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The stability of Minkowski space and its influence on the mathematical analysis of General Relativity

La stabilité de l'espace de Minkowski et son influence sur l'analyse mathématique de la Relativité Générale

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Abstract. The study of the stability of Minkowksi space has had a tremendous influence on the mathematical analysis of General Relativity and in a wider context, on geometric hyperbolic partial differential equations. This article proposes an overview of the context, ideas and part of the legacy surrounding this subject, focusing particularly on the original proof of Christodoulou–Klainerman.

Résumé. L'étude de la stabilité de l'espace de Minkowski a eu une influence majeure sur l'analyse mathématique de la Relativité Générale et, plus largement, sur les équations aux dérivées partielles hyperboliques géométriques. Cet article propose un aperçu du contexte, des idées et d'une partie de l'héritage entourant ce sujet, en mettant particulièrement l'accent sur la démonstration originale de Christodoulou–Klainerman.

Keywords. Einstein equations, General Relativity, Asymptotic stability.

Mots-clés. Équations d'Einstein, Relativité Générale, Stabilité asymptotique.

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1. Introduction

Since the work of Minkowski and its unification of space and time [1], Special Relativity is beautifully encoded in the geometry of the simplest Lorentzian manifold, the Minkowski space $(\mathbb{R}^{1+3}, \eta)$, where η is the Lorentzian metric whose matrix of components in the standard Cartesian coordinates (t,x,y,z) of $\mathbb{R}^{1+3}$ is given by $\left(\eta_{\alpha\beta}\right)_{0\le\alpha,\beta\le 3} = \mathrm{diag}(-1,1,1,1)$.

The notion of simplest Lorentzian manifold can even be formalized, since by the Killing–Hopf Theorem [2, Chapter 8], Minkowski space is the unique simply connected complete Lorentzian space of vanishing constant sectional curvature. For the vacuum Einstein equations, $\mathrm{Ric}(g) = 0$, it should therefore be regarded as the trivial solution, in a similar way that the scalar function $\phi := 0$ is the trivial solution to a non-linear homogeneous wave equation of the type $\Box\phi := -\partial_t^2\phi + \Delta\phi = F(\phi, \partial\phi, \partial^2\phi)$, for some non-linearity F vanishing at $(0,0,0)$.

The study of the stability properties of the Minkowski space is natural, both from a physical or a mathematical point of view, once the Einstein equations are understood as a system of evolution equations; this is the problem of understanding the dynamics in a neighborhood of an equilibrium, in fact, of the simplest equilibrium of General Relativity. The subject was also clearly of interest to Choquet–Bruhat and, as for many other mathematical problems of General Relativity, she produced several of the earlier works attempting to address it. In fact, the well-posedness statement of Choquet–Bruhat on the Einstein equations[1] already provides Cauchy stability, hence the local in time stability of the Minkowski space, as long as initial data close to Minkowski space solving the constraint equations could be constructed. This topic she addressed in [5] and [6], where she also mentioned as an open problem the question of long time evolution of such data:

> The important question whether any asymptotically flat excitations of the gravitational field remain nonsingular for all time is still open.

This question will of course only be fully resolved through the monumental work of Christodoulou–Klainerman, completed in 1989 and published in 1993 in [7] establishing the global-in-time stability of the Minkowski space.

Theorem 1.1 ([7]). *Under a global smallness assumption, any strongly asymptotically flat, maximal, initial data leads to a unique globally hyperbolic regular solution to the Einstein equations which is geodesically complete and globally asymptotically flat.*

The initial value problem for the Einstein equations, including the relevant notion of initial data are reviewed below in Section 2.2, while we recall the maximal foliation in Section 4.3. On top of constructing a global solution, detailed asymptotics for the solution were obtained in [7], such as decay estimates for the curvature or for the second fundamental form of the maximal foliation used.

Prior to [7], a work of Friedrich [8] had already established a form of future stability for hyperboloidal data close to Minkowski, but at the time of writing, it was not known how to reach the required hyperboloidal data from asymptotically flat initial data. The gluing constructions of Corvino [9], Chrusciel–Delay [10] and Corvino–Schoen [11] were eventually able to produce such data, bypassing the issue of understanding the dynamics of asymptotically flat data in a neighborhood of spatial infinity and the construction of a small piece of null infinity emanating from it, the *exterior problem.* Nonetheless, since the interior dynamics are heavily influenced by the asymptotics of the data, it is clear that one should aim to address the stability problem for the larger possible class of data, and in fact, several interesting physical phenomena can only be captured by data which are not strongly localized[2].

The work of Christodoulou–Klainerman [7] not only did provide the first complete proof of the stability of Minkowski space, but thanks to the precise asymptotics of the solutions obtained, also did contain the first rigorous derivation of the laws of gravitational radiation, such as the *Bondi mass loss formula* [13]. The non-linear *memory effect* of Christodoulou [14] was also obtained essentially as an outgrowth of the estimates obtained in the proof.

The stability of Minkowski space also constitutes a landmark in the field as the first proof in this domain of research which was too long to be published as a standard journal article; making harder the task of summarizing its content and highlighting its key ideas[3]; and nowadays one could say that it has become a sort of rite of passage to study or revisit (an excerpt of) the proof or write some form of extension.

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[1] In particular, the estimates of [3, Chapter III] leading to a reformulation of the existence theory into a fix point argument implicitly address an analysis of the difference of solutions. See also [4] for a subsequent treatment in Sobolev spaces which focuses on Cauchy stability and states the application to the construction of near Minkowskian spacetimes for finite time.

[2] See for instance [12] for historical details and a presentation of the physical relevance of slowly decaying asymptotic flat initial data.

[3] See nonetheless [15] for an introduction and background materials.

Starting in the late 70's, the study of the long-time behaviour of non-linear wave equations with small data had seen major development[4] and in particular, the standard overall strategy based on energy and decay estimates, wrapped up in a bootstrap argument, had already seen many applications for global or long-time existence. Nonetheless, using this in the context of the Einstein equations was still challenging a priori. Choquet–Bruhat had in particular identified in [17] a form of instability in wave coordinates, essentially proving that the components of the metric in wave coordinates near Minkowski cannot all be uniformly bounded in the standard linear spaces, such as the energy space. Thus, new important building blocks were needed, in particular

- A robust method to derive decay estimates for tensorial wave equations, including improved decay for certain components or derivatives of the tensor fields.
- A way to bypass the instability highlighted by Choquet–Bruhat. In [7], a key component to avoid the instability is a novel geometric framework which is adapted to the solution: the vector fields used for commutation and the foliation are not fixed a priori but constructed throughout the proof.

The first item above was initially developed in [18], following the development of commutation vector field methods for wave equations [19] as well as the understanding of the null condition [20,21]. The failure of the standard null condition is precisely what drove the instability mechanism identified by Choquet–Bruhat and thus, one of the aims of the second item was to provide a better framework to capture the null structural properties of the Einstein equations. Eventually, Lindblad–Rodnianski [22] identified that the Einstein equations in wave coordinates actually verify a weaker form of the null condition, that allows global existence and decay estimate, while the main effects of the instability of Choquet–Bruhat remain only as a small (logarithmic) growth for the worse component of the metric [23,24].

The aim of the present article is to review some ideas and context behind the stability of the Minkowski space as well as to present some recent developments concerning the study of the long time dynamics for the Einstein equations or, more generally, geometric hyperbolic partial differential equations, which uses or extend some important ideas of the proof.

1.1. *Overwiew*

In Section 2, we briefly review standard material concerning the Einstein equations. Besides fixing some notations, introducing the evolution problem in General Relativity and emphasizing the importance of gauge conditions for the Einstein equations, we also aim here to explain why the approach used for instance for orbital type stability results in the case of critical or subcritical non-linear Schrödinger or wave equations does not apply here. In Section 3, we review the problem of global existence for quasi-linear wave equations, provide a quick overview of the original Klainerman vector field method for scalar fields and the null condition in 3 spatial dimensions and end the section with a very brief discussion of the weak null condition and the approach of Lindblad–Rodnianski. Section 4 is devoted to the presentation of the work of Christodolou–Klainerman, focusing on geometric aspects. Though nowdays, there are several aspects of the proof that one would treat differently, we have chosen to focus on the original arguments. In Section 5, we review several extensions and developments on the theme of the stability of Minkowski space and we conclude in Section 6 with some works which were influenced by the analysis on the stability of Minkowski space but address other kinds of problems.

[4] See for instance [16] for a pertinent review.

1.2. *Convention*

We use standard Einstein summation convention. Greek indices $\alpha, \beta, \ldots$ are used for tensors defined on spacetimes. Latin indices i, j, k are used for tensors on spacelike hypersurfaces or the manifold of the initial data. The convention for the signature of Lorentzian metric is $- + + \ldots$. If $g_{\alpha\beta}$ denotes the components of a metric g, $g^{\alpha\beta}$ denotes the components of the inverse metric and we also used standard raising and lowering of indices on tensors throughout the article. Partial derivatives will be denoted either as $\partial_\alpha \psi$ or $\psi_{,\alpha}$. Starting from Section 4, we follow the convention of [7] that bold quantities such as $\mathbf{R}$, $\mathbf{D}$ are associated to the spacetime geometry and thus denotes the spacetime Riemann tensor and the Levi-Civita connexion, while R, ∇ refers to the geometry of a spacelike slice.

