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The nonlocal nature of the Reynolds stress

La nature non locale des contraintes de Reynolds

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Abstract. A nonlocal expression for the turbulent Reynolds stress $\overline{u'v'}$ solves in a unified manner for the velocity profiles in jets, shear layers, wakes, and boundary layers, offering predictions consistent with known measurements in these canonical flows. The framework is the random walk inspired closure of Prandtl (1925) since we write

$$\overline{u'v'} = -\tilde{v}\ell\partial_y u$$

with ℓ a mean free path but, by contrast with Prandtl, with $\tilde{v}$ a *nonlocal* transfer velocity resulting from an integration over an appropriate portion of space of the mean velocity profile $u(y)$. The status of ℓ is discussed and its value, a fraction $(10\pi)^{-1} \approx 0.031$ of the integral scale, is computed in opened shear flows. The closure is adapted to the special case of boundary layers, where the value of the von Kármán constant is found to be $\kappa = 6^{-1/2} \approx 0.41$.

Résumé. Une expression non locale de la contrainte de Reynolds turbulente $\overline{u'v'}$ permet de résoudre de manière unifiée les profils de vitesse turbulents dans les jets, les couches de cisaillement, les sillages et les couches limites. Les prédictions sont cohérentes avec les mesures connues dans ces écoulements canoniques. Le cadre du raisonnement est inspiré de la fermeture de marche aléatoire de Prandtl (1925) puisque nous écrivons

$$\overline{u'v'} = -\tilde{v}\ell\partial_y u$$

avec ℓ un libre parcours moyen mais, contrairement à Prandtl, avec $\tilde{v}$ une vitesse de transfert *non locale* résultant d'une intégration sur une portion appropriée de l'espace du profil de vitesse moyen $u(y)$. Le statut de ℓ est discuté et sa valeur, une fraction $(10\pi)^{-1} \approx 0{,}031$ de l'échelle intégrale, est calculée dans les écoulements cisaillés ouverts. La fermeture est adaptée au cas particulier des couches limites, où la valeur de la constante de von Kármán est prédite comme valant $\kappa = 6^{-1/2} \approx 0{,}41$.

Keywords. Turbulence, Reynolds stress, closures.

Mots-clés. Turbulence, contrainte de Reynolds, fermetures.

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1. Introduction: closures for the Reynolds stress

One possible way to represent a turbulent velocity field $U_i = u_i + u'_i$ is to split it into its mean u_i and fluctuating parts u'_i, a method inaugurated by Reynolds [1]. The interest of this approach is that in large Reynolds number (viscous term discarded), pressure gradient free flows (unconstrained opened flows like jets, wakes, etc.), the Navier–Stokes equations

$$\partial_t U_i + U_j \partial_j U_i = \partial_j\big(-(p/\rho)\delta_{ij} + \nu[\partial_j U_i + \partial_i U_j]\big) \quad \text{with } \partial_i U_i = 0, \tag{1}$$

when time-averaged as $\overline{(\cdot)} = \lim_{T\to+\infty} \frac{1}{T}\int_0^T (\cdot)\,\mathrm{d}t$, reduce to a form

$$u_j \partial_j u_i = -\partial_j \overline{u'_i u'_j} \tag{2}$$

interestingly singling-out the correlation of the fluctuations only.

The so-called Reynolds stress tensor $\overline{u'_i u'_j}$ which encodes all the information carried by the erratic motions has been the subject of several interpretations [2,3], starting with the one of [4]. Because it was a question of computing the value of a correlation between random variables, the temptation was great to make an analogy with the kinetic theory of gases, which Prandtl did. The construction is well known: one considers a turbulent flow as a medium equipped with a diffusivity $\mathscr{D}$ linking the stress tensor to the time-averaged velocity gradient (in analogy with gases and laminar flows) as $\overline{u'_i u'_j} = -\mathscr{D}(\partial_j u_i + \partial_i u_j)$, this effective coefficient being (by contrast) a function of the mean flow structure itself, namely of u_i and/or of its spatial derivatives $\partial_j u_i$. This description assumes that momentum transport is mediated by "liquid balls" (*Flüssigkeitsballen*) of mean free path ℓ (*Mischungsweglänge*) with short-time and space correlation as compared to those of the flow at scale L, an assumption which is questionable, in general. While not questioning the random walk construction *per se*, Taylor [5] suggested that, in two dimensional turbulent flows at least, it is not the momentum of the "eddies" which is conserved from one step to the other (unlike in the Reynolds–Prandtl theory), but their vorticity; we discuss this point of view and its relation to scalar transport in jets elsewhere [6].

For instance, if $U_i = \{U, V\} = \left\{u(x,y) + u'(x,y,t), v(x,y) + v'(x,y,t)\right\}$ in a plane jet, wake or boundary layer, or $\left\{u(x,r) + u'(x,r,t), v(x,r) + v'(x,r,t)\right\}$ as in an axisymmetrical round jet, Prandtl's closure starts with

$$\overline{u'v'} = -\mathscr{D}\partial_y u \quad \text{or} \quad -\mathscr{D}\partial_r u, \quad \text{with } \mathscr{D} = \overline{v'\ell} \tag{3}$$

where $\mathscr{D}$ remains to be interpreted.

Since, as opposed to thermally activated transport (which defines the fluid viscosity ν), there is no obvious intrinsic length scale in turbulence (see however Section 6), it is assumed that ℓ is, while smaller, proportional to the dimension of the flow L where the shear producing turbulence is sustained. In jets, L is the jet radius and $\ell \sim L$ thus depends on the axial distance x only (and not on y or r), but in boundary layers $\ell \sim y$ is taken as proportional to the distance to the wall y. It was even suggested by von Kármán [7] that ℓ could be related to the spatial structure of the mean flow u itself by $\ell \sim \partial_y u/\partial_y^2 u$ in pipe flow.

A number of *ad hoc* formulae have been proposed for $\mathscr{D}$, applied opportunistically (see [8] and [9] for a historical perspective) to different types of configurations (boundary layer, wake, jet, mixing layer…):

> "…agreement with experimental results can be obtained with various expressions for the turbulent viscosity",

as noted by Landau & Lifshitz [10]. Among them, two well-known proposals:

(1) The "mixing length" theory of Prandtl [4] for which $v' \sim u' \sim \ell\partial_y u$ in a plane jet ($\ell\partial_r u$ in an axisymmetrical jet) giving

$$\mathscr{D} \sim \ell^2|\partial_y u| \quad \text{or} \quad \ell^2|\partial_r u|. \tag{4}$$

This expression was originally designed for and successfully applied to turbulent boundary layers (the famous "Log-law of the wall") but was also recognized as "unsatisfactory" [8] for jets because $\mathscr{D}$ vanishes where the velocity gradient is zero that is, in particular, at the center of the jet in $y = 0$ where, on the contrary, the fluctuations intensities $\overline{u'^2}$ and $\overline{v'^2}$ are the largest but $\partial_y u = 0$ [11]. That flaw was circumvented by Prandtl himself [12] who proposed to use in that case

$$\mathscr{D} \sim \Delta u\,\ell, \tag{5}$$

a diffusivity taken as constant across the jet, proportional to the net velocity difference absorbed by the turbulent shear layer $\Delta u = u(x,0) - u(x,\infty)$. Prandtl's closure was, on the other hand, better suited to the plane mixing layer geometry where fluctuations and $\partial_y u$ are both the largest in $y = 0$, where $\partial_y^2 u = 0$ [13], although its solution for that flow using (4) suffers from an "esthetical deficiency" [8], to say the least. This lack of unity has certainly contributed to moderate the enthusiasm for the "gradient type" approach to turbulent dispersion, of which the quote above is a typical reflection (see also the critical appraisal by [14]).

(2) The question was approached earlier in very similar terms by Boussinesq [15] who proposed, for the flow in tubes and channels (see [16] for a recent study), an expression identical to (5) with ℓ being the tube radius. His idea, later reiterated by [1] is that "eddy agitations" (*agitation tourbillonaire*) mediate momentum transfer. One may push Boussinesq's logic one step further by applying the argument locally, stating that in a flow with "tumultuous movements", viscosity should be proportional to the intensity of the "mean agitation" of which $u(x,y)$ (or $u(x,r)$) is a measure, like in the kinetic theory of gases where temperature sets the thermal velocity (Maxwell formula, see e.g. [17]). In that perspective, $v' \sim u$ giving

$$\mathscr{D} \sim u\ell \tag{6}$$

where u actually means $u(x,r) - u(x,\infty)$ for Galilean invariance. This formulation is reminiscent of the $\mathrm{k}-\epsilon$ theory (with $\mathrm{k} \sim u^2$, $\epsilon \sim u^3/\ell$ and $\mathscr{D} \sim \mathrm{k}^2/\epsilon$), first proposed to model concentration fluctuations in jets [18]. But nothing guarantees that (6) should be universally true, in every flow.

