinear wave equations.

The classical local well-posedness result of Hughes–Kato–Marsden [8] in the Sobolev space H^s follows from the energy estimate

$$\|\boldsymbol{\partial}\Phi(t)\|_{H^{s-1}} \le c\|\boldsymbol{\partial}\Phi(0)\|_{H^{s-1}} \cdot \exp\left(\int_0^t \|\boldsymbol{\partial}\Phi(\tau)\|_{L_x^\infty} \mathrm{d}\tau\right), \tag{1.5}$$

as well as the Sobolev embedding and a standard iteration argument. This result holds for any $s > n/2+1$, since $H^s \subset L^\infty$ gives the control of $\|\boldsymbol{\partial}\Phi\|_{L_t^\infty L_x^\infty}$.

Theorem 1 ([8, 9]). *Let (Σ_0, g, k) be an initial data set for the Einstein vacuum equation (1.2) in $(3+1)$ spacetime. Assume Σ_0 can be covered by a locally finite system of coordinate charts, related to each other by C^1 diffeomorphisms, such that $(g,k) \in H^s_{\mathrm{loc}}(\Sigma_0) \times H^{s-1}_{\mathrm{loc}}(\Sigma_0)$ with $s > 5/2$. Then there exists a unique global hyperbolic development $(\mathscr{M}, \mathbf{g})$ verifying (1.2), for which Σ_0 is a Cauchy hypersurface.*

One can refer to [10, pages 304–310] for the classical local existence result for Einstein vacuum equations with maximal foliation, which means the mean curvature of the level set of t is trivial. For Einstein vacuum equations with constant mean curvature foliation, which means the mean curvature of the level set of t equals t, with $t < 0$, by setting on the level set of t the spatial harmonic gauge, Andersson–Moncrief proved the local well-posedness of the solution in [11], for data $(g,k) \in H^s \times H^{s-1}$ with $s > 5/2$.

In pursuit of an existence theorem for the Einstein vacuum equations with minimal regularity assumptions, Klainerman proposed the bounded L^2 curvature conjecture in [12], which, in brief, asserts that the local existence and uniqueness result for the Einstein vacuum equations can be extended to Cauchy data with locally finite L^2 curvature and locally finite L^2 norm of first order covariant derivatives of k (see Theorem 3 for a more precise statement). Additionally, in connection with the global stability of Minkowski spacetime established by Christodoulou and Klainerman in their seminal work [10], Klainerman posed the following problem in [13]:

Problem 1.1 (Strong stability of Minkowski space). *Does there exists a scale invariant smallness condition such that all developments, whose initial data sets (Σ, g, k) verify it, have complete maximal future developments?*

The bounded L^2 curvature conjecture was regarded as a highly challenging problem, a key step toward resolving the above question. Significant progress was made in the early 2000s to lower the regularity assumptions for local well-posedness results of quasilinear wave equations of type (1.4).

To improve the classical results, one key objective is obtaining a better estimate on $\|\boldsymbol{\partial}\Phi\|_{L_t^1 L_x^\infty}$. This boils down to deriving the Strichartz estimates for the wave operator $\mathbf{g}^{\alpha\beta}(\Phi)\partial_\alpha\partial_\beta$ which has rough coefficients, only as smooth as Φ. In this context, for $s < n/2+1$, Strichartz estimates provide a pathway to improving regularity, which is crucial for enhancing classical results. One can refer to [14–28] the works of Smith, Buhari, Chemin, Tataru, Klainerman, Rodnianki, Wang for results of $s < 5/2$ for the quasilinear wave equation of the type of (1.4). In particular Klainerman and Rodnianski gave the $s > 2$ result for Einstein vacuum equation under the wave coordinate gauge in [22–24], which is sharp due to the counter example given by Lindblad–Ettinger [29]. And the result of Smith–Tataru [28][2] and Wang [27] achieve the sharp $s > 2$ result for the general equation, (which takes the form as (1.4), but the metric does not have trivial Ricci curvature, see (2.2)). The counter example for the case of $s = 2$ is given by Lindblad [30].

[2] The result is for the dimension $2 \le n \le 5$, at least sharp in $n = 2, 3$.

Nevertheless, Strichartz estimates are established for the wave operator. When the nonlinearity $\mathcal{N}(\Phi, \boldsymbol{\partial}\Phi)$ exhibits a more delicate structure, better results are expected, which cannot be achieved solely through Strichartz estimates. For instance, for semilinear wave equations in $\mathbb{R}^{3+1}$

$$\Box\phi = \mathcal{N}(\phi, \boldsymbol{\partial}\phi), \quad \phi(0) = \phi_0, \partial_t\phi(0) = \phi_1,$$

using Strichartz estimate, the local-wellposedness was proved for $(\phi_0, \phi_1) \in H^s \times H^{s-1}$ with $s > 2$ by Ponce–Sideris [31] and the result is sharp due to the counter example by Lindblad [32]. However if the quadratic forms are null forms (see (3.23)), the result is improved by Klainerman and Machedon in [33] to $s = 2$ by establishing bilinear estimates.

For geometric wave equations, those are semilinear and exhibiting null conditions, including wave map, Maxwell–Klein–Gordon and Yang–Mills equations in Minkowski space, the study of low-regularity solutions is driven by investigating global-in-time behavior. An important example is the global finite energy solution for Maxwell–Klein–Gordon equations in Minkowski space, established in [34], as well as for Yang–Mills equations in [35]. For both cases, the energy of the curvature (together with the scalar field for Maxwell–Klein–Gordon equations) is conserved over time. In $\mathbb{R}^{3+1}$, Klainerman and Machedon [34, 35] proved in the 1990s that local solutions can be constructed for initial data with finite energy. The conservation of energy allows these local solutions to be extended globally in time. Their results were based on innovative bilinear estimates for null forms, improving classical results by more than half a derivative.

Further progress on well-posedness for lower regularity solutions was made by employing bilinear estimates in the wave-Sobolev space, developed independently by Klainerman–Machedon and Bourgain (see [33, 36, 37]). For instance, the scale-invariant, optimal regularity result on Maxwell–Klein–Gordon equations in $\mathbb{R}^{4+1}$ was achieved by Lührmann–Krieger [38] and Oh–Tataru [39], where global well-posedness for finite energy data was proven; one can refer to Krieger–Tataru [40] and Oh–Tataru [41] for the optimal regularity result in $\mathbb{R}^{4+1}$ for Yang–Mills equations.

While bilinear estimates rely heavily on Fourier analysis and the null form condition, extending these estimates to quasilinear wave equations poses a fundamental challenge. In the case of Einstein equations with maximal foliation, by using the Yang–Mills formalism and adopting the spatial Coulomb gauge, the equations exhibit the structure of null forms. The $s = 2$ result for Einstein vacuum equations is established by Klainerman, Rodnianski and Szeftel in [42] and [43–46] by taking advantage of this structure, which is so far the best regularity result for Einstein vacuum equations. The counter example in [29] implies that, under wave coordinate gauge, the so-called weak null form observed by Lindblad–Rodnianski [47] is insufficient to yield the result of the bounded L^2 curvature conjecture. In [42], it is suggested that, although the result is not optimal with respect to the standard scaling of the Einstein equation, this result may be sharp due to "null scaling", which is related to establishing a lower bound on the null radius of injectivity. This bound is crucial for their parametrix construction and establishing bilinear and trilinear estimates. To control the lower bound, it relies on a series of sharp trace estimates along null hypersurfaces. We will review this set of estimates in Sections 4 and 5.

In Section 2, we review the $s > 2$ result for the Einstein vacuum equations and similar types of equations. Section 3 provides a brief overview of work on resolving the bounded L^2 curvature conjecture. In Section 4, we discuss methods for controlling causal geometry in rough Einstein spacetimes. Finally, as applications of controlling null hypersurfaces with limited regularity, we review works on breakdown criteria for the Einstein equations in Section 5.

2. Rough solutions for Einstein equations

In the section, we review the result of Klainerman and Rodnianski in [22–24].

Theorem 2. *Consider the classical solution of (1.4), the reduced Einstein equation under the wave coordinates. Suppose* $\mathbf{g}(0)$ *is a continuous Lorentz metric, and* $\sup_{|x|=r}|\mathbf{g}_{\alpha\beta}(0)-\mathbf{m}_{\alpha\beta}|\to 0$ *as* $r\to\infty$*. The time* T *of the existence depends only on the size of the form* $\|\boldsymbol{\partial}\mathbf{g}_{\mu\nu}(0)\|_{H^{s-1}}$ *for any fixed* $s>2$.

One can refer to the counter example given in [29], which shows that Theorem 2 is sharp. The above result is extended to Einstein vacuum equation with CMC spatial harmonic gauge by Wang in [25, 26].

Since one can approximate a given H^s initial data set for the Einstein vacuum equations by classical initial data sets, i.e. $H^{s'}$ data sets with $s'>5/2$ for which the classical solutions exist and unique. The above theorem allows one to pass to the limit and derive existence of solutions for the given, rough, initial data set. In particular, one can refer to the result of Maxwell (which is for $s>3/2$) in [48] for justifying such approximation for the solutions of the constraint equations at the initial slice.

Recall the energy estimates for the solution of (1.4) in $n+1$ spacetime. To achieve the result for $s>2$, one need to control $\exp(\int_0^t\|\boldsymbol{\partial}\Phi(\tau)\|_{L_x^\infty})$. In terms of Sobolev embedding

$$\|\boldsymbol{\partial}\Phi\|_{L_x^\infty}\lesssim\|\boldsymbol{\partial}\Phi\|_{H_x^s},\quad s>\frac{n}{2}$$

which is consistent with the classical result. For the solution of the linear wave equation $\Box\phi=0$ in Minkowski spacetime $\mathbb{R}^{n+1}$, there holds the Strichartz estimate

$$\|\boldsymbol{\partial}\phi\|_{L^2_{[0,T]}L_x^\infty}\le cT^\sigma(\|\phi_0\|_{H^{\frac{n}{2}+\frac{1}{2}+\sigma}}+\|\phi_1\|_{H^{\frac{n}{2}-\frac{1}{2}+\sigma}})\tag{2.1}$$

with $\sigma>0$ arbitrarily small.

Consider the following quasilinear wave equation, similar in form to (1.4),

$$\begin{cases}-\partial_t^2\phi+g^{ij}(\phi)\partial_i\partial_j\phi=\mathcal{N}(\phi,\boldsymbol{\partial}\phi),\\ \phi|_{t=0}=\phi_0,\quad \partial_t\phi|_{t=0}=\phi_1\end{cases}\tag{2.2}$$

with g^{ij} bounded, uniformly elliptic and smooth about its variable. To achieve a better result than the classical one, it is important to recover the Strichartz estimate in (2.1) in the rough, curved spacetime with the metric $\mathbf{g}=-\mathrm{d}t^2+g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ depending on the solution. Strichartz estimate for wave equations with rough coefficients was first studied by Smith [14]. The breakthrough was then made by Bahouri–Chemin [18, 19] and by Tataru [15] using parametrix constructions. By establishing a Strichartz estimate for solutions to the linearized equation $-\partial_t^2\phi+g^{ij}(\phi)\partial_i\partial_j\phi=0$ of the form

$$\|\boldsymbol{\partial}\phi\|_{L_t^2L_x^\infty}\le c\left(\|\phi_0\|_{H^{2+\alpha}}+\|\phi_1\|_{H^{1+\alpha}}\right)$$

with a loss of $\alpha>1/4$, they obtained the well-posedness of (2.2) in H^s with $s>2+1/4$. This result was later improved to $s>2+1/6$ in [17].

An important next step was taken in [20] and [21], where in [20] Klainerman introduced the commuting vectorfields approach for Strichartz estimates, with which he reproduced the $s>2+1/6$ result. In particular he pointed out the regularity of null hypersurface can be improved by adopting the following decomposition for $\mathbf{R}_{LL}$, the component of Ricci curvature of the metric $\mathbf{g}$ contracted by the null vector-field L,

$$\mathbf{R}_{LL}=LP-\tfrac{1}{2}L^\mu L^\nu\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\mathbf{g}_{\mu\nu}+E\tag{2.3}$$

with P roughly $\mathbf{g}\cdot\boldsymbol{\partial}\mathbf{g}$ and $E=\mathbf{g}(\boldsymbol{\partial}\mathbf{g})^2$. This important geometric treatment introduced in [20] is crucially used in [21]. In [21], blended with the paradifferential calculus ideas initiated in [19], [16] and [17], Klainerman and Rodnianski improved the local well-posedness of (2.2) in $\mathbb{R}^{1+3}$ to data in the Sobolev space H^s with $s>2+(2-\sqrt{3})/2$. According to the counter-examples in [30], one can only expect to obtain the local well-posedness in H^s with $s>2$. Note that the metric of (2.2) is more general in the sense that it does not verify Einstein vacuum equation (1.2).

