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Review article / *Article de synthèse*

Burnett's conjecture in general relativity

La conjecture de Burnett en relativité générale

Arthur Touati [a]

[a] CNRS & IMB, Bordeaux University, France
E-mail: arthur.touati@math.u-bordeaux.fr

Abstract. We present the literature on Burnett's conjecture in general relativity, which relate weak limits of vacuum solutions to relativistic kinetic theory. A special care is put on relating these works with early Choquet-Bruhat's results on high-frequency gravitational waves and geometric optics.

Résumé. Nous passons en revue la littérature sur la conjecture de Burnett en relativité générale, qui relie les limites faibles des solutions du vide à la théorie cinétique relativiste. Une attention particulière est portée sur le lien entre ces travaux et les premiers résultats de Choquet-Bruhat concernant les ondes gravitationnelles haute fréquence et l'optique géométrique.

Keywords. Einstein equations, Backreaction, Relativistic kinetic theory, Compensated compactness, Geometric optics, High-frequency gravitational waves.

Mots-clés. Équations d'Einstein, Rétroaction, Théorie cinétique relativiste, Compacité compensée, Optique géométrique, Ondes gravitationnelles haute fréquence.

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1. Introduction

The author of the present text is more than honored to celebrate Yvonne Choquet-Bruhat's legacy in the field of mathematical general relativity, a field that, one could say, she launched in her seminal article of 1952 [1]. There, she built a local existence theory for the Einstein equations of general relativity. Of course, she did not stop there and went on to achieve an incredibly diverse career in general relativity, analysis and mathematical physics over more than 60 years. Her book [2] is the concrete proof of that, and fifteen years after its publication, this scientific testament still contains hidden treasures and research directions only waiting to be explored (see Section 6 for an example linked to this survey's topic). The goal of this survey is to present the mathematical literature on the so-called Burnett conjecture in general relativity. At this stage, we state the conjecture in the roughest way possible: weak limits of solutions to the Einstein vacuum equations can be identified with solutions to the massless Einstein–Vlasov system. Our first introductory task is to explicit the nature of the link between this conjecture and Choquet-Bruhat's work.

At first glance they seem rather unrelated, since the 785 pages of [2] don't contain any reference to Burnett himself, his conjecture or his original 1989 article [3]. However, and this may appease the reader rightfully questioning the presence of this text in this volume, Burnett's article [3] does contain a reference to Choquet-Bruhat's article [4] from 1969. In this article, Choquet-Bruhat, following an approach pioneered by some physicists [5–8], has put on a solid mathematical footing an approximate geometric optics construction for these equations. Since an oscillating behaviour in the limit where the frequency tends to infinity is the simplest example of weak convergence in the mathematical sense, the approximate solutions of [4] fit into Burnett's abstract framework and provide a precursory and neat illustration of the so-called backreaction phenomenon, i.e. the effect of small scale inhomogeneities on a background. This connects Choquet-Bruhat's [4] and Burnett's [3] through the description of a common physics. But the link between these two works has recently been strengthened. Indeed, after being almost completely silent for decades, the mathematical activity around Burnett's conjecture has been revived by the first rigorous result obtained by Green and Wald in 2011 [9]. Since then, Choquet-Bruhat's early approximate construction from [4] has been revisited and its rigorous justification has proved to be an efficient way of attacking Burnett's conjecture.

Because of these two links, an early one from physics and a modern one from mathematics, the narrative of Burnett's conjecture will start in Section 2 by a description of [4]. Another, more prosaic, explanation behind this choice lies in the author's personal interests and his will to honor in this volume a relatively unknown, at least from the mathematical community, aspect of Choquet-Bruhat's work (corresponding to the eleventh chapter of [2]). We also refer the reader to her autobiography [10], where her work on high-frequency gravitational waves is mentioned several times.

Needless to say that the Burnett conjecture and its literature are also connected to many more areas of the mathematical theory of general relativity and of the partial differential equations' world. To highlight the richness of these connections is also one of the objectives of this text, which should be viewed as a complement to other reviews on Burnett's conjecture like the recent [11].

2. Choquet-Bruhat's article from 1969

As early as [12], physicists have tried to linearize the equations of general relativity around simple exact solutions of the Einstein vacuum equations, such as the Minkowski spacetime. This has led early general relativity physicists to think that spacetime itself could be a potential medium for a new kind of waves, called gravitational waves. We refer to Part VIII of [13] for a rich discussion of how gravitational waves propagate, how they are generated and how one can detect them. Chapter 35 of [13] is particularly relevant for the present discussion.

The obvious drawback of any linearization procedure is that it assumes that the perturbation is small, as well as all its derivatives, thus forbidding the description of more interesting physical situations where the gravitational field is strong. Moreover, the linearization procedure rules out the possibility of describing the "energy" of the gravitational waves themselves, since such a quantity, though complicated to define in general relativity, must be of quadratic nature. This has led physicists to imagine various averaging schemes (see for instance [5–8]) unfortunately lacking a rigorous mathematical framework. Thanks to her (at the time) recent experience in nonlinear geometric optics (see [14,15][1]), Yvonne Choquet-Bruhat was the first to rigorously applied the WKB expansion method to the Einstein vacuum equations themselves in the article [4]. We will discuss in more details the general strategy of geometric optics in Section 4.1.1, but for now we summarize the approach and main outcomes of this pioneer article.

In [4], Choquet-Bruhat considers a family of metrics of the form

$$\mathbf{g}_\lambda = \mathbf{g}_0 + \lambda \mathbf{g}^{(1)}\left(\frac{\varphi}{\lambda}\right) + \lambda^2 \mathbf{g}^{(2)}\left(\frac{\varphi}{\lambda}\right), \tag{1}$$

[1] This paper appears to be related to two other versions by the same or similar title, one earlier and one contemporaneous with the present entry. They are, respectively: [16] and [17].

where $\mathbf{g}_0$ is another metric, the phase φ is a scalar function and where by $\mathbf{g}^{(i)}\left(\varphi/\lambda\right)$ we actually mean $\mathbf{g}^{(i)}\left(x,\varphi(x)/\lambda\right)$. Note that we adapt her original notations in order to ease the comparison with more modern results on Burnett's conjecture (in particular those of Section 4.1). We assume moreover that each $\mathbf{g}^{(i)}$ is periodic with respect to the second variable. The metric $\mathbf{g}_\lambda$ should be thought as describing a vacuum spacetime in which a gravitational wave of wavelength λ is propagating. Crucially, the geometric optics expansion (1) allows the perturbation to be small but its derivatives to be large in the high-frequency limit $\lambda \to 0$. The goal is to find conditions on φ, $\mathbf{g}_0$, $\mathbf{g}^{(1)}$ and $\mathbf{g}^{(2)}$ such that $\mathbf{g}_\lambda$ is an approximate solution of the Einstein vacuum equations at order 1, that is $R_{\mu\nu}(\mathbf{g}_\lambda) = O(\lambda)$. For that, we compute the Ricci tensor of the metric $\mathbf{g}_\lambda$ defined by (1), and because of the curvature being second order with respect to the metric, it is of the form

$$R_{\mu\nu}(\mathbf{g}_\lambda) = \frac{1}{\lambda}R^{(-1)}_{\mu\nu} + R^{(0)}_{\mu\nu} + O(\lambda), \tag{2}$$

where $R^{(-1)}_{\mu\nu}$ depends on $\mathbf{g}_0$ and $\mathbf{g}^{(1)}$ and $R^{(0)}_{\mu\nu}$ depends on $\mathbf{g}_0$, $\mathbf{g}^{(1)}$ and $\mathbf{g}^{(2)}$. In order to formulate the main result of [4], we introduce the polarization tensor of a tensor T with respect to a phase φ:

$$\mathbf{P}\left[T|\varphi\right]_\alpha := \mathbf{g}_0^{\mu\nu}\left(\partial_\mu\varphi T_{\nu\alpha} - \tfrac{1}{2}\partial_\alpha\varphi T_{\mu\nu}\right). \tag{3}$$

Theorem 1 ([4]). *Consider $\mathbf{g}_\lambda$ defined by (1) and $R^{(-1)}_{\mu\nu}$ and $R^{(0)}_{\mu\nu}$ as in (2).*

(i) $R^{(-1)}_{\mu\nu} = 0$ *if and only if* $\mathbf{g}_0^{-1}(\mathrm{d}\varphi,\mathrm{d}\varphi) = 0$ *and* $\mathbf{g}^{(1)}$ *satisfies the polarization condition*

$$\mathbf{P}\left[\mathbf{g}^{(1)}|\varphi\right] = 0. \tag{4}$$

(ii) *If* $R^{(-1)}_{\mu\nu} = 0$*, then* $R^{(0)}_{\mu\nu} = 0$ *if and only if* $\mathbf{g}_0$ *satisfies*

$$R_{\mu\nu}(\mathbf{g}_0) = \left\langle \tfrac{1}{4}\mathbf{g}_0^{\rho\sigma}\mathbf{g}_0^{\alpha\beta}\left(\partial_\theta\mathbf{g}^{(1)}_{\rho\alpha}\partial_\theta\mathbf{g}^{(1)}_{\sigma\beta} - \tfrac{1}{2}\partial_\theta\mathbf{g}^{(1)}_{\rho\sigma}\partial_\theta\mathbf{g}^{(1)}_{\alpha\beta}\right)\right\rangle_\theta \partial_\mu\varphi\partial_\nu\varphi, \tag{5}$$

$\mathbf{g}^{(1)}$ *satisfies the transport equation*

$$2\partial^\alpha\varphi\mathbf{D}_\alpha\mathbf{g}^{(1)} + (\Box_{\mathbf{g}}\varphi)\mathbf{g}^{(1)} = 0, \tag{6}$$

and $\mathbf{g}^{(2)}$ *satisfies a polarization condition of the form* $\mathbf{P}\left[\mathbf{g}^{(2)}|\varphi\right] = \mathbf{g}^{(1)}\star\mathbf{g}^{(1)}$*, where* $\star$ *denotes some tensor contractions.*

In (5), ∂_θ and $\langle\cdot\rangle_\theta$ denote respectively a derivative and the average with respect to the oscillating variable. Note that in Theorem 1, the case $\mathbf{g}_0^{-1}(\mathrm{d}\varphi,\mathrm{d}\varphi) \neq 0$ is excluded for physical reasons, since in this case a high-frequency change of variables can remove $\mathbf{g}^{(1)}$. Though Theorem 1 only produces approximate solutions to the Einstein vacuum equations, it provides a more rigorous averaging scheme than the results [5–8]. It shows how one can describe high-frequency gravitational waves in the nonlinear regime while still retaining some surprising linear features. For instance, as in the linearized gravity setting, the wave propagate along a null direction and the polarization condition 4 is actually equivalent to the TT-gauge conditions of linearized gravity which imply in particular that a gravitational wave has two degrees of freedom. Moreover, the (surprisingly linear) transport equation (6) is shown to propagate the polarization condition (4), so that the wave retains its form! Physically, this distinguishes greatly gravitational waves from other kind of waves, say in fluids, since it shows that the former don't self-steepen while the latter usually can accumulate and form shocks.