The article is written assuming standard knowledge of differential and Riemannian geometry as well as partial differential equations, but does not assume a priori knowledge concerning General Relativity or the Einstein equations.

2. Background material

2.1. *The Einstein equations and their hyperbolicity*

General Relativity can be understood as the study of 4-dimensional[5] Lorentzian manifolds $(\mathscr{M}, g)$ solving the Einstein equations

$$\mathrm{Ric}(g) - \tfrac{1}{2}\mathrm{S}(g)\cdot g + \Lambda g = T, \tag{2.1}$$

where $\mathrm{Ric}(g)$ and $\mathrm{S}(g)$ denotes respectively the Ricci and scalar curvature of the Lorentzian metric g, $\Lambda \in \mathbb{R}$ is the so-called cosmological constant and T is a source term, arising from matter fields, the so-called energy–momentum tensor. This source term, T, should not be considered as pure external data since the evolution of matter fields, whether they are constituted of electromagnetism fields, fluids, particles, ... are themselves subject to an evolution equation depending the metric g. Thus, unless $T = 0$, the Einstein equations need to be completed by the evolution equations for the matter to get a closed system. When $T = 0$, Equation (2.1) reduces to $\mathrm{Ric}(g) = \Lambda g$, the *vacuum Einstein equations*. Minkowski space is the trivial solution to (2.2) with $\Lambda = 0$, while the de-Sitter and Anti-de-Sitter space solve the equation with $\Lambda > 0$ and $\Lambda < 0$ respectively. In the rest of the article[6], we thus focus on $\Lambda = 0$ together the vacuum condition $T = 0$, in which case the equations reduce to the Ricci flat condition

$$\mathrm{Ric}(g) = 0. \tag{2.2}$$

Perhaps the most important property of the Einstein equations is that they are geometric: given $\phi : \mathscr{M} \to \mathscr{M}$ a diffeomorphism, g solves (2.2) if and only if $\phi^\star g$ solves (2.2), where $\phi^\star g$ denotes the pullback of the metric g by ϕ.

This already implies that one cannot expect to solve the Einstein equations based purely on a traditional, direct, application of the methods from partial differential equations. Indeed, those typically gives existence and uniqueness at the same time and here uniqueness can only hold in the geometric sense, that is to say up to the above diffeomorphism invariance.

More directly, the Ricci tensor of a metric g, viewed as a (non-linear) partial differential operator on the components of g, takes the form

$$\mathrm{Ric}(g)_{\alpha\beta} = -\tfrac{1}{2}\Box_g g_{\alpha\beta} + \partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha + Q_{\alpha\beta}(\partial g, \partial g), \tag{2.3}$$

where

[5] The theory can easily be written in other dimensions, but on top of being the physical dimension, dimension 4 plays a distinguished role for the problem of the stability of Minkowski space, cf. Section 3.4 below.

[6] Apart from Section 5.4, when we will review results concerning $T \neq 0$.

- $\Box_g$ is the (scalar) wave operator:

$$\Box_g \psi := \frac{1}{\sqrt{|\det g|}} \left(\partial_\alpha \sqrt{|\det g|} g^{\alpha\beta} \partial_\beta \psi\right).$$

- Γ_β are traces of the Christoffel symbols $\Gamma^\rho_{\alpha\gamma}$:

$$\Gamma_\beta := g_{\beta\rho} g^{\alpha\gamma} \Gamma^\rho_{\alpha\gamma} := g_{\beta\rho} g^{\alpha\gamma} \tfrac{1}{2} g^{\rho\delta} (g_{\delta\gamma,\alpha} + g_{\alpha\delta,\gamma} - g_{\alpha\gamma,\delta}).$$

- $Q_{\alpha\beta}(\partial g, \partial g)$ denotes a quadratic form in the first derivatives of g, with coefficients depending on g.

For a fixed metric g, $\Box_g$ is a second-order hyperbolic operator, but the terms $\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha$ contains second-order derivatives of g and thus a priori spoil the hyperbolicity of $\mathrm{Ric}(g)$. To retrieve the hyperbolicity, one needs to address the geometric invariance mentioned above, for instance by considering the equations in a particular coordinate system. The standard one, which of course was used by Choquet–Bruhat in her local well-posedness proof [3], is given by the so-callled *wave coordinates*.

Definition 1. A local coordinate system $\left(\mathscr{U} \subset \mathscr{M}, \phi := (x^\alpha)_{\alpha=0,1,2,3} : \mathscr{U} \to \mathbb{R}^4\right)$ of a Lorentzian manifold $(\mathscr{M}, g)$ is said to consist of *wave coordinates* if, $\forall \alpha = 0, \dots, 3$, the function x^α verifies the wave equation

$$\Box_g x^\alpha = 0. \tag{2.4}$$

Since in general, one has $\Box_g x^\alpha = -g^{\gamma\beta} \Gamma^\alpha_{\gamma\beta} = -g^{\beta\alpha} \Gamma_\beta$, we have as a consequence:

Lemma 2.1. *In wave coordinates, the vacuum Einstein equations (2.2) reduces to a system of the form*

$$\Box_g g_{\alpha\beta} = Q_{\alpha\beta}(\partial g, \partial g), \tag{2.5}$$

The Equations (2.5) are a system of *quasi-linear* wave equations. For such equations, standard energy methods can be applied to prove local existence[7] in high regularity classes.

Another important property, which again can be seen to arise from the geometric invariance, is that (2.5) are scale invariant: if $g_{\alpha\beta}(x^\alpha)$ solves (2.5), restricting appropriately the domain of definition, so does ${}^\lambda g(x) := g_{\alpha\beta}(\lambda x^\alpha)$, for any $\lambda > 0$. In three spatial dimensions, the Sobolev space left invariant by this scaling is the homogeneous space $\mathring{H}^{3/2}$. In particular, this implies that one cannot hope to study the Einstein equations, even locally and thus a fortiori globally in time, without controlling more than one derivative of the metric components, for instance by taking derivatives of the equations.

2.2. *The constraint equations and the initial value problem*

Given any spacelike hypersurface Σ, the Gauss–Codazzi equations implies, for any solution of (2.2), a system of constraint equations for the first and second fundamental forms h, k associated to Σ,

$$\mathrm{S}(h) + (\mathrm{tr}\, k)^2 - |k|^2 = 0, \tag{2.6}$$

$$\mathrm{div}\, k - \mathrm{d}(\mathrm{tr}\, k) = 0, \tag{2.7}$$

where $\mathrm{S}(h)$ denotes the scalar curvature of the Riemannian metric h, and operators and traces are taken with respect to h.

The initial value problem in General Relativity then consists, given a Riemannian manifold (Σ, h) and a 2-tensor k verifying (2.6)–(2.7), in the construction of a Lorentzian manifold $(\mathscr{M}, g)$

[7] See for instance [25, Chapter 1.4].

solving (2.2) together with an embedding $i : \Sigma \hookrightarrow \mathcal{M}$ such that (h,k) coincides with the first and second fundamental forms of i. Such an $(\mathcal{M}, g)$ is then called a *development* of the data (Σ, h, k).

The Choquet–Bruhat theorem [3], completed by the Choquet—Bruhat–Geroch construction of the maximal solution[8] [27], then provides, given any initial data set (Σ, h, k), the existence and uniqueness, up to the diffeomorphism invariance, of a *maximal globally hyperbolic*[9] development.

2.3. *Asymptotic flatness*

The Euclidean data $(\mathbb{R}^3, \delta, 0)$ form of course a solution to (2.6)–(2.7) and its maximal Cauchy development is given by the Minkowski spacetime. For any concept of perturbations of the Minkowski spacetime to make sense, one thus also needs to understand how to construct initial data sets which are arbitrary close to $(\mathbb{R}^3, \delta, 0)$. In particular, we would like the pertubations from the Euclidean data to be *localized*, i.e. to tend to 0 near infinity. Physically, this corresponds to *asymptotically flat* initial data sets and gives rise to spacetimes describing *isolated systems*. From the point of view of analysis, this localization is important, as it is needed for the derivation of decay estimates, which are at the core of the stability proofs. To introduce this quantitatively, we consider initial data sets (Σ, h, k) such that Σ possesses an asymptotic end, that is, the complement of a compact set of Σ is diffeomorphic to the complement of a ball in $\mathbb{R}^3$. This allows one to introduce a coordinate system (x^1, x^2, x^3) in a neighborhood of infinity $r := \sqrt{\sum_{i=1}^3 (x^i)^2} \to +\infty$. The original proof [7] then applies to *strongly asymptotically flat initial data*, which by definition verifies the following expansion[10]

$$h_{ij} = (1 + 2M/r)\delta_{ij} + o_4(r^{-3/2}), \tag{2.8}$$
$$k_{ij} = o_3(r^{-5/2}), \tag{2.9}$$

for a non-negative number $M \geq 0$. The positive mass-theorem [28,29] in fact guarantees that for such data, the mass M is strictly positive unless the initial data is flat. In particular, one cannot consider compactly supported perturbations of the flat data. The construction of asymptotically flat initial data near the Euclidean data was first considered in [5] and [6]. The problem of producing initial data with arbitrary decay has been more recently addressed in [30] and [31]. We refer to [32, Chapter 3] for a more precise introduction to asymptotic flatness in General Relativity.