The different proposals for $\mathscr{D}$ listed above have different levels of "locality": (4) is local because of the spatial derivative of u while (5) is comparatively more "integral" in the sense that it involves the total velocity difference across the whole flow; (6), where $\mathscr{D}$ is proportional to the local velocity rather than to its spatial derivative, is intermediate.

Turbulent flows in nature are generated by differences of velocity, density, temperature etc. maintained between distant boundaries. There is no reason to discard *a priori* the possibility that small scale features of the flow would depend on the large scale structure of its forcing; or at least that some aspects of the stress tensor may be sensitive to the general organisation of the flow, like the overall shape of the velocity profile, and not only to its local value or gradient. Nonlocal closures for fluxes have besides already been considered in sheared layers [19], in more general multi-scale flows [20], including turbulent flows explicitly [21].

In other sectors of condensed matter physics concerned with jammed, glassy materials or dense granular flows, this idea is commonplace: the nonlocality of the motion in sheared dense colloids and granular media has been recognized for some time [22,23] where the existence of large scale structures like vortices has been documented [24]. Newton's force chains, which mediate the stress in granular materials, mediate pressure in condensed normal fluids; for this reason, incompressible media are subjected to a long-range order.

Our approach, taking advantage of the qualitative remarks on the nonlocality in turbulence made above, first discusses the general form the Reynolds $\overline{u'v'}$ should have, introduces an effective, nonlocal transfer velocity $\tilde{v}$ and computes, from there, the mean velocity profiles of some canonical flows of fluid mechanics. We describe the degree of nonlocality for each flow (jets in Sections 2 and 3, mixing layers in Section 4, wakes in Section 5), including for the particular problem of the boundary layer (Section 7) a more "local" configuration than the one in open flows.

Our analysis also offers an interpretation of the mean free path ℓ (Section 6), as well as a value of the von Kármán constant in Section 7.

2. Relationships in jets

For stationary free (zero pressure gradient), plane or axisymmetrical jets at large Reynolds number, we note as in Section 1, $\{U, V\} = \{u + u', v + v'\}$. In practice, jets are slender (the opening angle of a round jet is 10°) and the axial gradient of the stress $\overline{u'v'}$ is negligible in front of the transverse one.

2.1. *Plane jet*

The representation above translates into $U\partial_x U + V\partial_y U = 0$ and $\partial_x U + \partial_y V = 0$, that is

$$u\partial_x u + v\partial_y u = -\partial_y \overline{u'v'}, \tag{7}$$

$$\partial_x u + \partial_y v = 0, \tag{8}$$

valid up to corrections of order $\mathcal{O}(\partial_x \overline{u'^2})$ which we neglect owing to slenderness. Far downstream from the injection region, the jet transverse size L broadens proportionally to the downstream distance x, and axial momentum flux $\sim u^2 x$ (per unit span-wise length and mass) conservation dictates that $u = \mathcal{O}(x^{-1/2})$, and $\mathcal{D} = \mathcal{O}(ux) \sim x^{1/2}$, see e.g. [2,8,10].

Introducing a stream function $\psi(\eta)$ consistent with the incompressibility condition in (8), we have

$$\begin{gathered} \psi \sim x^{1/2}\mathcal{F}(\eta), \quad \text{with } \eta = \frac{y}{x}, \\ \text{providing } u = \partial_y \psi \sim x^{-1/2}\mathcal{F}', \quad \text{and } v = -\partial_x \psi \sim x^{-1/2}\left(\eta\mathcal{F}' - \frac{1}{2}\mathcal{F}\right). \end{gathered} \tag{9}$$

A useful reformulation of Euler's equation was introduced by von Reichardt [25]. It amounts to extract a relationship between the Reynolds stress $\overline{u'v'}$ and the mean fields u and v. From (7) & (9) we find $\overline{u'v'} = u(\eta u - v)$, leading to

$$\overline{u'v'} \sim \frac{1}{2x}\mathcal{F}\mathcal{F}'. \tag{10}$$

2.2. *Round jet*

The above results are readily transposed to a round jet for which the balance equations are

$$u\partial_x u + v\partial_r u = -\partial_r (r\,\overline{u'v'})/r, \tag{11}$$

$$\partial_x u + \partial_r (r\,v)/r = 0. \tag{12}$$

Far downstream from the injection region, the jet radius L broadens proportionally to the downstream distance x, and axial momentum flux $\sim u^2 x^2$ (per unit mass) conservation dictates that $u = \mathcal{O}(x^{-1})$, and $\mathcal{D} = \mathcal{O}(ux) = \mathrm{C^{st}}$.

The stream function $\psi(\eta)$ is now

$$\begin{gathered} \psi \sim x\mathcal{F}(\eta), \quad \text{with } \eta = \frac{r}{x}, \\ \text{which provides } u = \partial_r \psi / r \sim \frac{\mathcal{F}'}{x\eta}, \quad \text{and } v = -\partial_x \psi / r \sim \frac{-\mathcal{F} + \eta\mathcal{F}'}{x\eta}. \end{gathered} \tag{13}$$

A similar von Reichardt's type of calculation now provides $\overline{u'v'} = u\big(\eta u - v\big)$ as for a plane jet, that is

$$\overline{u'v'} \sim \frac{1}{x^2}\frac{\mathcal{F}\mathcal{F}'}{\eta^2}. \tag{14}$$

3. The closure and its formulation in jets

Let us start by discussing the plane jet for which (10) has shown that

$$\overline{u'v'} \sim \frac{1}{2x}\mathscr{F}\mathscr{F}'. \tag{15}$$

The function $\mathscr{F}$ is defined from the stream function ψ, which itself arises from incompressibility; its dependence on the sole variable $\eta = y/x$ reflects the flow self-similarity, which is a separate issue (a flow may well be incompressible while non self-similar). The fact that the stress tensor is proportional to $\mathscr{F}$, that is to the integral over space (i.e. on η) of the velocity ($u \sim \mathscr{F}'$) shows that it incorporates ingredients which are necessarily *nonlocal*. This is not an artifact of a particular model since (10) is nothing but a reformulation of the basic conservation equations. This nonlocality of the stress, which arises from the dynamical equations of the flow, does not necessarily imply that, in general, the closure for $\overline{u'v'}$ should be nonlocal itself: when usual molecular, thermally activated viscosity ν is at play, the stress $-\nu(\partial_j u_i + \partial_i u_j)$ is *very local* since it involves the spatial derivatives of u_i. But here, viscous stresses are not extrinsic (i.e. thermally activated), but intrinsic; the effective viscosity $\mathscr{D}$ is the result of the flow itself, and since stresses in the flow are ruled by nonlocal effects, so should be $\mathscr{D}$.

Therefore, in looking for a model for the stress tensor in a flow with mean velocity profile $u(x,y)$, we need to have in mind that $\overline{u'v'}$ must:

(1) be Galilean invariant: when we write $u(x,y)$, we actually mean $u(x,y) - u(x,\infty)$, where $u(x,\infty)$ is an arbitrary velocity shift, taken here as zero;
(2) vanish when $\partial_y u = 0$, another way to express that fluxes are mediated by velocity *differences* only;
(3) be sensitive to the large-scale organization of the flow i.e. be *nonlocal* by involving, for example, an integral $\int u\,\mathrm{d}y$ between bounds which need to be discussed for each flow.

A possible form satisfying the constraints above may be derived from an extension of the familiar momentum transport model used by [4] in analogy with the kinetic theory of gases [17,26]. In the direction of the velocity gradient $\partial_y u$, the net axial momentum flux (per unit mass) crossing the plane y through instantaneous transverse fluctuations with mean free path ℓ at velocity v' is

$$\overline{u'v'} = \overline{v'\left[u\left(y+\frac{\ell}{2}\right) - u\left(y-\frac{\ell}{2}\right)\right]} = \overline{v'\ell}\partial_y u. \tag{16}$$

In a turbulent flow, both ℓ and v' are broadly distributed but may have a loose level of intercorrelation defining, in this construction, the intensity of the momentum transfer.