In addition to exhibiting this exceptional linear structure, Theorem 1 also shows the nonlinear effects caused by the wave on the background spacetime $\mathbf{g}_0$. Indeed, even though $\mathbf{g}_\lambda$ solves (approximately) the Einstein vacuum equations, this is not the case of $\mathbf{g}_0$ which "sees" the presence of the wave propagating, via the non-zero stress energy tensor in the right hand side of (5) (note that this expression, here derived rigorously, matches those found with other averaging schemes). In this sense, theorem 1 clearly prefigures Burnett's conjecture, which aims at characterizing completely the backreaction phenomenon (see Section 3 below).

We also mention subsequent works in the framework of [4], most of them being by Choquet-Bruhat herself. All these works show how important to her this direction of research was, from early to later stages of her career. High-frequency waves in relativistic fluids are considered in [14,15,18]. The coupling between gravitational waves and electromagnetic or fluid waves are the objects of [19] and [20] respectively. The case of a charged scalar field is considered in [21] and the one of the Yang–Mills equations in [22]. She even considers high-frequency gravitational waves with Gauss–Bonnet corrections or with stringy effects in [23] and [24] respectively. As we will see, the results on Burnett's conjecture discussed in Section 4.1 can also be thought as extensions of the pioneer work of Yvonne Choquet-Bruhat on high-frequency gravitational waves.

3. Statement of Burnett's conjecture

In this Section, we first introduce the massless Einstein–Vlasov system, state and discuss conceptualy Burnett's conjecture, and present the first rigorous result on the conjecture.

3.1. *Relativistic kinetic theory*

Relativistic kinetic theory is the name given to the physical description of a system of particles living in a spacetime described by general relativity, see the tenth chapter of [4] or [25]. While in classical kinetic theory the particles in consideration are most often gas or liquid particles, in general relativity, this perspective is adopted to describe "particles" at very different scales, such as stars, galaxies or even clusters of galaxies. Nevertheless, the framework of relativistic kinetic theory is identical to its classical counterpart: particles are described by a density function defined in phase space and solving a PDE encoding the interactions between the particles, and integrating this density with respect to the momentum variable provides macroscopic information on the system of particles.

If $(\mathscr{M},\mathbf{g})$ is a Lorentzian manifold, we first look for an equivalent of the classical phase space encoding both position in space and time and velocity or momentum. Since tangent vectors to curves are represented by vector fields in differential geometry, the phase space in relativistic kinetic theory will be a particular subset of the tangent bundle $T\mathscr{M}$, the latter consisting of pairs (x,p) where $x\in\mathscr{M}$ and $p\in T_x\mathscr{M}$. More precisely, for $m\geq 0$ we define the mass shell at a point x by

$$\mathscr{P}_{m,x} := \left\{p\in T_x\mathscr{M} \mid \mathbf{g}\right\}_{|_x}(p,p) = -m^2 \quad \text{and} \quad p \text{ is future oriented}.$$

If we set $\mathscr{P}_m := \bigsqcup_{x\in\mathscr{M}}\mathscr{P}_{m,x}$, then a density f describing particles of (rest) mass m is simply a function $f:\mathscr{P}_m\longrightarrow\mathbb{R}_+$, and $f(x,p)$ is the "number" of particles at position $x\in\mathscr{M}$ with momentum $p\in T_x\mathscr{M}$. To lighten the notations, in what follows we don't write down the measure on $\mathscr{P}_{m,x}$ induced by the metric $\mathbf{g}$.

We will assume here the absence of collisions between the particles in the system, so that the particles interact with each other only through their interaction with other fields. If moreover we neglect the charges of the particles, the only reasonable such field is the gravitational one and such a matter model is sometimes referred to as self-gravitating. This means that each particle is following a geodesic on $\mathscr{M}$, which is equivalent to the following Vlasov-type equation for the distribution function f:

$$p^\alpha\partial_\alpha f - \Gamma^\alpha_{\mu\nu}p^\mu p^\nu\partial_{p^\alpha}f = 0. \tag{7}$$

If $(\mathscr{M},\mathbf{g})$ is given, the behaviour of the solutions to this equation is of tremendous physical importance. However, as always in general relativity, we can also describe the effect of a matter model on the Lorentzian metric $\mathbf{g}$ itself by plugging its stress energy tensor into the right hand side of the Einstein equations. In the case of kinetic theory, the stress energy tensor is simply the second moment of the density f, which leads us to the so-called Einstein–Vlasov system with rest mass m:

$$\begin{cases} R_{\mu\nu}(\mathbf{g}) - \dfrac{1}{2}R(\mathbf{g})\mathbf{g}_{\mu\nu} = \displaystyle\int_{\mathscr{P}_{m,x}} f p_\mu p_\nu, \\ p^\alpha \partial_\alpha f - \Gamma^\alpha_{\mu\nu} p^\mu p^\nu \partial_{p^\alpha} f = 0. \end{cases} \tag{8}$$

In the context of (8), the Vlasov equation for f can also be interpreted as the condition for the stress energy tensor to have zero divergence. In the case $m = 0$, i.e. when the particles in consideration have zero rest mass, the system (8) is called the massless Einstein–Vlasov system. Note that in this case, the mass shell $\mathscr{P}_0$ is simply the collection of null cones for the metric $\mathbf{g}$ over each point of the manifold.

The literature on the system (8) is very rich and is nowadays concerned with global existence results, stationary solutions etc. Note that an important step was the proof of well-posedness in wave coordinates in [26]. Finally, as is standard in general relativity, we expect the limit $c \to +\infty$ to correspond to the classical Newtonian gravity. This is indeed true for relativistic kinetic theory, since it is proved in [27] that the famous Vlasov–Poisson system is the Newtonian limit of (8).

3.2. *First discussion of the conjecture*

We are now ready to state Burnett's conjecture from [3]. For that, fix a manifold $\mathscr{M}$ and consider sequences of metrics $(\mathbf{g}_\lambda)_{\lambda\in(0,1]}$ solving the Einstein vacuum equations

$$R_{\mu\nu}(\mathbf{g}_\lambda) = 0, \tag{9}$$

and such that there exists another metric $\mathbf{g}_0$ such that in some coordinates system $(x^\alpha)_{\alpha=0,1,2,3}$ on $\mathscr{M}$ we have

$$(\mathbf{g}_\lambda)_{\alpha\beta} - (\mathbf{g}_0)_{\alpha\beta} \longrightarrow 0 \quad \text{uniformly}, \quad \partial_\mu\left((\mathbf{g}_\lambda)_{\alpha\beta} - (\mathbf{g}_0)_{\alpha\beta}\right) \longrightarrow 0 \quad \text{weakly}, \tag{10}$$

when the parameter λ tends to 0. Note that instead of the second assumption in (10), one can ask the sequence of first order derivatives $\left(\partial_\mu(\mathbf{g}_\lambda)_{\alpha\beta}\right)_{\lambda\in(0,1]}$ to be bounded.

Conjecture 2 (Burnett's conjecture). *One conjectures that:*

- *If* $(\mathbf{g}_\lambda)_{\lambda\in(0,1]}$ *and* $\mathbf{g}_0$ *are as above, then* $\mathbf{g}_0$ *is a solution to the massless Einstein–Vlasov system (8) for a suitable density.*
- *Conversely, if* $\mathbf{g}_0$ *is a solution to the massless Einstein–Vlasov system (8), then there exists a sequence* $(\mathbf{g}_\lambda)_{\lambda\in(0,1]}$ *as above and with limit* $\mathbf{g}_0$.

The above conjecture aims at completely describing backreaction for the Einstein vacuum equations, i.e. the effect of small scales inhomogeneities onto a background spacetime. These inhomogeneities are described by the lack of strong convergence of the first derivatives $\partial_\mu\left((\mathbf{g}_\lambda)_{\alpha\beta} - (\mathbf{g}_0)_{\alpha\beta}\right)$. In his original article, Burnett refers to this weak convergence as being the trace of some high-frequency oscillations in the metric itself. However, this regime also includes concentration effects, and only two references in the sequel of this text will allow for concentration effects, while all the others will be concerned with a true high-frequency regime, meaning where (10) is replaced by the stronger

$$\left|\partial^k\left((\mathbf{g}_\lambda)_{\alpha\beta} - (\mathbf{g}_0)_{\alpha\beta}\right)\right| \leq \lambda^{1-k}, \quad k = 0, \ldots, K,$$

for some $K \geq 2$.

