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Research article

Evaluation of the incremental ERR in interface cracks with frictional contact and its application in the coupled criterion of finite fracture mechanics

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This article is dedicated to the memory of Dr. Dominique Leguillon

Abstract. A crack located in a straight and perfectly bonded interface between dissimilar isotropic linear elastic materials with a frictional contact zone adjacent to the crack tip is considered under plane strain conditions. Assuming the Coulomb friction law, the crack-tip stress singularity in such a crack is weaker than the classical square-root singularity. The main difficulty in predicting propagation of such an interface crack is that the Energy Release Rate (ERR) is zero, which is a direct consequence of this weak stress singularity at the crack tip. Therefore, the Griffith fracture criterion, which assumes infinitesimal crack advances, cannot be applied in this case. To overcome this problem a new approach to predict the propagation of an interface crack with a frictional contact zone at the crack tip, based on the Coupled stress and energy Criterion (CC) of Finite Fracture Mechanics (FFM), is proposed and analyzed. In contrast to previous approaches, the critical finite crack advance Δa_{c} is determined by the CC as a structural parameter given by the overall problem configuration. Two methods for calculating the incremental ERR $G_{\mathrm{II}}(\Delta a)$ are considered which differ in the treatment of the frictional energy dissipated along the crack advance Δa. Closed-form expressions for $G_{\mathrm{II}}(\Delta a)$ are derived for sufficiently large interface cracks when the most singular term of the asymptotic expansion of the elastic solution at the crack tip is dominant along the path of crack advance Δa before the crack propagation occurs. In this case, closed-form expressions for the critical crack advance Δa_{c} and the critical stress intensity factor K_{IIc} are derived.

Keywords. Interface crack, Comninou contact model, Coulomb friction law, weak singularity, energy release rate, Irwin integral, VCCT.

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1. Introduction

Interface cracks are present in many types of heterogeneous materials and structural elements, such as composites, multilayers, and polycrystals (e.g., metals, ceramics, rocks, ice, and solar cells), at different scales, from nanoscale to macroscale. Interface cracks in laminates and multilayers are commonly referred to as delaminations, and in polycrystals as intergranular fractures. As follows from the seminal contributions to the theory of interface cracks by Comninou and co-workers [1–4], there is often a relevant contact zone adjacent to an interface crack tip. Such configurations are usually analyzed by means of the so-called contact model of interface cracks, contrary to the widely used open model of interface cracks introduced by Williams [5], and further developed in [6–11] among others. It should be noted that the crack tip solution of the open model has a very peculiar oscillatory character associated with the complex singularity exponent λ for non-zero values of the Dundurs bimaterial parameter $\beta \neq 0$, i.e. for dissimilar materials, with an infinite number of traction oscillations and an infinite number of overlapping zones very near the crack tip. For a comprehensive review of both the contact and open models of interface cracks, see [12–16]. Recall that in all these works a perfectly bonded interface is considered, i.e., the traction equilibrium and displacement compatibility are fulfilled along the undamaged interface part. The same hypothesis is considered in the present work.

In many engineering problems the contact zone adjacent to the interface crack tip is negligibly small, so the Small-Scale Contact (SSC) assumption proposed by Rice [9] is adequate, allowing to apply the open model of interface cracks, as comprehensively studied by Hutchinson and Suo [10] for isotropic bimaterials and Banks-Sills [17] for anisotropic bimaterials. However, there are also many practical applications where the size of the near-tip contact zone is relevant in comparison with some characteristic length of the problem, e.g., the interface crack size or the adjacent lamina thickness, and the SSC hypothesis is not valid, so the Comninou contact model of interface cracks must be applied.

In the frictionless case, see Leblond [18] for a review of this problem, the size of the contact zone adjacent to the crack tip can be estimated quite accurately as shown in [12,19,20].