3. Global existence for quasi-linear wave equations and the basic vector field method

3.1. *A strategy for quasi-linear wave equations*

In view of the form of the reduced Equations (2.5), it is natural to first think of the stability of the Minkowski spacetime as a specific case of global existence for quasi-linear wave equations of the form

$$(-\partial_t^2 + \Delta)u_i + F^{\alpha\beta}(u)\partial_\alpha\partial_\beta(u_i) = Q_i(\partial u, \partial u), \tag{3.1}$$
$$u(t=0) = u_0, \tag{3.2}$$
$$\partial_t u(t=0) = u_1, \tag{3.3}$$

[8]See also the modern treatment by Sbierski [26] for an improved construction of the maximal solution, which contrary to [27] does not rely on Zorn's lemma.

[9]Here by *globally hyperbolic*, we mean the existence of a *Cauchy hypersurface*, that is an achronal hypersurface such that any inextendible causal curve register once and only once on this hypersurface, cf. [2, Chapter 14].

[10]Recall that $f = o_m(r^{-k})$ if $\partial_l f = o(r^{-k-|l|})$, for any multi-index l of length $|l| \leq m$.

for an unknown $u : (t,x) \to (u_i)(t,x) \in \mathbb{R}^N$, non-linearities such that $F^{\alpha\beta}(u) = F^{\beta\alpha}(u)$ are smooth functions of u vanishing at 0, $Q_i(\partial u, \partial u)$ which are quadratic forms in the first derivatives of u and initial data (u_0, u_1) which are small, sufficiently regular and localized.

By now, the standard approach for this type of problems is essentially based on

- A scale of weighted energy spaces to control the data and the solutions. We denote the top order energy norm at time t of a solution u by $E_N[u(t)]$, the index N referring to the order in the scale, i.e. $E_N[u]$ controls up to N derivatives of u in a weighted L^2-type space.
- Energy estimates allowing to propagate global bounds of the solutions, in particular at top order of regularity, of the form
$$E_N[u(t)] \le CE_N[u(0)] + \int_0^t \mathrm{NL}(s)\,\mathrm{d}s, \tag{3.4}$$
where NL denotes error terms arising from the non-linearities and $C > 0$ is independent of u.
- Decay estimates that allow to control the non-linearities, leading, for instance, to an estimate of the form
$$|\mathrm{NL}(s)| \lesssim \frac{E_N[u(s)]}{(1+s)^q}, \tag{3.5}$$
with $q > 0$.
- A bootstrap argument: by local existence, for initial data such that $E_N[u(0)] \le \epsilon$, a solution u exists on some interval $[0,T]$, for some $T > 0$, and such that $E_N[u(t)] \le 2C\epsilon$, for all $t \in [0,T]$. Assuming $q > 1$ for the time being in (3.5), a Grönwall argument combined with the energy estimate (3.4) and the decay estimate (3.5) then allows to improve the bootstrap assumption to $E_N[u(t)] \le C\epsilon + O(\epsilon^2) \le 3/2C\epsilon$. By a standard continuity argument, it then follows that the solution exists for all times and verifies, for all $t \ge 0$,
$$E_N[u(t)] \le C\epsilon + O(\epsilon^2).$$

To be more specific and check under which conditions, one can obtain the estimate (3.5) with $q > 1$, we recall below the standard vector field method, as introduced in [19], that provides both a first definition of the spaces E_N and the decay estimate (3.5).

3.2. *The vector field method of Klainerman*

We consider the wave operator $\Box := -\partial_t^2 + \Delta$, where Δ is the usual Laplacian in $\mathbb{R}^n$ and recall first the basic energy estimate.

For ϕ and S defined on $\mathbb{R}_t \times \mathbb{R}^n_x$, decaying sufficiently fast in x and solving $\Box\phi = S$, the energy
$$E[\phi](t) := \int_{x\in\mathbb{R}^n} (\partial_t\phi(t,x))^2 + |\nabla_x\phi(t,x)|^2\,\mathrm{d}x$$
verifies for all t,
$$E[\phi](t) = E[\phi](0) + \int_0^t \int_{\mathbb{R}^n} |\partial\phi\cdot S|\,\mathrm{d}x\,\mathrm{d}s,$$
which one can (at least formally) derive by multiplying the wave equation by $\partial_t\phi$ and integration by parts. Moreover, if ϕ solves instead the perturbed wave equation
$$(-\partial_t^2 + \Delta)\phi + F^{\alpha\beta}\partial_\alpha\partial_\beta\phi = S,$$
with $F^{\alpha\beta} = F^{\beta\alpha}$ sufficiently small ($\|F\|_{L^\infty_{t,x}} < 1/4$), then the above is replaced by the energy estimate:
$$E[\phi](t) \le 2E[\phi](0) + 2\int_0^t \int_{\mathbb{R}^n} |\partial\phi|\cdot(|S| + |\partial\phi||\partial F|)\,\mathrm{d}x\,\mathrm{d}s. \tag{3.6}$$

To construct higher order norms that are still propagated by the equation, one then considers vector fields that commute with the linear wave operator:

Lemma 3.1. *Let Z be any of the vector fields*

- *Translations* $\partial_t, \partial_{x^i}$,
- *Rotations* $\Omega_{ij} = x^i \partial_{x^j} - x^j \partial_{x^i}$,
- *Hyperbolic rotations* $\Omega_{0i} = t\partial_{x^i} + x^i \partial_t$.
- *Scaling* $S = t\partial_t + \sum_i x^i \partial_{x^i}$.

Then, $[\Box, Z] = 0$, *apart from the scaling which verifies* $[\Box, S] = 2\Box$.

Denote by $\mathscr{A}$ the set of all such vector fields and recall that the Z vector fields form an algebra: $[Z_1, Z_2] \in \operatorname{span}\mathscr{A}$, for all $Z_1, Z_2 \in \mathscr{A}$. We introduce an ordering on $\mathscr{A}$ and use standard multi-index notations below. In particular, for a multi-index α, $|\alpha|$ denotes its length and Z^α denotes a differential operator of order $|\alpha|$ obtained as a composition of $|\alpha|$ vector fields within $\mathscr{A}$.

The decay estimate is then essentially incoporated in the following weighted Sobolev inequality.

Proposition 2. *Let $\psi := \psi(t,x)$ defined on $[0,T] \times \mathbb{R}^n_x$ be a sufficiently regular function. Then,* $\forall (t,x) \in [0,T] \times \mathbb{R}^n_x$,

$$|\psi(t,x)| \lesssim \frac{1}{(1+t+|x|)^{(n-1)/2}(1+|t-|x||)^{1/2}} \sum_{|\alpha| \le (n+2)/2,\, Z^\alpha \in \mathscr{A}^{|\alpha|}} \|Z^\alpha \psi(t,.)\|_{L^2_x}. \tag{3.7}$$

Let ψ be a solution to the linear wave equation $\Box\psi = 0$. Applying (3.7) to $\partial\psi$, we obtain

$$\begin{aligned}
|\partial\psi(t,x)| &\lesssim \frac{1}{(1+t+|x|)^{(n-1)/2}(1+|t-|x||)^{1/2}} \sum_{|\alpha| \le (n+2)/2,\; Z \in \mathscr{A}^{|\alpha|}} \|Z^\alpha \partial\psi(t,.)\|_{L^2_x} \\
&\lesssim \frac{1}{(1+t+|x|)^{(n-1)/2}(1+|t-|x||)^{1/2}} \sum_{|\alpha| \le (n+2)/2,,\; Z \in \mathscr{A}^{|\alpha|}} \|\partial Z^\alpha \psi(t,.)\|_{L^2_x} \\
&\lesssim \frac{1}{(1+t+|x|)^{(n-1)/2}(1+|t-|x||)^{1/2}} \sum_{|\alpha| \le (n+2)/2,,\; Z \in \mathscr{A}^{|\alpha|}} E[Z^\alpha \psi]^{1/2}(0),
\end{aligned}$$

where we have used that $\Box Z^\alpha \psi = 0$ and hence $E[Z^\alpha\psi](t) = E[Z^\alpha\psi](0)$ from energy conservation.

Thus, we have obtain a proof of *decay* for solutions to the homegeneous wave equation. Moreover, we see that a natural scale of energy norms is given by

$$E_N[u(t)] := \sum_{|\alpha| \le N,,\; Z \in \mathscr{A}^{|\alpha|}} E[Z^\alpha u(t)].$$

Remark 3.2. We see from (3.7) that away from the the light cone region $t = |x|$, for instance, in the interior region, $|x| < (1/2)t$, the strong decay $|\partial\phi| \lesssim t^{-3/2}$ holds. Provided the initial data enjoy enough decay, this estimate can be improved[11]. One way to capture extra decay is to consider a stronger weighted energy norm that also make use of the multiplier $K_0 := 2tS + (-t^2 + |x|^2)\partial_t$. This vector field, first used by Morawetz in [33], also played an important role in the estimates of [7], cf. Section 4.4 below.