The core idea behind Conjecture 2 is the following. If first order derivatives of $\mathbf{g}_\lambda$ converge only weakly, then products $\partial\mathbf{g}_\lambda\partial\mathbf{g}_\lambda$ of these derivatives don't necessarily converge to $\partial\mathbf{g}_0\partial\mathbf{g}_0$, and since the Ricci tensor contains such semilinear terms, passing to the limit in (9) does not lead a priori to $R_{\mu\nu}(\mathbf{g}_0) = 0$ but rather to

$$R_{\mu\nu}(\mathbf{g}_0) - \frac{1}{2}R(\mathbf{g}_0)(\mathbf{g}_0)_{\mu\nu} =: T^{\text{effective}}_{\mu\nu},$$

where $T^{\text{effective}}_{\mu\nu}$ can be called the effective stress energy tensor of the inhomogeneities. The first part of Conjecture 2 can be thus rephrased as: there must exist some density function f defined on $\mathscr{P}_0$ (defined with the metric $\mathbf{g}_0$) such that

$$T^{\text{effective}}_{\mu\nu} = \int_{\mathscr{P}_{0,x}} f p_\mu p_\nu. \tag{11}$$

The effective stress energy tensor $T^{\text{effective}}_{\mu\nu}$ is thus caused by the lack of convergence of the semilinear terms $\partial\mathbf{g}_\lambda\partial\mathbf{g}_\lambda$. This shows that in the particular case (obviously included in Conjecture 2) where the convergence of $\partial_\mu\left((\mathbf{g}_\lambda)_{\alpha\beta} - (\mathbf{g}_0)_{\alpha\beta}\right)$ is actually strong, then we can pass to the limit in (9) and $\mathbf{g}_0$ thus solves the Einstein vacuum equations.

As explained in the introduction of [3], the fact that backreaction is necessarily of kinetic nature can be first observed on plane wave solutions to (9), where $\mathbf{g}_\lambda$ is explicitly given by

$$-\mathrm{d}u\mathrm{d}v + B_\lambda(u)^2\left(\mathrm{e}^{\lambda\alpha(u)\cos\left(\frac{u}{\lambda}\right)}\mathrm{d}x^2 + \mathrm{e}^{-\lambda\alpha(u)\cos(u/\lambda)}\mathrm{d}y^2\right),$$

where α is a given function of u and where B_λ needs to solve an ODE. One can show that $\mathbf{g}_\lambda$ converge in the sense (10) to $\mathbf{g}_0$ satisfying $R_{\mu\nu}(\mathbf{g}_0) = 1/2\alpha^2\partial_\mu u\partial_\nu u$. Formally summing such plane waves gives rise to the integrated expression in (11). However, and though Burnett does not mention it, the same kind of "discrete" effective stress energy tensor is obtained in the approximate geometric optics construction of [4] described in Section 2 (recall (5)). Burnett's explicit example has the benefit of being a family of exact solutions to (9), while Choquet-Bruhat's approximate solutions have the benefit of living outside any symmetry class. Moreover, as Section 4.1 will show, Choquet-Bruhat's strategy can be extended to produce exact solutions. However, Burnett's and Choquet-Bruhat's concrete examples of backreaction have one thing in common: the background kinetic spacetime metric $\mathbf{g}_0$ is a particular measure-valued solution of the massless Einstein–Vlasov system. More precisely, by considering several solutions of the eikonal equations $(u_{\mathbf{A}})_{\mathbf{A}\in\mathscr{A}}$ for $\mathscr{A}$ a finite set, we can consider the density function $f(x,p) = \sum_{\mathbf{A}\in\mathscr{A}} F^2_{\mathbf{A}}(x)\otimes\delta_{p=(\mathrm{d}u_{\mathbf{A}})^\#}$, where $\delta_{p=(\mathrm{d}u_{\mathbf{A}})^\#}$ denotes the Dirac measure at $(\mathrm{d}u_{\mathbf{A}})^\#$ in each tangent space and where the $F_{\mathbf{A}}$'s are scalar functions. In this case the massless Einstein–Vlasov becomes the so-called Einstein-null dusts system:

$$\begin{cases} R_{\mu\nu}(\mathbf{g}_0) = \sum_{\mathbf{A}\in\mathscr{A}} F^2_{\mathbf{A}}\partial_\mu u_{\mathbf{A}}\partial_\nu u_{\mathbf{A}}, \\ \mathbf{g}_0^{-1}(\mathrm{d}u_{\mathbf{A}},\mathrm{d}u_{\mathbf{A}}) = 0, \quad \text{for all } \mathbf{A}\in\mathscr{A}, \\ 2\mathbf{g}_0^{\alpha\beta}\partial_\beta u_{\mathbf{A}}\partial_\alpha F_{\mathbf{A}} + \left(\Box_{\mathbf{g}_0}u_{\mathbf{A}}\right)F_{\mathbf{A}} = 0, \quad \text{for all } \mathbf{A}\in\mathscr{A}, \end{cases} \tag{12}$$

The system (12) can be viewed as a "discrete" version of the massless Einstein–Vlasov system (8), but it also corresponds to the Einstein equations coupled to a perfect null fluid with zero pressure. Other examples of weak limits of vacuum spacetimes have been considered with two-dimensional symmetry, such as the $\mathbb{T}^2$ symmetry in [28].

3.3. *What does it say about the structure of the Einstein vacuum equations?*

Even before discussing actual results on Conjecture 2, we wish to reflect here on its significance. We start by noticing that, as it is apparent from its statement, Conjecture 2 is a double conjecture. Its two parts are of very different mathematical nature.

The first part, referred to as the *direct Burnett conjecture*, aims at describing the generic behaviour of sequences of solutions to the Einstein vacuum equations. In particular, it shows that this set is not closed for the topology of (10). Since this backreaction is due to a lack of strong convergence, results on the direct Burnett conjecture fall into the scope of homogenization theory (we refer to [29] for an introduction to this field, which can be roughly defined as the study of averaging processes in disordered media), and the direct Burnett conjecture is an instance of compensated compactness. On the other hand, the second part of Conjecture 2, referred to as the *reverse Burnett conjecture*, aims at constructing solutions to the Einstein vacuum equations that converge in the sense of (10) to a given kinetic target. If it holds together with the direct part, then the set of massless kinetic spacetimes completely characterizes the closure of the set of solutions to the Einstein vacuum equations for the topology of (10). Therefore, if Burnett's conjecture is true, the Einstein vacuum equations stand in between two kinds of PDEs:

- PDEs for which the set of solutions is weakly closed, such as semilinear wave systems with the classical null condition of [30,31]. In this case, compensated compactness can be seen as a consequence of the famous div-curl lemma (see [32,33]). Such PDEs are said to display *rigidity*.
- PDEs for which weak limits of solutions are somehow arbitrary, such as incompressible fluid equations. In the case of incompressible Euler, weak limits of solutions are shown to solve the Euler–Reynolds system (see [34] or the survey [35]), which plays there a similar role as the massless Einstein–Vlasov. As opposed to the latter, the Euler–Reynolds system is underdetermined, thus leading to the display of *flexibility* (see the resolution of Onsager's conjecture in [36] for a consequence of this flexibility).

Therefore, one could say that the Einstein vacuum equations display some *flexible rigidity* or *rigid flexibility* (we let the reader decide which one is more accurate).

Not only does Burnett's conjecture distinguish the Einstein vacuum equations from incompressible fluid equations, in the sense that the failure of convergence satisfies a transport equation, it also reveals a hidden linear behaviour, despite the equations being very much nonlinear (already observed by Choquet-Bruhat in [4], see the discussion after Theorem 1). Indeed, the fact that backreaction can be described by the massless Einstein–Vlasov system, i.e. a self-gravitating matter model, shows that small scale inhomogeneities don't interact directly but only through their impact on the background metric. In other words, the linearity of the Vlasov equation with respect to the density, or the fact that no collision operator shows up in the right hand side, is a second feature of the Einstein vacuum equations. Depending on the context, these surprising nonlinear cancellations can be related to compensated compactness again (for instance in [11] it is described as a secondary form of compensated compactness), to the classical null condition and Shatah's normal form approach (see [37]) in semilinear waves or to transparency in geometric optics (see [38]). The deep link between all these approaches is discussed in depth in the Bourbaki seminar [39].

However, it is very well-known from the works on stability of Minkowski (see [40,41]) that the Einstein vacuum equations don't satisfy the null condition in wave coordinates. It is complicated to directly compare the structure of the Einstein vacuum equations in different gauges, but if Burnett's conjecture were to be true in full generality, it would show that they don't satisfy the null condition in any gauge or setting, since otherwise backreaction would not occur and vacuum spacetimes would not approach anything else but other vacuum spacetimes. Burnett's conjecture thus hints at a very fine structure: semilinear terms in the Einstein vacuum equations fail to satisfy the null condition (otherwise there would be no backreaction) but the failure of the null condition has a unique structure still leading to some cancellations (otherwise the Vlasov equation would be nonlinear).

In [42], Choquet-Bruhat introduced gauge-invariant definitions of the null condition and of a relaxation of it, but perhaps the most famous relaxed version of the null condition in wave coordinates is the weak null condition introduced in [43]. One could then wonder if the aforementioned unique structure of the failure of the null condition is precisely Lindblad and Rodnianski's weak null condition. This is in fact not the case and if true, Burnett's conjecture thus demonstrates that the Einstein vacuum equations possess an even finer structure, strictly in between the null condition and the weak null condition. This last fact actually brings Burnett's conjecture closer to the bounded L^2 curvature theorem proved in [44], which (to the best of the author's knowledge) is the only other instance where the Einstein vacuum equations behave better than a system satisfying the weak null condition. The link between Burnett's conjecture and low-regularity solutions in general relativity is actually even stronger, but we postpone this discussion to Section 3.5.

3.4. *First rigorous result in the context of cosmology*

Before discussing in the rest of this survey the "PDE" results on Conjecture 2, we start with a result coming from the physics community. Obtained by Green and Wald in [9], this is the first mathematically rigorous result on Conjecture 2 since Burnett's original article [3].