In the friction case, frictional sliding takes place in the near-tip contact zone, which leads to a specific crack tip solution studied for the first time by Comninou [2] for isotropic bimaterials, considering the Coulomb law of friction. See [4,16,21–36], for subsequent studies on various aspects of this problem. A remarkable feature of this interface crack tip solution with frictional sliding at the crack tip is that, for dissimilar materials with a nonzero value of the Dundurs parameter $\beta \neq 0$, the stress singularity is weaker than that appearing in a classical crack in a homogeneous material, as shown by Comninou [2] for a stationary crack under monotonic loading, and in the general case of an interface crack propagating quasi-statically by Audoly [27]. See also [26,35,36], for an analysis of interface cracks in anisotropic bimaterials. This is consistent with the physical intuition that friction can only make the stress state at the crack tip less severe, i.e. less singular. Thus, the singularity exponent λ for an interface crack between dissimilar materials with a sliding frictional contact zone at the crack tip is greater than 0.5, $0.5 < \lambda < 1$, whereas $\lambda = 0.5$ for a crack in a homogeneous material or an interface crack between similar materials with $\beta = 0$. Following [37] we will refer to this kind of stress singularity as weak singularity.

A fundamental consequence of this weak stress singularity in frictional interface cracks is that the Energy Release Rate (ERR) vanishes for such cracks. Considering the two basic options for computing ERR, either the Irwin [38] crack closure integral to compute the incremental ERR $G(\Delta a)$ due to a small crack advance $\Delta a > 0$, or the Rice[39] J-Integral along a small circular path of radius $\delta > 0$ enclosing the crack tip, both lead to essentially the same conclusion, namely, the vanishing incremental ERR [24,25]

$$\lim_{\Delta a \to 0} G(\Delta a) = 0, \tag{1}$$

and the vanishing J-integral, which is path-dependent due to the frictional dissipation in the contact zone [21,22,27–29],

$$\lim_{\delta \to 0} J(\delta) = 0. \tag{2}$$

In addition, Leblond and Frelat [30] showed that the Stress Intensity Factors (SIFs) tend to zero for vanishing extension of a kink crack from such a frictional interface crack.

These observations raise the fundamental question of how to predict the propagation of such cracks and indeed how such interface cracks can propagate, see [21,22,24,25,28,30]. In the past, several proposals have been introduced to address this issue by establishing a suitable fracture criterion. Deng [22], in view of the vanishing J-integral with decreasing radius of the integration path $\delta \to 0$, suggested that the SIF K_{II} should be used instead as a better measure of the fracture driving force.

Sun and Qian [24] and Qian and Sun [25] considered the released energy due to a fixed finite crack extension Δa in their crack closure integral. A similar assumption of a fixed crack extension was made by Graciani et al. [33], although a different approach was used for the crack closure integral computation. In fact, these works [24,25,33] are related to the so-called Theory of Critical Distances (TCD) [40], which could also be applied to assess the initiation of propagation of such a frictional interface crack, although to the best of the authors' knowledge, no work has been reported in the literature attempting this. Audoly [28] proposed to use the $J(\delta)$-integral for a usually very small Barenblatt's [41] length δ_{c}. However, the meaning of using $J(\delta_{\mathrm{c}})$ is not clear as it represents the dissipation due to the frictional sliding behind the crack tip along the a priori existing crack faces (extrinsic toughness) and does not include the energy available for breaking interface bonds in front of the crack tip (intrinsic toughness).

Another widely developed and thoroughly studied way to overcome this difficulty is to consider an imperfect interface with a cohesive constitutive law [42–49], assuming a gradual increase of the friction effect with the gradual increase of the interface damage. See [50], for a review of coupling Cohesive Zone Models (CZM) and frictional contact.

Focusing hereinafter on perfectly bonded interfaces, the key observation regarding the J-integral is that, considering an infinitesimal growth of a frictional interface crack between dissimilar materials ($\beta \neq 0$), the energy flowing into the crack tip region circumvented by the integration path is completely consumed by the frictional dissipation along the parts of crack faces inside this crack tip region. Thus, such J-integral is path independent assuming fixed endpoints of the integration path, cf. [31,51]. In view of the above, we can conclude that an infinitesimal growth of such interface crack with frictional sliding contact zone adjacent to the crack tip, and thus with the vanishing ERR, $G = 0$, is not possible.

As mentioned above, by relaxing the hypothesis of classical fracture mechanics of an infinitesimal crack growth by allowing finite crack advances Δa, and calculating the so-called incremental ERR $G(\Delta a)$, makes it possible to avoid the above difficulty associated with the null ERR resulting from an infinitesimal advance of such frictional interface crack in (1). Therefore, the aim of the present article is to introduce some new ideas by further developing the original proposal of a finite crack extension Δa by Sun and Qian [24], see also Graciani et al. [33]. The main concept is based on:

(1) Considering a finite crack advance, instead of the infinitesimal crack advance assumed in the classical Linear Elastic Fracture Mechanics (LEFM), as Hashin [52] proposed in the framework of Finite Fracture Mechanics (FFM), see also [24,33].