Remark 3.3. In the strategy described above, vector fields play two different kind of roles. First, the overall energy estimates can be obtained by *multiplying* the wave equation by $X(\phi)$, for $X = X^\alpha \partial_{x^\alpha}$ a vector field, and integration over suitable regions. For instance, the basic energy estimate (3.6) relies on the vector field multiplier $X = \partial_t$, while we introduce just above the multiplier $X = K_0$, leading to weighted energy estimates. The other occurrence of the vector fields is via *commutation*, cf. Lemma 3.1 and Proposition 2. The existence of these commutation vector fields is highly connected with the symmetry of the equations, hence is less likely to be useful outside from questions related to the stability of Minkowski. On the other hand, the use of multiplier vector fields has an extremely wide range of applications. In fact, it is part

[11] In fact, for compactly supported initial data, Huygens' principle ensures that the support of the solution is located in a cylindrical strip around the light-cone.

of the *general method of energy currents*, cf. [33,34], and many recent developments in General Relativity and geometric hyperbolic partial equations rely on some applications or extensions of these principles, cf. Section 5.5 below.

3.3. *Global existence with* $n \geq 4$

Returning to the scheme introduced in Section 3.1, commuting the equations N-times and applying the energy estimate (3.6), we obtain the top order energy estimate (3.4) with an error term NL which can be estimated as

$$|NL|(s) \lesssim E_N[u(s)] \cdot \|\partial^\alpha u(s)\|_{L^\infty_x},$$

with $|\alpha| \leq N/2$. If $N/2 \geq (n+2)/2$, we can then appeal to the inequality (3.7), to obtain (3.5) with $q = (n-1)/2$. In particular, with $n \geq 4$, we have $q > 1$ and we can improve the bootstrap assumptions as explained in Section 3.1, while we see that we only have $q = 1$ if $n = 3$, so that this strategy a priori fails.

3.4. *The case* $n = 3$*: the null condition*

With $n = 3$, the linear decay and a Grönwall argument in (3.6) a priori gives a logarithmic growth. That this growth can in fact lead to blow up for certain systems of quasi-linear wave equations was proved in the celebrated works of John [35,36]. Nonetheless, global existence can be recovered provided the non-linearities verify certain structural properties, in particular the *null condition* [21,37], which guarantees that the non-linear terms always contain at least a derivative *tangential* to the light-cone.

For instance, for the semi-linear terms Q_i in (3.1), such a condition would imply, for $t \geq 0$, that they can be written as a linear combination with bounded coefficients of products of the form

$$\underline{\partial} u \cdot \partial u,$$

where ∂ denote all possible derivatives, while $\underline{\partial} \in \{\partial_t + \partial_r, e_A, A = 1,2\}$, with e_A derivatives tangential to the spheres of constant (t,r). The relevance of these tangential derivatives is that they verify an improved decay estimate. In fact, for any tangential derivative $\underline{\partial}$, one has a decomposition of the form

$$(1+|t|+|x|)\underline{\partial} = \sum_{Z^k \in \mathscr{A}} c_k Z^k,$$

for some uniformly bounded coefficients c_k, and the Z vector fields commute with the wave operator from Lemma 3.1, the improved decay is clear. Using the extra decay, an estimate similar to (3.5) and global existence can then be derived with some minor modifications[12].

3.5. *The Einstein equations in wave gauge*

Unfortunately, as noted by Choquet–Bruhat in [17], the Einstein equations in the wave gauge fail to verify the classical null condition. In fact, she showed later that there is no generalization of the classical null condition for the Einstein equations [40], independently of the choice of coordinates.

[12]At top order, it is harder to fully exploit the null condition, so the top order norm E_N is often allowed to grow slowly in t, while lower order norms stay uniformly bounded. See for instance [38, inequality (5.17) in Chapter 5, p. 78]. Stronger top order estimates without growth can in fact also be proven using additional spacetime estimates. Those are related to the so-called *ghost weight* of Alinhac [39]. Although the equations take on a different form, the top order estimates in [7] are also free of additional growth.

The failure of the null condition for the $Q_{\alpha\beta}$ quadratic terms in (2.5) is actually relatively easy to capture. Indeed, the $Q_{\alpha\beta}$ terms can be decomposed into a sum of null forms, that verify the null condition, cubic terms and higher order terms which will decay fast, and extra P quadratic forms of the form

$$P_{\alpha\beta}(\partial h, \partial h) = \eta^{\rho\rho'}\eta^{\sigma\sigma'}\partial_\alpha h_{\rho\sigma}\partial_\beta h_{\rho'\sigma'} - \frac{1}{2}\eta^{\rho\rho'}\eta^{\sigma\sigma'}\partial_\alpha h_{\rho\rho'}\partial_\beta h_{\sigma\sigma'},$$

where $h = g - \eta$ is the pertubation of the metric g compared to the Minkowski metric η. The above expression for P does not verify the classical null condition, for instance P_{tt} will contain products of t-derivattives that do not factorize with a tangential derivative to the light-cone. Nonetheless, it still contains some structure, apparent in the various contractions between η and the two copies of ∂h. Moreover, from the wave gauge condition, one derive

$$\eta^{\alpha\beta}\partial_\alpha h_{\beta\mu} = \eta^{\alpha\beta}\frac{1}{2}\partial_\mu h_{\alpha\beta} + \text{h.o.t},$$

from which one can rewrite non-tangential derivatives of certain components of h as tangential derivatives. All together, one can show that while the system does not verify the null condition, and as a consequence, that some components of h will not behave as in the linear case and suffer a small (at least logarithmic) growth, this growth does not propagate through the whole system thanks to a *nilpotent* type structure. These weaker structural properties, called the *weak null structure* by Lindblad–Rodnianski [22], allowed them to eventually provide an alternative proof of stability of Minkowski spacetime, first for Schwarzschildean initial data in [41] and then general asymptotically flat initial data in [42].

Remark 3.4. This wave gauge approach explained above is less geometrical than the original proof of Christodoulou–Klainerman. In particular, geometric conclusions of the stability of Minkowski spacetime, such as curvatures estimates, the Bondi Mass law, or even the completeness of null infinity[13], which are rather direct consequences of the global existence and the estimates obtained in [7], require an extra analysis on top of the global existence of solutions in wave coordinates. On the other hand, the fact that global existence (from the point of view of pdes) can be obtained independently of a strong control on the geometry is somehow a testament of the flexibility of the method and can be advantageous. See Section 5 below for many developments based on either of the two strategies.

4. The Christodoulou–Klainerman geometric approach to the stability of Minkowski space

We describe in this section some aspects of the proof of [7]. This proof is arguably too complex to summarize accurately in a few pages, so we have chosen to focus this section mostly on geometric notions and the overall strategy.

4.1. *The metric and the time foliation*

As explained in Section 2, one is forced to make gauge choices to study the Einstein equations as an evolution problem and we start by introducing the one used in [7]. To this end, we consider $(\mathcal{M}, g)$, the maximal globally hyperbolic development of evolution of strongly asymptotically flat initial data posed on $\mathbb{R}^3$ as introduced in Sections 2.2 and 2.3. By global hyperbolicity, it follows that $\mathcal{M} \simeq \mathbb{R}\times\mathbb{R}^3$ and in particular, one can introduce a time function t and coordinates (x^1, x^2, x^3) relative to which the metric takes the form

$$g = -\phi^2(t,x)\,\mathrm{d}t \otimes \mathrm{d}t + g_{ij}(t,x)\,\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

[13] See Chapter 17 in [7].

The function t can be chosen so that its level sets $\Sigma_t := \{p \in \mathcal{M},\ t(p) = t\}$ provide a foliation of $\mathcal{M}$, the normalized condition $\phi \to 1$ as $x \to \infty$ is verified on each Σ_t, and such that $\Sigma_0 = \{t = 0\}$ coincides with the[14] initial data set. We denote by $\mathbf{D}$ the Levi-Civita connexion of g and by ∇ the one associated to the spacelike metric g_{ij}.

Let k_{ij} denotes the component of the second fundamental form associated to the Σ_t foliation

$$k_{ij} := \frac{1}{2\phi}\partial_t g_{ij}. \tag{4.1}$$

To decompose the tensor fields, it is useful to introduced the orthonormal frame (T, e_i), defined by $T = (1/\phi)\partial_t$ while the e_i are tangential to Σ_t and Fermi propagated along T, so that

$$\mathbf{D}_T e_i = \phi^{-1}(\nabla_i \phi)T.$$

4.2. *The Weyl tensor*

The Weyl tensor associated to g is the traceless part of the curvature tensor

$$\mathbf{C}_{\alpha\beta\gamma\delta} := \mathbf{R}_{\alpha\beta\gamma\delta} - \tfrac{1}{2}(g_{\alpha\gamma}\mathbf{R}_{\beta\delta} + g_{\beta\delta}\mathbf{R}_{\alpha\gamma} - g_{\beta\gamma}\mathbf{R}_{\alpha\delta} - g_{\alpha\delta}\mathbf{R}_{\beta\gamma}) + \tfrac{1}{6}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})\mathbf{S}(g).$$

In particular, for a solution to the Einstein equations, the Weyl tensor coincides with the curvature tensor. It verifies the Bianchi equations,

$$\mathbf{D}_{[\epsilon}\mathbf{C}_{\alpha\beta]\gamma\delta} = 0, \tag{4.2}$$

and, on top of being traceless by construction, enjoys the same symmetry properties as the curvature tensor.