Our model is the following: in shear flows like jets, fluctuations are driven by a sustained large-scale shear, namely by the velocity difference between the core of the jet, and the outside; this is, in essence, the nature of the Kelvin–Helmholtz instability. The correlation length of the motion is the jet radius itself and thus ℓ should on average scale with the jet radius $L = \alpha x$. For the same reason, fluctuations of any extensity (momentum, vorticity, concentration) are likely to be transported from the rich core of the jet to any off-center depleted location y. The velocity v' driving the transfer in $y < L$ can thus be anything between its value at the jet centerline, and the one at that location y. We take arbitrarily a flat average of u in the interval $[0, y]$ as a representative value of the transfer velocity which we call $\widetilde{v}$ and, instead of $\widetilde{v} \sim u$ [15] or $\widetilde{v} \sim \ell\partial_y u$ [4], we write

$$\widetilde{v} = \frac{1}{y}\int_0^y u(y')\,\mathrm{d}y', \tag{17}$$

so that the stress is

$$\overline{u'v'} = -\widetilde{v}\ell\partial_y u, \quad \text{with } \ell = ax, \tag{18}$$

fulfilling the requirements of Galilean invariance (when u is interpreted as the velocity difference with the velocity far away from the jet), vanishing when $\partial_y u = 0$ and involving a spatial integration of u through the *partial mean* $\widetilde{v}$. In this conjecture, the only ajustable parameter is a, which is nevertheless constrained to be much smaller than α, for the effective mean free path ℓ to be smaller than the jet radius L (see Section 6).

The integration bounds in (17) suit to a jet, where the momentum reservoir is in its core. In a mixing layer, or a wake, the situation is opposite and, according to the same principle, the same formula for $\widetilde{v}$ holds but with y' now running from y to an arbitrarily large distance h, say infinite, thus defining the outside mean velocity u_∞ as (see Sections 4 & 5)

$$\widetilde{v} = \frac{1}{h}\int_y^{y+h\to\infty} u\,\mathrm{d}y \approx u_\infty. \tag{19}$$

More generally in a flow-field $\{u_i\}$, a formal expression of the transfer velocity $\widetilde{v}_i$ perpendicular to u_i, aligned with the directions of $\partial_j u_i$ and $\partial_k u_i$ is

$$\widetilde{v}_i = \max\left\{\frac{1}{\mathscr{A}}\int_{\mathscr{A}} u_i\,\mathrm{d}x_j\,\mathrm{d}x_k,\ u_\infty\right\}, \tag{20}$$

where the integration area $\mathscr{A}(x_j,x_k) = \int \mathrm{d}x_j\,\mathrm{d}x_k$ is computed along the axis/direction perpendicular to u_i (Figure 1).

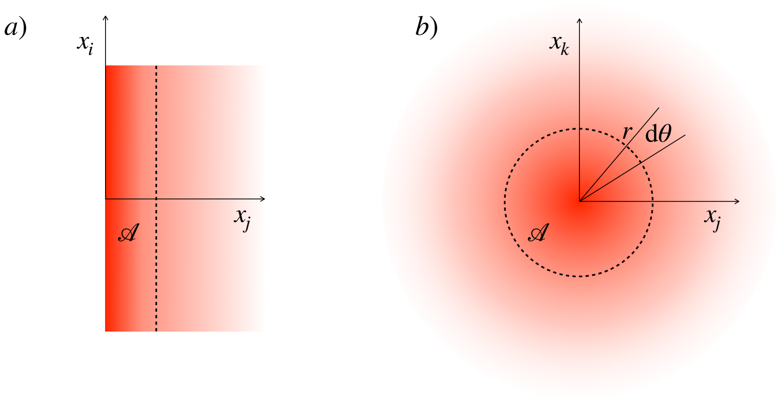

Figure 1. Sketch of the averaging area $\mathscr{A}(x_j,x_k)$ defining $\widetilde{v}_i$. The red area feature momentum rich cores feeding an outside depleted region. a) In a plane jet (see (17)), the flow with velocity $u_i = u$ is in the direction $x_i = x$ perpendicular to $x_j = y$ and $\mathscr{A} = \int_0^y \mathrm{d}y'$. b) In a round jet (see (23)), the flow direction with velocity $u_i = u$ is perpendicular to the plane $\{x_j,x_k\}$ and we have $\mathrm{d}x_j\,\mathrm{d}x_k = r\,\mathrm{d}\theta\,\mathrm{d}r$ so that $\mathscr{A} = \int_0^{2\pi}\mathrm{d}\theta\int_0^r r'\,\mathrm{d}r'$.

3.1. *The plane jet*

We may now close the problem, and find an expression for the velocity profile $u = x^{-1/2}\mathscr{F}'$. Euler equations translate into (15) which is equivalent to (one checks that $\widetilde{v} = x^{-1/2}\mathscr{F}/\eta$)

$$\overline{u'v'} = \frac{1}{2}\eta\widetilde{v}u. \tag{21}$$

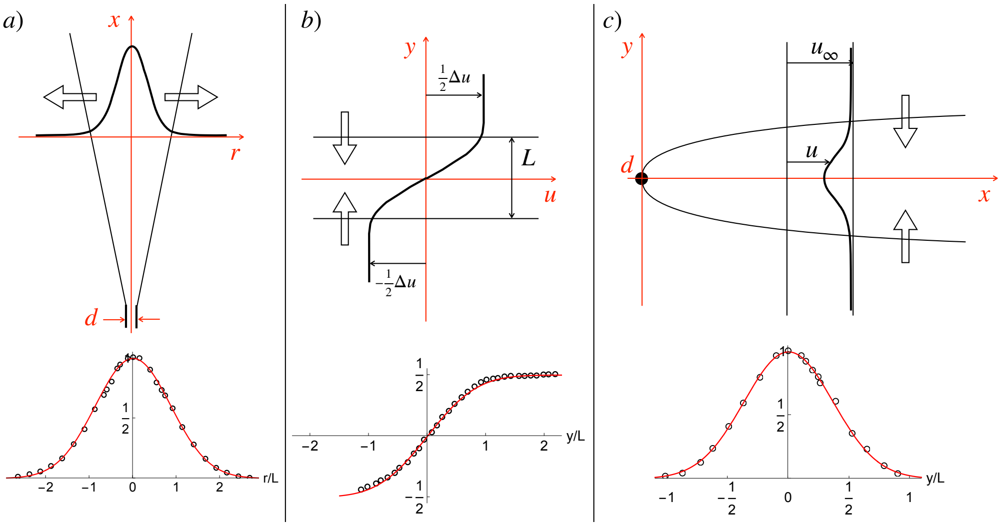

Figure 2. Sketches of the different mean turbulent velocity profiles studied here, and the corresponding experimental profiles reported in [8], with page numbers: a) round jet (measurements by Reichardt, p. 749, fit by (25)), b) mixing layer (measurements by Reichardt, p. 738, fit by (34)) and c) wake (measurements by Schlichting, p. 741, fit by (53)). The big empty arrows indicate the direction of the momentum transfer: towards $r > 0$ (or larger $|y|$ in two-dimensions) for jets, and from large $|y|$ towards $y = 0$ in mixing layers and wakes, explaining the sign of $\tilde{v}$ and the integration interval of its definition ($\int_0^y u\,\mathrm{d}y$ for jets and $\int_y^\infty u\,\mathrm{d}y$ for mixing layers and wakes).

When put in relation with the closure in (17), this provides $2a\partial_\eta u = -\eta u$, that is

$$\mathscr{F}'' = -\frac{1}{2a}\eta\mathscr{F}' \\ \text{or } \mathscr{F} = \sqrt{\pi a}\,\mathrm{erf}\left(\frac{\eta}{2\sqrt{a}}\right), \quad \text{giving } u \sim \mathscr{F}' = e^{-\frac{\eta^2}{4a}}. \tag{22}$$

The velocity profile is Gaussian, as many experiments, since the seminal ones displayed in Figure 2, have shown.

3.2. *The round jet*

In a round jet, our closure, identically to (17), is $\overline{u'v'} = -\tilde{v}\ell\partial_r u$ with $\ell = ax$ except that now the velocity $\tilde{v}$ must be understood as an average over the jet cross section since there are two degrees of freedom for radial transport, and involves a surface integration ($\mathrm{d}x_j\,\mathrm{d}x_k = r\,\mathrm{d}\theta\,\mathrm{d}r$, $\theta \in [0, 2\pi]$, see Figure 1 and (20))

$$\tilde{v} = \frac{1}{\pi r^2}\int_0^r 2\pi r' u(r')\,\mathrm{d}r' = \frac{2}{x}\frac{\mathscr{F}}{\eta^2}. \tag{23}$$

Since $\partial_r u = (\mathscr{F}'/\eta)'/x$ and that, from Euler equation in (14)

$$\overline{u'v'} = \frac{1}{x^2}\frac{\mathscr{F}\mathscr{F}'}{\eta^2}, \tag{24}$$

one sees that $\mathscr{F}$ must solve (with $\mathscr{F}(0)=0$ and $\mathscr{F}'/\eta \xrightarrow[\eta\to 0]{} 1$)

$$\begin{aligned} &2a\left(\frac{\mathscr{F}'}{\eta}\right)' = -\mathscr{F}' \\ &\text{giving } \mathscr{F} = 2a(1-e^{-\frac{\eta^2}{4a}}), \quad \text{that is } u \sim \mathscr{F}'/\eta = e^{-\frac{\eta^2}{4a}}. \end{aligned} \tag{25}$$

The velocity profile is also Gaussian, as again experiments show (Figure 2).