(2) The Coupled Criterion (CC), coupling the classical stress criterion (in terms of the interface shear strength τ_c) and the incremental energy criteria (in terms of the fracture energy in Mode II G_{IIc}) in the framework of FFM, proposed by Leguillon [53] and later by Cornetti et al. [54], see Weissgraeber et al. [55] and Doitrand et al. [56], for a review of CC-FFM. Thus, only finite crack extensions with sufficiently high stresses acting along the path of this extension are considered when looking for the minimum load fulfilling both stress and incremental energy criteria. Thus, the length of such crack extension becomes a structural parameter, a function of the whole problem configuration, instead of being fixed as considered in previous works [24,33].

Noteworthy, the present work differs from the first application of CC-FFM to the propagation of interface cracks, in mixed mode and with a non-zero Dundurs parameter $\beta \neq 0$, developed by Mantič [57], because in that work the open model of interface cracks was considered, with an oscillating singularity associated with a complex singularity exponent λ.

In Section 2, the most singular term in the asymptotic expansion of the crack tip solution is analyzed. Several approaches to compute the incremental ERR are presented and compared in Section 3. Section 4 discusses the application of the fracture criterion considering a finite crack extension and different methods for the incremental ERR calculation. The Coupled Criterion (CC) is used in Section 5 to predict the propagation of a frictional interface crack. Finally, some concluding remarks are made in Section 6.

2. Asymptotic solution in the vicinity of an interfacial crack tip with friction

Consider two isotropic and linearly elastic adherents perfectly bonded along a straight interface, except for a debonded region where an interface crack of length a is located. Assuming the plane-strain hypotheses, a 2D model can be used to study such configuration. For simplicity, unit thickness is assumed. Focusing on the right crack tip with a frictional sliding contact zone on the left and a perfectly bonded interface on the right, the Cartesian and polar coordinates systems, (x, y) and (r, θ), centered at this crack tip denoted as O, will be used as shown in Figure 1.

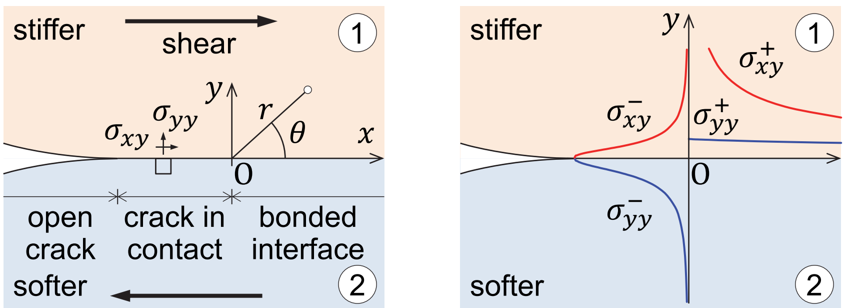

Figure 1. Near-tip elastic solution in the vicinity of a frictional interfacial crack tip.

Considering the polar coordinate system, Comninou [2] obtained the following asymptotic elastic solution in plane strain, for the interface tractions and the relative tangential displacement, in the vicinity of the frictional interfacial crack tip, for $r \to 0$,

$$\sigma_{xy}(r,0) = K_{\mathrm{II}}(2\pi r)^{(\lambda-1)}, \tag{3}$$

$$\sigma_{xy}(r,\pm\pi) = K_{\mathrm{II}}\cos[(1-\lambda)\pi](2\pi r)^{(\lambda-1)}, \tag{4}$$

$$\sigma_{yy}(r,\pm\pi) = -K_{\mathrm{II}}\beta\sin[(1-\lambda)\pi](2\pi r)^{(\lambda-1)}, \tag{5}$$

$$\Delta u_x(r) = K_{\mathrm{II}}\frac{\sin[(1-\lambda)\pi]}{\widehat{E}\lambda(2\pi)^{1-\lambda}}r^{\lambda} = u_x(r,\pi) - u_x(r,-\pi), \tag{6}$$

where K_{II} is the SIF. The real singularity exponent λ can be obtained from the interfacial friction coefficient f and the Dundurs bimaterial parameter β by