Instead of considering all the components of the Weyl tensor, it is preferable to introduce an irreducible decomposition, the *electro-magnetic* decomposition, which takes into account its symmetries. Given a vector field, here the unit normal T, one considers the tensors E and H, defined by

$$\begin{aligned} E_{\alpha\beta} &= \mathbf{C}_{\mu\alpha\nu\beta}T^\mu T^\nu, \\ H_{\alpha\beta} &= {}^\star\mathbf{C}_{\mu\alpha\nu\beta}T^\mu T^\nu, \end{aligned}$$

where ${}^\star C$ is the *Hodge* dual of C

$${}^\star\mathbf{C}_{\mu\alpha\nu\beta} = \tfrac{1}{2}\varepsilon_{\mu\alpha\rho\sigma}\mathbf{C}^{\rho\sigma}{}_{\nu\beta},$$

with $\varepsilon_{\mu\alpha\rho\sigma}$ denoting the components of the volume form.

From the symmetric tensors E and H, one can reconstruct the whole Weyl tensor and moreover, from the Bianchi equations, it follows that (E, H) verifies a symmetric hyperbolic system similar to the Maxwell equations. For instance, the evolution equation for E takes the form

$$-\phi^{-1}\partial_t E_{ab} + \operatorname{curl} H_{ab} = Q_{ab}(\nabla\phi, E, H, k),$$

where $\operatorname{curl} H_{ab} := (1/2)(\varepsilon_a^{cd}\nabla_d H_{cb} + \varepsilon_b^{cd}\nabla_d H_{ca})$ and Q_{ab} is a quadratic in $\nabla\phi, E, H, k$, which is similar to the differential Ampère–Maxwell law.

[14] More precisely, Σ_0 coincides with the image of the initial data set by the embedding introduced in the definition of a development, cf. Section 2.2.

4.3. *Maximal foliations and elliptic systems*

One important ingredient of the setup is to consider the Weyl tensor as one of the the main unknown and only recover a control on the metric via the definition of k, Equation (4.1) and the Gauss and Codazzi equation associated to the hypersurfaces Σ_t. However, one still needs to fix the foliation, that is to constrain the hypersurfaces Σ_t. Here, one considers *maximal* hypersurfaces[15], i.e. one imposes $\operatorname{tr} k = 0$ on each Σ_t. With this choice, the lapse function ϕ verifies the elliptic equation

$$\Delta\phi = |k|^2\phi.$$

Moreover, k also verifies an elliptic div-curv type system

$$\begin{aligned} \operatorname{curl} k_{ij} &= H_{ij}, \\ \operatorname{div} k &= 0, \end{aligned}$$

where H is the magnetic part of the electro-magnetic decomposition of the Weyl tensor. Thus, one sees that a good control on the Weyl tensor leads via elliptic estimates to a good control of the second fundamental form and all the metric quantities.

4.4. *The Bel–Robinson tensor and the vector field method applied to the Bianchi equations*

At the linearized level, one thus needs to understand the following problem: given a tensor field $W := W_{\alpha\beta\gamma\delta}$ that verifies all the symmetry and traceless properties of the Weyl tensor as well as the Bianchi equations

$$\partial_{[\epsilon} W_{\alpha\beta]\gamma\delta} = 0,$$

on Minkowski space, how can one prove estimates on W? In fact, to prove a stability statement, similar to the approach highlighted in Section 3.1, one needs both energy type estimates at the top order and decay estimates. Moreover, having in mind the importance of the null condition (cf. Section 3.4) and thus of structural properties of the non-linearity, we will see here, already at the level of the components of W, that different components of W (akin to different derivatives for scalar waves) enjoy different decay properties.

First, the energy estimates are proven using the so-called *Bel–Robinson* tensor Q

$$Q_{\alpha\beta\gamma\delta} := W_{\alpha\mu\beta\nu} W^{\mu\ \nu}_{\ \gamma\ \delta} + {}^\star W_{\alpha\mu\beta\nu} {}^\star W^{\mu\ \nu}_{\ \gamma\ \delta}.$$

The Bel–Robinson tensor plays the role of the standard energy–momentum tensor for matter fields. It is divergence free for W solving the Bianchi equations and it is coercive whenever contracted with 4 identically directed timelike vector fields.

On Minkowski space, a vector field method leading to energy estimates and weighted Sobolev inequalities, which encodes the decay of the fields, similar to the one for the scalar field presented in Section 3.2 can then be implemented. These ideas were first presented in [18]. Compared with the simplest case of a scalar field of Section 3.2, a notable difference comes the fact that, since W is a tensor field, it has different components, and the different components enjoy different decay properties. These different decay properties are related to the directions of propagation of the evolution equations, and thus to the null conditions. To capture them, it is thus necessary to decompose W into components tangential or transversal to the null cones. Let thus $L = \partial_t + \partial_r$,

[15] In particular, [7] requires the initial data to already be maximal, i.e. that $\operatorname{tr} k = 0$ on the initial surface. The existence of such complete maximal hypersurfaces in general, asymptotically flat Lorentzian manifold (under a barrier assumption) was established by Bartnik in [43].

$\underline{L} = \partial_t - \partial_r$ and complete $L, \underline{L}$ by a pair of orthonormal vector fields e_A, $A = 1,2$, tangential to the spheres of constant (t,r). We then decompose W on the null frame $(L, \underline{L}, e_A)$:

$$\begin{aligned} \underline{\alpha}_{AB} &:= W_{A\underline{L}B\underline{L}}, \quad \alpha_{AB} := W_{ALBL}, \\ \underline{\beta}_A &:= \tfrac{1}{2} W_{A\underline{L}\underline{L}L}, \quad \beta_A := \tfrac{1}{2} W_{AL\underline{L}L}, \\ \rho &:= \tfrac{1}{4} W_{\underline{L}L\underline{L}L}, \quad \sigma := \tfrac{1}{4} \epsilon^{\alpha\beta} W_{\alpha\beta\underline{L}L}. \end{aligned}$$

Using the Morawetz vector field K_0 (cf. Remark 3.2), which rewrites in the null frame as $K_0 = (1/2)(v^2 L + u^2 \underline{L})$ and the Bel–Robinson tensor Q, one can then compute the energy density

$$Q(K_0, T, T, T) = \tfrac{1}{8} u^2 |\underline{\alpha}|^2 + \tfrac{1}{8} v^2 |\alpha|^2 + \tfrac{1}{4}(3u^2 + v^2)|\underline{\beta}|^2 + \tfrac{1}{4}(3v^2 + u^2)|\beta|^2 + \tfrac{3}{4}(u^2 + v^2)(\rho + \sigma),$$

where $u = t - r$, $v = t + r$, from which the different weights for the different components are already apparent.

4.5. *The Eikonal equation and geometrically defined vector fields*

In implementing the above estimates on an asymptotically flat spacetime which is not Minkowski space, one important difficulty arises from the long-range effect of the $1/r$ term of the initial data, cf. (2.8), the *Schwarzschildean* tail of the metric. Indeed, the geodesic null cones of the spacetime then diverge logarithmically from the corresponding ones of Minkowski space, and their shear, which vanishes in Minkowski space, also has bad asymptotic properties. To avoid these divergences from spoiling the estimates, the strategy of [7] is to adapt the vector fields which are used in the construction of the energy norms, using the geometry of the current spacetime rather than that of the fixed Minkowski space. For instance, the vector field K_0 is replaced by

$$K := \tfrac{1}{2}(v^2 e_+ + u^2 e_-), \tag{4.3}$$

where

(1) u is an *optical* function, that is to say, for $r \geq r_0/2$, it solves[16] the Eikonal equation,

$$g(\mathrm{d}u, \mathrm{d}u) = 0, \tag{4.4}$$

(2) $e_\pm$ are null vectors: $e_\pm = T \pm N$, where T is the forward unit normal to Σ_t and N is the outward unit normal, tangential to Σ_t, to the spheres $S_{t,u}$ of constant (t,u).

(3) $v = 2r - u$, where r is the area radius of the $S_{t,u}$ spheres.

Similarly, the angular momentum vector fields Ω_{ij} replacing the flat ones (cf. Lemma 3.1) are also modified, for instance, we want $\Omega_{ij}(u) = 0$, and thus since u is now constructed dynamically, so must the Ω_{ij}.