3.3. *The dispersion coefficients*

Rather than being at the source of the computation providing the form of u, the dispersion coefficient $\mathscr{D}$ is a consequence of it. We have not conjectured a particular form of $\mathscr{D}$ (various forms are compared in Figure 3), we have instead discussed what features the stress $\overline{u'v'}$ should have, and obtained u right away. We can now compute $\mathscr{D}$. It is true that we have argued in (17) why $\overline{u'v'}$ should be proportional to the velocity gradient and this implies, assuming a gradient type form for the flux

$$\begin{aligned} &\overline{u'v'} = -\mathscr{D}\partial_y u \ \text{ or } \ -\mathscr{D}\partial_r u, \\ &\text{that } \mathscr{D} = \tilde{v}\ell, \end{aligned} \tag{26}$$

but that was not our starting point. In a plane jet, we have $\overline{u'v'} = -\mathscr{D}\partial_y u$ so that

$$\begin{aligned} \mathscr{D} &\sim -x^{1/2}\frac{\mathscr{F}\mathscr{F}'}{2\mathscr{F}''} \\ &= a\sqrt{\pi x}\,\frac{\operatorname{erf}\left(\frac{\eta}{2\sqrt{a}}\right)}{\eta/\sqrt{a}}. \end{aligned} \tag{27}$$

The dispersion coefficient increases with downstream distance as $x^{1/2}$, is constant close to the jet center for $\eta \ll 1$ and decays far away from it like η^{-1} for $\eta \gg 1$.

In a round jet, where $\overline{u'v'} = -\mathscr{D}\partial_r u$ we have

$$\begin{aligned} \mathscr{D} &\sim \frac{\mathscr{F}\mathscr{F}'}{\mathscr{F}' - \eta\mathscr{F}''} \\ &= a\frac{1-e^{-\frac{\eta^2}{4a}}}{(\eta/2\sqrt{a})^2}, \end{aligned} \tag{28}$$

showing that $\mathscr{D}$ is independent of x, and again that it is constant close to the jet center for $\eta \ll 1$ and decays far away from it like η^{-2} this time, for $\eta \gg 1$.

It would have been, admittedly, difficult to intuit these relatively complicated formulae for $\mathscr{D}$ in order to obtain the ubiquitous Gaussian form for u. Although qualitatively similar in plane and round jets, the detailed form of $\mathscr{D}$ is besides different in each case. This illustrates how guessing $\mathscr{D}$ to infer u is a risky exercise, if not an *a priori* lost battle.

Although the velocity profiles are both Gaussian in plane and round jets, the dispersion coefficients are different in each case. This is at contrast with standard molecular diffusion where the dispersion coefficient is intrinsic, because thermally activated (the viscosity ν in fluids); the originality of turbulence is, on the contrary, that the dispersion coefficient depends on the structure of the flow itself and has therefore no reason to be universal; the velocity profiles, interestingly, are.

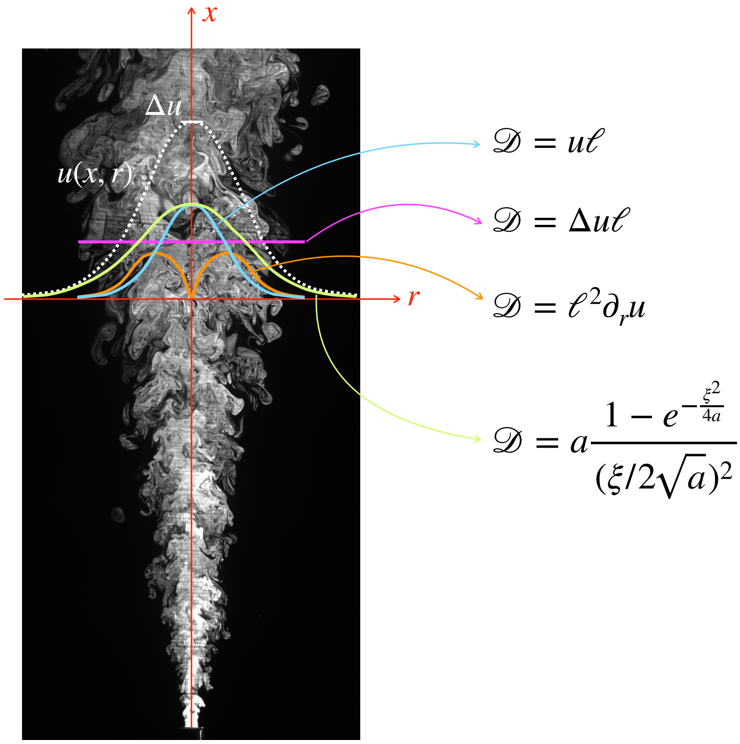

Figure 3. Different models of the turbulent dispersion coefficient $\mathscr{D}$ for a turbulent round jet (Section 1). The closure $\mathscr{D} \sim \Delta u\ell$ of Boussinesq [15] where Δu is the amplitude of the velocity profile u across the jet, its local version $\mathscr{D} \sim u\ell$, the Prandtl [4] version $\mathscr{D} \sim u\ell^2\partial_r u$, initially conceived for boundary layers, clearly at odd for velocity profiles with a maximum (see also [12]), and the one derived from the present conjecture of the stress tensor in (17), leading to a Gaussian velocity profile $u \sim e^{-\frac{\eta^2}{4a}}$, which is intermediate between the Boussinesq local, and constant forms (see (28)).

3.4. *Relation to experiments*

These Gaussian mean velocity profiles fit well the observed ones, including on the more stringent log-scale compared to the lin-lin scaling routinely used in this context [27], in any case better than those derived by Schlichting [8,28] under the assumption of a constant, and uniform dispersion coefficient $\mathscr{D}$ across the jet, as suitable for a laminar jet [10,29]. Indeed, (22) & (25) compare favorably well with experiments, unlike the corresponding predictions of Schlichting, namely

$$u \sim 1 - \tanh^2\eta \quad \text{for a plane jet,}$$
$$\text{and} \quad u \sim \left(1 + \frac{\eta^2}{2}\right)^{-2} \quad \text{for a round jet,} \tag{29}$$

which both present much too heavy tails, obvious in logarithmic units (see [6]).

We have, on purpose, mentioned in Figure 2 the early seminal measurements reported in [8] only, but the empirical adequacy of the Gaussian fit in jets has been since then reported consistently [30–32]. In the present representation, $\mathscr{D}$ decays (slowly) on the edge of the jet, while dispersion is intense close to the center of the jet [33]. This results in a more “localized” profile around the jet centerline than those assuming a constant $\mathscr{D}$ in the transverse direction (a fact true for both plane and round jets). Prandtl's parametrization in (4) has also a vanishing $\mathscr{D}$ at the jet edge (where the velocity gradient weakens), but dramatically fails on the centerline, predicting oddly that $\mathscr{D} = 0$ there (see [34] for an application of Prandtl's mixing length theory to the turbulent round jet).

For the diffusive description of the phenomenon to be consistent, the mean free path $\ell = ax$ in (17) must be appreciably smaller than the jet radius $L = \alpha x$, implying that $a \ll \alpha$. In a round jet, $\alpha = 1/6 \approx 0.16$ and $a \approx 0.004$ fit well the experimental profiles, thus fulfilling the condition of a short mean free path, when seen at the scale of the jet radius.

4. Mixing layers

4.1. *The temporal mixing layer*

The mixing layer has for a long time been considered as the *Drosophila* of turbulent shear flows: a benchmark to exercise new ideas, methods and concepts. We work out the closure in (17) in a two-dimensional, temporally growing mixing layer between two streams of opposite velocities $\pm\Delta u/2$ initially separated by a sharp discontinuity. The flow is uniform along x, and the layer grows in the y-direction so that from the Euler equations $\partial_t U + \partial_x U^2 + \partial_y (UV) = 0$ with $u(y,t)$ and $v = 0$, we have

$$\partial_t u = -\partial_y \overline{u'v'} \tag{30}$$

where now the average $\overline{(\cdot)}$ must be taken over a time-period T large compared to the correlation time of u' and v' (taken as zero in the pure brownian limit), but small compared to the characteristic time of the flow evolution $u/\dot{u}$.