Their motivation comes from cosmology, where a wealth of literature has been produced on the question of the effect of small scale inhomogeneities in the universe we live in. In particular, backreaction from these inhomogeneities onto the cosmological background has been believed to be a potential explanation of the presence of dark matter in our universe, in replacement of the standard explanation via the presence of (so far undetected) weakly-interacting "dark particles" or modifications of Einstein's gravity, see for instance [45,46]. However, if small scale inhomogeneities are described by sequences of metrics with the convergence regime (10), then Burnett's Conjecture 2 contradicts this belief, since the effect of backreaction is characterized by a massless Vlasov field. This motivated the work [9], where the authors get the following theorem:

Theorem 3 ([9]). *If $(\mathbf{g}_n)_{n\in\mathbb{N}}$ is a sequence of solutions to*

$$R_{\mu\nu}(\mathbf{g}_n) - \frac{1}{2}R(\mathbf{g}_n)(\mathbf{g}_n)_{\mu\nu} + \Lambda(\mathbf{g}_n)_{\mu\nu} = (T_n)_{\mu\nu},$$

where each T_n is trace free and satisfy the weak energy condition with respect to $\mathbf{g}_n$, and if $(\mathbf{g}_n)_{n\in\mathbb{N}}$ converges to some $\mathbf{g}_0$ as in (10), then

$$R_{\mu\nu}(\mathbf{g}_0) - \frac{1}{2}R(\mathbf{g}_0)(\mathbf{g}_0)_{\mu\nu} + \Lambda(\mathbf{g}_0)_{\mu\nu} = (T_0)_{\mu\nu},$$

where T_0 is trace free and satisfy the weak energy condition with respect to $\mathbf{g}_0$.

The conclusion of Theorem 3 completely rules out the possibility of dark matter being described by backreaction under the convergence regime (10) (see also [47] where the authors provide interesting explicit examples of such regime). Note also that the conclusion of Theorem 3 are in accordance with Burnett's conjecture, since the stress energy tensor of a massless Vlasov field is indeed traceless and satisfies the weak energy condition. Moreover, Theorem 3 fails to describe precisely the structure of the effective stress energy tensor, or the fact that the Vlasov transport equation is satisfied. However, some features distinguish it from the "PDE" results described in Sections 4 and 5 below: it allows for both cosmological constant and sequences of non-vacuum spacetimes, it does not require any symmetry nor gauge choice, it allows for both concentration and oscillation. For all these reasons, Theorem 3 reaches a high level of generality compared to the PDE results.

In the physics community, the article [9] has led to strong debates on the nature of backreaction, see for instance [48,49]. It also provoked an important increase in the number of citations of Burnett's original article [3], and it raised the interest of mathematicians and analysts working on the Einstein equations.

3.5. *Low-regularity and two different strategies*

In Sections 4 and 5 below, we will present the results [50–57] on Burnett's Conjecture 2. In order to motivate the splitting of the results between these two sections and their inner structures, we need to address a crucial feature of Burnett's conjecture, already hinted at in Section 3.3. To put it plainly, Burnett's conjecture is intrinsically a low-regularity phenomenon from the point of view of the Einstein vacuum equations. Indeed, the most generic well-posedness result for the Einstein vacuum equations is the celebrated bounded L^2 curvature theorem, which can be formulated as follows:

Theorem 4 ([44]). *The time of existence, with respect to a maximal foliation, of a classical solution to the Einstein vacuum equations depends only on the L^2 norm of the curvature and second fundamental form of the initial data set and on a lower bound of its volume radius.*

In other words, the Einstein vacuum equations are well-posed in H^s if $s \geq 2$. Theorem 4 is expected to be sharp, meaning that the *if* in the previous sentence is supposedly an *if and only if*[2]. However, in order to produce a nontrivial backreaction, a sequence $(\mathbf{g}_\lambda)_{\lambda\in(0,1]}$ of solutions to the Einstein vacuum equations must necessarily be unbounded in any H^s for $s > 1$, since otherwise the convergence of $\partial_\mu\big((\mathbf{g}_\lambda)_{\alpha\beta} - (\mathbf{g}_0)_{\alpha\beta}\big)$ would be strong. This shows that Burnett's conjecture is a low-regularity phenomenon and that the mere existence of sequences $(\mathbf{g}_\lambda)_{\lambda\in(0,1]}$ of solutions converging to some $\mathbf{g}_0$ on a time interval uniform in λ and being unbounded in H^s for $s > 1$ is not at all guaranteed by the general theory (said differently, the best we can do a priori is a time of existence of order λ). Since such sequences are the starting point of any theorems on the direct conjecture, they might be empty theorems. This leaves us with two possible strategies:

- First, if in a given setting (either a symmetry class or a particular gauge choice), no improvement of Theorem 4 is available, then one needs to construct concrete examples of such sequences unbounded in H^s for $s > 1$ bypassing the general obstruction, which is equivalent to proving the reverse conjecture! This motivates discussing first the reverse conjecture in the two frameworks fitting this category, and then the corresponding results on the direct conjecture (which are then ensured to be non-empty).
- Second, if in a given setting it is possible to somewhat improve Theorem 4, in the sense that the time of existence of a solution in this setting can be shown to depend only on the H^1 norm of the data in *some* directions, then the existence of the unbounded sequences would be ensured. As we will see, in this case proving the full Conjecture 2 actually reduces to proving its direct part.

Section 4 will present the results fitting into the first category, i.e. [50–54,56,57], in which Burnett's conjecture is studied in $\mathbb{U}(1)$ symmetry and generalized wave coordinates. Section 5 will present the only result fitting into the second catagory, i.e. [55], which is concerned with angularly regular spacetimes in double null gauge.

4. The $\mathbb{U}(1)$ symmetry and generalized wave coordinates

In this section, we discuss first the results [51,54,56,57], which attack the reverse conjecture via multiphase geometric optics constructions, and then the results [50,52,53], which prove the direct conjecture by means of microlocal defect measures.

[2] The presence of symmetries can lower the regularity threshold, see for instance Christodoulou's BV theory in spherical symmetry [58].

4.1. *The reverse conjecture*

As explained in Section 3.5, without any improvement of the general theory, proving the reverse conjecture amounts to the construction of particular solutions defined by special ansatz. As mentioned in Section 3.2, oscillation and concentration are the two different phenomena that, if present at the level of the first order derivatives of the metric, can cause boundedness in H^1 and unboundedness in H^s for $s > 1$ as required by the Burnett regime (10). If one chooses oscillation, then one actually studies the geometric optics approximation for the Einstein vacuum equations already considered by Choquet-Bruhat in [4].

4.1.1. *Geometric optics*

Broadly speaking, the field of geometric optics is concerned with the propagation of waves, and in particular with the description of rays out of it (see [59,60] for a more in depth introduction to this field). Concretely, the goal is to construct high-frequency solutions to nonlinear hyperbolic systems. These solutions are defined via multiscale asymptotic expansions of the form

$$u^{\varepsilon} \sim \varepsilon^{p} \sum_{n\geq 0} \varepsilon^{n} U_{n}\left(t, x, \frac{\varphi}{\varepsilon}\right), \tag{13}$$

where ε is a small parameter and where the U_n are periodic with respect to their third variable. This ansatz is very much in the spirit of WKB expansions (after Wentzel, Kramers and Brillouin) from the 1920's used to recover classical mechanics from quantum mechanics. In the case considered in [61,62] of a linear hyperbolic system, the oscillating profiles U_n can be shown to solve a hierarchy of transport equations along the "rays" defined by the phase φ, which needs to solve the eikonal equation associated to the differential operator. The first to have considered nonlinear systems is actually Yvonne Choquet-Bruhat herself, after having followed Leray's ("mon maitre", as she calls him in [10]) lectures at Princeton, see [14,15]. Of course, the article [4] is another instance of a geometric optics being applied to a nonlinear system, but with the subtlety that for the Einstein vacuum equations true hyperbolicity comes at a cost of a gauge choice. See also [63,64] for examples of geometric optics constructions on gauge-invariant semilinear systems. Given a nonlinear system, the basic questions are: can we construct formal solutions of the form (13)? are these approximate solutions stable, i.e. can we solve for a remainder in the expansion and transform them into exact solutions? In general, this is a very difficult problem strongly depending on the system and on the strength of the wave.

The parameter $p \geq 0$ in (13) describes the strength of the wave. If p is too large, no nonlinear effects will be observed in the transport hierarchy for the profiles. If one decreases the value of p, one can reach a particular value where nonlinear interactions terms enter the transport equation for the first profile, thus making it nonlinear. This is the regime of weakly nonlinear geometric optics, see [65,66] for classical results. For quadratic systems such as the Einstein equations, the critical value is $p = 1$. We remark crucially that [4] considers precisely the weakly nonlinear regime for the Einstein vacuum equations, and that if $p = 1$ then an ansatz of the form (13) precisely enters Burnett's convergence regime (10). In conclusion, we see that proving the reverse Burnett conjecture via oscillations is equivalent to proving the stability of the geometric optics approximation for the Einstein vacuum equations in the weakly nonlinear regime.