For (2.2), the optimal $s > 2$ result is achieved in [28] and [27], both of which take advantage of the decomposition (2.3) to control the causal geometry.

Next we briefly summarize the main reduction steps for the proof of Theorem 2.

To prove Theorem 2, under the bootstrap assumption

$$\|\boldsymbol{\partial}\Phi\|_{L^2_{[0,T]}L^\infty_x} \le 1, \tag{2.4}$$

one needs to prove the energy estimate and Strichartz estimate below

$$\|\boldsymbol{\partial}\Phi\|_{L^\infty_{[0,T]}H^{1+\gamma}} \lesssim \|\boldsymbol{\partial}\Phi(0)\|_{H^{1+\gamma}}, \quad \|\boldsymbol{\partial}\Phi\|_{L^2_{[0,T]}L^\infty_x} \le CT^\delta, \tag{2.5}$$

with some $\delta > 0$, $C > 0$ a universal constant depending on the bound of $\|\boldsymbol{\partial}\Phi(0)\|_{H^{1+\gamma}}$. The energy estimate follows from (1.5) by using (2.4). Assuming the Strichartz estimate in (2.5), with T sufficiently small, (2.4) is improved.

Step 1. Reduction to dyadic Strichartz estimate. To prove the Strichartz estimate in (2.5), it suffices to prove

$$\|P_\lambda \boldsymbol{\partial}\Phi\|_{L^2_{[0,T]}L^\infty_x} \lesssim c_\lambda T^\delta \|\boldsymbol{\partial}\Phi(0)\|_{H^{1+\gamma}} \tag{2.6}$$

where P_λ is the Littlewood–Paley projection with the frequency $\lambda \in 2^{\mathbb{Z}}$, and $\sum_\lambda c_\lambda \le 1$. In fact, it suffices to focus on a fixed dyadic frequency with $\lambda > \Lambda_0$ with Λ_0 sufficiently large.

With ϵ_0 fixed and $0 < \epsilon_0 < \gamma/5$, divide $[0,T]$ into subintervals I with size $\approx T\lambda^{-8\epsilon_0}$. On an interval I consider the linearized equation

$$\mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha\partial_\beta\psi = 0 \tag{2.7}$$

where $\mathbf{g}^{\alpha\beta}_{<\lambda} = \sum_{0<\mu\le 2^{-M_0}\lambda} P_\mu \mathbf{g}^{\alpha\beta}$, with $M_0 > 0$ a fixed large constant.

For the frequency localized data, satisfying

$$(2^{-10}\lambda)^m \|\boldsymbol{\partial}\psi(0)\|_{L^2_x} \le \|\nabla^m \boldsymbol{\partial}\psi(0)\|_{L^2_x} \le (2^{10}\lambda)^m \|\boldsymbol{\partial}\psi(0)\|_{L^2_x}$$

prove that with $\delta > 0$ such that $5\epsilon_0 + \delta < \gamma$ there holds

$$\|P_\lambda \boldsymbol{\partial}\psi\|_{L^2_I L^\infty_x} \lesssim |I|^\delta \|\boldsymbol{\partial}\psi(0)\|_{\dot H^{1+\delta}}. \tag{2.8}$$

Step 2. Rescaling. With the metric

$$H_{(\lambda)}(t,x) = \mathbf{g}_{<\lambda}(\lambda^{-1}t, \lambda^{-1}x)$$

consider the rescaled equation of (2.7)

$$H^{\alpha\beta}_{(\lambda)}(t,x)\partial_\alpha\partial_\beta\psi = 0$$

in the region $[0,t_*] \times \mathbb{R}^3$, where $t_* \le \lambda^{1-8\epsilon_0}$. Then (2.8) can be obtained by showing

$$\|P\boldsymbol{\partial}\psi\|_{L^2_{[0,t_*]}L^\infty_x} \lesssim t_*^\delta \|\boldsymbol{\partial}\psi(0)\|_{L^2_x},$$

where $P = P_1$.

Step 3. Reduction to an $L^1 - L^\infty$ decay estimate. By running a TT^* type argument, the proof of the dyadic strichartz estimate in Step 2 can be further reduced to an $L^1 - L^\infty$ decay estimate. Let ψ be the solution of

$$\Box_{H_{(\lambda)}}\psi = 0 \tag{2.9}$$

with data at $t = t_0 \in [0, t_*]$. Prove

$$\|P\boldsymbol{\partial}\psi\|_{L^\infty_x} \lesssim \left(\frac{1}{(1+|t-t_0|)^{1-\delta}} + d(t)\right)\sum_{k=0}^m \|\nabla^k \boldsymbol{\partial}\psi(t_0)\|_{L^1_x} \tag{2.10}$$

for some integer $m > 0$, where $\delta > 0$ is sufficiently small, and $t_*^{1/q}\|d\|_{L^q([0,t_*])} \lesssim 1$ for some $q > 2$ sufficiently close to 2.

Step 4. Reduction to a localized L^2-L^∞ estimate. The next step is to further localize in space and reduce (2.10) to the following result

$$\|P\boldsymbol{\partial}\psi(t)\|_{L_x^\infty} \lesssim \left(\frac{1}{(1+|t-t_0|)^{2/q}}+d(t)\right)\sum_{k=0}^{m-2}\|\nabla^k\boldsymbol{\partial}\psi(t_0)\|_{L_x^2}, \tag{2.11}$$

where ψ is the solution of (2.9) and with data supported within a geodesic ball of radius 1/2, centered at the origin at Σ_{t_0}.

In the spacetime slab $[0,t_*]\times\mathbb{R}^3$, let u be the optical function satisfying $H_{(\lambda)}^{\alpha\beta}\partial_\alpha u\partial_b u=0$, with initial condition properly chosen, and $\underline{u}=2t-u$. Let

$$L'=-H_{(\lambda)}^{\alpha\beta}\partial_\beta u\partial_\alpha,\quad \mathbf{b}^{-1}=-\langle L',\mathbf{T}\rangle=\mathbf{T}(u).$$

Define the canonical null pair L and $\underline{L}$ by

$$L=\mathbf{b}L',\quad \underline{L}=2\mathbf{T}-L,$$

where $\mathbf{T}$ is the future-directed time-like unit normal of Σ_t in the spacetime. Let

$$Q_{\alpha\beta}[f]=\partial_\alpha f\partial_\beta f-\tfrac{1}{2}H_{(\lambda)\alpha\beta}H_{(\lambda)}^{\mu\nu}\partial_\mu f\partial_\nu f.$$

It is then reduced to bound the conformal energy, defined with the energy density constructed by contracting the standard current $\mathscr{P}^\alpha[\psi]=Q_{\alpha\beta}[\psi]\mathbf{T}^\beta$ with the Morawetz vector field $K=(1/2)n(u^2\underline{L}+\underline{u}^2L)$, followed with a proper normalization.

To control the propagation of such energy, it is important to control the deformation tensor of K, defined by ${}^{(k)}\pi_{\alpha\beta}=(\mathbf{D}_\alpha K)_\beta+(\mathbf{D}_\beta K)_\alpha$. As a result of rescaling, the original spacetime slab stretches by the high frequency $\lambda>\Lambda_0$, so the analysis is carried out in a spacetime with a metric close to Minkowski. The derivatives of the mollified metric $H_{(\lambda)}$ are bounded by powers of the frequency λ. Since K is conformal killing in Minkowskian space, it is expected to be nearly conformal killing in the spacetime slab with the metric $H_{(\lambda)}$, with the error displaying smallness in the form of negative powers of λ.

The level surfaces of u, denoted by $\mathscr{H}_u$, are the outgoing lightcones. Using an arbitrary orthonormal frame $(e_A)_{A=1}^2$ on $S_{t,u}=\Sigma_t\cap\mathscr{H}_u$, define the connection coefficients

$$\begin{aligned}\chi_{AB}&=\langle\mathbf{D}_A L,e_B\rangle,\quad \underline{\chi}_{AB}=\langle\mathbf{D}_A\underline{L},e_B\rangle\\ \zeta_A&=\tfrac{1}{2}\langle\mathbf{D}_{\underline{L}}L,e_A\rangle,\quad \underline{\eta}_A=\tfrac{1}{2}\langle\mathbf{D}_L\underline{L},e_A\rangle.\end{aligned} \tag{2.12}$$

It is necessary to obtain sufficient decay in terms of λ for $\mathrm{tr}\chi-2/n(t-u),\hat{\chi},\zeta,\underline{\eta}$ and for the first order derivatives of $\mathrm{tr}\chi$. This can be accomplished by using a set of null structure equations, such as the Raychaudhuri equation:

$$L\mathrm{tr}\chi+\tfrac{1}{2}(\mathrm{tr}\chi)^2=-|\hat{\chi}|^2-\mathbf{R}_{LL}(H_{(\lambda)})+\cdots$$

However, applying (2.3) with metric $H_{(\lambda)}$ is insufficient to give the desired bound on the first order derivatives of $\mathrm{tr}\chi$. To improve the result of Klainerman–Rodnianski in [21] to the $s>2$ result in Theorem 2, it is crucial to use the Einstein vacuum equation (1.2) to achieve sufficient decay for the connection coefficients. To this end, it is shown in [24] that the Ricci tensor of $H_{(\lambda)}$ and its derivatives are close to zero in a certain sense, given that the original metric is Einstein vacuum.

To avoid establishing the delicate comparison estimates in [24], [25] adapts the reduction procedure to bound the conformal energy of the lowest order within the domain of influence of a unit ball in the original metric. The difficulty is reduced to obtain the necessary control of the connection coefficients in the rough Einstein spacetime. This set of estimates is derived by directly taking advantage of the geometric structure of Einstein vacuum spacetime in [26]. Additional challenges arise in the CMCSH gauge, where certain estimates, such as the time derivative of the shift of the metric cannot be obtained. It is crucial in particular for providing sufficient control of the null cones in the rough spacetime. This issue is circumvented by normalizing the basic energy current.

The adapted reduction procedure is used in [27] to prove the sharp $s > 2$ result for the general equation (2.2). In [27] the construction of conformal energy is further adapted by using the method of Dafermos–Rodnianski in [49], which additionally achieves the crucial conformal flux. The added control relaxes slightly the requirements on the causal geometry in the rough spacetime. By using the conformal invariance of the null hypersurface, introduce a conformal change of the spacetime metric with a well chosen conformal factor. Then with respect to the normalized metric, the required control on the causal geometry to bound the newly constructed conformal energy can be obtained in the rough spacetime. In [50], Wang further adapts the methods in [25,27] to give the low-regularity well-posedness result for compressible Euler equation in 3-D.

3. The bounded L^2 curvature conjecture for Einstein vacuum equations

The bounded L^2 curvature conjecture for Einstein vacuum equations in $(3+1)$ spacetime is resolved in [42] and [43–46]. The conjecture states that

Theorem 3 ([42]). *Let $(\mathcal{M}, \mathbf{g})$ be an asymptotically flat solution to the Einstein vacuum equations (1.2) together with a maximal foliation by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Assume that the initial slice $(\Sigma_0, \mathbf{g}, k)$ is such that the Ricci curvature $Ric, \nabla k \in L^2(\Sigma_0)$ and Σ_0 has a strictly positive volume radius on scales ≤ 1, i.e. $r_{\text{vol}}(\Sigma_0, 1) > 0$. Then, there exists a time $T = T(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{\text{vol}}(\Sigma_0, 1)) > 0$ and a constant $C = C(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{\text{vol}}(\Sigma_0, 1)) > 0$ such that the following control holds on $0 \le t \le T$:*

$$\|\mathbf{R}, \nabla k\|_{L^\infty_{[0,T]} L^2_{\Sigma_t}} \le C, \quad \textit{and} \quad \inf_{0 \le t \le T} r_{\text{vol}}(\Sigma_t, 1) \ge 1/C.$$

The above result can be regarded as a continuation principle for Einstein vacuum equation under the maximal foliation gauge. Moreover, the above result is used by Czimek and Graf in [51] to give the low regularity existence result for the spacelike-characteristic initial data problem in Einstein spacetime.