$$\lambda = 1 - \frac{1}{\pi}\,\mathrm{arc}\cot(f\beta). \tag{7}$$

The Dundurs bimaterial parameter β is defined as

$$\beta = \frac{\mu_1(\kappa_2-1) - \mu_2(\kappa_1-1)}{\mu_1(\kappa_2+1) + \mu_2(\kappa_1+1)}, \tag{8}$$

where μ_1 and μ_2 are the shear moduli of the materials, and the Kolosov constants κ_1 and κ_2 are defined by the Poisson ratios ν_1 and ν_2 as $\kappa_m = 3 - 4\nu_m$, with $m = 1,2$. Recall that $\beta = 0$ for identical or similar materials, and $\beta \neq 0$ for dissimilar materials.

The bimaterial stiffness parameter $\widehat{E}$ in (6) is defined as

$$\frac{1}{\widehat{E}} = \frac{1-\beta^2}{2}\left[\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right], \tag{9}$$

where E_1 and E_2 are the Young moduli of the materials.

As shown in [2,26,27,30] there is only one allowable direction of sliding near the crack tip given by the condition $f\beta > 0$. Then, the expression (7) leads to $0.5 < \lambda < 1$, which means that stress singularity at the crack tip is weak, i.e. weaker than in a classical crack in a homogeneous material, cf. [2,37]. Some of the consequences of this weak stress singularity were discussed in the introduction.

For the sake of simplicity and without loss of generality, it will be assumed that $\beta > 0$, which can be interpreted as the subscript 1 denoting the stiffer material and the subscript 2 denoting the softer material. Then, assuming this material definition, the friction coefficient should be positive $f > 0$. Notice that in this case $K_{\mathrm{II}} > 0$.

3. Incremental ERR in an interfacial crack with friction

In this section, the so-called incremental ERR associated to a finite crack extension Δa along the interface of two materials will be derived. The aim is to apply it in the prediction of such crack propagation by the CC-FFM. Let us consider Problems A and B, depicted in Figure 2, respectively, corresponding to an interface crack of length a, Problem A, and the same interface crack after a relatively small crack extension of length Δa has taken place, Problem B. For simplicity, a straight crack under plane strain conditions is considered as in the previous section.

As shown in Figure 2, a local Cartesian reference system located at the crack tip is employed in each problem, with the x coordinate oriented in the direction of crack propagation, to define the components of the stress tensor and the relative displacement, while a polar reference system, also located at the crack tip, will be employed to define the point locations. For the sake of simplicity, the following notation will be used to denote the near-tip stresses and relative displacements, with $i, j = x, y$ and $0 < \rho < \Delta a$,

$$\sigma_{ij}^{A+} = \sigma_{ij}^{A}(\rho, 0), \quad \sigma_{ij}^{A-} = \sigma_{ij}^{A}(\Delta a - \rho, \pi), \tag{10}$$

$$\sigma_{ij}^{B+} = \sigma_{ij}^{B}(\rho, 0), \quad \sigma_{ij}^{B-} = \sigma_{ij}^{B}(\Delta a - \rho, \pi), \tag{11}$$

$$\Delta u_i^{B} = u_i^{B}(\Delta a - \rho, \pi) - u_i^{B}(\Delta a - \rho, -\pi). \tag{12}$$

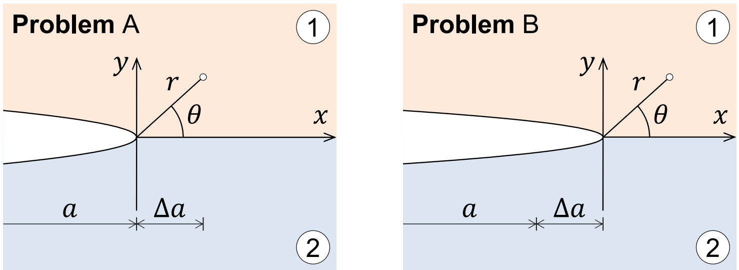

Figure 2. Interfacial crack and reference systems. Problem A before crack extension. Problem B after crack extension.