4.6. *Curvature estimates: commutation, error terms and the principle of signature*

As explained in Section 4.4, the components of the Weyl curvature tensor are estimated using the Bel–Robinson tensor. Moreover, to obtain higher order estimates as well as obtained improved decay, one also needs to estimate components of derivatives of W such as $\widehat{\mathscr{L}}_T W$ where $\widehat{\mathscr{L}}_X W$ for a vector field X is the modified Lie derivative[17] defined by

$$\begin{aligned} \widehat{\mathscr{L}}_X W_{\alpha\beta\gamma\delta} := {} & \mathscr{L}_X W_{\alpha\beta\gamma\delta} - \tfrac{1}{2}\left({}^{(X)}\pi^\mu_\alpha W_{\mu\beta\gamma\delta} + {}^{(X)}\pi^\mu_\beta W_{\alpha\mu\gamma\delta} + {}^{(X)}\pi^\mu_\gamma W_{\alpha\beta\mu\delta} + {}^{(X)}\pi^\mu_\delta W_{\alpha\beta\gamma\mu}\right) \\ & - \tfrac{3}{8} \mathrm{tr}^{(X)}\pi W_{\alpha\beta\gamma\delta}, \end{aligned}$$

[16] To avoid caustics, u is actually constructed as a solution to (4.4) in an exterior region, $r \geq r_0/2$, for some $r_0(t) > 0$ and matched to an another optical in the supplementary interior region.

[17] The point of the modified Lie derivative is that it has the property of preserving the tracefreeness of Weyl fields: $\widehat{\mathscr{L}}_X W^\gamma_{\beta\gamma\delta} = 0$.

where ${}^{(X)}\pi_{\alpha\beta} := (1/2)\mathscr{L}_X g_{\alpha\beta}$, with $\mathscr{L}_X$ the standard Lie derivative acting on tensor fields, is the deformation tensor of the vector field X.

In the case of the T vector field, commuting the Bianchi equations $\mathbf{D}^\alpha W_{\alpha\beta\gamma\delta} = 0$, one obtains

$$\mathbf{D}^\alpha(\widehat{\mathscr{L}}_T W_{\alpha\beta\gamma\delta}) = J(T;W)_{\beta\gamma\delta},$$

for some error term $J(T;W)$ which is a quadratic form in the deformation tensor of T and the components of W.

Consider the Bel–Robinson tensor $Q(\widehat{\mathscr{L}}_T W)$ associated to $\widehat{\mathscr{L}}_T W$. Because of the additional T derivatives, the initial data of $\widehat{\mathscr{L}}_T W$ decay faster than that of W, so that $Q(\widehat{\mathscr{L}}_T W)(K,K,K,T)$, where K is the Morawetz vector field, has finite integral on the initial hypersurface, despite the strong weights in the definition of K, cf. (4.3). The divergence theorem applied between the hypersurfaces Σ_t and Σ_0 leads to

$$\begin{aligned}\int_{\Sigma_t} Q(\widehat{\mathscr{L}}_T W)(K,K,K,T) &= \int_{\Sigma_0} Q(\widehat{\mathscr{L}}_T W)(K,K,K,T) + \int_0^t \mathrm{d}t' \int_{\Sigma_{t'}} \phi \operatorname{div} Q(\widehat{\mathscr{L}}_T W)_{\beta\gamma\delta} K^\beta K^\gamma K^\delta \\ &\quad + \frac{3}{2}\int_0^t \mathrm{d}t' \int_{\Sigma_{t'}} \phi\, Q(\widehat{\mathscr{L}}_T W)_{\alpha\beta\gamma\delta}\, {}^{(K)}\pi^{\alpha\beta} K^\gamma K^\delta,\end{aligned}$$

where the first spacetime error term comes from commuting the Bianchi equations and thus depends on $J(T;W)$ while the second comes from the fact that the vector field K is not an exact conformal Killing field. Similar integral formulas hold for other derivatives of W with the amount of K vector fields contracted with the Bel–Robinson tensor depending on how fast the data decay. To prove energy estimates for the components of W and its derivatives, one thus needs to control the error terms using decay estimates, and to avoid logarithmic divergences, to exploit the special structure of certain products, similar to Sections 3.1 and 3.4. Each of these error terms can be expressed as a product of null components of the deformation tensors of K or T and the null components of $\widehat{\mathscr{L}}_T W$ and one needs to exploit the different decay or integrability rates of each term. Since there are many of those, a convenient, systematic, approach is given by the *principle of conservation of signature*: given any tensor tangential to the spheres $S_{t,u}$, obtained by contraction of a spacetime tensor via contraction with the null vectors e_3 or e_4 and projections, one assigns a signature defined as the difference between the total number of contractions with the outgoing null vector e_4 and the total number of contractions with the ingoing vector e_3. For instance, $\alpha_{AB} = W(e_4, e_A, e_4, e_B)$ is given the signature $s(\alpha) = +2$ and $\underline{\alpha}_{AB} = W(e_3, e_A, e_3, e_B)$ the signature $s(\underline{\alpha}) = -2$.

By the principle of conservation of signature, given any tensor field U which can be written as a multilinear form in an arbitrary number of tensors $U_1, \dots, U_p$ with coefficients depending only the metric, the signature of any null term of U, expressed in terms of the null components of $U_1, \dots, U_p$ is equal to the sum of the signatures of each term in the decomposition. As an illustration, the component Q_{AB34} of the Bel–Robinsor tensor has signature 0. By definition, Q is a quadratric form in the null curvature components of a Weyl field, and by the principle of conservation of signature, all quadratic terms must have signature 0, leaving as only option products of the form $\alpha\cdot\underline{\alpha}$, $\beta\cdot\underline{\beta}$, $\rho\cdot\rho$, $\rho\cdot\sigma$ and $\sigma\cdot\sigma$. This can used in a (quasi)-systematic to obtain the correct form of each error term.

4.7. *Bootstrap assumptions and the last slice argument*

Similar to Section 3.1, the overall strategy is based on a bootstrap argument. Thus, one assume the existence of $t^\star$ such that within $[0, t^\star]$, quantitative estimates, both pointwises and in energy type norms holds. As an illustration, the L^2 exterior top order bootstrap assumptions on α and $\underline{\alpha}$ take the form

$$\|(1+u^2)^2 r^q \nabla^q \underline{\alpha}\|_{L^2(r\geq r_0/2)} + \|r^{q+2}\nabla^q \alpha\|_{L^2(r\geq r_0/2)} \leq \epsilon_0, \quad q=0,1,2,$$

and the aim is then to improve these bounds and all the other bootstrap assumptions.

One important twist on the bootstrap argument concerns the optical function u. The Eikonal equation is a non-linear transport equation and thus, solutions are determined by their initial data. In [7], in the exterior region, u is initialized from the future, that is to say, from the last slice $\Sigma_{t^\star}$. Indeed, fixing u on a spacelike slice is equivalent to choosing a radial foliation of the slice. If this is done on the initial slice, since u verify a transport equation, there is no reason why the $S(t,u)$ would get rounder as $t\to+\infty$ which would forbid the improvement of certain bootstrap assumptions. Instead, one initializes u on the last slice $t^\star$ by first solving for a radial foliation on $\Sigma_{t^\star}$ verifying improved bounds. In particular, the optical function ${}^{(t^\star)}u$ depends a priori on the time-slab $[0,t^\star]$. After the bootstrap assumptions are improved, the local existence provides an extended time-slab $[0,t^\star+\delta]$, $\delta>0$, and thus a new optical function ${}^{(t^\star+\delta)}u$ must be constructed with initialization on $\Sigma_{t^\star+\delta}$. In the limit, one then also needs to prove the convergence of the optical functions ${}^{(t)}u$ as $t\to+\infty$.

5. Developments and extensions of the stability of Minkowski space

We present here a survey of result which extends or complete the stability of the Minkowski space. The exposition is thematical rather than historical.

5.1. *Stability under weaker assumptions on the initial data: Bieri and Shen works*

The initial data in the original work of Christodoulou–Klainerman were assumed to be strongly asymptotically flat, cf. Section 2.3. In [44], Bieri proved the stability of the Minkowski space for initial data verifying

$$\begin{aligned} g_{ij} &= \delta_{ij} + o_{q+1}(r^{-\frac{s-1}{2}}),\\ k_{ij} &= o_{q+1}(r^{-\frac{s+1}{2}}), \end{aligned}$$

with $s=2$. The number of derivatives on the initial data required was also lower to $q=2$, compared to up to 4 derivatives in [7]. Note that under this assumption, one cannot apriori define the ADM mass, which requires $s>2$. This result has been recently improved in the work of Shen [45], which requires only $s>1$. A main difference of the analysis in [45] compared to [44] and [7] comes from the construction of the optical function, which is no longer initialized on a last slice, but rather on the initial slice and a timelike curve playing the role of the axis of symmetry. Another analytical difference comes from abandoning of the classical vector field method, in particular, the weighted norms obtained via contraction with the K vector field, which is replaced by the r^p-weighted method of Dafermos–Rodnianski [46].