Next we summarize the main idea and the main steps of the proof.

3.1. *The Yang–Mills formalism*

Let $\{e_\alpha\}_{\alpha=0}^3$ be an orthonormal frame on $\mathcal{M}$, i.e.

$$\mathbf{g}(e_\alpha, e_\beta) = \mathbf{m}_{\alpha\beta} = \text{diag}(-1, 1, \dots, 1),$$

with $e_0 = \mathbf{T}$. Consistent with Cartan formalism, define the connection 1-form

$$\mathbf{A}_{\alpha\beta}(X) = \mathbf{g}(\mathbf{D}_X e_\beta, e_\alpha)$$

where X is an arbitrary vectorfield in $T\mathcal{M}$. Then

$$\mathbf{R}(e_\alpha, e_\beta, \partial_\mu, \partial_\nu) = \partial_\mu(\mathbf{A}_\nu)_{\alpha\beta} - \partial_\nu(\mathbf{A}_\mu)_{\alpha\beta} + [\mathbf{A}_\mu, \mathbf{A}_\nu]_{\alpha\beta} \tag{3.1}$$

where

$$([\mathbf{A}_\mu, \mathbf{A}_\nu])_{\alpha\beta} = (\mathbf{A}_\mu)_\alpha{}^\gamma(\mathbf{A}_\nu)_{\gamma\beta} - (\mathbf{A}_\nu)_\alpha^\gamma(\mathbf{A}_\mu)_{\gamma\beta}.$$

Denote

$$(\mathbf{F}_{\mu\nu})_{\alpha\beta} = \mathbf{R}_{\alpha\beta\mu\nu}.$$

Using (1.2), it follows from the Bianchi identity that

$$\mathbf{D}^\mu \mathbf{F}_{\mu\nu} + [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] = 0.$$

It then follows by using the above identity, (1.2), (3.1) and the fact that $\partial_\mu(\mathbf{A}_\nu) - \partial_\nu(\mathbf{A}_\mu) = \mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu$ that

$$\square_{\mathbf{g}} \mathbf{A}_\nu - \mathbf{D}_\nu(\mathbf{D}^\mu \mathbf{A}_\mu) = \mathbf{J}_\nu \tag{3.2}$$

where

$$\mathbf{J}_\nu = \mathbf{D}^\mu([\mathbf{A}_\mu, \mathbf{A}_\nu]) - [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}],$$

and it is direct to check that $\mathbf{D}_\mu \mathbf{J}^\mu = 0$.

Let $A_0 = \mathbf{A}_\mu e_0^\mu$ and $A_i = \mathbf{A}_\mu e_i^\mu$. It is straightforward to see that

$$\begin{aligned}(A_i)_{0j} &= (A_j)_{0i} = -k_{ij}, \quad i,j = 1,2,3\\ (A_0)_{0i} &= -n^{-1}\nabla_i n, \quad i = 1,2,3.\end{aligned}$$

Note that there is freedom to choose a frame e_1, e_2, e_3 such that the corresponding connection A satisfies the Coulomb gauge condition (see [42, Lemma 4.3])

$$\nabla^i (A_i)_{jk} = 0.$$

With $\boldsymbol{\partial} = (\partial_0, \partial)$[3] and $\mathbf{A} = (A_0, A_i)$, (3.2) is then reduced to

$$\Delta A_0 = \mathbf{A}\boldsymbol{\partial}\mathbf{A} + \mathbf{A}\partial A_0 + \mathbf{A}^3 \tag{3.3}$$

$$\Box_{\mathbf{g}} A_i + \partial_i(\partial_0 A_0) = A^j \partial_j A_i + A^j \partial_i A_j + A_0 \boldsymbol{\partial}\mathbf{A} + A\boldsymbol{\partial} A_0 + \mathbf{A}^3. \tag{3.4}$$

The spatial derivative estimates of A_0 can be obtained by using elliptic estimates and (3.3). It is more important to control $A = A_i$, for which one may need to employ the wave equation (3.4). For this purpose, one needs eliminate the term $\partial_i(\partial_0 A_0)$ in (3.4), which would be done by projecting the equation onto divergence free vector-fields with the help of a non-local operator if following the treatment in [35]. However, adapting this approach to the Einstein vacuum equations is too complicated. In [42], by introducing

$$B_i = (-\Delta)^{-1}(\mathrm{curl}\,(A)_i), \tag{3.5}$$

instead of directly using (3.4) to control A, the problem reduces to controlling ∂B, based on the comparison estimate [42, Lemma 6.5]

$$A = \mathrm{curl}\, B + E \tag{3.6}$$

where E is a controllable error. In [42], the authors rely on $\Box_{\mathbf{g}} B$ to obtain derivative estimates of B.

3.2. *Bilinear and Trilinear estimates*

The proof of Theorem 3 relies on bilinear and trilinear estimates, which will be outlined in this subsection.

The first step is to reduce the proof of Theorem 3 to a small data problem (see [42, Section 2.3]). Under the assumption in Theorem 3, one can use [52, 53] to obtain the lower bound of harmonic radius depending on $\|R\|_{L^2(\Sigma_0)}$ and the lower bound of initial volume radius. Within the geodesic ball of radius no more than the lower bound, there are coordinates with respect to which the metric is comparable to Euclidean metric and is bounded in H^2. Using this result, by localizing, rescaling and adapting the gluing process in [54, 55], Theorem 3 is reduced to

Theorem 4 ([42, Theorem 2.10]). *Let $(\mathcal{M}, \mathbf{g})$ be an asymptotically flat solution to the Einstein vacuum equations (1.2) together with a maximal foliation by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Assume that the initial slice (Σ_0, g, k) is such that:*

$$\|R\|_{L^2(\Sigma_0)} \le \varepsilon, \quad \|k, \nabla k\|_{L^2(\Sigma_0)} \le \varepsilon, \quad r_{\mathrm{vol}}(\Sigma_0, 1) \ge \tfrac{1}{2},$$

then there exists a small universal constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, the following control holds on $0 \le t \le 1$

$$\|\mathbf{R}, k, \nabla k\|_{L^\infty_{[0,1]} L^2(\Sigma_t)} \lesssim \varepsilon, \quad r_{\mathrm{vol}}(\Sigma_t, 1) \ge \tfrac{1}{4}.$$

[3] $\partial_0 = \partial_{e_0}, \partial_i = \partial_{e_i}$, and denote for simplicity $\partial = \partial_i$.

Theorem 4 is proved by using a bootstrap argument.

Assumption 3.1. *Let $M \geq 1$ be a large enough constant, to be chosen later in terms only of universal constants. By choosing $\varepsilon > 0$ small, $M\varepsilon$ can be small enough.*

Assume in the spacetime slab $\cup_{t\in[0,1]}\Sigma_t$,

$$\|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} \leq M\varepsilon, \quad \|\mathbf{R}\cdot L\|_{L^2(\mathscr{H})} \leq M\varepsilon \tag{3.7}$$

$$\|A\|_{L_t^\infty L^2(\Sigma_t)} + \|\boldsymbol{\partial} A\|_{L_t^\infty L^2(\Sigma_t)} \leq M\varepsilon \tag{3.8}$$

where $\mathscr{H}$ denotes the null hypersurface in the spacetime slab, with future directed normal L, normalized by $\langle L, \mathbf{T}\rangle = -1$; assume the following bilinear and trilinear estimates,

$$\|k_{ij}\partial^i\phi\|_{L^2(\mathscr{M})} \leq M^2\varepsilon \sup_{\mathscr{H}} \|\nabla\phi\|_{L^2(\mathscr{H})} + M\varepsilon\|\partial\phi\|_{L_t^\infty L^2(\Sigma_t)} \tag{3.9}$$

$$\left|\int_{\mathscr{M}} Q_{ij\gamma\delta}k^{ij}e_0^\gamma e_0^\delta \mathrm{d}\mathscr{M}\right| \leq \mathscr{M}^4\varepsilon^3 \tag{3.10}$$

with $Q_{ij\gamma\delta}$ components of the Bel–Robinson tensor; and assume a set of estimates on A_0 and its derivatives in the spacetime slab. (See the full set of bootstrap assumptions in [42, Section 5.3].)

Next we follow the steps given in [42] to improve the bootstrap assumptions.

3.2.1. *Improvement of (3.7)*

Under the bootstrap assumptions, one can obtain by elliptic estimates that

$$\|n-1, \nabla n\|_{L^\infty(\mathscr{M})} \lesssim M\varepsilon. \tag{3.11}$$

To improve (3.7), one may use the Bel–Robinson tensor

$$Q_{\alpha\beta\gamma\delta} = \mathbf{R}_\alpha{}^\lambda{}_\gamma{}^\delta \mathbf{R}_{\beta\lambda\delta\sigma} + {}^*\mathbf{R}_\alpha{}^\lambda{}_\gamma{}^{\sigma *}\mathbf{R}_{\beta\lambda\delta\sigma}.$$

Let $P_\alpha = Q_{\alpha\beta\gamma\delta}e_0^\beta e_0^\gamma e_0^\delta$. Then

$$\mathbf{D}^\alpha P_\alpha = 3Q_{\alpha\beta\gamma\delta}\pi^{\alpha\beta}e_0^\gamma e_0^\delta, \tag{3.12}$$

where $\pi_{\alpha\beta} = \mathbf{D}_\alpha\mathbf{T}_\beta + \mathbf{D}_\beta\mathbf{T}_\alpha$. With $h_{\alpha\beta} = \mathbf{g}_{\alpha\beta} + 2(e_0)_\alpha(e_0)_\beta$, define the norm $|\cdot|$ for a spacetime tensor U by

$$|U|^2 = U_{\alpha_1\cdots\alpha_k}U_{\alpha_1'\cdots\alpha_k'}h^{\alpha_1\alpha_1'}\cdots h^{\alpha_k\alpha_k'}.$$

Note the following standard property of Bel–Robinson tensor

$$P_\alpha e_0^\alpha = |\mathbf{R}|^2, \quad P_\alpha L^\alpha = |\mathbf{R}\cdot L|^2.$$

Integrating (3.12) over a spacetime region, bounded by Σ_0, Σ_t and $\mathscr{H}$, it is direct to obtain

$$\int_{\Sigma_t}|\mathbf{R}|^2 + \int_{\mathscr{H}}|\mathbf{R}\cdot L|^2 \lesssim \|\mathbf{R}\|^2_{L^2(\Sigma_0)} + \left|\int_{\mathscr{M}} Q_{\alpha\beta\gamma\delta}\pi^{\alpha\beta}e_0^\gamma e_0^\delta\right| \lesssim \varepsilon^2 + \left|\int_{\mathscr{M}} Q_{\alpha\beta\gamma\delta}\pi^{\alpha\beta}e_0^\gamma e_0^\delta\right|. \tag{3.13}$$

Noting that the nontrivial components of $\pi_{\alpha\beta}$ are

$$\pi_{ij} = -2k_{ij}, \quad \pi_{0i} = n^{-1}\nabla_i n, \mathrm{tr}k = 0, \tag{3.14}$$

the last term on the right-hand side of (3.13) can be treated by using (3.10) and (3.11)

$$\begin{aligned}\left|\int_{\mathscr{M}} Q_{\alpha\beta\gamma\delta}\pi^{\alpha\beta}e_0^\gamma e_0^\delta\right| &\lesssim \left|\int_{\mathscr{M}} Q_{ij\gamma\delta}k^{ij}e_0^\gamma e_0^\delta\right| + \left|\int_{\mathscr{M}} Q_{0i\gamma\delta}n^{-1}\nabla_i n e_0^\gamma e_0^\delta\right| \\ &\lesssim M^4\varepsilon^3 + \|\nabla n\|_{L^\infty}\|\mathbf{R}\|^2_{L_t^\infty L^2(\Sigma_t)} \\ &\lesssim M^4\varepsilon^3 + (M\varepsilon)^3. \end{aligned} \tag{3.15}$$

Substituting the above estimate into (3.13) yields

$$\int_{\Sigma_t}|\mathbf{R}|^2 + \int_{\mathscr{H}}|\mathbf{R}\cdot L|^2 \lesssim \varepsilon^2 + M^4\varepsilon^3 + (M\varepsilon)^3. \tag{3.16}$$

Here we remark that schematically

$$\left|\int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta\right| \approx \left|\int_{\mathcal{M}} k\mathbf{R}^2\right|.$$

With $\|k\|_{L_t^1 L_x^\infty}$ bounded, we can control the right-hand side by

$$\left|\int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta\right| \lesssim \|k\|_{L_t^1 L_x^\infty} \|\mathbf{R}\|^2_{L_t^\infty L^2(\Sigma_t)}.$$

Nevertheless one would not expect to control $\|k\|_{L_t^1 L_x^\infty}$, since this requires the initial data to satisfy $(g,k) \in H^{2+\epsilon}(\Sigma_0) \times H^{1+\epsilon}(\Sigma_0)$. Therefore it is crucial to rely on the assumption (3.10) instead.