3.1. *Incremental ERR by Graciani et al. [33]*

By generalizing the Irwin crack closure integral procedure [38] to interface cracks with friction, cf. [51], the balance of mechanical energy leads to the following expression giving the energy available for crack propagation can be evaluated as

$$\widetilde{G}_{\mathrm{II}}(\Delta a)\Delta a = U^{A} - U^{B} + \Delta W_{e+f}^{A\to B}, \tag{13}$$

where $\widetilde{G}_{\mathrm{II}}(\Delta a)$ is the incremental ERR in Mode II and U^A and U^B are the elastic strain energies per unit thickness, respectively, stored in the system in Problems A and B, and $\Delta W_{e+f}^{A\to B}$ is the work per unit thickness carried out by the external forces, including the frictional forces along the (existing) crack faces, during crack propagation.

In this section $\widetilde{G}_{\mathrm{II}}(\Delta a)$ will be evaluated from the near-tip elastic solution, for an interface crack with frictional sliding contact zone, depicted in Figure 3, using a virtual crack propagation technique in presence of frictional sliding contact and a generalization of the Irwin crack closure integral [38], cf. [33]. Notice that the crack faces are in frictional contact during the virtual extension of the crack, thus only the tangential components of the virtual stresses contribute to the variation of the strain energy in the process. Therefore, only the shear component of the stresses and the relative sliding between crack faces are depicted in Figure 3. However, for the sake of clarity, the Mode II propagation has been represented in Mode I fashion.

Starting from the situation described in Problem A depicted in Figure 3, in a first stage, the interface ahead of the crack tip is virtually clamped along a certain length Δa.

Subsequently, in the second stage, the interface is externally cut, thus the interfacial stresses along the virtually clamped length denoted as σ_{ij}^{A+} are transformed into external stresses applied by the virtual grips. No energy variation is observed at these two stages since no change in the displacements takes place.

Finally, in the third stage, the virtual grips are released in a way that the applied stresses σ_{ij}^{A+} are linearly transformed into σ_{ij}^{B-}, i.e., into the near-tip contact stresses corresponding to Problem B. Given that the final situation is identical to Problem B, the relative displacements at the virtual

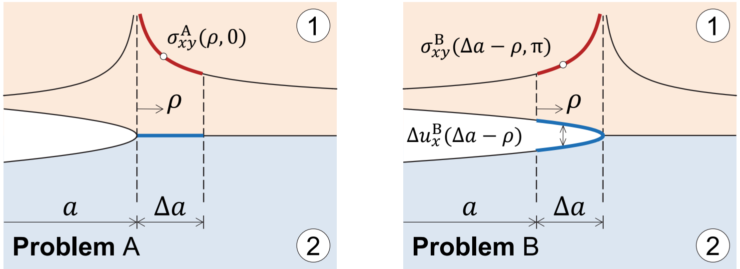

Figure 3. Local elastic solution along the near-tip interface region. Problem A before crack extension. Problem B after crack extension.

grips are given by Δu_i^{B}. Note that the receding nature of this frictional contact problem has been tacitly considered in the previous analysis. It is well known that in general in a receding contact problem its solution is a homogeneous function of degree 1 of a positive scalar factor governing some proportional boundary conditions, i.e., the problem solution behaves linearly with respect to such scalar factor.

The procedure described above permits evaluating the elastic strain energy stored in Problem B as:

$$U^{\mathrm{B}} = U^{\mathrm{A}} + \Delta W_{\mathrm{e+f}}^{\mathrm{A}\to\mathrm{B}} + \Delta W_{\mathrm{v}}^{\mathrm{A}\to\mathrm{B}}, \tag{14}$$

where $\Delta W_{\mathrm{v}}^{\mathrm{A}\to\mathrm{B}}$ is the (negative) work carried out by the virtual stresses during the virtual crack extension.

Introducing (14) into (13) gives the following expression for $\widetilde{G}_{\mathrm{II}}(\Delta a)$

$$\widetilde{G}_{\mathrm{II}}(\Delta a) = -\frac{1}{\Delta a}\Delta W_{\mathrm{v}}^{\mathrm{A}\to\mathrm{B}}. \tag{15}$$