4.1.2. *The* $\mathbb{U}(1)$ *symmetry*

The first setting where the reverse Burnett conjecture has been studied is the one of $\mathbb{U}(1)$ symmetry, that is when the spacetime has a spacelike Killing vector field. The reduction of the 3+1 Einstein vacuum equations in this case can be deduced from the general Kaluza–Klein theory (found for instance in Appendix VII of [2]). A general metric on a $3+1$ manifold with ∂_{x^3} Killing is of the form

$$\mathbf{g} = \mathrm{e}^{-2\varphi} g + \mathrm{e}^{2\varphi}\left(\mathrm{d}x^3 + A_\alpha \mathrm{d}x^\alpha\right),$$

where g is a $2+1$ Lorentzian metric, φ is a scalar function and A a 1-form. One can prove that the Einstein vacuum equations for $\mathbf{g}$ as above are equivalent to the following Einstein-wave map system in $2+1$ dimension

$$\begin{cases} \Box_g \varphi = -\frac{1}{2}\mathrm{e}^{-4\varphi} g^{-1}(\mathrm{d}\omega, \mathrm{d}\omega), \\ \Box_g \omega = 4 g^{-1}(\mathrm{d}\omega, \mathrm{d}\varphi), \\ R_{\mu\nu}(g) = 2\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}\mathrm{e}^{-4\varphi}\partial_\mu\omega\partial_\nu\omega, \end{cases} \tag{14}$$

where ω is related to A. This system has been studied for instance in [67,68], motivated by its connection with cosmology, and the constraint equations for (14) are solved in [69]. Several choices of gauge are possible for (14), for instance generalized wave coordinates in [70] or null geodesic in [71], but it is the elliptic gauge which proves to be useful for Burnett's conjecture (or other low-regularity problems such as [72,73]). It is defined in the following way: after having decomposed g with lapse and shift, we ask its spacelike part to be conformal to the Euclidean metric (thanks to the uniformization theorem of surfaces) and each time slice to be maximal, i.e. with zero mean curvature. We now state the main result of [51] by Huneau and Luk:

Theorem 5 ([51]). *If $\left(g_0, \varphi_0, \omega_0, F_{\mathbf{A}}, u_{\mathbf{A}}\right)$, with $\mathbf{A}$ belonging to a finite set, is a generic solution on $[0,1]\times\mathbb{R}^2$ of the Einstein-null dusts system in $\mathbb{U}(1)$ symmetry in the elliptic gauge and close to Minkowski, then there exists a sequence $(g_\lambda, \varphi_\lambda, \omega_\lambda)_{\lambda\in(0,1]}$ of solutions of (14) in the elliptic gauge on $[0,1]\times\mathbb{R}^2$ and such that $(g_\lambda, \partial g_\lambda, \varphi_\lambda, \omega_\lambda) \longrightarrow (g_0, \partial g_0, \varphi_0, \omega_0)$ uniformly and $(\partial\varphi_\lambda, \partial\omega_\lambda) \longrightarrow (\partial\varphi_0, \partial\omega_0)$ weakly when λ tends to 0.*

The Einstein-null dusts system in $\mathbb{U}(1)$ symmetry mentioned in Theorem 5 is the equivalent of (14) with the addition of the stress energy tensor of the null dusts system, see (12). The existence of the background solution $\left(g_0, \varphi_0, \omega_0, F_{\mathbf{A}}, u_{\mathbf{A}}\right)$ follows from a low-regularity existence result in elliptic gauge from [74]. If we denote by $\mathfrak{g}$ either the lapse, the shift or the conformal factor, and by ψ either φ or ω, the system (14) in elliptic gauge can be rewritten in the following schematic form

$$\begin{cases} \Box_g \psi = g^{-1}(\mathrm{d}\psi, \mathrm{d}\psi), \\ \Delta\mathfrak{g} = (\partial\psi)^2 + (\nabla\mathfrak{g})^2. \end{cases} \tag{15}$$

The sequence $(\mathfrak{g}_\lambda, \psi_\lambda)_{\lambda\in(0,1]}$ is defined by a high-frequency ansatz, a simplified version of which being

$$\begin{aligned} \psi_\lambda &= \psi_0 + \lambda\sum_{\mathbf{A}} \psi^{(1)}_{\mathbf{A}}\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \lambda^2\left(\sum_{\mathbf{A}} \psi^{(2)}_{\mathbf{A}}\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \sum_{\mathbf{A}\neq\mathbf{B},\pm} \psi^{(2,\pm)}_{\mathbf{AB}}\left(\frac{u_{\mathbf{A}} \pm u_{\mathbf{B}}}{\lambda}\right)\right) + \text{remainder}, \\ \mathfrak{g}_\lambda &= \mathfrak{g}_0 + \lambda^2\left(\sum_{\mathbf{A}} \mathfrak{g}^{(2)}_{\mathbf{A}}\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \sum_{\mathbf{A}\neq\mathbf{B},\pm} \mathfrak{g}^{(2,\pm)}_{\mathbf{AB}}\left(\frac{u_{\mathbf{A}} \pm u_{\mathbf{B}}}{\lambda}\right)\right) + \text{remainder}, \end{aligned} \tag{16}$$

where, for instance, by $f(u_{\mathbf{A}}/\lambda)$ we mean that f is a quantity depending in a periodic manner on $u_{\mathbf{A}}/\lambda$ (as the profiles U_n in (13)). By plugging (16) in (15), we derive transport equations for the profiles $\psi^{(i)}$ and "algebraic" equations for the profiles $\mathfrak{g}^{(i)}$ (the solvability of the latter requires a coherence assumption on the $u_{\mathbf{A}}$'s, as it is common in multiphase geometric optics). Solving for the remainders in (16) over the required time scale is the most technical part of the proof and relies crucially on the structure of (15). From the present perspective, this structure can be summarized as follows: the dynamics and the geometry are completely decoupled into a quasilinear hyperbolic system with the null condition and a semilinear elliptic system without the null condition. This observation explains why there is no λ term in $\mathfrak{g}_\lambda$, or equivalently why $\partial\mathfrak{g}_\lambda$ converges strongly, as opposed to $\partial\psi_\lambda$. It also sheds light on the discussion of Section 3.3: the lack of null condition is compensated here by an elliptic operator, very much in the spirit of the

Coulomb gauge used in [44]. To conclude the discussion of Theorem 5, let us clarify how dust-like backreaction can arise. If say $\psi_{\mathbf{A}}^{(1)}(u_{\mathbf{A}}/\lambda) = \cos(u_{\mathbf{A}}/\lambda)\tilde{\psi}_{\mathbf{A}}^{(1)}$, then the quadratic term $\partial_\mu\psi_\lambda\partial_\nu\psi_\lambda$ in (14) or (15) becomes

$$\begin{aligned}\partial_\mu\psi_\lambda\partial_\nu\psi_\lambda = {} & \partial_\mu\psi_0\partial_\nu\psi_0 + \frac{1}{2}\sum_{\mathbf{A},\mathbf{B}}\left(\tilde{\psi}_{\mathbf{A}}^{(1)}\right)^2\partial_\mu u_{\mathbf{A}}\partial_\nu u_{\mathbf{A}} \\ & -\frac{1}{2}\sum_{\mathbf{A}}\sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right)\tilde{\psi}_{\mathbf{A}}^{(1)}\partial_{(\mu}\psi_0\partial_{\nu)}u_{\mathbf{A}} - \frac{1}{2}\sum_{\mathbf{A},\mathbf{B}}\left(\tilde{\psi}_{\mathbf{A}}^{(1)}\right)^2\cos\left(\frac{2u_{\mathbf{A}}}{\lambda}\right)\partial_\mu u_{\mathbf{A}}\partial_\nu u_{\mathbf{A}} \\ & + \sum_{\mathbf{A}\neq\mathbf{B}}\sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right)\sin\left(\frac{u_{\mathbf{B}}}{\lambda}\right)\tilde{\psi}_{\mathbf{A}}^{(1)}\tilde{\psi}_{\mathbf{B}}^{(1)}\partial_\mu u_{\mathbf{A}}\partial_\nu u_{\mathbf{B}} + O(\lambda).\end{aligned}$$

Thanks to the coherence assumption, the last two lines converge weakly to 0 when λ tends to 0, showing how a new resonant term appears in the high-frequency limit.

In the forthcoming [54], Huneau and Luk prove that it is possible to rigorously send the number of dusts to infinity. Relying on an improved existence result in this setting [75], they perform a double limit argument, i.e. construct ansatz of the form (16) with uniform estimates with respect to both λ and the number of dusts. In particular, this allows to target general solutions to the massless Einstein–Vlasov system, by approximating any measure which is absolutely continuous with respect to the Lebesgue measure by sums of Dirac measures.

4.1.3. *Generalized wave coordinates*

In this section, we discuss results on the reverse Burnett conjecture for the Einstein vacuum equations in generalized wave coordinates. To define them, we give the expression of the Ricci tensor in any coordinates system:

$$2R_{\alpha\beta}(\mathbf{g}) = -\tilde{\Box}_{\mathbf{g}}\mathbf{g}_{\alpha\beta} + \mathbf{g}_{\rho(\alpha}\partial_{\beta)}H^\rho + H^\rho\partial_\rho\mathbf{g}_{\alpha\beta} + P_{\alpha\beta}(\mathbf{g})(\partial\mathbf{g},\partial\mathbf{g}), \tag{17}$$

where $\tilde{\Box}_{\mathbf{g}} := \mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu$, $H^\rho := \mathbf{g}^{\mu\nu}\Gamma^\rho_{\mu\nu}$ and $P_{\alpha\beta}(\mathbf{g})(\partial\mathbf{g},\partial\mathbf{g})$ contains every quadratic semilinear terms. Constructing coordinates such that $H = 0$ while simultaneously solving the reduced equations $\tilde{\Box}_{\mathbf{g}}\mathbf{g}_{\alpha\beta} = P_{\alpha\beta}(\mathbf{g})(\partial\mathbf{g},\partial\mathbf{g})$ has been shown to be possible in the pioneering [1]. Since $H^\rho = -\Box_{\mathbf{g}}x^\rho$, such coordinates are called wave coordinates. They are in fact a special instance of the so-called generalized wave coordinates, where instead of asking $H = 0$ we allow H to be equal to a non-zero quantity depending on $\mathbf{g}$ and not on $\partial\mathbf{g}$. In the literature, this extra freedom has been used for instance to help determine some spacelike or timelike asymptotic behaviour in stability proofs such as [70,76–78] or even in numerical simulations [79]. In the context of Burnett's conjecture, the role of H is somewhat different. First, the main result of [57] by the author of this review:

Theorem 6 ([57]). *If $(\mathbf{g}_0, F_{\mathbf{A}}, u_{\mathbf{A}})$, with $\mathbf{A}$ belonging to a finite set, is a generic solution on $[0,1]\times\mathbb{R}^3$ of (12) in wave coordinates and close to Minkowski, then there exists a sequence $(\mathbf{g}_\lambda)_{\lambda\in(0,1]}$ of solutions of the Einstein vacuum equations in generalized wave coordinates on $[0,1]\times\mathbb{R}^3$ and such that $\mathbf{g}_\lambda \longrightarrow \mathbf{g}_0$ uniformly and $\partial\mathbf{g}_\lambda \longrightarrow \partial\mathbf{g}_0$ weakly when λ tends to 0.*