3.2.2. *Improvement of (3.8)*

Next we discuss the improvement over (3.8), which is reduced to control derivatives of B, based on the following result

Proposition 5 ([42, Proposition 4.4, Lemma 6.5 and Section 11.1.1]). *The error E in (3.6) satisfies*

$$\|\partial E\|_{L_t^\infty L^2(\Sigma_t)} + \|E\|_{L_t^2 L^\infty(\Sigma_t)} \lesssim M^2\varepsilon^2. \tag{3.17}$$

There hold

$$\Box_{\mathbf{g}} B = F \tag{3.18}$$

with

$$F = (-\Delta)^{-1}[\Box_{\mathbf{g}}, \Delta]B + (-\Delta)^{-1}\Box_{\mathbf{g}}(\operatorname{curl} A),$$

and

$$\|A, \partial B\|_{L^2(\Sigma_0)} + \|\partial(\boldsymbol{\partial} B)\|_{L^2(\Sigma_0)} \lesssim \varepsilon, \quad \|\partial F\|_{L^2(\mathcal{M})} \lesssim M^2\varepsilon^2. \tag{3.19}$$

We will treat (3.18) by using the following set of estimates

Lemma 6. *Let F be a scalar function on $\mathcal{M}$, and let ϕ_0 and ϕ_1 be two scalar functions on Σ_0. Let ϕ be the solution of the following wave equation on $\mathcal{M}$:*

$$\begin{cases} \Box_{\mathbf{g}}\phi = F, \\ \phi|_{\Sigma_0} = \phi_0, \quad \partial_t\phi|_{\Sigma_0} = \phi_1. \end{cases} \tag{3.20}$$

Then ϕ satisfies the following energy estimate

$$\begin{aligned} \|\boldsymbol{\partial}\phi\|_{L_t^\infty L^2(\Sigma_t)} &+ \sup_{\mathcal{H}}(\|\nabla\!\!\!/\phi\|_{L^2(\mathcal{H})} + \|L\phi\|_{L^2(\mathcal{H})}) \\ &\lesssim \|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)} + \|F\|_{L^2(\mathcal{M})}, \end{aligned} \tag{3.21}$$

where $\mathcal{H}$ denotes null hypersurface with future directed normal L such that $\langle L, \mathbf{T}\rangle = -1$, $\nabla\!\!\!/$ denotes the Levi-Civita connection on $S_t = \mathcal{H} \cap \Sigma_t$ with respect to the induced metric γ. There also hold the higher order estimates,

$$\begin{aligned} &\|\partial\boldsymbol{\partial}\phi\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial_0^2\phi\|_{L^2(\mathcal{M})} + \sup_{\mathcal{H}}\left(\|\nabla\!\!\!/(\partial\phi)\|_{L^2(\mathcal{H})} + \|L(\partial\phi)\|_{L^2(\mathcal{H})}\right) \\ &\lesssim \|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\nabla F\|_{L^2(\mathcal{M})}; \end{aligned}$$

and

$$\|\Box_{\mathbf{g}}\partial\phi\|_{L^2(\mathcal{M})} \lesssim M\varepsilon(\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\nabla F\|_{L^2(\mathcal{M})}).$$

Proof of Lemma 6. Recall that the standard energy momentum tensor $Q_{\alpha\beta}$ on $\mathcal{M}$ is given by

$$Q_{\alpha\beta}[f] = \partial_\alpha f \partial_\beta f - \tfrac{1}{2}\mathbf{g}_{\alpha\beta}\mathbf{g}^{\mu\nu}\partial_\mu f \partial_\nu f.$$

Let $\mathscr{P}_\alpha = Q_{\alpha 0}[\phi]$. In view of (3.20), it is straightforward to derive

$$\mathbf{D}^\alpha \mathscr{P}_\alpha = \mathbf{D}^\alpha Q_{\alpha 0}[\phi] + Q_{\alpha\beta}[\phi]\mathbf{D}^\alpha \mathbf{T}^\beta = F\partial_0\phi + \tfrac{1}{2}Q_{\alpha\beta}[\phi]\pi^{\alpha\beta}.$$

Integrating in the region bounded by Σ_0, Σ_t and $\mathscr{H}$ gives

$$\|\boldsymbol{\partial}\phi\|^2_{L^\infty_t L^2(\Sigma_t)} + \sup_{\mathscr{H}} \left(\|\nabla\!\!\!/ \phi\|^2_{L^2(\mathscr{H})} + \|L\phi\|^2_{L^2(\mathscr{H})} \right)$$
$$\lesssim \|\nabla\phi_0\|^2_{L^2(\Sigma_0)} + \|\phi_1\|^2_{L^2(\Sigma_0)} + \left| \int_{\mathscr{M}} F\partial_0\phi \,\mathrm{d}\mathscr{M} \right| + \left| \int_{\mathscr{M}} Q_{\alpha\beta}[\phi]\pi^{\alpha\beta} \mathrm{d}\mathscr{M} \right| \tag{3.22}$$

where $\mathrm{d}\mathscr{M} = n\,\mathrm{d}t\,\mathrm{d}\mu_g$ denotes the volume element in the spacetime $\mathscr{M}$, with $\mathrm{d}\mu_g$ the area element in Σ_t. Using (3.14) the last term can be computed further

$$\begin{aligned} \int_{\mathscr{M}} Q_{\alpha\beta}[\phi]\pi^{\alpha\beta}\mathrm{d}\mathscr{M} &= -2\int_{\mathscr{M}} Q_{ij}[\phi]k^{ij}\mathrm{d}\mathscr{M} + \int_{\mathscr{M}} n^{-1}\nabla^i n Q_{0i}[\phi]\mathrm{d}\mathscr{M} \\ &= -2\int_{\mathscr{M}} \partial_i\phi\partial_j\phi k^{ij}\mathrm{d}\mathscr{M} + \int_{\mathscr{M}} n^{-1}\nabla^i n\partial_i\phi\partial_0\phi\,\mathrm{d}\mathscr{M}. \end{aligned}$$

One would not expect to control $\|k\|_{L^1_t L^\infty_x}$. Similar to the analysis for improving (3.7), to control the standard energy of ϕ, one has to rely on (3.9) and also use (3.11) to derive

$$\begin{aligned} \left| \int_{\mathscr{M}} Q_{\alpha\beta}[\phi]\pi^{\alpha\beta}\mathrm{d}\mathscr{M} \right| &\lesssim \|k_i.\partial^i\phi\|_{L^2(\mathscr{M})}\|\partial\phi\|_{L^2(\mathscr{M})} + \|\nabla n\|_{L^\infty(\mathscr{M})}\|\boldsymbol{\partial}\phi\|^2_{L^2(\mathscr{M})} \\ &\lesssim M^2\varepsilon \sup_{\mathscr{H}} \|\nabla\!\!\!/\phi\|_{L^2(\mathscr{H})}\|\boldsymbol{\partial}\phi\|_{L^2(\mathscr{M})} + M\varepsilon\|\boldsymbol{\partial}\phi\|_{L^2(\mathscr{M})}\|\partial\phi\|_{L^\infty_t L^2_{\Sigma_t}}. \end{aligned}$$

Substituting the above estimate into (3.22) gives (3.21). To prove the higher order estimates, it requires more bilinear estimates than listed in Assumption 3.1. For simplicity, we skip the proof of the higher order estimates in Lemma 6. □

It follows by applying Lemma 6 to $\phi = B$ with the help of (3.19) that

$$\|\partial\boldsymbol{\partial}B\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon + M^2\varepsilon^2.$$

Using the above estimate, (3.6) and (3.17) gives

$$\|\partial A\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \|\partial \mathrm{curl}\, B\|_{L^\infty_t L^2(\Sigma_t)} + \|\partial E\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon + M^2\varepsilon^2.$$

In view of the schematic formula $\partial_0 A_j = \partial_j A_0 + \mathbf{R} + \mathbf{A}\cdot\mathbf{A}$ and using (3.16) and (3.8)

$$\|\partial_0 A_j\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \|\partial_j A_0\|_{L^\infty_t L^2(\Sigma_t)} + M^2\varepsilon^2 + M^4\varepsilon^3.$$

Assuming the estimates for A_0

$$\|A_0\|_{L^\infty L^4(\Sigma_t)} \lesssim M\varepsilon, \quad \|\partial_j A_0\|_{L^\infty_t L^2(\Sigma_t)} \lesssim (M\varepsilon)^{\frac{3}{2}}$$

then we arrive at

$$\|\partial A_0\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon + M^2\varepsilon^{\frac{3}{2}}.$$

By integrating the above bound in $t \in [0,1]$ with the help of (3.8), we obtain

$$\|A\|_{L^2(\Sigma_t)} \lesssim \varepsilon + M^2\varepsilon^{\frac{3}{2}},$$

as desired. We refer the reader to [42, Section 9.3] for more details.

3.3. *Construction of parametrix*

To prove the bilinear estimate (3.9) and the trilinear estimate (3.10), it relies on constructing parametrix in the curved spacetime. In [6], Klainerman and Rodnianski gave the construction of parametrix for the linear wave equation (3.20) in Lorentzian spacetime. By applying the construction in Minkowski space, they proved the standard bilinear estimates for free waves,

Proposition 7 ([6]). *Consider ϕ, ψ solutions of the flat wave equation $\Box\phi = \Box\psi = 0$ and $Q(\phi,\psi)$ one of the null forms in*

$$Q_0(\phi,\psi) = \partial_\alpha\phi \cdot \partial^\alpha\psi, \quad Q_{\alpha\beta}(\phi,\psi) = \partial_\alpha\phi \cdot \partial_\beta\psi - \partial_\beta\phi \cdot \partial_\alpha\psi, \forall \alpha \neq \beta, \tag{3.23}$$

then

$$\|Q(\phi,\psi)\|_{L^2(\mathbb{R}^{3+1})} \lesssim \|\phi[0]\|_{H^2(\mathbb{R}^3)} \|\psi[0]\|_{H^1(\mathbb{R}^3)},$$

where for $f[0] = (f(0), \partial_t f(0))$,

$$\|f[0]\|_{H^\alpha(\mathbb{R}^3)} := \|f(0)\|_{H^\alpha(\mathbb{R}^3)} + \|\partial_t f(0)\|_{H^{\alpha-1}(\mathbb{R}^3)}.$$

Moreover, they also provided the bilinear estimate for parametrix and the solution of the wave equations in the Lorentzian spacetime.

In the sequel we sketch the approach to improve the bilinear and trilinear bootstrap assumptions (3.9) and (3.10) by constructing parametrix for solution of (3.20) (see [42, Section 10]).