5.2. *The double null gauge and the exterior problem*

With spatial dimension 3, due to the slow decay of waves near the light-cone $t=|x|$, many steps in the analysis of quasi-linear wave equations, including for the stability of Minkowski space, must treat differently an interior region of the form $r\leq(r_0/2)(t)$, from the exterior region $r\geq(r_0/2)(t)$, where $r_0(t)$ is the area-radius of the spheres of intersection between the initial cone $u=0$ and Σ_t. In the exterior region, it can be advantageous to not consider the Einstein equations in the maximal gauge as in [7], but in the *double null gauge*, where the spacetime metric can be put under the form

$$g := -2\Omega^2(\mathrm{d}u \otimes \mathrm{d}\underline{u} + \mathrm{d}\underline{u} \otimes \mathrm{d}u) + \gamma_{ab}(\mathrm{d}\omega^a - X^a\,\mathrm{d}u) \otimes (\mathrm{d}\omega^b - X^b\,\mathrm{d}u).$$

Here, both u and $\underline{u}$ are optical functions, i.e. solutions to the Eikonal equations, while ω^a are coordinates on the spheres $S_{u,\underline{u}}$ of constants $(u,\underline{u})$. The level sets of u and $\underline{u}$, $C(u)$, $\underline{C}(\underline{u})$ are characteristic cones and thus the double null gauge is naturally adapted to the equations. In particular, it captures better the null properties of the equations and it is also advantageous when studying radiation phenomena taking place on the asymptotic null cone $C_{\underline{u}}$, $\underline{u} \to +\infty$.

The exterior problem can then be understood as follows: given an asymptotically flat initial data set (Σ, h, k), such that for some compact set K, the initial data on $\Sigma \setminus K$ verifies a smallness assumption. Let $(\mathscr{M}, g)$ be the maximal Cauchy development of (Σ, h, k), and let $\mathscr{M}_+$ be the domain of dependence of $\Sigma \setminus K$. Show that $\mathscr{M}_+$ can be foliated by cones $(C(u), \underline{C}(\underline{u}))$ such that the outgoing cones $C(u)$ are complete and such that the curvature and Ricci coefficients are controlled in a quantitative manner. For instance, for strongly asymptotically flat data, α and $\underline{\alpha}$ must verify the decay estimates

$$r^{7/2}|\alpha| \lesssim C, \quad r|u|^{5/2}|\underline{\alpha}| \lesssim C,$$

where C depends on the initial data and r is the area radius of the spheres $S_{u,\underline{u}}$.

The exterior problem was first addressed in the work of Klainerman–Nicolò [47] and was recently revisited in the work of Shen [48], which in particular generalized the class of initial data considered, as well as Hintz [49], whose work is based on generalized wave coordinates instead of the double null gauge.

The double null foliation is particularly well-adapted to the study of characteristic initial value problems, cf. [50]. Together with the maximal gauge and the wave gauge and their generalizations, it constitutes now one of the classical ways to write the Einstein equations, with numerous spectacular applications, such as [51].

As a complement to the exterior problem, let us also mention the *spacelike-characteristic* initial value problem, which one could coin the interior problem, in which the the initial data is posed on a compact 3-disk and on a future complete null hypersurface emanating from its boundary. The global stability of the Minkowski space in this context was proved by Graf [52] based on a gauge which is partially initialized on a timelike curve, as in [45].

5.3. *The* $1+2$*-dimensional case*

In [53] and [54], Huneau studied the stability of the Minkowski space with a translational symmetry. Due to the translational invariance, the data are no longer asymptotically flat as in Section 2.3 since they enjoy no decay in the direction of symmetry. Thanks to the symmetry, the equations reduce to a quasi-linear system of wave equations in $1+2$ dimensions. Since the decay of waves improves with the number of spatial dimensions, the setting is harder than the $1+3$ case. General quasi-linear wave equations verifying the null condition were first studied by Alinhac [39]. The work of Huneau must of course address all the extra difficulties coming from the Einstein equations. In the $1+3$ dimensional case, to address the presence of the Schwarzschildean tail of the metric, the term of order $1/r$ proportional to the mass, in [24], an ansatz for the perturbation was introduced to solve only for the remainder terms. In the work of Huneau, another ansatz must be introduced, since the explicit Schwarzchild solution no longer captures the asymptotics. Instead, here, the background metrics are no longer explicit and Huneau constructs them based on the (radial) Einstein–Rosen waves. Another important modification compared to [24] concerns the usage of *dynamical, generalized* wave coordinates, where the coordinates x^α now verify inhomogeneous wave equations instead of (2.4), with an inhomogeneous term that depends on a non-trivial background metric as well as the actual metric (but not its derivatives). This allows to substract off some of worse source terms in the wave equations for the metric perturbations.

5.4. *The stability of Minkowski space for Einstein-matter systems*

5.4.1. *The Einstein equations coupled to wave equations*

One natural way to extend the stability results of the Minkowski space is to consider the Einstein equations coupled to matter. The matter models which are also governed by wave or tensorial wave equations are clear first candidates to be studied, since the additional fields can a priori be treated by the methods already developed for the Einstein equations. The Maxwell case was addressed by Zipser in [44] based on the Christodoulou–Klainerman framework and subsequently Loizelet [55] using the wave coordinates approach of Lindblad–Rodnianski. The Einstein-scalar field is actually contained in the analysis of Lindblad–Rodnianski [24], which interestingly, thanks to the possible matter fields, shows that the positivity of the ADM mass is logically independent from the stability properties of the Minkowski space. Indeed, one can produce initial data with negative mass by considering the Einstein-scalar field equations with an negative sign in front of the energy–momentum tensor in (2.1) and yet the main theorem of [24] still applies.

In [56], Speck studied the Einstein equations coupled to non-linear electromagnetism, which contains additional difficulties due to the intrinsic non-linearities of the matter equations, even on a fixed background. Recently, Kauffman and Lindblad have considered the case of the Einstein equations coupled to a charged scalar field [57]. To treat the case of non-vanishing charge, very precise estimates on certain metric coefficients are needed, that go beyond those contained in [24]. In the vacuum case, those were derived by Lindblad [58] where a full set of sharp asymptotics for the metric components is obtained in generalized wave coordinates.

5.4.2. *Coupling between massless and massive wave equations: the Einstein–Klein–Gordon system*

In the Einstein–Klein–Gordon system (EKG) system, the Einstein equations are coupled to a massive wave equation. At the linear level, the Klein–Gordon field ψ verifies the Klein–Gordon equation

$$\Box\psi + m^2\psi = 0, \tag{5.1}$$

with mass $m > 0$. Equation (5.1), while still sharing with the same principal symbol with the wave equation, does not respect the conformal symmetries of Minkowski space due to the extra mass term. As a consequence, the conformal vector fields, such as the scaling S, do not enjoy good commutation properties with the equation. Yet, an alternative vector field approach for the study of (5.1) was developed in [59], exploiting in particular a novel foliation of the forward light-cone by hyperboloidal spacelike hypersurfaces. The coupling to the Einstein equations is still non-trivial: if one were to strictly apply the methods of either [7] or [55] in (2.1), commuting vector fields would naturally land over to the energy–momentum tensor T and force us a priori to also commute the matter equations via the same vector fields. As a consequence, one is forced to also modify the analysis of the Einstein equations in a non-trivial way. The stability of the Minkowski space for the EKG system was first obtained in the work of LeFloch–Ma in [60] for Schwarschildean initial data and then in [61] for data with more general asymptotics, using an extension of the Klainerman hyperboloidal approach. An independent proof was obtained by

Ionescu–Pausader [62], which includes slowly decaying data, based on a novel framework for the study of quasi-linear wave equations, incorporating both Fourier analysis and physical space methods. In [63], Wang initiated an approach to this problem based on intrinsic hyperboloids, instead of using the hyperboloidal foliation of the background Minkowski space.

The methods developed for the study of systems of wave-Klein–Gordon equations have found further applications, such as in the work of LeFloch–Ma addressing $f(R)$ theories of gravity [64], where the Einstein equations are replaced by a fourth order system of wave-Klein–Gordon equations, or the analysis of the Kaluza–Klein spacetime by Huneau, Stingo and Wyatt on the stability of the Kaluza–Klein spacetime [65]. Here the analysis reduces to a quasi-linear system of wave equations on a product space $\mathbb{R}^{1+3}\times\mathbb{S}^1$, corresponding to $1+4$ dimensional Minkowski spacetime with one direction compactified on a circle. The resulting system can then be viewed as coupling wave equations to an infinite sequence of Klein–Gordon equations with different masses.

5.4.3. *The Einstein–Vlasov system*

Colissionless kinetic matter is a popular matter model widely used in astropphysics or cosmology. In this case, the Einstein equations are coupled to a kinetic equation, of unknown f called a *Vlasov* field, representing a distribution of particles, which is defined on a hypersurface of the tangent bundle

$$f:\mathscr{P}\to\mathbb{R}_+,\quad \mathscr{P}:=\{(x,p)\in T\mathscr{M},\ g_x(p,p)=-m^2,\ p\text{ future-oriented}\}. \tag{5.2}$$

Here, m denotes the mass of the particles and the massless case $m=0$ has different analytical properties from the massive one $m>0$. The evolution equation for f is that it must be propagated by geodesics, so that it verifies the transport equation

$$v^\alpha\partial_{x^\alpha}f-p^\alpha p^\gamma\Gamma^\beta_{\alpha\gamma}\partial_{p^\beta}f=0. \tag{5.3}$$

The energy–momentum tensor in (2.1) is given by the second moment of f. For instance, in the flat case $g=\eta$, $\mathscr{P}$ can be parametrized by

$$\mathbb{R}^{1+3}_x\times\mathbb{R}^3_p\to\mathscr{P} \tag{5.4}$$

$$(x,p)\to\left(x,(p^0,p^i):=\left(\sqrt{m^2+|p|},p\right)\right) \tag{5.5}$$

and the energy–momentum tensor of a Vlasov field can then be written as

$$T_{\alpha\beta}[f]:=\int_{(p_i)\in\mathbb{R}^3}p_\alpha p_\beta f\frac{\mathrm{d}p}{p^0}.$$

A novel difficulty comes here from the fact that the Einstein equations and the kinetic one (5.3) are of quite different nature a priori, and thus, it is not clear that they can be treated by a common approach. For instance, if one tries to commute (2.1) with vector fields, it is clear that we will be forced to consider derivatives of the energy–momentum tensor, and thus, a priori of f and (5.3).