The work per unit area $\mathscr{W}_{\mathrm{v}}^{\mathrm{A}\to\mathrm{B}}$ and the released energy per unit area $\widetilde{\mathscr{G}}_{\mathrm{II}}(\Delta a)$ at each interface point within the crack extension Δa are defined as

$$\Delta W_{\mathrm{v}}^{\mathrm{A}\to\mathrm{B}} = \int_0^{\Delta a} \mathscr{W}_{\mathrm{v}}^{\mathrm{A}\to\mathrm{B}}\,\mathrm{d}\rho, \tag{16}$$

$$\widetilde{G}_{\mathrm{II}}(\Delta a) = \frac{1}{\Delta a}\int_0^{\Delta a} \widetilde{\mathscr{G}}_{\mathrm{II}}(\Delta a)\,\mathrm{d}\rho. \tag{17}$$

In view of Figure 4, $\widetilde{\mathscr{G}}_{\mathrm{II}}(\Delta a)$ and $\mathscr{W}_{\mathrm{v}}^{\mathrm{A}\to\mathrm{B}}$ can be written as

$$\widetilde{\mathscr{G}}_{\mathrm{II}}(\Delta a) = -\mathscr{W}_{\mathrm{v}}^{\mathrm{A}\to\mathrm{B}} = \frac{1}{2}\left[\sigma_{xy}^{\mathrm{A}+} + \sigma_{xy}^{\mathrm{B}-}\right]\Delta u_x^{\mathrm{B}}, \tag{18}$$

and, therefore, the incremental ERR is given by

$$\widetilde{G}_{\mathrm{II}}(\Delta a) = -\frac{1}{\Delta a}\Delta W_{\mathrm{v}}^{\mathrm{A}\to\mathrm{B}} = \frac{1}{2\Delta a}\int_0^{\Delta a}\left[\sigma_{xy}^{\mathrm{A}+} + \sigma_{xy}^{\mathrm{B}-}\right]\Delta u_x^{\mathrm{B}}\,\mathrm{d}\rho. \tag{19}$$

If the crack extension Δa_0 is sufficiently small in comparison with the crack length, $\Delta a_0 \ll a$, the near tip solutions of Problem A and Problem B can be identified as

$$\sigma_{xy}^{+} \equiv \sigma_{xy}^{\mathrm{A}+}, \tag{20}$$

$$\sigma_{xy}^{-} \equiv \sigma_{xy}^{\mathrm{A}-} \cong \sigma_{xy}^{\mathrm{B}-}, \tag{21}$$

$$\Delta u_x \equiv u_x^{\mathrm{A}}(\Delta a_0 - \rho, \pi) - u_x^{\mathrm{A}}(\Delta a_0 - \rho, -\pi) \cong \Delta u_x^{\mathrm{B}}. \tag{22}$$

Consequently, $\widetilde{G}_{\mathrm{II}}(\Delta a_0)$ can be obtained from the solution of a single problem as

$$\widetilde{G}_{\mathrm{II}}(\Delta a_0) = \frac{1}{\Delta a_0}\int_0^{\Delta a_0}\widetilde{\mathscr{G}}_{\mathrm{II}}(\Delta a_0)\,\mathrm{d}\rho = \frac{1}{2\Delta a_0}\int_0^{\Delta a_0}\left[\sigma_{xy}^{+} + \sigma_{xy}^{-}\right]\Delta u_x\,\mathrm{d}\rho. \tag{23}$$

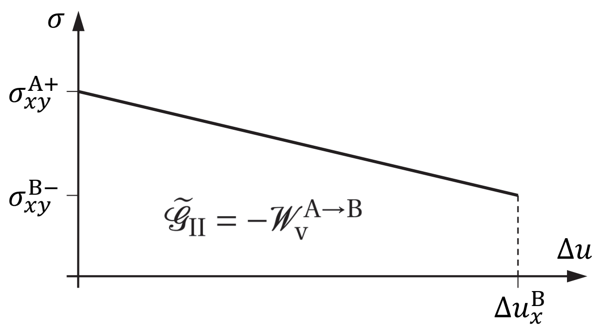

Figure 4. Work per unit area done by the virtual stresses at each interfacial point within the crack extension Δa.