Let $u_\pm$ be two families of scalar functions defined on the spacetime $\mathcal{M}$ and indexed by $\omega \in \mathbb{S}^2$, satisfying the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u_\pm \partial_\beta u_\pm = 0$ for each $\omega \in \mathbb{S}^2$. The initial data for $u_\pm$ are set in [43] such that $u_\pm$ at Σ_0 asymptotically approach $x \cdot \omega$. Let $\mathcal{H}^\omega{}_{u_\pm}$ denote the null level hypersurfaces of ${}^\omega u_\pm$. Let ${}^\omega L_\pm$ be their null normals, fixed by the condition that $\mathbf{g}({}^\omega L_\pm, \mathbf{T}) = \mp 1$. For any pair of functions $f_\pm$ on $\mathbb{R}^3$, define

$$\psi[f_+, f_-](t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\, {}^\omega u_+(t,x)} f_+(\lambda\omega)\lambda^2 d\lambda\, d\omega + \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\, {}^\omega u_-(t,x)} f_-(\lambda\omega)\lambda^2 d\lambda\, d\omega.$$

We refer to the following theorem for constructing parametrix

Theorem 8 ([43, Theorem 2.11] and [45, Theorem 2.17]). *Let ϕ_0 and ϕ_1 be two scalar functions on Σ_0. Then there is a unique pair of functions (f_+, f_-) such that*

$$\psi[f_+, f_-]|_{\Sigma_0} = \phi_0, \quad \partial_t(\psi[f_+, f_-])|_{\Sigma_0} = \phi_1.$$

And $f_\pm$ satisfy the following estimates[4]*,*

$$\|\lambda^i f_\pm\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla^i\phi_0\|_{L^2(\Sigma_0)} + \|\nabla^{i-1}\phi_1\|_{L^2(\Sigma_0)}, \quad i = 1,2.$$

$\Box_{\mathbf{g}}\psi[f_+, f_-]$ satisfies the following estimates

$$\|\partial^i \Box_{\mathbf{g}}\psi[f_+, f_-]\|_{L^2(\mathcal{M})} \lesssim M\varepsilon(\|\nabla^{i+1}\phi_0\|_{L^2(\Sigma_0)} + \|\nabla^i\phi_1\|_{L^2(\Sigma_0)}), \quad i = 0,1.$$

Due to the above theorem, associated to any pair of functions ϕ_0, ϕ_1 on Σ_0 the function $\Psi_{om}[\phi_0, \phi_1]$ defined for $(t,x) \in \mathcal{M}$

$$\Psi_{om}[\phi_0, \phi_1] = \psi[f_+, f_-].$$

Moreover, for any pair of functions $f_\pm$ on $\mathbb{R}^3$, and for any $s \in \mathbb{R}$, define the following scalar function on $\mathcal{M}$:

$$\psi_s[f_+, f_-](t,x,s) = \int_{\mathbb{S}^2}\int_0^t e^{i\lambda\, {}^{\omega,s} u_+(t,x)} f_+(\lambda\omega)\lambda^2 d\lambda\, d\omega + \int_{\mathbb{S}^2}\int_0^t e^{i\lambda\, {}^{\omega,s} u_-(t,x)} f_-(\lambda\omega)\lambda^2 d\lambda\, d\omega, \tag{3.24}$$

where ${}^{\omega,s}u_\pm$ are the optical functions similarly defined as for ${}^\omega u_\pm$, except that they are initialized at Σ_s as in [43]. There are analogous estimates for $\psi_s[f_+, f_-]$ as those for $\psi[f_+, f_-]$ in Theorem 8.

Next for any $s \in \mathbb{R}$, associated to $(\phi_0, \phi_1) = (0, -nF)$ a pair of functions (f_+, f_-) on Σ_s such that

$$\psi_s[f_+, f_-]|_{\Sigma_s} = 0, \quad \partial_t(\psi_s[f_+, f_-]|_{\Sigma_s}) = -nF.$$

Define

$$\Psi(t,s)F = \psi_s[f_+, f_-](t).$$

[4]The superscript on ∇ or ∂ denotes the order of the derivative.

Clearly

$$\Box_{\mathbf{g}}\left(\int_0^t \Psi(t,s)F(s)\mathrm{d}s\right) = F(t) + \int_0^t \Box_{\mathbf{g}}\Psi(t,s)F(s)\mathrm{d}s. \tag{3.25}$$

Similar to Theorem 8 and using (3.11), there holds

$$\|\partial^i \Box_{\mathbf{g}}\Psi(t,s)F\|_{L^2(\mathcal{M})} \lesssim M\varepsilon\|\nabla^i F\|_{L^2(\Sigma_s)}, \quad i=0,1. \tag{3.26}$$

Based on the above notations and estimates, we are ready to give the representation formula for the solution of (3.20) (see [42, Theorem 10.8]).

Proposition 9. *There holds the following representation formula for the solution of (3.20). Let*

$$\phi^{(0)} = \Psi_{om}[\phi_0,\phi_1] + \int_0^t \Psi(t,s)F^{(0)}(s,\cdot)\mathrm{d}s, \quad F^{(0)} = F$$

and for $j \ge 1$

$$\phi^{(j)} = \int_0^t \Psi(t,s)F^{(j)}(s,\cdot)\mathrm{d}s, \quad F^{(j)} = -\Box_{\mathbf{g}}\phi^{(j-1)} + F^{(j-1)}$$

then

$$\phi = \sum_{j=0}^{\infty} \phi^{(j)}.$$

Proof. We first derive from (3.25) that

$$F^{(j)} = -\int_0^t \Box_{\mathbf{g}}\Psi(t,s)F^{(j-1)}(s), \quad j \ge 2.$$

Due to (3.26), we have

$$\|\partial^i F^{(j)}\|_{L^2(\mathcal{M})} \lesssim M\varepsilon\|\nabla^i F^{(j-1)}\|_{L^2(\mathcal{M})}, \quad j \ge 2, \tag{3.27}$$

and Theorem 8 implies

$$\|\partial^i F^{(1)}\|_{L^2(\mathcal{M})} \lesssim M\varepsilon(\|\nabla^{i+1}\phi_0\|_{L^2(\Sigma_0)} + \|\nabla^i\phi_1\|_{L^2(\Sigma_0)} + \|\partial^i F\|_{L^2(M)}). \tag{3.28}$$

It follows by using Lemmas 6, (3.27), (3.28) and the construction of $\phi^{(j)}$ that

$$\begin{aligned}&\|\partial^i \boldsymbol{\partial}\phi^{(j)}\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial^i F^{(j)}\|_{L^2(\mathcal{M})}\\ &\quad\lesssim (M\varepsilon)^j(\|\nabla^{i+1}\phi_0\|_{L^2(\Sigma_0)} + \|\nabla^i\phi_1\|_{L^2(\Sigma_0)} + \|\partial^i F\|_{L^2(\mathcal{M})}), \quad i=0,1.\end{aligned} \tag{3.29}$$

Since

$$\Box_{\mathbf{g}}\left(\sum_{j=0}^{N}\phi^{(j)}\right) = F - F^{(N+1)},$$

(3.29) implies as $N \to \infty$,

$$\Box_{\mathbf{g}}\left(\sum_{j=0}^{\infty}\phi^{(j)}\right) = F.$$

Noting that

$$\phi^{(0)}|_{\Sigma_0} = \phi_0, \quad \partial_t\phi^{(0)} = \phi_1, \quad \phi^{(j)}|_{\Sigma_0} = 0, \quad \partial_t\phi^{(j)} = 0, \quad j \ge 1$$

we thus conclude the representation in Proposition 9. $\square$

Using Proposition 9, we will prove

Proposition 10. *Denote by* $\mathscr{C}(U,\nabla\phi)$ *the contraction with respect to one index between a tensor* U *and* $\nabla\phi$*, for* ϕ *a solution of the scalar wave equation (3.20) with* F,ϕ_0,ϕ_1 *satisfying the estimate*

$$\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})} \lesssim M\varepsilon,$$

there holds

$$\|\mathscr{C}(U,\nabla\phi)\|_{L^2(\mathcal{M})} \lesssim M\varepsilon \sup_{\omega\in\mathbb{S}^2}\|\mathscr{C}(U,{}^{\omega}\mathbf{N})\|_{L^\infty_{\omega_u}L^2(\mathscr{H}_{\omega_u})}.$$

To see the above result, we let

$$\mathfrak{C}^{\pm}[U,f]=\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda^{\omega}u_{\pm}(t,x)\omega}\mathbf{b}_{\pm}^{-1}\mathscr{C}(U,{}^{\omega}\mathbf{N}_{\pm})f(\lambda\omega)\lambda^3 d\lambda d\omega, \tag{3.30}$$

where ${}^{\omega}\mathbf{N}_{\pm}=\nabla^{\omega}u_{\pm}/|\nabla^{\omega}u_{\pm}|$ and ${}^{\omega}\mathbf{b}_{\pm}^{-1}=|\nabla({}^{\omega}u_{\pm})|$ is the null lapse. For convenience, we may drop $+$ or $-$ signs in (3.30) simultaneously.

Similar to [6, Theorem 3.3], one can derive the following result (see [42, Section 11.1])

Lemma 11. *Assuming* $|{}^{\omega}\mathbf{b}^{-1}|\lesssim 1$ [5], *there holds*

$$\|\mathfrak{C}[U,f]\|_{L^2(\mathscr{M})}\lesssim\|\lambda^2 f\|_{L^2(\mathbb{R}^3)}\sup_{\omega\in\mathbb{S}^2}\|\mathscr{C}(U,{}^{\omega}\mathbf{N})\|_{L^\infty_{\omega_u}L^2(\mathscr{H}_{\omega_u})}.$$

Proof of Proposition 10. In view of Proposition 9, we write

$$\mathscr{C}(U,\nabla\phi)=\mathscr{C}(U,\nabla(\Psi_{om}[\phi_0,\phi_1]))+\sum_{j=0}^{\infty}\int_0^t\mathscr{C}(U,\nabla\Psi(t,s)F^{(j)}(s,\cdot))ds.$$

Note

$$\mathscr{C}(U,\nabla(\Psi_{om}[\phi_0,\phi_1]))=\mathfrak{C}^+[U,f_+]+\mathfrak{C}^-[U,f_-].$$

Applying Lemma 11 to the first term in the above and also using Theorem 8 gives

$$\|\mathscr{C}(U,\nabla(\Psi_{om}[\phi_0,\phi_1]))\|_{L^2(\mathscr{M})}\lesssim\sup_{\omega\in\mathbb{S}^2}\|\mathscr{C}(U,{}^{\omega}\mathbf{N})\|_{L^\infty_{\omega_u}L^2(\mathscr{H}_{\omega_u})}(\|\nabla^2\phi_0\|_{L^2(\Sigma_0)}+\|\nabla\phi_1\|_{L^2(\Sigma_0)}.$$

Similarly, it follows by using Theorem 8 and (3.29) that

$$\begin{aligned}&\left\|\int_0^t\mathscr{C}(U,\nabla\Psi(t,s)F^{(j)}(s,\cdot))ds\right\|_{L^2(\mathscr{M})}\\&\lesssim\|\partial F^{(j)}\|_{L^2(\mathscr{M})}\sup_{\omega\in\mathbb{S}^2}\|\mathscr{C}(U,{}^{\omega}\mathbf{N})\|_{L^\infty_{\omega_u}L^2(\mathscr{H}_{\omega_u})}\\&\lesssim(M\varepsilon)^j(\|\nabla^2\phi_0\|_{L^2(\Sigma_0)}+\|\nabla\phi_1\|_{L^2(\Sigma_0)}+\|\partial F\|_{L^2(\mathscr{M})})\sup_{\omega\in\mathbb{S}^2}\|\mathscr{C}(U,{}^{\omega}\mathbf{N})\|_{L^\infty_{\omega_u}L^2(\mathscr{H}_{\omega_u})}.\end{aligned}$$

Summing over $j=0,\dots,$ we conclude Proposition 10. □

Next, we use Propositions 10 to improve (3.9) and (3.10).

3.4. *Improvement of (3.9) and (3.10)*

We will first consider (3.9). Noting that $k_{j\cdot}=A_j$, we need control A. In view of (3.6), to improve (3.9), we first bound the leading term $\|(\mathrm{curl}B)^j\partial_j\phi\|_{L^2(\mathscr{M})}$.