The only length scale in this problem is $\Delta u\, t$, we thus look for a solution of the form

$$u = \Delta u \mathscr{F}(\eta) \quad \text{with } \eta = \frac{y}{\Delta u\, t}. \tag{31}$$

As usual we compute the stress $\overline{u'v'}$ as a function of u and we have from (30), making use of $\partial_t u = -\Delta u \eta \mathscr{F}'/t$,

$$\begin{aligned} \overline{u'v'} &= \overline{u'v'}\big|_0 - \int_0^y \partial_t u \,\mathrm{d}y \\ &= \overline{u'v'}\big|_0 + (\Delta u)^2 \int_0^\eta \eta \mathscr{F}' \,\mathrm{d}\eta \\ &= -(\Delta u)^2 \int_\eta^\infty \eta \mathscr{F}' \,\mathrm{d}\eta \end{aligned} \tag{32}$$

since $\overline{u'v'} \xrightarrow[\eta \to \pm\infty]{} 0$ and that therefore $\overline{u'v'}\big|_0 = -(\Delta u)^2 \int_0^\infty \eta \mathscr{F}' \,\mathrm{d}\eta$.

Now comes the closure: the mean free path $\ell = a\Delta u\, t$ is proportional to the layer thickness, $\partial_y u = \mathscr{F}'/t$ and (17) & (31) give $\overline{u'v'} = -a\Delta u \widetilde{v} \mathscr{F}'$, the minus sign reflecting the direction of the momentum transfer, from the (infinite) upper layer above the sheared region (in $y > 0$, say, the profile is anti-symmetrical in $y < 0$), towards the core of the layer in $y = 0$. That direction is opposite to the one in inertial jets diluting their momentum in a quiescent environment, as those examined before. The momentum reservoir thickness h being practically infinite, we have

$$\widetilde{v} = \frac{1}{h} \int_y^{y+h\to\infty} u \,\mathrm{d}y \approx \Delta u/2 \tag{33}$$

since the integral extends to infinity. Then, from (32), we find

$$\begin{aligned} &\mathscr{F}' = \frac{2}{a} \int_\eta^\infty \eta \mathscr{F}' \,\mathrm{d}\eta \quad \text{or, differentiating, } \mathscr{F}'' = -\frac{2}{a} \eta \mathscr{F}', \\ &\text{finally providing } \mathscr{F} = \frac{1}{2} \operatorname{erf}\left(\frac{\eta}{\sqrt{a}}\right), \end{aligned} \tag{34}$$

which fulfills the boundary conditions $u(\eta=\pm\infty)=\pm\Delta u/2$.

The dispersion coefficient $\mathscr{D}$ in a turbulent mixing layer, such that $\overline{u'v'}=-\mathscr{D}\partial_y u$, or

$$\mathscr{D}=\frac{(\Delta u)^2\int_\eta^\infty \eta\mathscr{F}'\,\mathrm{d}\eta}{\mathscr{F}'}\,t=\frac{a}{2}(\Delta u)^2\,t, \tag{35}$$

is proportional to the product of the mean free path $\ell=a\Delta u\,t$ times the velocity difference Δu sustained across the layer. It is independent of y, increases linearly in time in proportion of ℓ, while the turbulent stresses $\overline{u'v'}\sim\sqrt{a}\,e^{-\eta^2/a}$ remain localized over a region, called the mixing layer, whose width increases accordingly. If one chooses the "vorticity thickness" $L=\Delta u/\partial_y u\big|_{\max}$, equal to $\Delta u t/\mathscr{F}'(0)$ in the present theory (since the maximum of the shear rate $\partial_y u$ is in $y=0$) to quantify the layer thickness, we find that

$$L=\sqrt{\pi a}\,\Delta u t. \tag{36}$$

As already recalled, the growth of the layer thickness (as well as the growth of jets widths) is driven by the Kelvin–Helmholtz instability of the sustained large-scale shear $\Delta u/L$. Since this instability is at the core of the phenomenon, one may expect to link our description, which still lacks a fundamental understanding of the—free at this stage—parameter a, with the fundamentals of the Kelvin–Helmholtz instability. The (temporal) growth rate of a disturbance amplitude with wavenumber k along the layer is given by (see [35] for a comprehensive review, and [36, Appendix] for a synthetic formulaire)

$$\frac{\Delta u}{L}\mathscr{J}(kL) \tag{37}$$

where $\mathscr{J}(kL)$ is a function describing the amplification of mode k, which is tangent to kL for $kL\to 0$ and falls to zero for $k_c L=\mathscr{O}(1)$. For instance, $\mathscr{J}(kL)=\frac{1}{2}\sqrt{e^{-2kL}-(1-kL)^2}$ for a broken-line profile absorbing a total velocity difference Δu over a uniform crossover of thickness L, first studied by [37]. In that generic case (the precise shape of $\mathscr{J}(kL)$ is very little sensitive to the detailed shape of $u(y)$, see [35]), the maximal amplification rate is

$$0.2\frac{\Delta u}{L} \tag{38}$$

obtained at $k_m L\approx 0.8$. How does this result connect with the linear growth of L in (36)? The increase of L reflects the amplification of a superposition of all the modes k in the unstable range $0<k<k_c$, and in that sense its growth rate $\dot{L}/L$ must be proportional to the one of the preferred mode $0.2\Delta u/L$. Having said that, we see that we have already understood the structure of (36): because the growth rate is itself inversely proportional to the layer thickness, $\dot{L}$ is constant, hence the linear growth in t. Now, what about the pre-factor? The instability analysis tells us that in the growing phase of the layer, the vorticity is contained in the layer L, plus two adjacent evanescent zones of thickness such that $ky=\mathscr{O}(1)$ in $y=\pm L/2$ (the velocity perturbations decay like $e^{\mp ky}$ from there). Thus, since the preferred mode for this instability is also such that $kL=\mathscr{O}(1)$, the effective layer thickness in (38) is approximately

$$L+2\times L/2=2L, \tag{39}$$

giving finally an effective layer thickness growth

$$L\approx 0.1\Delta u t, \tag{40}$$

as some numerical simulations have suggested [38]. Comparing with (36), we see that $a=(0.1)^2/\pi\approx 0.0032$. The ratio of the mean free path ℓ to the layer thickness is $\sqrt{a/\pi}=0.032$, consistently smaller than unity.

The effect described above, linking linear stability and effective growth of the layer is dominant over another effect, related to the transient character of base state inherent to its growth: since the layer thickness is itself time-dependent, there is no reason for the linear stability, which assumes a given L, to apply in every details. In particular, there is no reason that the system, under growth, picks up the amplification rate of the most amplified mode k_m in (38). When the layer has grown by ΔL from a given state L, it is easy to see that the most amplified mode shifts to $k_m L\left(1-\frac{1}{2}\frac{\Delta L}{L}\right)$ so that the growth rate is diminished by a factor $1-\frac{1}{4}\left(\frac{\Delta L}{L}\right)^2$. This second order correction, which does cooperate in the damping of the growth, is however weak in comparison to the one due to the thickening of the vorticity layer due to the development of the instability itself, and which accounts for the factor 2 in (39).

4.2. *The spatial mixing layer*

The spatial development of a steady mixing layer between two parallel streams of different velocities $u_1 < u_2$ is described along the very same lines as above. Let $\Delta u = u_2 - u_1$ be the net velocity jump across the layer, in a fluid of uniform density (the method can be generalized to streams presenting a density contrast), and $u_0 = (u_1 + u_2)/2$ be the average velocity with, say, $\Delta u \ll u_0$.

The steady form of the Euler equations $\partial_t U + \partial_x U^2 + \partial_y (UV) = 0$ with $u(x, y)$ and $|v\partial_y u|/|u\partial_x u| \ll 1$ amounts to

$$u\partial_x u = -\partial_y \overline{u'v'}. \tag{41}$$

The only length scale in this problem is the downstream distance x from the contact location of the streams (a splitter plate, typically), we thus look for a solution of the form

$$u = u_0 + \Delta u \mathscr{F}(\eta) \quad \text{with } \eta = \frac{y}{x}. \tag{42}$$

The stress $\overline{u'v'}$ is given, from (41) making use of $u\partial_x u \approx -u_0 \Delta u \eta \mathscr{F}'/x$ (when $\Delta u \ll u_0$), by

$$\begin{aligned}\overline{u'v'} &= \overline{u'v'}\big|_0 - \int_0^y u\partial_x u \,\mathrm{d}y \\ &= \overline{u'v'}\big|_0 + u_0 \Delta u \int_0^\eta \eta \mathscr{F}' \,\mathrm{d}\eta \\ &= -u_0 \Delta u \int_\eta^\infty \eta \mathscr{F}' \,\mathrm{d}\eta\end{aligned} \tag{43}$$

since $\overline{u'v'} \xrightarrow[\eta\to\pm\infty]{} 0$ and that therefore $\overline{u'v'}\big|_0 = -u_0 \Delta u \int_0^\infty \eta \mathscr{F}' \,\mathrm{d}\eta$.