Remark. In this section, a linear transition between the states A and B is considered, despite the generally non-linear character of a frictional contact problem. However, as will be demonstrated below, this linear transition can be considered as a valid approximation of the actual fracture process under the assumption that the interface crack is frictionally sliding in the same direction, as given by the Comninou contact model in the states A and B, in a neighbourhood of the interface crack tip, denoted as V, which is sufficiently large compared to the new crack segment Δa. In this situation, the frictional contact conditions along the interface contact zone in V become linear conditions. Consider a scalar parameter α changing from 0 to 1, $\alpha \in \langle 0, 1\rangle$, which can be used to define a convex combination of the displacement and stress solutions in these states as

$$u_i^\alpha = (1-\alpha)u_i^{\mathrm{A}} + \alpha u_i^{\mathrm{B}}, \tag{24}$$

$$\sigma_{ij}^\alpha = (1-\alpha)\sigma_{ij}^{\mathrm{A}} + \alpha\sigma_{ij}^{\mathrm{B}}. \tag{25}$$

A similar convex combination of the relative displacement along the new crack segment Δa can be defined as

$$\Delta u_i^\alpha = (1-\alpha)\Delta u_i^{\mathrm{A}} + \alpha\Delta u_i^{\mathrm{B}} = \alpha\Delta u_i^{\mathrm{B}}, \tag{26}$$

where in the last equation it has been considered that $\Delta u_i^{\mathrm{A}} = 0$ because of the perfect interface bonding along Δa in the state A.

Consider Δu_i^α as the imposed relative displacements along the new crack segment Δa, see Section 3.1 for details. Since the frictional sliding contact conditions are satisfied in V by both elastic solutions in the states A and B, these conditions are also satisfied by their convex combinations u_i^α and σ_{ij}^α due to the linear nature of these contact conditions in V. Applying the Saint-Venant principle to the self-balanced load changing with α along the relatively small segment Δa, we can assume that the stress solutions in the states A and B, as well as their convex combinations (σ_{ij}^{A}, σ_{ij}^{B} and σ_{ij}^α), are approximately the same along the outer boundary of V, which is sufficiently far from Δa. Therefore, these convex combinations are sufficiently accurate approximations of the exact solution of the full problem, for a given Δu_i^α, within V, especially near the crack tip. This seems to justify the linear transition assumed in this section. □

3.2. *Incremental ERR by Sun and Qian [24]*

Following the Irwin crack closure integral approach [38], Sun and Qian [24] proposed a slightly different approach to evaluate the incremental ERR associated to a small but finite characteristic interface crack extension with friction, denoted in the following as $\widehat{G}_{\rm II}(\Delta a)$, respecting the notation in [24]. The approach proposed in [24] is somewhat related to the analysis in [51] for frictional cracks in a homogeneous material. According to [24], see also [16], $\widehat{G}_{\rm II}(\Delta a_0)$ can be obtained as

$$\widehat{G}_{\rm II}(\Delta a) = \frac{1}{\Delta a}\int_0^{\Delta a} \widehat{\mathscr{G}}_{\rm II}(\Delta a)\,{\rm d}\rho = \frac{1}{2\Delta a}\int_0^{\Delta a} \left[\sigma_{xy}^{\rm A+} - \sigma_{xy}^{\rm B-}\right]\Delta u_x^{\rm B}\,{\rm d}\rho. \tag{27}$$

Sun and Qian [24] use an energy balance in which external forces and the frictional forces are considered separately

$$\widehat{G}_{\rm II}(\Delta a) = \frac{U^{\rm A} - U^{\rm B}}{\Delta a} + \frac{\Delta W_{\rm e}^{\rm A\to B}}{\Delta a} - G_{\rm d}(\Delta a), \tag{28}$$

where $U^{\rm A}$ and $U^{\rm B}$ are the elastic strain energies, respectively, stored in the system in Problems A and B, $\Delta W_{\rm e}^{\rm A\to B}$ is the work carried out by the external forces and $G_{\rm d}(\Delta a)$ is the dissipation energy rate, associated with the frictional sliding of the existing and newly created crack faces in contact, during crack propagation. The dissipation energy rate due to frictional sliding is given by

$$G_{\rm d}(\Delta a) = G_{\rm d}^{\rm N}(\Delta a) + G_{\rm d}^{\rm e}(\Delta a), \tag{29}$$

where

$$G_{\rm d}^{\rm N}(\Delta a) = \frac{1}{\Delta a}\int_0^{\Delta a} \mathscr{G}_{\rm d}^{\rm N}(\Delta a)\,{\rm d}\rho \tag{30}$$

is associated with the newly formed crack surface, and $G_{\rm d}^{\rm e}(\Delta a)$ is associated with the existing crack surfaces that are in contact. Therefore,

$$\Delta W_{\rm e+f}^{\rm A\to B} = \Delta W_{\rm e}^{\rm A\to B} - G_{\rm d}^{\rm e}(\Delta a)\,\Delta a. \tag{31}$$

and

$$\widetilde{G}_{\rm II}(\Delta a) = \widehat{G}_{\rm II}(\Delta a) + G_{\rm d}^{\rm N}(\Delta a). \tag{32}$$