Since B satisfies (3.18), we can apply the representation formula in Proposition 9 to B,

$$B=\Psi_{om}[\phi_0,\phi_1]+\sum_{l=0}^{\infty}\int_0^t\Psi(t,s)F^{(l)}(s,\cdot)ds. \tag{3.31}$$

Note

$$(\mathrm{curl}B)^j\partial_j\phi=\partial_m B_n\epsilon^{jmn}\partial_j\phi,$$

where ϵ^{jmn} is the volume form on Σ_t. In this case, applying Proposition 10 with $U=\epsilon^{jmn}\partial_j\phi$, we calculate

$$\mathscr{C}(U,{}^{\omega}\mathbf{N})=\partial_j\phi\epsilon^{jmn\omega}\mathbf{N}_m=\epsilon_{AB}\nabla\!\!\!/_A\phi,$$

and then derive by using (3.19) that

$$\begin{aligned}\|(\mathrm{curl}B)^j\partial_j\phi\|_{L^2(\mathscr{M})}&\lesssim(\|\nabla^2\phi_0\|_{L^2(\Sigma_t)}+\|\nabla\phi_1\|_{L^2(\Sigma_t)}+\|\partial F\|_{L^2(\mathscr{M})})\sup_{\omega\in\mathbb{S}^2}\|\mathscr{C}(U,{}^{\omega}\mathbf{N})\|_{L^\infty_{\omega_u}L^2(\mathscr{H}_{\omega_u})}\\&\lesssim(\|\nabla^2\phi_0\|_{L^2(\Sigma_t)}+\|\nabla\phi_1\|_{L^2(\Sigma_t)}+\|\partial F\|_{L^2(\mathscr{M})})\sup_{\omega\in\mathbb{S}^2}\|\nabla\!\!\!/\phi\|_{L^\infty_{\omega_u}L^2(\mathscr{H}_{\omega_u})}\\&\lesssim M\varepsilon\sup_{\omega\in\mathbb{S}^2}\|\nabla\!\!\!/\phi\|_{L^\infty_{\omega_u}L^2(\mathscr{H}_{\omega_u})}.\end{aligned}$$

[5] We assume this estimate in the sequel for simplicity. This estimate can be proved via the bootstrap argument.

Using (3.6) and (3.17), we conclude

$$\|k_\cdot^j \partial_j \phi\|_{L^2(\mathcal{M})} \lesssim M\varepsilon \sup_{\omega\in\mathbb{S}^2} \|\nabla\phi\|_{L^\infty_{\omega_u} L^2(\mathcal{H}_{\omega_u})} + M^2\varepsilon^2 \|\partial\phi\|_{L_t^\infty L^2(\Sigma_t)}$$

which improves (3.9).

To improve (3.10), it relies on the observation that

$$\epsilon^{jmn} \cdot {}^\omega\mathbf{N}_m Q_{j\cdots} = \mathbf{R}\cdot(\mathbf{R}\cdot L) \tag{3.32}$$

where $Q_{j\cdots}$ denotes the Bel–Robinson tensor with one component being j. Using (3.6), we derive

$$\left|\int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta\right| = \left|\int_{\mathcal{M}} Q_{ij\gamma\delta} (\operatorname{curl} B + E)^j e_0^\gamma e_0^\delta\right|. \tag{3.33}$$

Note symbolically $Q_{j\cdots} = \mathbf{R}\cdot\mathbf{R}$

$$\left|\int_{\mathcal{M}} Q_{ij\gamma\delta}\cdot E e_0^\gamma e_0^\delta\right| \lesssim \|E\|_{L_t^2 L_x^\infty} \|\mathbf{R}\|^2_{L_t^\infty L^2(\Sigma_t)} \lesssim M^4\varepsilon^4 \tag{3.34}$$

where we used (3.7) and (3.17) to derive the last inequality.

For the part of $\operatorname{curl} B$ contributed from the first term on the right-hand side of (3.31), it is reduced bound the term

$$I = \left|\int_{\mathcal{M}} \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda^\omega u(t,x)\omega} \mathbf{b}^{-1} \epsilon^{jmn} \cdot {}^\omega\mathbf{N}_m Q_{j\cdots} f(\lambda\omega)\lambda^3 d\lambda d\omega\, d\mathcal{M}\right|$$

where, in view of (3.19), $\|\lambda^2 f\|_{L^2(\mathbb{R}^3)} \lesssim M\varepsilon$.

Similar to improving (3.9), it is direct to bound

$$\begin{aligned}
I &\lesssim \int_{\mathbb{S}^2} \|{}^\omega\mathbf{b}^{-1}(\epsilon^{jmn}\cdot{}^\omega\mathbf{N}_m Q_{j\cdots}) \int_0^\infty e^{i\lambda^\omega u(t,x)} f(\lambda\omega)\lambda^3 d\lambda\|_{L^1(\mathcal{M})} d\omega \\
&\lesssim \sup_{\omega\in\mathbb{S}^2} \|{}^\omega\mathbf{b}^{-1}\|_{L^\infty(\mathcal{M})} \sup_{\omega\in\mathbb{S}^2} \|\epsilon^{jmn}\cdot{}^\omega\mathbf{N}_m Q_{j\cdots}\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})} \int_{\mathbb{S}^2} \|\lambda^3 f(\lambda\omega)\|_{L^2_\lambda} d\omega \\
&\lesssim \sup_{\omega\in\mathbb{S}^2} \|\epsilon^{jmn}\cdot{}^\omega\mathbf{N}_m Q_{j\cdots}\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})} \|\lambda^2 f\|_{L^2(\mathbb{R}^3)} \\
&\lesssim M\varepsilon \sup_{\omega\in\mathbb{S}^2} \|\epsilon^{jmn}\cdot{}^\omega\mathbf{N}_m Q_{j\cdots}\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})}.
\end{aligned}$$

Substituting into (3.33) the remaining part of $\operatorname{curl} B$ contributed by the remaining term of (3.31), we can similarly obtain

$$\begin{aligned}
&\left|\int_{\mathcal{M}} \int_0^t Q_{ij\gamma\delta} (\operatorname{curl}\Psi(t,s) F^{(l)}(s,\cdot))^j e_0^\gamma e_0^\delta ds\, d\mathcal{M}\right| \\
&\quad \lesssim \|\partial F^{(l)}\|_{L^2(\mathcal{M})} \sup_{\omega\in\mathbb{S}^2} \|\epsilon^{jmn}\mathbf{N}_m Q_{j\cdots}\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})}.
\end{aligned}$$

Thus, with the help of (3.29) and (3.19), we arrive at

$$\left|\int_{\mathcal{M}} Q_{ij\gamma\delta} (\operatorname{curl} B)^j e_0^\gamma e_0^\delta\right| \lesssim M\varepsilon \sup_{\omega\in\mathbb{S}^2} \|\epsilon^{jmn}\mathbf{N}_m Q_{j\cdots}\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})}.$$

By using (3.7), (3.32) and (3.34), we deduce from the above that

$$\left|\int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta\right| \lesssim M\varepsilon \sup_{\omega\in\mathbb{S}^2} \|\mathbf{R}\cdot(\mathbf{R}\cdot L)\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})} + (M\varepsilon)^3 \lesssim (M\varepsilon)^3$$

which improves (3.10).

It is highly nontrivial and challenging to prove Theorem 8. The control of parametrix is established by Szeftel in [43–45]. Note that with

$$\phi_f(0,x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda^\omega u(0,x)} f(\lambda\omega)\lambda^2 d\lambda\, d\omega,$$

the error is

$$Ef(t,x) = \Box_{\mathbf{g}} \phi_f(t,x) = i \int_{\mathbb{S}^2} \int_0^\infty \Box_{\mathbf{g}} {}^\omega u e^{i\lambda^\omega u(0,x)} f(\lambda\omega)\lambda^3 d\lambda\, d\omega.$$

Note that the quantity ${}^{\omega}\mathbf{b}\Box_{\mathbf{g}}{}^{\omega}u = \mathrm{tr}\chi$ is the null expansion of $\mathscr{H}_{{}^{\omega}u}$. In Minkowski space $\Box_{\mathbf{g}}{}^{\omega}u = 0$, nevertheless $\mathrm{tr}\chi$ is nontrivial in the curved spacetime $\mathscr{M}$. Thus, controlling the null expansion is crucial for proving Theorem 8. In particular, it is necessary to derive at least the bound

$$\|\mathrm{tr}\chi\|_{L^\infty(\mathscr{M})} \lesssim \varepsilon$$

under the bootstrap assumptions. The control of causal geometry under regularity assumptions consistent with the bounded L^2 curvature conjecture was first given in [56–58], while an extensive set of such control is established in [44]. In the sequel, we briefly review the result of causal geometry in the rough Einstein vacuum spacetime.

4. Causal geometry of Einstein spacetime with finite curvature flux

One of the central challenges in proving the bounded L^2 curvature conjecture is controlling the causal geometry in the rough spacetime. Consistent with Theorem 3, only a bound on the curvature $\|\mathbf{R}\cdot L\|_{L^2(\mathscr{H})}$ is expected. Given this bound, Klainerman and Rodnianski provided the control of $L^\infty(\mathscr{H})$ bound of $\mathrm{tr}\chi$ in [56–58]. Unlike the set-up in [42], their result applies to truncated null cones rather than null hyperplanes, as used in [42] and [43–46]. The result from [56] was extended to null cones including the vertex in [59, 60], and to null cones with time foliation in [61] (a companion paper to [62]). It was also extended to the Einstein equations with certain matter fields (see [63]) and to null hyperplanes in the Einstein vacuum spacetime with canonical foliation in [64], a companion paper to [51].

Consider a null hypersurface $\mathscr{H}$ in Einstein vacuum space. Let L be the null geodesic generator of $\mathscr{H}$

$$\langle L, L\rangle = 0, \quad \mathbf{D}_L L = 0.$$

The null geodesics are parametrized by s such that $L(s) = 1$ and $s = 0$ at the initial 2-D surface S_0, which is diffeomorphic to 2-sphere.

Theorem 12 ([56]). *Consider an outgoing null hypersurface $\mathscr{H}$ intiating on a closed 2-surface S_0 diffeomorphic to $\mathbb{S}^2$, foliated by level set of s with $0 \le s \le 1$. Assume that both the set of initial data $\mathscr{I}_0$ and the curvature flux $\mathscr{R}_0 = \|\mathbf{R}\cdot L\|_{L^2(\mathscr{H})}$ are sufficiently small. Then*

$$\left\|\mathrm{tr}\chi - \frac{2}{r}\right\|_{L^\infty(\mathscr{H})} \lesssim \mathscr{I}_0 + \mathscr{R}_0,$$

with $r = \sqrt{(4\pi)^{-1}|S_s|}$, $|S_s|$ the area of S_s with respect to the induced metric on S_s, and additional estimates hold for $\hat{\chi}, \zeta, \underline{\chi}$.

Unlike the choice of $\underline{L}$ in Section 2, let $\underline{L}$, the conjugate null vector field, satisfy

$$\langle \underline{L}, L\rangle = -2, \langle \underline{L}, X\rangle = 0, \quad \forall X \in TS_s,$$

where S_s is the level set of s in the null hypersurface $\mathscr{H}$. Recall (2.12) for the definition of the set of connection coefficients (see also [10, Section 13.1]). Under the geodesic foliation, $\zeta + \underline{\eta} = 0$.

For $(e_a)_{a=1}^2$ the orthonormal basis on TS_s, the null components of curvature tensor $\mathbf{R}$ are given by

$$\alpha_{ab} = \mathbf{R}(L, e_a, L, e_b), \quad \beta_a = \tfrac{1}{2}\mathbf{R}(e_a, L, \underline{L}, L),$$
$$\rho = \tfrac{1}{4}\mathbf{R}(L, \underline{L}, L, \underline{L}), \quad \sigma = \tfrac{1}{4}{}^{\star}\mathbf{R}(L, \underline{L}, L, \underline{L}).$$

Below we recall a set of null structure equations for the connection coefficients and a sets of null Bianchi identities (see the detailed derivations in [10, Chapter 7]).

$$L\mathrm{tr}\chi = -\tfrac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2 \tag{4.1}$$

$$\nabla\!\!\!/_L \hat{\chi} = -\mathrm{tr}\chi\hat{\chi} - \alpha \tag{4.2}$$

$$
\begin{aligned}
\mathrm{d}\mkern-9mu/\mathrm{iv}\hat{\chi} &= -\beta + \tfrac{1}{2}\nabla\!\!\!/\,\mathrm{tr}\chi + \tfrac{1}{2}\mathrm{tr}\chi\zeta - \zeta\cdot\hat{\chi} && (4.3)\\
L(\mathrm{tr}\underline{\chi}) &= 2\mathrm{d}\mkern-9mu/\mathrm{iv}\underline{\eta} + 2\rho - \tfrac{1}{2}\mathrm{tr}\chi\cdot\mathrm{tr}\underline{\chi} - \hat{\chi}\cdot\underline{\hat{\chi}} + 2|\underline{\eta}|^2 && (4.4)\\
\nabla\!\!\!/_L\beta + 2\mathrm{tr}\chi\beta &= \mathrm{d}\mkern-9mu/\mathrm{iv}\alpha + (2\zeta + \underline{\eta})\alpha && (4.5)\\
L\rho + \tfrac{3}{2}\mathrm{tr}\chi\rho &= \mathrm{d}\mkern-9mu/\mathrm{iv}\beta - \tfrac{1}{2}\underline{\hat{\chi}}\cdot\alpha + \zeta\cdot\beta + 2\underline{\eta}\cdot\beta && (4.6)\\
L\sigma + \tfrac{3}{2}\mathrm{tr}\chi\sigma &= -\mathrm{c}\mkern-9mu/\mathrm{url}\,\beta + \tfrac{1}{2}\underline{\hat{\chi}}\wedge\alpha - \zeta\wedge\beta - 2\underline{\eta}\wedge\beta. && (4.7)
\end{aligned}
$$

Using (4.1), the $L^\infty(\mathcal{H})$ bound on $\mathrm{tr}\chi - 2/r$ can be obtained if $\int_0^s |\hat{\chi}|^2 \mathrm{d}s'$ can be bounded. Here the integral is taken along an outgoing null geodesic initiated from S_0. Since α is merely in $L^2(\mathcal{H})$, this quantity can not be controlled by using (4.2).