The existence of the background solution $(\mathbf{g}_0, F_{\mathbf{A}}, u_{\mathbf{A}})$ is proved in [80], and the case of a single null dust is considered in [56,81]. Theorem 6 is the strict equivalent of Theorem 5 without any symmetry assumption and the multiphase geometric optics strategy is also followed as $\mathbf{g}_\lambda$ is of the form

$$\mathbf{g}_\lambda = \mathbf{g}_0 + \lambda\sum_{\mathbf{A}}\mathbf{g}_{\mathbf{A}}^{(1)}\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \lambda^2\left(\sum_{\mathbf{A}}\mathbf{g}_{\mathbf{A}}^{(2)}\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \sum_{\mathbf{A}\neq\mathbf{B},\pm}\mathbf{g}_{\mathbf{AB}}^{(2,\pm)}\left(\frac{u_{\mathbf{A}}\pm u_{\mathbf{B}}}{\lambda}\right)\right) + \text{remainder}. \tag{18}$$

Note that if we forget about the remainder and consider only one dust, the ansatz (18) is precisely the one considered in the approximate construction [4]. By plugging (18) into (17) and asking for $R_{\alpha\beta}(\mathbf{g}_\lambda) = 0$, we derive a system of transport equations and polarization conditions for the profiles $\mathbf{g}^{(i)}$, the former coming from the truely hyperbolic terms in (18) and the latter from the gauge terms ∂H. Contrary to the elliptic gauge in $\mathbb{U}(1)$ symmetry and because of the absence of decoupling, the transport equations and polarization conditions must be shown to be compatible at any order in λ. The polarization conditions are of the schematic form

$$\mathbf{P}\left[\mathbf{g}_{\mathbf{A}}^{(1)}|u_{\mathbf{A}}\right]=0,\quad \mathbf{P}\left[\mathbf{g}_{\mathbf{A}}^{(2)}|u_{\mathbf{A}}\right]=\left|\mathbf{g}_{\mathbf{A}}^{(1)}\right|^2_{\mathbf{g}_0}\mathrm{d}u_{\mathbf{A}}\otimes\mathrm{d}u_{\mathbf{A}},\quad \mathbf{g}_{\mathbf{AB}}^{(2,\pm)}-\mathbf{P}\left[\mathbf{g}_{\mathbf{AB}}^{(2,\pm)}|u_{\mathbf{A}}\pm u_{\mathbf{B}}\right]=\mathbf{g}_{\mathbf{A}}^{(1)}\star\mathbf{g}_{\mathbf{B}}^{(1)},\qquad(19)$$

where the $\star$ denotes some tensor contractions and $\mathbf{P}$ has been defined in (3). The first condition in (19) is the equivalent of Choquet-Bruhat's (4) and the non-trivial RHS's of the last two conditions are due to the lack of null condition of the semilinear terms in (17). The solvability of the last two conditions in (19) is a structural miracle, in some sense equivalent to the ellipticity of the equations for $\mathfrak{g}$ in (15). However, as opposed to [51], here the metrics $\mathbf{g}_\lambda$ don't satisfy the same gauge conditions as their limit $\mathbf{g}_0$, precisely because of the last two conditions in (19), showing that H plays the role of a bin in which we hide undesired terms. Note that the first condition in (19) still implies that

$$H(\mathbf{g}_\lambda)\longrightarrow H(\mathbf{g}_0),\quad\text{uniformly,}\quad\text{and}\quad \partial H(\mathbf{g}_\lambda)\longrightarrow\partial H(\mathbf{g}_0)\quad\text{weakly,}\qquad(20)$$

when λ tends to 0.

As explained in Section 3.5, the existence of an exact solution of the form (18) up to time 1 does not follow from any general result. This can be directly seen on the wave equation satisfied by the remainder in (18), which, due to the quasilinearity of the wave operator in (17) is coupled to the transport equation for $\mathbf{g}^{(2)}$, ultimately leading to the apparent loss of one derivative (in consistence with Burnett's regime living one derivative lower than what Theorem 4 provides). Regaining this derivative makes crucial use of the high-frequency ansatz (18).

4.2. *The direct conjecture*

In this section, we review the results in [50,52,53] on the direct conjecture in $\mathbb{U}(1)$ symmetry or in generalized wave coordinates. These results will assume the existence of a sequence of vacuum metrics $(\mathbf{g}_\lambda)_{\lambda\in(0,1]}$ converging in some sense, actually in a stronger compared to (10), to a given metric $\mathbf{g}_0$. Their goal is then to compute the Einstein tensor of $\mathbf{g}_0$ and show that it satisfies a transport equation. The main tool are microlocal defect measures and the structural miracle is compensated compactness.

4.2.1. *Microlocal defect measures*

Microlocal defect measures (sometimes also called H-measures) have been introduced to detect the lack of strong convergence of a sequence in L^2. We refer to [82] for a nice introduction to these measures and to their semi-classical counterparts. Introduced in [83,84], microlocal defect measures are associated to sequences $(u_k)_{k\in\mathbb{N}}$ of functions in $L^2(\mathbb{R}^{d+1},\mathbb{C})$ and converging weakly to 0, and in these references it is shown that there exists a non-negative complex-valued Radon measure μ (the so-called microlocal defect measure) such that

$$\lim_{k\to+\infty}\langle Au_k,u_k\rangle_{L^2(\mathbb{R}^{d+1},\mathbb{C})}=\int_{S^*\mathbb{R}^{d+1}}a(x,\xi)\mathrm{d}\mu,\qquad(21)$$

for all order 0 pseudo-differential operator A with principal symbol a homogeneous of degree 0, and where $S^*\mathbb{R}^{d+1}$ is the cosphere bundle of $\mathbb{R}^{d+1}$. For instance, if $u_k(x)=k^{d+1/2}\chi(k(x-x_0))$ (with χ compactly supported) then $\mathrm{d}\mu$ is of the form $\delta_{x_0}\otimes\mathrm{d}\nu$, and if $u_k(x)=\varphi(x)\cos(x\cdot\omega)$ then $\mathrm{d}\mu=|\varphi|^2\mathrm{d}x\otimes\delta_\omega$. These two simple examples illustrate respectively lack of strong convergence by concentration and oscillation and the second one reminds us of the backreaction's dust nature obtained via geometric optics (see Section 4.1). Microlocal defect measures satisfy several important properties, the first of them being localization: if P is an order m differential operator with principal symbol p, and if $(P(u_k))_{k\in\mathbb{N}}$ is relatively compact in H^{-m}_{loc} then $p\mathrm{d}\mu=0$. This shows that $\mathrm{d}\mu$ is supported (in phase space) on the zero set of the symbol p. In the context of Burnett's conjecture, P will always be a wave operator associated to a Lorentzian metric, and the zero set of p in this case is precisely $\mathscr{P}_0$, the zero mass shell! The second key property of microlocal defect measures in the context of wave equations is that they propagate. This has been first proved in the linear case in [85]. As a toy model, consider a sequence u_k such that ∂u_k and $\Box_{\mathbf{m}} u_k$ (where $\mathbf{m}$ is the Minkowski metric) converge respectively weakly and strongly to 0 in L^2, it can be shown that the microlocal defect measure associated to derivatives of u_k satisfy

$$\left\{\mathbf{m}^{\alpha\beta}\xi_\alpha\xi_\beta, a\right\}\frac{\mathrm{d}\nu}{|\xi|^2} = 0 \tag{22}$$

for all a homogeneous of degree 1, and where $\{\cdot,\cdot\}$ denotes the Poisson bracket in the (x,ξ) variables. The relation (22) is a weak formulation of the Vlasov equation (7).

Note that microlocal defect measures are only sensible to the direction $\xi/|\xi|$ of the lack of strong convergence, and in this regard semi-classical defect measures are more precise. Both types of defect measures are powerful tools with applications in the theory of homogenization but also to the control of wave equations, see for instance [86,87]. However, as mentioned earlier in this text, the phenomenon at the heart of Burnett's conjecture is compensated compactness, the most famous example of which is the div-curl lemma (see [32,33]). Generalized in [83], it stipulates that one can pass to the limit in particular quadratic expressions. For the application to Burnett's conjecture, one would like a similar statement for cubic interactions. Prior to the results discussed in the sections below, the only known example of trilinear compensated compactness was the result [88], where it is shown that $u_k^{(1)}u_k^{(2)}u_k^{(3)}$ converges weakly to $u_\infty^{(1)}u_\infty^{(2)}u_\infty^{(3)}$ if $u_k^{(i)}$ converges weakly to $u_\infty^{(i)}$ in L^2 and if there exists vector fields $X^{(i)}$ satisfying a particular coherence condition and such that $X^{(i)}u_k^{(i)}$ is bounded in L^2.

As geometric optics is the perfect tool to prove Burnett's reverse conjecture, it turns out that microlocal defect measures are the perfect tools to prove Burnett's direct conjecture, as we will now show.