Within spherical symmetry, the nonlinear stability of the Minkowski space for the Einstein–Vlasov system was initiated by Rein and Rendall [66] in the massive case and Dafermos in the case of massless particules [67]. Outside from spherical symmetry, the massless case has been addressed first by Taylor, in which the Einstein equations are treated as in [7,51], in the double null gauge, while the analysis of the the Vlasov field relied on the construction of Jacobi fields and assumed compact support on the initial Vlasov field. Another proof was given in [68] by Bigorgne–Fajman–Joudioux–Smulevici–Thaller, in wave coordinates, in the case of non-compact support for the Vlasov field. The massive case has first been addressed in [69,70] by Fajman–Joudioux–Smulevici and Lindblad–Taylor and more recently by Wang in [71], using an extension of the

Fourier analysis techniques of Ionescu–Pausader to address both the Einstein and the Vlasov part of the equations.

To address the issue of the coupling between the Einstein and the Vlasov equations, a systematic, vector field approach was first developped in [72] and form the core of the strategy of [69] and [68]. Another point of interest comes from the analysis of the non-linear wave/kinetic field interactions, where structures similar to the null condition or the weak null condition have been discovered. For instance, given g a solution to the wave equation and f a Vlasov field, the non-linear terms $\partial_{x^i} g.\partial_{p_i} f$ and $[\partial_{x^i} + (v^i/\sqrt{1+|v|^2})\partial_t](g).\partial_{p_i} f$ verifies stronger estimates compared to generic products $\partial_x g.\partial_p f$, due to identities such as

$$\partial_{x^i} g \partial_{p_i} = \frac{Z(g)}{t}\partial_v - \frac{x}{t}\frac{\partial_t g}{\sqrt{1+|v|^2}}\widehat{Z} + \frac{\partial_t g}{\sqrt{1+|v|^2}}\left(S + \frac{|x|^2-t^2}{t}\partial_t\right), \tag{5.6}$$

where Z, S (respectively $\widehat{Z}$) are vector fields commuting with the wave operator (respectively, the relativistic transport operator on Minkowski space).

5.5. *New and newer vector field methods and other approaches for stability*

After Minkowski space, one would naturally like to understand the stability properties of the other stationary solutions to the Einstein equations, which, in $3+1$ dimensions[18] are given by the Schwarzschild and Kerr family. To describe all the recent progress on this fundamental problem goes well beyond the scope of this article, but we wish only to highlight some developments concerning the approach of Section 3.

First, since these geometries have less symmetry than the Minkowski space, the commutation vector field approach to decay and the strategy presented in Sections 3.1 and 3.2 needs to be considerably altered, see Remark 3.3.

In the context of the linear scalar wave equation $\Box_{g_0}\psi = 0$ on a stationary asymptotically flat geometry, such as the Kerr or Schwarzshild spacetime, a first important new tool is the derivation of local integrated decay estimates, which in simple setting can be stated as

$$\int_0^\infty \int_{r\le R} \mathscr{E}_{r\le R,\chi} \lesssim \mathscr{E}(0), \tag{5.7}$$

where $\mathscr{E}_{r\le R,\chi}(\tau)$ denotes a degenerate energy flux supported in a compact region $r \le R$ which controls the first derivative of ψ in L^2 multiplied by a function $0 \le \chi(r) \le 1$ and $\mathscr{E}(0)$ denotes the initial energy of the solution. The point of the function χ is that it degenerates in the so-called *trapped region*, the physical space location of null geodesics which never escapes to infinity or fall in the black hole in the case of Schwarzschild and Kerr. Starting from an estimate of the form (5.7), one can then try to derive pointwise estimates. The r^p-weighted vector field [46], mentioned already in Section 5.1, provides now a systematic and robust way to derive these pointwise estimates. In particular, one advantage compared to previous methods is that it avoids completely the use of commutator or multiplier vector fields having t weights.

In the recent [74] by Holzegel–Dafermos–Rodnianski–Taylor, yet a new and more robust strategy for the stability of stationary states has been developed. The main feature is that the analysis is essentially localized in dyadic time slabs and the fact that two main ingredients for stability are a "black box" linear inhomogeneous energy estimate on exactly stationary metrics, together with a novel physical space top order identity that need not capture the structure of trapping and is robust to perturbation. On Schwarschild and slowly rotating Kerr, this estimate

[18] Together with the Minkowski space, it is indeed conjectured that these are the only stationary solutions of the vacuum Einstein equations [73] with $\Lambda = 0$ while the picture in higher dimension or with non-vanishing Λ is considerably more complex.

relies essentially on the red-shift effect [75]. For a more precise description of the development of this new strategy (cf. the work of Mavrogiannis [76]), see the introduction of [74].

5.6. *Development of microlocal analysis and other tools for stability problems*

The black hole geometries have all in common the presence of trapped rays. In the high frequency approximation, waves can be localized arbitrarily close to these specific geodesics cf. [77], a clear obstacle to decay, which is then linked to the size and stability properties of the trapped set[19]. One way to capture this analytically is via microlocal and semi-classical analysis, and thus, many recent progress on the stability properties of black holes relies on frequency decomposition and pseudo-differential calculus[20].

Interestingly, at least in slowly rotating Kerr, physical space techniques, exploiting the existence of a Killing tensor field (replacing the role of Killing vector fields), can still be used to prove decay, cf. Andersson–Blue [80], and in fact, the work of Giorgi–Klainerman–Szeftel [81], o ne of the pieces in the recent proof of non-linear stability of slowly rotating Kerr by Klainerman and Szeftel [82], relied on the Andersson–Blue approach.

Beyond pseudo-differential calculus, the *b-calculus* of Melrose has found many applications in general relativity. For instance, in [83], Hintz and Vasy have given a novel proof of the stability of Minkowski space, based on generalized wave coordinates, the weak null structure of Lindblad–Rodnianski and the b-calculus, obtaining as a by-product of these techniques polyhomogeneous expansions of the metric components. The proof of the stability of slowly Kerr–Sitter spacetimes [84] by Hintz–Vasy also relies heavily on such methods.

6. Conclusions: beyond stability in relativity and small data problems

The methods introduced in [7] are now the basic blocks of a large literature concerning evolution problems in hyperbolic geometric partial differential equations, including many works which go beyond the problem of stability of stationary states. For instance, the use of curvature estimates, of the null structure equations and the construction and analysis of optical functions are essential in the other two monumental works of Christodoulou, the formation of trapped surfaces [51] and the formation of shocks [85].

It is also interesting to note that the null condition, which is a key element of the small data theory of quasi-linear waves, as explained in Section 3.4 is also a key structural property for many low-regularity well-posedness issues, starting from the null form estimates of Machedon, [86], to the resolution of the bounded L^2 curvature conjecture by Klainerman–Rodnianski–Szeftel [87]. In fact, the Bel–Robinson tensor is also used in [87] to estimate the curvature, and the null structure[21] which allowed in [7] to control the error terms via decay is now used to prove bilinear and trilinear estimates with improved regularity. Finally, we also remark that [87] also relies on the maximal foliation.

The Lindblad–Rodnianski approach has also had numerous developements. On top of those mentioned in the previous section, which concerns stability problems of stationary solutions,

[19] In particular, the trapped set of Schwarschild or Kerr is *normally hyperbolic*, cf. [78] for a recent analysis on general asymptotically flat spacetimes.

[20] We note also that local integrated decay estimates in this setting are typically replaced by resolvent estimates. See for instance [79] for a recent application of microlocal analysis to the Teukolsky equation on Kerr and the analysis of the resolvent.

[21] Many more ingredients are of course used in [87] to exploit the null condition, such as a quasi-linear Yang–Mills type formulation of the equations.

in [88], Luk–Oh have addresed the stability of large dispersive solutions to the Einstein–scalar–field, which are very far from the Minkowski metric, relying in particular on the understanding of the precise asymptotic behavior of the metric components in generalized wave coordinates, as in [89]. In the recent work of Touati [90] concerning the geometric optics approximation of the Einstein equations, the nilpotent structure and the the weak null condition of the Einstein equations in generalized wave coordinates, as identified by Lindblad–Rodnianski, also plays a key role in the analysis.

In conclusion, as we hope to have convey in this survey article, the global non-linear analysis of the Minkowski space is sufficiently complex that a thorough analysis and a full proof required a deep understanding of geometric quasilinear wave equations, the Einstein equations and their fine structure. Thanks to this, the main ideas introduced in its resolution have and will continue to deeply influence many of the subsequent, important, developments in General Relativity.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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