As a Galilean invariant, the velocity $\widetilde{v}$ of the closure in (17) must remain unchanged whatever the mean drift u_0 may be, therefore

$$\widetilde{v} = \frac{1}{h}\int_y^{y+h\to\infty} (u - u_0)\,\mathrm{d}y \approx \Delta u/2. \tag{44}$$

In the present view, if one compares (30) with (41), a spatially growing mixing layer is nothing but a translated (at the mean drift velocity u_0) temporally growing layer for which we had $\ell = a\Delta u\, t$. Relating time to space by $t = x/u_0$ (a kind of Gaster Transformation, see [39]), we thus expect that the mean free path, which is a (small) fraction of the layer width at location x, will be $\ell = a\Delta u/u_0 x$ so that $\overline{u'v'} = -\frac{1}{2} a\mathscr{F}'(\Delta u)^3/u_0$ which, together with (43) provides

$$\begin{aligned}&\mathscr{F}' = \frac{2}{a}\left(\frac{u_0}{\Delta u}\right)^2 \int_\eta^\infty \eta \mathscr{F}' \,\mathrm{d}\eta \quad \text{or, differentiating, } \mathscr{F}'' = -\frac{2}{a}\left(\frac{u_0}{\Delta u}\right)^2 \eta \mathscr{F}', \\ &\text{so that } \mathscr{F} = \frac{1}{2}\operatorname{erf}\left(\frac{\eta}{\sqrt{a}\Delta u/u_0}\right),\end{aligned} \tag{45}$$

fulfilling the boundary conditions $u(\eta = \pm\infty) = u_0 \pm \Delta u/2$.

The dispersion coefficient $\mathscr{D} = -\overline{u'v'}/\partial_y u$ of the spatially growing layer is, again, computed *a posteriori* as

$$\mathscr{D} = \frac{u_0 \int_\eta^\infty \eta \mathscr{F}' \, \mathrm{d}\eta}{\mathscr{F}'} x = \frac{a}{2} \frac{(\Delta u)^2}{u_0} x. \tag{46}$$

It is proportional to the product of the mean free path ℓ times the velocity difference Δu sustained across the layer. It is independent of y, increases linearly with x in proportion of ℓ, while the turbulent stresses $\overline{u'v'} \sim \sqrt{a}\Delta u/u_0 \, e^{-\eta^2/(\sqrt{a}\Delta u/u_0)^2}$ remain localized over a region—the mixing layer—whose standard deviation in the y-direction increases accordingly like $\sqrt{a}\Delta u/u_0 x$. These expressions for u and $\mathscr{D}$ are formally identical to the ones for the temporal mixing layer changing, *mutatis mutandis*, time to space $t \to x/u_0$.

The celebrated study by [40] offers a precise relationship describing the "vorticity thickness" of the layer $L = \Delta u/\partial_y u\big|_{\max}$, equal to $x/\mathscr{F}'(0)$ in the present theory since the maximum of the shear rate $\partial_y u$ is in $y = 0$. They find empirically that

$$\frac{L}{x} = 0.18 \frac{u_2 - u_1}{u_2 + u_1} = 0.09 \frac{\Delta u}{u_0} \tag{47}$$

which allows us to infer what the pre-factor a should be in mixing layers. Since $1/\mathscr{F}'(0) = \sqrt{\pi a}\Delta u/u_0$, we find from (47) above that $a = (0.09)^2/\pi \approx 0.00257$. The ratio of the mean free path ℓ to the layer thickness is $\sqrt{a/\pi} = 0.028$, consistently smaller than unity, identically to the temporal mixing layer above.

In this respect, the 0.18 in Brown–Roshko's law has to be understood as reflecting the 0.2 in (38), which itself fundamentally derives from the linear stability of the layer. Indeed, the 0.1 in $L \approx 0.1 \Delta u t$ of (40) actually means 0.2/2 and with $t = x/u_0$ and $u_0 = (u_1 + u_2)/2$, (40) restores pretty closely (47). The interpretation of [40]'s measurements was already contained in a calculation made by [37]!

5. The wake behind an obstacle

An obstacle of size d is placed in a steady stream with velocity u_∞ along x. The obstacle has a large aspect ratio (a rod, a bar), its largest dimension being perpendicular to the flow direction so that the resulting wake behind the obstacle is two-dimensional. The velocity deficit $u_\infty - u$ in the wake reflects the momentum flux deficit $\sim \rho u_\infty^2 d$ absorbed by the obstacle (the force exerted on it per unit length at large Reynolds number). In a free wake (no superimposed body forces), that momentum deficit spreads-out in the transverse y direction, but is globally conserved so that [8]

$$\begin{aligned} u_\infty^2 d &\sim \int_{-\infty}^{\infty} u(u_\infty - u)\, \mathrm{d}y, \\ \text{but since } u \lesssim u_\infty \text{ in the far field,} \quad u_\infty d &\sim \int_{-\infty}^{\infty} (u_\infty - u)\, \mathrm{d}y. \end{aligned} \tag{48}$$

Unlike in jets, the drift velocity is now constant and we expect the wake width to be $L \sim (xd)^{1/2}$, and a velocity deficit as $u_\infty - u \sim (x/d)^{-1/2}$. We thus look for a solution of the form

$$\begin{aligned} u_\infty - u = u_\infty \left(\frac{x}{d}\right)^{-1/2} \mathscr{F}(\eta), \quad \text{where } \eta = \frac{y}{(xd)^{1/2}} \\ \text{with, owing to the integral constraint in (48),} \quad \int_{-\infty}^{\infty} \mathscr{F}\, \mathrm{d}\eta = 1. \end{aligned} \tag{49}$$

Euler equations for $u(x, y)$ write at dominant order ($u \lesssim u_\infty$ and $|v\partial_y u|/|u_\infty \partial_x u| \ll 1$)

$$u_\infty \partial_x u = -\partial_y \overline{u'v'} \tag{50}$$

so that the stress $\overline{u'v'}$ is

$$\begin{aligned}\overline{u'v'} &= -u_\infty \int_0^y \partial_x u \,\mathrm{d}y \\ &= -\frac{u_\infty^2}{2}\frac{d}{x}\eta\mathscr{F}.\end{aligned} \tag{51}$$

Contrary to jets, but like in mixing layers, the inertia reservoir is, in a wake, outside of it, far from its depleted core, but in the rich free stream. The Galilean invariant velocity $\widetilde{v}$ in our closure $\overline{u'v'} = -\widetilde{v}\ell\partial_y u$ is thus

$$\widetilde{v} = \frac{1}{h}\int_y^{y+h\to\infty}\big(u-u(0)\big)\,\mathrm{d}y \approx u_\infty\left(\frac{x}{d}\right)^{-1/2}. \tag{52}$$

The mean free path $\ell = a(xd)^{1/2}$ is a fraction of the wake transverse width L and $\partial_y u = -u_\infty\mathscr{F}'/x$ finally giving, making use of (51)

$$\begin{aligned}\mathscr{F}' &= -\frac{1}{2a}\eta\mathscr{F} \\ \text{or } \mathscr{F} &= \frac{1}{2\sqrt{\pi a}}e^{-\frac{\eta^2}{4a}}, \quad \text{fulfilling } \int_{-\infty}^{\infty}\mathscr{F}\,\mathrm{d}\eta = 1.\end{aligned} \tag{53}$$

The velocity deficit has a Gaussian profile, whose width increases downstream as $L = (axd)^{1/2}$ and the mean free path is indeed a small fraction of it since $\ell/L = \mathscr{O}(a^{1/2}) \ll 1$ when $a \ll 1$, as we expect. The dispersion coefficient $\mathscr{D} = -\overline{u'v'}/\partial_y u$ in the wake is

$$\mathscr{D} = -\frac{1}{2}u_\infty d\frac{\eta\mathscr{F}}{\mathscr{F}'} = au_\infty d, \tag{54}$$

a constant, independent of η. Because a wake results from a momentum deficit caused by an obstacle, its size d remains apparent all along, including in the far field structure of the velocity profile.

Unlike in mixing layers, the flow in a wake is not caused by an instability but is instead a pure relaxation problem towards the uniform state $u = u_\infty$, mediated by turbulent fluctuations. In that sense, it is a "pure" dispersion problem, not surprisingly described by a Gaussian, and a uniform dispersion coefficient. Consequently, we have however no additional ingredient to set, in wakes, the value of the unknown small pre-factor a, besides the one explained in the next Section 6. Comparison with the measurements in [8, p. 743] indicates that $a \approx 0.011 \times \frac{1}{2}C_D$ with C_D the obstacle drag coefficient, a value indeed small compared to unity.

6. The status of the mean free path

The existence of a mean free path ℓ in the turbulent flows we have studied is an assumption. It was systematically chosen as being proportional (by a factor a) to the largest length-scale of the flow L (jet radius, mixing layer thickness, wake width), and independent of the Reynolds number. This assumption provides relevant predictions (the velocity profiles) which are self-consistent: the pre-factor a, which must be smaller than unity by construction, is indeed found to be as such when it can be computed from first principles in mixing layers. The order of magnitude of ℓ is a few percent of the outer flow dimensions. For instance, we have found that

$$a = (0.2/2)^2/\pi \approx 0.0032 \qquad \text{and} \qquad \frac{\ell}{L} = \sqrt{\frac{a}{\pi}} = 0.032, \tag{55}$$

an exact result in temporal mixing layers; the same order of magnitude is found for jets and wakes.