4.2.2. *The* $\mathbb{U}(1)$ *symmetry*

In this section, we describe the work [52] and the subsequent [50], which consider spacetimes enjoying $\mathbb{U}(1)$ symmetry in the elliptic gauge, a framework already described in Section 4.1.2. To state the main theorems of these works, we first define the kinetic spacetimes under consideration. If $\mathrm{d}\nu$ is a non-negative Radon measure on $S^*\mathbb{R}^{2+1}$, we say that $(g,\varphi,\omega,\mathrm{d}\nu)$ is a radially-averaged measure solution to the massless Einstein–Vlasov system in $\mathbb{U}(1)$ symmetry if the first two equations in (14) hold, if for all vector field X smooth and compactly supported we have

$$\int_{\mathbb{R}^{2+1}} R_{\mu\nu}(g)X^\mu X^\mu = \int_{\mathbb{R}^{2+1}}\left(2(X\varphi)^2 + \frac{1}{2}\mathrm{e}^{-4\varphi}(X\omega)^2\right) + \int_{S^*\mathbb{R}^{2+1}} g(\xi,X)^2\frac{\mathrm{d}\nu}{|\xi|^2}, \tag{23}$$

if for all f compactly supported on $\mathscr{M}$ we have

$$\int_{S^*\mathbb{R}^{2+1}} f g^{\alpha\beta}\xi_\alpha\xi_\beta\frac{\mathrm{d}\nu}{|\xi|^2} = 0, \tag{24}$$

and if for all a defined on $T^*\mathbb{R}^{2+1}$ and homogeneous of degree 1 we have

$$\int_{S^*\mathbb{R}^{2+1}}\left(g^{\alpha\beta}\xi_\alpha\partial_\beta a - \frac{1}{2}\partial_\mu g^{\alpha\beta}\xi_\alpha\xi_\beta\partial_{\xi_\mu}a\right)\frac{\mathrm{d}\nu}{|\xi|^2} = 0. \tag{25}$$

Note that, when put together, Equations (23)–(25) are a weak formulation of the massless Einstein–Vlasov system in $\mathbb{U}(1)$ symmetry. The following is the main result of [52] by Huneau and Luk and of [50] by Guerra and Teixeira da Costa:

Theorem 7 ([50,52]). *Let* $(g_n,\varphi_n,\omega_n)_{n\in\mathbb{N}}$ *a sequence of solutions to (14) in the elliptic gauge such that there exists* (g_0,φ_0,ω_0) *satisfying the elliptic gauge conditions and such that* (g_n,φ_n,ω_n)

converges to (g_0,φ_0,ω_0) *uniformly and weakly in* $W^{1,p}$ *for some* $p>2$*, then there exists* $\mathrm{d}\nu$ *a non-negative Radon measure on* $S^*\mathbb{R}^{2+1}$ *such that* $(g_0,\varphi_0,\omega_0,\mathrm{d}\nu)$ *satisfy (23) and (24). If moreover there exists a sequence* $(\lambda_n)_{n\in\mathbb{N}}$ *converging to 0 such that for all compact* $K\subset\mathbb{R}^{2+1}$ *we have*

$$\sum_{k=0}^{q}\lambda_n^{k-1}\left\|\partial^k\left((g_n,\varphi_n,\omega_n)-(g_0,\varphi_0,\omega_0)\right)\right\|_{L^\infty(K)}\leq C_K, \tag{26}$$

with $q=4$ *in [52] and* $q=2$ *in [50], then* $(g_0,\varphi_0,\omega_0,\mathrm{d}\nu)$ *satisfy (25).*

Note that the existence of sequences $(g_n,\varphi_n,\omega_n)_{n\in\mathbb{N}}$ as in Theorem 7 and moreover unbounded in $W^{s,p}$ if $s>1$ (so that the measure $\mathrm{d}\nu$ obtained has a chance to be non-zero) is a consequence of [51]. As it can be seen in Theorem 7, the existence of the measure $\mathrm{d}\nu$ such that $(g_0,\varphi_0,\omega_0,\mathrm{d}\nu)$ satisfy (weakly) the Einstein equations with a massless kinetic stress energy tensor only follows from the Burnett convergence regime (10) (allowing both concentration and oscillation), while the fact that $\mathrm{d}\nu$ solves (weakly) a Vlasov equation requires the stronger high-frequency assumption (26). We now discuss these two parts of Theorem 7 successively, using again our concise notations $(\mathfrak{g},\psi)$ from Section 4.1.2.

The existence of the measure $\mathrm{d}\nu$ follows from the general existence results [83,84] applied to the sequence $\left(\partial(\psi_n-\psi_0)\right)_{n\in\mathbb{N}}$, which by assumption converge to 0 only weakly. We wish now to pass to the limit in the system (15) that $(\mathfrak{g}_n,\psi_n)$ satisfy. For the wave map system, this is possible thanks to standard bilinear compensated compactness applied to the null form $g^{-1}(\mathrm{d}\psi_n,\mathrm{d}\psi_n)$. For the Ricci equation, the elliptic gauge allows to improve weak convergence to strong convergence for $\nabla\mathfrak{g}_n$, so that we can deal with $(\nabla\mathfrak{g})^2$, and thanks to the equivalent of (21) here, the term $(\partial\psi_n)^2$ converges weakly to $(\partial\psi_0)^2+\xi^2\mathrm{d}\nu/|\xi|^2$. The support property of $\mathrm{d}\nu$ follows again from bilinear compactness used together with the localization property of microlocal defect measure and the wave maps system for $\psi_n-\psi_0$. In conclusion, we have proved the first part of Theorem 7.

The most difficult part of [50,52] is to deduce the weak Vlasov equation (25) from the additional high-frequency assumption (26). We repeat that a similar statement was known for linear wave equations, while here a quasilinear hyperbolic-elliptic system is considered and trilinear compensated compactness needs to be uncover. The derivation relies fully on the wave map structure of the equations for ψ, both for the semilinear terms satisfying the null condition and for the coupling between the two components of the wave map, and on the coupling to the elliptic equations for the metric in order to handle quasilinear terms.

4.2.3. *Generalized wave coordinates*

We describe now the work [53], which proves Burnett's direct conjecture in generalized wave gauge. This framework has already been discussed in Section 4.1.3 above. In consistence with Theorem 6 and in particular (20), [53] considers sequences $(\mathbf{g}_n)_{n\in\mathbb{N}}$ of solutions to the Einstein vacuum equations satisfying two sets of assumptions (27) and (28) below, with $\mathbf{g}_0$ a given metric

$$\left\|\mathbf{g}_n-\mathbf{g}_0\right\|_\infty+\left\|H(\mathbf{g}_n)-H(\mathbf{g}_0)\right\|_\infty\leq\lambda_n,\quad\left\|\partial\mathbf{g}_n\right\|_\infty\leq C, \tag{27}$$

$$\lambda\left\|\partial^2\mathbf{g}_n\right\|_\infty+\left\|\partial H(\mathbf{g}_n)\right\|_\infty\leq C, \tag{28}$$

where $C>0$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a given sequence converging to 0. The following result from [53] is due to Huneau and Luk:

Theorem 8 ([53]). *Let* $(\mathbf{g}_n)_{n\in\mathbb{N}}$ *be a sequence of solutions to the Einstein vacuum equations and* $\mathbf{g}_0$ *a metric. If (27) holds, then there exists a non-negative Radon measure* $\mathrm{d}\mu$ *on* $T^*\mathbb{R}^{3+1}$ *supported on the zero mass shell of* $\mathbf{g}_0$ *and such that*

$$\int_{\mathbb{R}^{3+1}}\psi R_{\mu\nu}(\mathbf{g}_0)=\int_{S^*\mathbb{R}^{3+1}}\psi\xi_\mu\xi_\nu\mathrm{d}\mu$$

for all function ψ smooth and compactly supported. If moreover (28) holds, then for any a homogeneous of degree 1 we have $\{\mathbf{g}_0^{\alpha\beta}\xi_\alpha\xi_\beta, a\}\mathrm{d}\mu = 0$.

Theorem 8 is the equivalent of Theorem 7 in generalized wave coordinates, which in particular removes any symmetry assumption. Its conclusions are the same, and it splits similarly into two parts. First, getting the existence of $\mathrm{d}\mu$ and the Ricci equation for $(\mathbf{g}_0, \mathrm{d}\mu)$ requires only (27) and follows again from [83,84]. More precisely we set $\mathrm{d}\mu := \mathbf{g}_0^{\alpha\rho}\mathbf{g}_0^{\beta\sigma}\left(1/4\mathrm{d}\mu_{\rho\beta\alpha\sigma} - 1/2\mathrm{d}\mu_{\rho\alpha\beta\sigma}\right)$ where each $\mathrm{d}\mu_{\rho\beta\alpha\sigma}$ is associated to $\partial(\mathbf{g}_n - \mathbf{g}_0)$ in the following way

$$\lim_{n\to+\infty}\left\langle \partial_\gamma(\mathbf{g}_n - \mathbf{g}_0)_{\alpha\beta}, A\partial_\delta(\mathbf{g}_n - \mathbf{g}_0)_{\rho\sigma}\right\rangle_{L^2} = \int_{S^*\mathbb{R}^{3+1}} a\xi_\gamma\xi_\delta \mathrm{d}\mu_{\alpha\beta\rho\sigma}.$$

The measure $\mathrm{d}\mu$ is tuned to match the exact expression of the semilinear terms in the Ricci tensor that don't satisfy the null condition. Thanks to the div-curl lemma applied to the null form $\mathbf{g}^{-1}(\mathrm{d}\varphi, \mathrm{d}\psi)$ and to estimates from [89] applied to the null forms $\partial_\mu\varphi\partial_\nu\psi - \partial_\mu\psi\partial_\nu\varphi$ (which are new compared to the $\mathbb{U}(1)$ symmetry setting), one shows that the semilinear terms without the null condition are the only one contributing when passing to the limit in $R_{\mu\nu}(\mathbf{g}_n) = 0$. The proof of the propagation property of $\mathrm{d}\mu$, i.e. the fact that under the stronger assumption (28) we have $\{\mathbf{g}_0^{\alpha\beta}\xi_\alpha\xi_\beta, a\}\mathrm{d}\mu = 0$, requires to compute the weak limit of terms of the form $\left\langle \partial(\mathbf{g}_n - \mathbf{g}_0), A\left(\partial(\mathbf{g}_n - \mathbf{g}_0), \partial(\mathbf{g}_n - \mathbf{g}_0)\right)\right\rangle_{L^2}$. One of the new issue compared to the $\mathbb{U}(1)$ symmetry are the semilinear terms not satisfying the null condition, and trilinear compensated compactness for these terms can be shown using the generalized wave coordinates condition $\|\partial H(\mathbf{g}_n)\|_\infty \leq C$. This shows again that the failure of null condition can be tamed by gauge terms in the Ricci tensor.

5. Angularly regular spacetimes

As explained in Section 3.5, the article [55] deserves its own section because it is very different in spirit from the articles discussed in Section 4 above. In particular, it provides an analytical framework where both parts of Burnett's conjecture can be studied, and in fact the reverse part will be deduced from the direct part.