Using $\beta \in L^2(\mathcal{H})$, in view of the Codazzi equation (4.3), it is only expected that $\nabla\!\!\!/\,\hat{\chi} \in L^2(\mathcal{H})$. Suppose the following trace inequality holds for U being $\mathrm{tr}\chi$ and $\hat{\chi}$ [6],

$$\|U\|_{L^\infty_\omega L^2_S} \lesssim \|\nabla\!\!\!/\,U\|_{L^2(\mathcal{H})}. \tag{4.8}$$

Together with using (4.1), one could obtain the estimate in Theorem 12. However the trace inequality (4.8) does not hold. To have the valid bound on $\|U\|_{L^\infty_\omega L^2_S}$, one needs a stronger bound than the right-hand side of (4.8).

In [56], Klainerman and Rodnianski relied on the Besov norm and the structures of (4.3) and (4.6) to provide a sharp trace inequality to achieve the control of $\|\hat{\chi}\|_{L^\infty_\omega L^2_s(\mathcal{H})}$.

Define the following norms[7]

$$
\begin{aligned}
\|F\|_{\mathcal{P}^0} :&= \sum_{\lambda\geq 1} \|P_\lambda F\|_{L^2(\mathcal{H})} + \|F\|_{L^2(\mathcal{H})}\\
\mathcal{N}_1(F) :&= \|\nabla\!\!\!/_L F\|_{L^2(\mathcal{H})} + \|\nabla\!\!\!/\,F\|_{L^2(\mathcal{H})} + \|F\|_{L^2(\mathcal{H})}\\
\|F\|_{\mathcal{B}^0} :&= \sup_{\lambda\geq 1} \|P_\lambda F\|_{L^\infty_s L^2(S_0)} + \|F\|_{L^\infty_s L^2(S_0)}.
\end{aligned}
$$

Remark 13. For tensor field F, the standard Littlewood–Paley decomposition for F is applied to the components of F with respect to a set of suitably chosen parallel-transported frames. Klainerman and Rodnianski in [58] developed an intrinsic Littlewood–Paley decomposition that is directly applicable to tensor fields. For the definitions of $\mathcal{B}^0$ and $\mathcal{P}^0$, the two types of Littlewood–Paley decompositions are equivalent on a rough null hypersurface (see [59,60] for the comparison estimates between them). As we shall see when discussing the sharp trace inequality, introducing the intrinsic Littlewood–Paley decomposition is essential to completing the proof.

With $f = \int_0^s |\hat{\chi}|^2 \mathrm{d}s'$, schematically, there holds

$$\nabla\!\!\!/_L \nabla\!\!\!/\,f = 2\nabla\!\!\!/\,\hat{\chi}\cdot\hat{\chi} + \cdots.$$

Hence

$$\|\nabla\!\!\!/\,f\|_{\mathcal{B}^0} \lesssim \left\| \int_0^s \nabla\!\!\!/\,\hat{\chi}\cdot\hat{\chi} \right\|_{\mathcal{B}^0} + \cdots.$$

Similarly, differentiating (4.1), we can derive

$$\nabla\!\!\!/_L \nabla\!\!\!/\,\mathrm{tr}\chi = -\mathrm{tr}\chi\nabla\!\!\!/\,\mathrm{tr}\chi - 2\nabla\!\!\!/\,\hat{\chi}\cdot\hat{\chi} + \cdots.$$

Hence

$$\|\nabla\!\!\!/\,\mathrm{tr}\chi\|_{\mathcal{B}^0} \lesssim \mathcal{I}_0 + \left\| \int_0^s \nabla\!\!\!/\,\hat{\chi}\cdot\hat{\chi} \right\|_{\mathcal{B}^0} + \cdots$$

---

[6] The norm $\|\cdot\|_{L^\infty_\omega L^2_s}$ means first taking the L^2 norm along a null geodesic initiated from S_0 along $\mathcal{H}$, followed with taking the supremum over the initial point in S_0.

[7] Along the truncated null cone $\mathcal{H}$, the area element on S_s is comparable to the one in S_0 for $0 < s \leq 1$. Therefore, we regard $L^2(S_s)$ as $L^2(S_0)$ for simplicity.

Instead of relying on $\|\nabla\hat{\chi}\|_{\mathscr{P}^0}$ to control the right-hand side, which is unobtainable with merely the bounded curvature flux, they first use (4.3) to write that

$$\nabla\hat{\chi} = \nabla\mathscr{D}_2^{-1}\beta + \nabla\mathscr{D}_2^{-1}\nabla \mathrm{tr}\chi\cdots$$

where $\mathscr{D}_2$ is the operator that takes a 2-covariant, symmetric, traceless tensor ξ into the 1-form $\mathrm{d}\!/\mathrm{iv}\xi$. Denote by $\mathscr{D}_1$ the operator that takes any 1-form ξ on S_s into the pair of functions $(\mathrm{d}\!/\mathrm{iv}\xi, \mathrm{cu}\!/\mathrm{rl}\,\xi)$. They observe that, in view of (4.6) and (4.7), using the normalized pair of functions $(\check{\rho},\check{\sigma})$ which are constructed to eliminate the quadratic term containing the bad term $\underline{\hat{\chi}}$ on the right-hand side,

$$\beta = \mathscr{D}_1^{-1}\{L(\check{\rho},\check{\sigma}) + \zeta\cdot\beta) + \cdots\}$$

which, symbolically, gives

$$\begin{aligned}\nabla\hat{\chi} &= \nabla\mathscr{D}_2^{-1}\mathscr{D}_1^{-1}L(\check{\rho},\check{\sigma}) + \nabla\mathscr{D}_2^{-1}\nabla\mathrm{tr}\chi + \cdots\\ &= \nabla_L\nabla\mathscr{D}_2^{-1}\mathscr{D}_1^{-1}(\check{\rho},\check{\sigma}) + \nabla\mathscr{D}_2^{-1}\nabla\mathrm{tr}\chi + [\nabla\mathscr{D}_2^{-1}\mathscr{D}_1^{-1}, L](\check{\rho},\check{\sigma}) + \cdots.\end{aligned}$$

Hence for $\nabla\hat{\chi}$, they obtain the decomposition

$$\nabla\hat{\chi} = \nabla_L P + E \tag{4.9}$$

where

$$P = \nabla\mathscr{D}_2^{-1}\mathscr{D}_1^{-1}(\check{\rho},\check{\sigma}) + \cdots, \quad E = \nabla\mathscr{D}_2^{-1}\nabla\mathrm{tr}\chi + \cdots$$

are tensors of the same type as $\nabla\hat{\chi}$.

To take advantage of such a decomposition, in [57], they establish the following sharp trace inequality with the help of integration by parts along the null geodesic.

Theorem 14 ([57, Theorem 4.1, Theorem 4.3]). *Consider the transport equation*

$$\nabla_L W = \nabla_L P\cdot F$$

with W, P, F all tensor fields tangent to S_s. There holds the sharp trace inequality

$$\|W\|_{\mathscr{B}^0} \lesssim \|W|_{S_0}\|_{B^0_{2,1}(S_0)} + \mathscr{N}_1(P)(\mathscr{N}_1(F) + \|F\|_{L^\infty_\omega L^2_s}). \tag{4.10}$$

For S_s tangent tensor fields W, E, F satisfying

$$\nabla_L W = E\cdot F$$

there holds

$$\|W\|_{\mathscr{B}^0} \lesssim \|W|_{S_0}\|_{B^0_{2,1}(S_0)} + \|E\|_{\mathscr{P}^0}(\mathscr{N}_1(F) + \|F\|_{L^\infty_\omega L^2_s}). \tag{4.11}$$

Applying Theorem 14 to $F = \hat{\chi}$ and P, E in (4.9) gives

$$\|\nabla\mathrm{tr}\chi\|_{\mathscr{B}^0} \lesssim \mathscr{I}_0 + (\mathscr{N}_1(\hat{\chi}) + \|\hat{\chi}\|_{L^\infty_\omega L^2_s})(\mathscr{N}_1(P) + \|E\|_{\mathscr{P}^0}).$$

Due to Calderon–Zygmund theorem, $\|E\|_{\mathscr{P}^0} \approx \|\nabla\mathrm{tr}\chi\|_{\mathscr{P}^0} + \cdots$, it follows by using Sobolev embedding $\|\hat{\chi}\|^2_{L^\infty_\omega L^2_s} \lesssim \|f\|_{\mathscr{B}^0}$ and Theorem 14 that

$$\|\hat{\chi}\|^2_{L^\infty_\omega L^2_s} \lesssim (\mathscr{N}_1(\hat{\chi}) + \|\hat{\chi}\|_{L^\infty_\omega L^2_s})(\mathscr{N}_1(P) + \|\nabla\mathrm{tr}\chi\|_{\mathscr{P}^0} + \cdots).$$

We then derive

$$\|\hat{\chi}\|_{L^\infty_\omega L^2_s} \lesssim \mathscr{N}_1(\hat{\chi}) + \mathscr{N}_1(P) + \|\nabla\mathrm{tr}\chi\|_{\mathscr{P}^0} + \cdots$$

Substituting the above estimate to the estimate of $\|\nabla\mathrm{tr}\chi\|_{\mathscr{B}^0}$ yields

$$\|\nabla\mathrm{tr}\chi\|_{\mathscr{B}^0} \lesssim \mathscr{I}_0 + (\mathscr{N}_1(\hat{\chi}) + \mathscr{N}_1(P) + \|\nabla\mathrm{tr}\chi\|_{\mathscr{P}^0})(\mathscr{N}_1(P) + \|\nabla\mathrm{tr}\chi\|_{\mathscr{P}^0})$$

where we dropped the additional error terms for simplicity.

$\mathscr{N}_1(\hat{\chi})$ can be bounded by using (4.2) and (4.3). $\mathscr{N}_1(P)$ can be controlled by using (4.6), (4.7), Calderon–Zygmund theorem with the help of proper commutation. Since these bounds are expected to be small, $\|\nabla\mathrm{tr}\chi\|_{\mathscr{B}^0}$ can be controlled, which gives the estimate of $|\mathrm{tr}\chi - 2/r|$.

In the above sketched proof, we neglected all the terms involved with ζ. In fact, ζ and the mass aspect function $\mu = -\text{d}\!\!/\text{iv}\zeta - \check{\rho} + |\zeta|^2$ are related in a similar pattern as the pair of quantities $(\hat{\chi}, \nabla\!\!\!\!/\,\text{tr}\chi)$. To control them, Klainerman and Rodnianski provided a trace decomposition for $\nabla\!\!\!\!/\,\zeta$ as the one for $\nabla\!\!\!\!/\,\hat{\chi}$. Having such a particular structure allows them to apply Theorem 14 to obtain the control of $\|\zeta\|_{L^\infty_\omega L^2_s}$ and $\|\mu\|_{\mathcal{B}^0}$, which are necessary to complete the proof for Theorem 12.