This ubiquitous fact must relate to a fundamental feature of turbulence for which we have, at this stage, no clue, but for which we may imagine a mechanism: we must find a microscopic process which selects, in a turbulent flow at large Reynolds number, a length-scale ℓ which is a small fraction of the stirring (integral) scale of the flow L while being independent of viscosity. We will loosely identify this length-scale with a mean free path because, not being affected by viscous drag, the corresponding fluid elements are free to transport momentum.

We conduct the discussion in two-dimensions (for simplicity) and consider a set of adjacent slabs of fluid spaced by s_0 equipped with alternate opposite velocities. The initial flow pattern is for instance (in a frame moving at $\langle u_i \rangle$, the translational velocity of the slabs bundle)

$$u(y, t=0) = u_0 \cos(k_0 y) \quad \text{with} \ \ k_0 = 2\pi/s_0. \tag{56}$$

This flow-field is intended to mimic the expansion of a shear instability in the bulk of the flow, as a result of a velocity difference Δu established across the whole integral size of the system L (like whirling Kelvin–Helmholtz billows in jets, for instance). This velocity difference is absorbed within an internal boundary, or shear layer of thickness λ, and we know that this internal shear layer is liable to trigger an internal Kelvin–Helmholtz instability provided [41,42]

$$R_\lambda = \frac{\Delta u \lambda}{\nu} \gtrsim 100, \tag{57}$$

a condition below which the instability is damped by viscous spreading, and above which the layer destabilizes with the inviscid growth rate in (38). Just below the transition, when the velocity difference is persistent over the transit time $L/\Delta u$ across the system, the internal layer can grow up to $\lambda = (\nu L/\Delta u)^{1/2} \sim L(\Delta u L/\nu)^{-1/2}$ which identifies with the Taylor microscale [43], the correlation length of the velocity gradient. At the transition, namely when the criterion in (57) is met, the value of the Reynolds number based on Δu (itself a measure of the large scale velocity differences in the flow) is

$$\mathrm{Re} = \frac{\Delta u L}{\nu} = 10^4, \tag{58}$$

and above this limit, the internal shear layers destabilize, enhancing considerably the growth of material lines and surfaces. This is called the "Mixing transition" [44,45] and it is an experimental fact that an "inertial range" with marked separation between the stirring scale L and the dissipative range of scales (around λ) requires $R_\lambda \gtrsim 100$ [46].

Above the transition, the later evolution of u (along x, in a frame moving at $\langle u_i \rangle$) in (56) is ruled by

$$\partial_t u + u\partial_x u + v\partial_y u = \nu(\partial_x^2 u + \partial_y^2 u), \tag{59}$$

and we further caricature the v-component as a compression velocity $v = -\gamma y$, a consequence of the instability development (expansion in x implies compression in y owing to incompressibility) with $\gamma \sim \Delta u/\lambda$, scaling like the instability growth rate. Since $\partial_x u = 0$, we have $\partial_t u - \gamma y \partial_y u = \nu \partial_y^2 u$ which reads $\partial_\tau u = \partial_\xi^2 u$ in the variables $\{\xi, \tau\}$ such that

$$\xi = k_0 y e^{\gamma t} \quad \text{and} \quad \tau = \nu \int_0^t \frac{\mathrm{d}t'}{(s_0 e^{-\gamma t'})^2} = \frac{1}{2R_s}(e^{2\gamma t} - 1), \quad \text{with} \ \ R_s = \frac{\gamma s_0^2}{\nu}. \tag{60}$$

The velocity field in the slabs bundle becomes

$$u(\xi, \tau) = u_0 e^{-\tau} \cos \xi, \tag{61}$$

as solution well known for scalar concentrations, as well as the transformation in (60) in Mixing [47].

Whatever it may be initially, the amplitude $u_0 e^{-\tau}$ of the velocity differences in the bundle is damped, at large R_s, within a time such that $\tau = \mathcal{O}(1)$, that is

$$t_s = \frac{1}{2\gamma} \ln R_s. \tag{62}$$

In the meantime, the compressed substrate has shrunk by an amount $e^{-\gamma t_s} = R_s^{-1/2}$ to form a coherent fluid element (no residual velocity fluctuations), moving ballistically (at velocity $\langle u_i \rangle$) free from viscous damping; this is Prandtl's "liquid ball" whose size is given by

$$\ell = L R_s^{-1/2} = \frac{L}{s_0}\left(\frac{\nu}{\gamma}\right)^{1/2}. \tag{63}$$

At onset of the internal shear layer instability, $\lambda/L = 100/10^4 = 1/100$ for a shear rate $\gamma \approx 100\Delta u/L$ (see (57) & (58)). The initial slabs spacing s_0 and the layer thickness λ are related by the nature of the Kelvin–Helmholtz instability: we know that the selected mode is such that $k_0\lambda/2 = \mathcal{O}(1)$ so that $s_0 = \pi L(\Delta u L/\nu)^{-1/2}$ giving finally

$$\begin{aligned}\ell &\approx \frac{L}{\pi L(\Delta u L/\nu)^{-1/2}}\left(\frac{\nu}{100\Delta u/L}\right)^{1/2}\\ &= \frac{L}{10\pi} = 0.031L,\end{aligned} \tag{64}$$

a value nearly identical to the one found in mixing layers.

There is a compensation effect: the viscous fusion, or mixing operation between the slabs restores a Reynolds number invariant length-scale ℓ about a 30^{th} of L from slabs whose size s_0 decreases while their aggregation rate increases as the Reynolds number $\mathrm{Re} = \Delta u L/\nu$ gets larger. In this respect, ℓ is equivalent to the *coarsening scale* in the scalar field [47]. The Reynolds number of the critical layer $R_s = \gamma s_0^2/\nu = 100\pi^2 \approx 10^3$ is larger than unity, consistent with our claim that the slabs fusion process is a stretching enhanced viscous diffusion problem.

7. The special case of boundary layers

The free turbulent shear flows, developing in opened spaces we have examined so far (jets, wakes, shear layers) have all enjoyed a unified description: one seeks for a velocity profile $u(x,y)$ under the hypothesis that there is a mean free path $\ell(x)$ (or $\ell(t)$ in the temporal mixing layer) independent of y, and that the momentum exchanges are nonlocal in the y-direction; the momentum transfer velocity v' may assume any values between the extrema in these flows, hence the spatially averaged effective velocity $\widetilde{v}$.

The presence of a confining wall in $y = 0$ where $U_i = 0$ is constrained owing to impermeability and the adherence principle makes that particular configuration—the boundary layer—different from opened flows.

The friction at the wall is a sink of stress, transported along y through the boundary layer where it is absorbed to produce a force in the x-direction. The stress at the wall is $\tau/\rho = \nu\partial_y u\big|_{y=0}$ and Euler equations, with this source term, now write $u\partial_x u + v\partial_y u = \partial_y(-\overline{u'v'} + \tau/\rho)$. Since the boundary layer thickness (in y) is much smaller than the dimension (in x) over which u varies appreciably, we have $\partial_y(-\overline{u'v'} + \tau/\rho) \approx 0$ with τ approximately constant along y so that [8]

$$\frac{\tau}{\rho} = \overline{u'v'} = u_\star^2, \tag{65}$$

which defines the friction velocity $u_\star$.