This framework is the one of angularly regular spacetimes in double null gauge. Requiring no symmetry assumption, the double null gauge has proved very flexible over the years, see for instance the important works [90–92]. If S is a compact 2-surface and $u_*, \underline{u}_* > 0$, the Lorentzian manifold $\left([0, u_*] \times [0, \underline{u}_*] \times S, \mathbf{g}\right)$ is said to be in double null gauge if

$$\mathbf{g} = -4\Omega^2 \mathrm{d}u\mathrm{d}\underline{u} + \gamma_{AB}\left(\mathrm{d}\theta^A - b^A\mathrm{d}u\right)\left(\mathrm{d}\theta^B - b^B\mathrm{d}u\right), \tag{29}$$

where $\Omega > 0$ is a function, γ is a Riemannian metric on S, (θ^1, θ^2) are coordinates on S and b is vector field tangent to S. The local existence of spacetimes in double null gauges is usually formulated with a characteristic initial value problem, where characteristic data are put on two intersecting null hypersurfaces $H_0 = \{0\} \times [0, \underline{u}_*] \times S$ and $\underline{H}_0 = [0, u_*] \times \{0\} \times S$, see [93]. In [94], motivated by the study of impulsive gravitational waves (see also [95]), Luk and Rodnianski prove a low-regularity local existence result in double null gauge where the ∂_u and $\partial_{\underline{u}}$ derivatives of the metric coefficients in (29) are allowed to be only in L^2, while the ∂_{θ^A} derivatives must be more regular than H^1. Compared to Theorem 4, the spacetimes produced by [94] are more regular in the angular directions θ^A but require one derivative less in the null directions u and $\underline{u}$, the higher regularity compensating for the lower regularity. The details of the proof of the main result in [94] are out of the scope of this text, let us just say that the structure of the Einstein vacuum equations is exploited at its fullest and that ellipticity in the angular directions plays again a significant role.

Since in the framework of [94] derivatives in the null directions u and $\underline{u}$ are allowed to be only in L^2, the Burnett convergence regime (10) (and thus Burnett's conjecture itself) is in some sense no longer a low-regularity phenomenon, if we restrict ourself to the situation where derivatives in the null directions are the only ones lacking strong convergence. This allows for a very soft proof of both sides of the conjecture with the only restriction that the kinetic spacetimes under consideration must be two null dusts propagating in the u and $\underline{u}$ directions. More precisely, there target kinetic spacetimes are angularly regular weak solutions to the Einstein-null dusts system of the form $(\mathbf{g}, \mathrm{d}\nu, \mathrm{d}\underline{\nu})$ where $\mathrm{d}\nu$ and $\mathrm{d}\underline{\nu}$ are two non-negative Radon measures such that

$$\begin{cases} \int_{\mathcal{M}} \left(\mathrm{div}X\mathrm{div}Y - D_\mu X^\nu D_\nu Y^\mu\right)\mathrm{dVol}_{\mathbf{g}} = \int_{\mathcal{M}} (Xu)(Yu)\mathrm{d}\nu + \int_{\mathcal{M}} (X\underline{u})(Y\underline{u})\mathrm{d}\underline{\nu}, \\ \int_{\mathcal{M}} \mathbf{g}^{-1}(\mathrm{d}u, \mathrm{d}\varphi)\mathrm{d}\nu = \int_{\mathcal{M}} \mathbf{g}^{-1}(\mathrm{d}\underline{u}, \mathrm{d}\varphi)\mathrm{d}\underline{\nu} = 0, \end{cases} \tag{30}$$

for all smooth X, Y and φ. Note that the equations in (30) are respectively the weak formulation of the first and second equations in (8) in the case of two null dusts. We gather in the following statement rough versions of the main results of [55] by Luk and Rodnianski:

Theorem 9 ([55]). *The following hold:*

(i) *Any sequence of characteristic angularly regular vacuum initial data satisfying uniform bounds gives rise to a sequence of angularly regular vacuum spacetimes in double null gauge (with a common region of existence). Up to a subsequence, this sequence converges in the sense (10) to an angularly regular weak solution to (30).*

(ii) *Any angularly regular weak solution of (30) in double null gauge can be approximated in the sense (10) by a sequence of angularly regular vacuum spacetimes in double null gauge.*

(iii) *Any characteristic angularly regular initial data for (30) gives rise to a unique angularly regular weak solution of (30).*

The three points of Theorem 9 address respectively Burnett's direct conjecture, Burnett's reverse conjecture and the local theory for angularly regular measure-valued null dusts. As we will see, points (ii) and (iii) are consequences of point (i), so we first discuss point (i). The existence part of point (i) is a direct consequence of the local existence result of [94], as does the existence of the limit, with the help of standard compactness arguments using the uniform bounds provided by [94]. The fact that the limit is an angularly regular weak solution to (30) follows from a careful analysis of each quadratic products $\Gamma^{(1)}\Gamma^{(2)}$ in the structure equations of the double null gauge, where the $\Gamma^{(i)}$'s denote Christoffel symbols in the null frame. It turns out that the only such products for which[3]

$$\text{w-lim}\left(\Gamma^{(1)}\Gamma^{(2)}\right) \neq \left(\text{w-lim}\Gamma^{(1)}\right)\left(\text{w-lim}\Gamma^{(1)}\right)$$

are $|\hat{\chi}|^2$ and $|\underline{\hat{\chi}}|^2$ in the Raychaudhuri equations, where $\hat{\chi}$ and $\underline{\hat{\chi}}$ are the shears of the null hypersurfaces. These products, which fully characterize the lack of null condition for the Einstein vacuum equations in this setting, have the additional property to have a sign and to satisfy elliptic equations (the so-called Codazzi equations), two facts that should not surprise the reader anymore. The Radon measures representing the two dusts are then roughly defined by

$$\mathrm{d}\nu := \text{w}^*\text{-lim}|\hat{\chi}|^2 - |\text{w-lim}\hat{\chi}|^2, \quad \mathrm{d}\underline{\nu} := \text{w}^*\text{-lim}|\underline{\hat{\chi}}|^2 - |\text{w-lim}\underline{\hat{\chi}}|^2.$$

Once point (i) of Theorem 9 is proved, points (ii) and (iii) are reduced to initial data statements. Indeed, given a solution of (30), if we are able to explicitly (say with oscillations) approach its data by vacuum data, then point (i) and a uniqueness low-regularity statement implies point (ii). Point (iii) follows from similar arguments, and is the first local theory for measure-valued null dusts, in particular allowing the construction of null dust shell solutions (see [96,97] for their significance).

Appart from the fact that the kinetic spacetimes considered in the framework of angularly regular spacetimes in double null gauge are restricted to only two null dusts, Theorem 9 is much more powerful then the statements in Section 4 and we would like to discuss further some aspects

[3] Here w-lim denotes the weak limit.

of it. In addition to providing a unified treatment of both parts of Burnett's conjecture, Theorem 9 completely drops geometric optics as a tool to construct examples. As a consequence, the target kinetic spacetime can be very rough, while the justification of geometric optics expansions in Section 4.1 relies strongly on the smoothness of the target. Moreover, the target kinetic spacetime can be far from Minkowski, while some smallness is required in Theorems 5 and 6. Lastly, as opposed to Theorems 7 and 8 who needed oscillations in order to get the Vlasov equation, lack of strong convergence by concentration is allowed in Theorem 9.

6. Conclusion

In this section we highlight interesting future directions of research on Burnett's conjecture. The review [11] already offers a very rich list of such directions, and we start by briefly presenting some of them.

Even though they clearly give Burnett's conjecture some validity, the various results presented in Sections 4 and 5 have serious limitations. Most of these results concern kinetic solutions close to Minkowski spacetime, with exception Theorem 9 but which is restricted to two null dusts, and dropping the smallness assumption in $\mathbb{U}(1)$ symmetry or generalized wave coordinates would be very interesting. All of these results are local in time, and one could imagine considering as target kinetic spacetimes a global perturbation of Minkowski as constructed in [98,99] and obtaining a global-in-time Burnett approximation result. Opposite to this scenario, the approximation by vacuum spacetimes of the static spherically symmetric self-gravitating solutions of [100] doesn't seem to be true globally in time, in view of the final state conjecture, but perhaps it is for large times. More generally, given any "exotic" solution to the massless Einstein–Vlasov with particular properties, the question of its approximation by vacuum in the spirit of Burnett's conjecture is an interesting one.

By transforming the early [4] into more analytical theorems, the results of Section 4.1 bring mathematical general relativity closer to the vast geometric optics literature. However they represent only the very first step and many more phenomena remain to be explored, such as caustics, diffractive effects etc. Because of the beautiful transparency properties of the Einstein vacuum equations, one could also try to go beyond Burnett's regime and construct stronger oscillations by concretely lowering the value of p in (13). This would correspond to the strongly nonlinear geometric optics regime, used for instance to prove ill-posedness of some supercritical wave equations by Lebeau [101]. In this spirit, the approximate constructions of large amplitudes gravitational waves in [102,103] are definitely of interest, also since they seem to be stronger short pulses than what exists in the mathematical general relativity literature (see [71,90,104,105]).

We conclude this text by coming back to Yvonne Choquet-Bruhat, in honor of which this text has been conceived. In the 90's, she explored with her collaborators alternative formulations of the Einstein vacuum equations, and in particular higher order formulations built with first or second order derivatives of the Ricci tensor. For instance in [106] they give a fourth-order system (with respect to the metric), equivalent to the Einstein vacuum equations, and show that without any choice of gauge this system still retains a very weak form of hyperbolicity [107]. In Remark 6 of the review [108], Choquet-Bruhat even hints that this fourth-order system would enjoy better structural semilinear properties than the Einstein equations, in link with the relaxed null condition introduced in [42]. She repeats this remark in Chapter VIII Section 4 of [2], sending the reader to Chapter XI where high-frequency waves are studied. However, no mention to this exotic formulation is made in Chapter XI, let alone to its application to high-frequency waves, the nature of which thus remains a mystery.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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