Now consider (4.10) in Theorem 14. If P is a tensor, we assume without loss of generality that P is a one form. Relative to a set of parallel-transported frames $\{X^l\}_{l=1}^m$ (see [57, Proposition 3.28]), note that

$$\|\nabla\!\!\!\!/\,(P(X^l))\|_{L^2(\mathcal{H})} \le \|\nabla\!\!\!\!/\,P \cdot X^l\|_{L^2(\mathcal{H})} + \|P \cdot \nabla\!\!\!\!/\,X^l\|_{L^2(\mathcal{H})},$$

and one can obtain by transport equations that $X^l \in L^\infty$ and $\nabla\!\!\!\!/\,X^l \in L^2_{S_0} L^\infty_s$ with bounded curvature flux. To establish the equivalence between $\|\nabla\!\!\!\!/\,P\|_{L^2(\mathcal{H})}$ and $\sum_{l=1}^m \|\nabla\!\!\!\!/\,(P(X^l))\|_{L^2(\mathcal{H})}$, it requires that $P \in L^\infty_{S_s} L^2_s$. With $\mathcal{D}$ either $\mathcal{D}_1$, or $\mathcal{D}_2$, from (4.9), schematically, $P \approx \mathcal{D}^{-1}(\mathbf{R}\cdot L)$. With bounded curvature flux, P would not be bounded in $L^\infty_{S_s} L^2_s$. Therefore, proving (4.10) without scalarizing the tensor field P becomes crucial. However, in the proof, one must perform a Littlewood–Paley projection, which is typically defined for scalar functions. To solve this issue, in [58], Klainerman and Rodnianski introduced a geometric Littlewood–Paley decomposition using heat flow on weakly regular 2-D closed surfaces, which can be directly applied to tensor fields. They recovered the essential analytic properties of the standard Littlewood–Paley theory for their geometric version in weakly regular 2-surfaces, such as S_s. Further simplifications for proving Theorem 14 were developed in [59,60] and [65]. In particular, a nice alternative treatment to the geometric Littlewood–Paley theory was given in [65], based on deriving improved regularity for the parallel frames.

5. The breakdown criterion of Einstein spacetime

Theorem 12 was initially motivated by the goal of proving the bounded L^2 curvature conjecture. In [7] it also provides control over the null radius of conjugacy, which played a key role in [5] in establishing the breakdown criterion for Einstein spacetimes. It is not immediately clear how controlling causal geometry under low regularity assumptions on the null cone relates to establishing a breakdown criterion for classical solutions of the Einstein vacuum equations. We will explain this connection while introducing the results of [5, 7].

Recall the standard energy argument for bounding the Bel–Robinson energy in (3.15), assuming

$$\int_{t_0}^{t_*} \{\|k\|_{L^\infty_x} + \|\nabla \log n\|_{L^\infty_x}\} \mathrm{d}t < \infty \tag{5.1}$$

the Bel–Robinson energy at Σ_t with $t \in (t_0, t_*)$ can be directly bounded in terms of its value at the initial slice Σ_0 and the bound of the quantity in (5.1).

In [5], Klainerman–Rodnianski proved that

Theorem 15 ([5]). *The Einstein vacuum spacetime with CMC foliation Σ_t with $t < 0$ can be extended beyond at $t_* < 0$ provided that*

$$\sup_{t\in[t_0,t_*)} (\|k\|_{L^\infty_x} + \|\nabla \log n\|_{L^\infty_x}) = \mathcal{K}_0 < \infty. \tag{5.2}$$

Clearly (5.2) implies (5.1) which gives the control of Bel–Robinson energy on Σ_t for $t \in (t_0, t_*)$. However only the lowest order energy is directly bounded by using (5.2) and the initial data. To extend the solution beyond t_*, they managed to bound $\|\mathbf{D}^{\le 2}\mathbf{R}\|_{L^2(\Sigma_t)}$ for $t_0 < t < t_*$ under the assumption (5.2). To control the higher order energy, they bounded $\|\mathbf{R}\|_{L^\infty}$ in the spacetime slab with the help of the equation

$$\Box_{\mathbf{g}}\mathbf{R} = \mathbf{R} \star \mathbf{R}$$

which is obtained by using Bianchi equation and (1.2).

In the same spirit to Sobolev [66] and Choquet-Bruhat [2], Klainerman and Rodnianski established a Kirchoff formula in the curved spacetime in [67] (see [68] for a simplification), by which they represented at a point p in the spacetime by

$$\mathbf{R}(p) = -\int_{\mathcal{N}^-(p,\tau)} \mathbf{A}\cdot\mathbf{R}\star\mathbf{R}+\mathscr{E} \tag{5.3}$$

where $\mathscr{E}$ denotes all other terms, $\mathbf{A}$ is a 4-covariant tensor defined as a solution of a transport equation along the backward light-cone $\mathcal{N}^-(p,\tau)$ initiated from p in the time interval $[t(p)-\tau, t(p)]$.

The representation holds within the null radius of injectivity $i_*(p)$. To define $i_*(p)$, we denote the backward null geodesic initiated from p by $\Upsilon(\omega,s)$, with $\omega\in\mathbb{S}^2$, $s(0)=p$ and $L(s)=1$ for the null geodesic generator L, normalized by $\langle L,\mathbf{T}\rangle(p)=-1$. With $\omega\in\mathbb{S}^2$ fixed, we parameterize s using the temporal parameter $\tau = t_p - t$ so that $s = s(\omega,\tau)$. $i_*(p)$ is the supremum of τ such that the exponential map sending $(\omega,\tau)\to\Upsilon(\omega,s(\tau,\omega))$ by the past null geodesics remains a global diffeomorphism between $\mathbb{S}^2\times(0,\tau)$ and its image along the backward null cone $\mathcal{N}^-(p)$. The null radius of injectivity $i_*(p)=\min(s_*(p),l_*(p))$, where $s_*(p)$ denotes the null radius of conjugacy and $l_*(p)$ is the radius of past null cut locus at p [8]. To control the higher order energies of $\mathbf{R}$, it is crucial to obtain a uniform lower bound of null radius of injectivity for all $p\in\Sigma\times(t_0,t_*)$, which is given in [7].

Next we briefly sketch the proof of [7]. We first note that only Bel-Robison energy on Σ_t, $t\in(t_0,t_*)$ and the curvature flux on $\mathcal{N}^-(p,\tau)$ with $\tau\le i_*(p)$ is bounded by universal constants. These constants depend only on the initial Bel–Robinson energy, the initial volume $|\Sigma_{t_0}|$, the initial metric bound I_0 such that $I_0^{-1}\le(g_{ij})\le I_0$, $\mathcal{K}_0$ and t_*. Consequently, only limited regularity control is obtained directly on the null cone. Prior to [7], existing results in the literature relied on pointwise bounds of the Riemann curvature tensor for controlling the injectivity radius. However, such results do not provide a uniform lower bound on the null injectivity radius in terms of universal constants.

The crucial next step is to show that

$$s_*(p) > \min(l_*(p),\delta_*)$$

where $\delta_*>0$ is a universal constant. This is achieved by showing that

$$\sup_{\mathcal{N}^-(p,\tau)}\left|\mathrm{tr}\chi-\frac{2}{s}\right|\le C \tag{5.4}$$

with $\tau=\min(l_*(p),\delta_*)$ and C a universal constant. The proof is done by rescaling the result of causal geometry analysis in [56] and [59, 60] and also by a continuity argument.

The next step is to find a good system of local space-time coordinates under which $\mathbf{g}$ is comparable with the Minkowski metric. More precisely, for a sufficiently small constant $\epsilon>0$, one needs to show that there exists a constant[9] $\delta_*>0$, depending only on ϵ and some universal constants, for which each geodesic ball $B_{\delta_*}(p)$ with $p\in\Sigma_t$ admits local coordinates $x=(x^1,x^2,x^3)$ such that under the corresponding transport coordinates $x^0=t, x^1,x^2,x^3$ the metric $\mathbf{g}=-n^2\mathrm{d}t^2+g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ with

$$|n-n(p)|\le\epsilon \quad\text{and}\quad |g_{ij}-\delta_{ij}|\le\epsilon \tag{5.5}$$

on $B_{\delta_*}(p)\times[t(p)-\delta_*, t(p)]$. Note that the first estimate can be achieved by controlling $\partial_t n$ via elliptic estimates; the second one is obtained by using (5.2) and

$$\partial_t g_{ij} = -2nk_{ij}. \tag{5.6}$$

[8] $s_*(p)$, $l_*(p)$ are both measured in terms of the temporal parameter.

[9] The constant is no greater than δ_* appeared in the above. We still use δ_* to denote it.

The existence of such local coordinates together with (5.4) will enable one to show that $\mathcal{N}^-(p,\delta_*)$ is close to the flat cone and consequently $l_*(p) > \delta_*$ [10]. Therefore the null radius of injectivity verifies

$$i_*(p) > \min(\delta_*, t(p) - t_0).$$

Then going back to (5.3), there is a uniform lower bound $\delta_* > 0$ of τ for the formula to hold. Due to the structure of $\mathbf{R} \star \mathbf{R}$, one of the curvature component can be controlled by the curvature flux along the backward null cone. Moreover one can achieve $\|\mathbf{A}\|_{L^2(\mathcal{N}^-(p,\tau))} \lesssim \tau^{1/2}$. This leads to

$$\|\mathbf{R}(t)\|_{L^\infty_\Sigma} \lesssim \tau^{1/2} \sup_{t' \in (t-\tau,t)} \|\mathbf{R}(t')\|_{L^\infty_\Sigma} + \|\mathscr{E}\|_{L^\infty_\Sigma}.$$

For $\mathscr{E}$, suppose[11]

$$|\mathscr{E}| \lesssim \sup_{t' \in [t-\tau, t-\frac{1}{2}\tau]} \tau^{-1} \|\mathbf{D}^{\leq 2}\mathbf{R}\|_{L^2(\Sigma_{t'})}.$$

Combining the above estimates implies

$$\|\mathbf{R}(t)\|_{L^\infty} \lesssim \tau^{-1} \sup_{t-2\tau \leq t' \leq t - \frac{\tau}{2}} (\|\mathbf{R}\|_{L^2(\Sigma_{t'})} + \|\mathbf{D}R\|_{L^2(\Sigma_{t'})} + \|\mathbf{D}^2\mathbf{R}\|_{L^2(\Sigma_{t'})})$$

with $\tau > 0$ sufficiently small and fixed.

Then with the step length $(1/2)\tau$, after finitely many steps, it follows from the above estimate that

$$\|\mathbf{R}\|_{H^2(\Sigma_t)} \lesssim \|\mathbf{R}\|_{H^2(\Sigma_0)}, \quad t_0 < t < t_*,$$

which enables the continuation of the solution beyond t_*.

The results on the breakdown criterion and the lower bound for the radius of injectivity of null hypersurfaces were improved in [61, 62] to depend on the weaker assumption (5.1) rather than (5.2). To control both the null radius of conjugacy and the radius of the null cut locus, the proof of Klainerman–Rodnianski relied on L^∞ bounds for k and $\nabla \log n$. With careful analysis, and assuming (5.1), it still requires universal bounds on $\int_{t(p)-\tau}^{t(p)} |k(x,t')|^2 \mathrm{d}t'$ and $\|\pi_{LL}\|_{L^\infty_\omega L^2_s(\mathcal{N}^-(p,\tau))}$ for all p in the spacetime slab to control the null radius of injectivity. In view of (5.6), bounding the first quantity yields the second estimate in (5.5). Based on a delicate bootstrap argument, [62] achieves the first bound by representing k using the wave equation for k. To obtain the second bound, it relies on decomposing $\nabla(\pi_{LL})$ into the form of $\nabla_L P + E$, followed with applying the sharp trace estimates given in Theorem 14. The latter is achieved together with controlling $\mathrm{tr}\chi - 2/s$ and other connection coefficients in [61], using the wave equation for k.

The result of Klainerman–Rodnianski in [5] was also extended by Shao in [63] to Einstein-scalar field and Einstein–Maxwell equations.

Declaration of interests

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[10] See [7] for the detailed geometric argument and [62, Section 4] for the reduction to quantitative control.

[11] Here for simplicity we skip a cutoff step, which is involved with giving the desired bound of $\mathscr{E}$.

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