In a boundary layer, the exchanges along y are necessarily more *local* than in opened flows. A velocity fluctuation $v'(y,t)$ with initial value, say, $v'(y_0,0) = y_0/t_0$ evolves, in a frame moving along x at velocity $u(y)$, according to $\partial_t v' = -\partial_y v'^2$, or $v'(y,t) = y/(2t + t_0)$. This fluctuation relaxes in a time t_0 given by its initial position, and intensity. The mean free path distance of a motion starting in y is therefore given by y itself. This constitutes the originality of that flow, an artifact of the presence of an impermeable wall. Unlike in opened flows, the mean free path is not an intrinsic feature of turbulence (see Section 6), but is geometrically constrained. The intuition of [4], who imagined the concept of turbulent viscosity in analogy with the kinetic theory of gases with a mean free path $\ell \sim y$, was utmost correct. That the motion at distance y from the wall can only involve y-sized objects has led to the concept of hierarchical, space-filling "attached eddies" [48,49]. Since the wall is *de facto* a rigid condition constraining the amplitude of the motion, we have exactly

$$\ell = y. \tag{66}$$

We may now adapt our closure $\overline{u'v'} = -\widetilde{v}\ell\partial_y u$ to a situation where the exchanges are bounded by a ℓ-distant wall, in a frame moving at $u(y)$ so that

$$\begin{aligned}\widetilde{v} &= \frac{1}{2\ell}\int_{-\ell}^{\ell}\big(u(y+\xi)-u(y)\big)\,\mathrm{d}\xi\\ &= \frac{1}{2\ell}\int_{-\ell}^{\ell}\left(u(y)+\xi\partial_y u+\frac{1}{2}\xi^2\partial_y^2 u+\text{h.o.t.}-u(y)\right)\mathrm{d}\xi\\ &= \frac{1}{6}\ell^2\partial_y^2 u+\text{h.o.t.},\end{aligned} \tag{67}$$

with the next non-zero term being of order $\ell^4\partial_y^4 u$. The velocity $\widetilde{v}$ is the average velocity difference about $u(y)$ in a neighborhood of width 2ℓ (an interpretation of the Laplacian first given by [50], see also [51,52]). At this stage (67) holds whatever ℓ may be, but now making use of (65) & (66), we have

$$\begin{aligned} u_\star^2 &= -\frac{1}{6}y^3\partial_y u\,\partial_y^2 u\\ &= -\frac{1}{12}y^3\partial_y(\partial_y u)^2,\\ \text{integrating into }\ \partial_y u &= \frac{u_\star}{\kappa y},\quad \text{with }\ \kappa^{-1}=6^{1/2},\\ \text{giving finally }\ \frac{u}{u_\star} &= \kappa^{-1}\ln y+\mathrm{C^{st}}.\end{aligned} \tag{68}$$

The velocity profile is a "Log at first sight" (see [53,54] and references therein) since the h.o.t. in the expansion of (67) obviously make the profile deviate slightly from a pure logarithm.

For a given $u_\star$, itself fixed by the stress at the wall τ where a no-slip boundary condition applies, viscous stresses are dominant within the "viscous sublayer" of thickness $\sim \nu/u_\star$. Denoting $y^+ = u_\star y/\nu$, a commonly admitted empirical form of the "Log-law of the wall" [2,10] is

$$\frac{u}{u_\star} = \kappa^{-1}\ln y^+ + 5.2 \tag{69}$$

with $\kappa = 0.41$ the so-called Kármán constant, which receives here a straightforward interpretation, namely $\kappa = 6^{-1/2} \approx 0.41$. That interpretation and meaning are different from the usually admitted ones: in [4] and [7] visions, who also both use a closure $u_\star^2 = (\ell\partial_y u)^2$ different from ours in (68), κ is an unknown pre-factor linking the mean free path ℓ and y, i.e. $\ell = \kappa y$. We have argued in (66) why this pre-factor should be equal to unity, and shown in (67) that κ reflects instead the proportionality between the Laplacian of the velocity and the transfer velocity $\widetilde{v}$. These two interpretations, beyond the fact that they rely on different microscopic momentum balances, are very different. In one case κ is an adjustable parameter while in the other it stems out of an exact calculation, admittedly contingent to our conjecture stating that $\ell = y$ in wall bounded flows; a proportionality coefficient different from unity would obviously alter the value of κ.

One may along the same lines compute the velocity profile across a turbulent channel flow of total width $2R$. The profile $u(y)$ is now symmetrical with respect to the channel centerline in $y = R$ if $y = 0$ is positioned at one of the confining walls, and we thus have $\partial_y u\big|_{y=R} = 0$. Denoting $\xi = y/R$, the velocity profile $u(\xi) = u/u_\star$ must solve according to (68)

$$\partial_\xi(\partial_\xi u)^2 = -\frac{12}{\xi^3},\quad \text{with }\ \partial_\xi u\big|_{\xi=1} = 0, \tag{70}$$

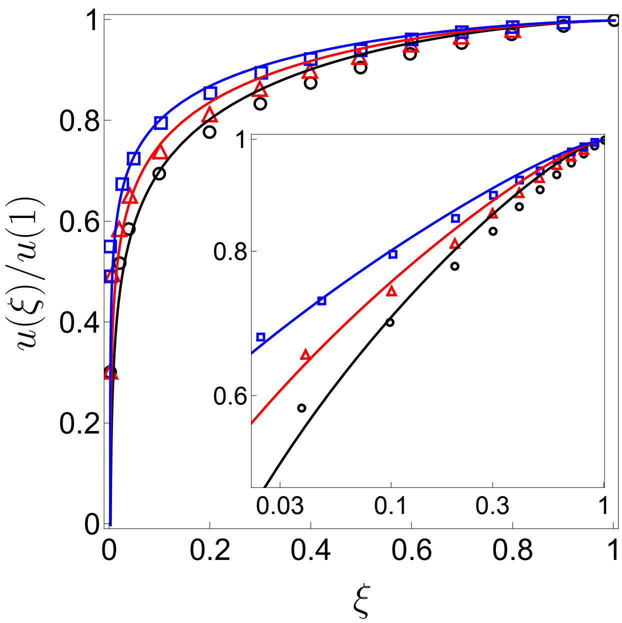

Figure 4. The radial velocity profiles in a turbulent pipe flow with mean axial velocity $\overline{u}$ and diameter $d = 2R$ measured by Nikuradse (reported in [8, p. 598]), for different Reynolds numbers $\mathscr{R} = \overline{u}d/\nu$. The profiles $u(\xi)$ are normalized by their maximal value in $\xi = y/R = 1$. The pipe Reynolds number is connected to $\mathscr{R}_\star$ in (71) by $\mathscr{R}_\star \approx (\mathscr{R}/14)^{7/8}$ [8]. Black circles: $\mathscr{R} = 4\times 10^3$. Red triangles: $\mathscr{R} = 2\times 10^4$. Blue squares: $\mathscr{R} = 1\times 10^6$. The lines are predictions from (71) & (72). Inset: near wall close-up in log-log units.

giving

$$u(\xi) = u(1) + \kappa^{-1}\left(\sqrt{1-\xi^2} - \operatorname{arctanh}\sqrt{1-\xi^2}\right). \tag{71}$$

The maximal velocity $u(1)$ at the centerline is set by the friction at the wall. Since $y^+ = \xi\mathscr{R}_\star$ with $\mathscr{R}_\star = u_\star R/\nu$ we have $u(\mathscr{R}_\star^{-1}) = 5.2$ owing to (69) evaluated in $y^+ = 1$, and therefore

$$\begin{aligned} u(1) &= 5.2 - \kappa^{-1}\left(\sqrt{1-\mathscr{R}_\star^{-2}} - \operatorname{arctanh}\sqrt{1-\mathscr{R}_\star^{-2}}\right) \\ &\approx 4.44 + \kappa^{-1}\ln\mathscr{R}_\star, \end{aligned} \tag{72}$$

which, together with (71), solves for the velocity profile in a channel flow.

Figure 4 shows a comparison between the above prediction with the experimental profiles recorded by Nikuradse in a circular *pipe* at different Reynolds numbers. Although, because of the geometry difference, a quantitative mismatch can be expected, the agreement is more than qualitative, especially close to the pipe wall. The rescaled profiles tend towards a plug flow at large Reynolds number, a trend well captured by (71) & (72).

8. Conclusion

We have introduced a new form of the turbulent Reynolds stress arguing that, because of incompressibility, a closure must be nonlocal, meaning that instead of implying derivatives of the mean velocity field, it should on the contrary be a function of an integral of that field. The spatial integration range depends on the flow configuration, and we have shown how the velocity profiles in jets, shear layers and wakes are solved in a unified manner.

The solutions all involve Gaussian (jets, wakes), or error functions (mixing layers), a signature of the fundamental diffusive nature of momentum dispersion in these flows, consistent with the early random walk vision of [4]. Based on the celebrated closure in (4) which is different from ours, none of Prandtl's solutions, all listed in [8], coincide with the ones derived here. We have also provided a mechanistic construction for Prandtl's momentum carrying "liquid balls" and their until then conjectured mean free path, which appears to be a coarsening scale having, in the velocity field, the same status as for random mixtures in the scalar field [47]. The mean free path is a fraction $(10\pi)^{-1} \approx 0.031$ of the integral scale in opened shear flows.

In boundary layers, the integration range is rigidly bounded by the presence of the wall, imposing the value of the mean free path. A prediction of our closure is the value of the von Kármán constant ("…the only constant included in the theory of turbulence" dixit [7]), found to be equal to $\kappa \approx 6^{-1/2} \approx 0.41$.

Finally, if the present discussion has concentrated on the nature of momentum dispersion in turbulent flows to compute velocity profiles, an adaptation of [25] remark used in Section 2 allows to discuss the concentration profiles of scalars transported by these flows. This is the subject of a separate work which explores the closely related problem of the turbulent dispersion of passive scalars, and the status of the turbulent Schmidt number [6].

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Declaration of interests

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