Then, in each point of the newly formed crack surface it holds that

$$\mathscr{G}_{\rm d}^{\rm N}(\Delta a) = \sigma_{xy}^{\rm B-}\Delta u_x^{\rm B}, \tag{33}$$

as it is depicted in Figure 5, and

$$G_{\rm d}^{\rm N}(\Delta a) = \frac{1}{\Delta a}\int_0^{\Delta a} \sigma_{xy}^{\rm B-}\Delta u_x^{\rm B}\,{\rm d}\rho. \tag{34}$$

If the crack extension is sufficiently small in comparison with the crack length, $\Delta a_0 \ll a$, then in view of (20) and (21)

$$\widehat{G}_{\rm II}(\Delta a_0) = \frac{1}{\Delta a_0}\int_0^{\Delta a_0} \widehat{\mathscr{G}}_{\rm II}(\Delta a_0)\,{\rm d}\rho = \frac{1}{2\Delta a_0}\int_0^{\Delta a_0} \left[\sigma_{xy}^{+} - \sigma_{xy}^{-}\right]\Delta u_x\,{\rm d}\rho, \tag{35}$$

where σ_{xy}^{+}, σ_{xy}^{-} and Δu_x are defined in (20)–(22).

Remark. When analyzing the Finite Element (FE) procedure proposed in [16,24] for energy calculation, the following expressions can be inferred for the incremental ERR, and the dissipated energy rate due to friction along the newly formed crack advance,

$$\widehat{G}_{\rm II}^{\rm FE}(\Delta a) = \frac{1}{2\Delta a}\int_0^{\Delta a} \left[\sigma_{xy}^{\rm A+} + f\sigma_{yy}^{\rm A+}\right]\Delta u_x^{\rm B}\,{\rm d}\rho, \tag{36}$$

and

$$G_{\rm d}^{\rm N,FE}(\Delta a) = \frac{1}{2\Delta a}\int_0^{\Delta a} \left[-f\sigma_{yy}^{\rm A+} - f\sigma_{yy}^{\rm B-}\right]\Delta u_x^{\rm B}\,{\rm d}\rho = \frac{1}{2\Delta a}\int_0^{\Delta a} \left[-f\sigma_{yy}^{\rm A+} + \sigma_{xy}^{\rm B-}\right]\Delta u_x^{\rm B}\,{\rm d}\rho, \tag{37}$$

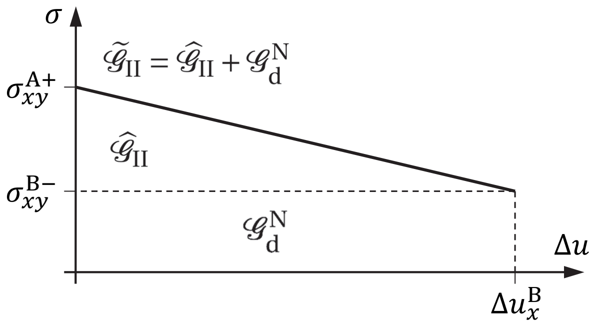

Figure 5. Energies released and dissipated at each interfacial point within the crack extension Δa.

where $f > 0$ is the friction coefficient, and compressive stresses $\sigma_{yy}^{\mathrm{A}+} < 0$ and $\sigma_{yy}^{\mathrm{B}-} < 0$ are assumed. Somewhat surprisingly, these expressions differ from those in (27) and (34), respectively. Nevertheless, it is easy to check that the relationship (32) remains valid, i.e., $\widetilde{G}_{\mathrm{II}}(\Delta a) = \widehat{G}_{\mathrm{II}}^{\mathrm{FE}}(\Delta a) + G_{\mathrm{d}}^{\mathrm{N,FE}}(\Delta a)$. Thus, the expressions (34) and (37) represent different estimations of the dissipation energy rate due to frictional sliding along the newly forming crack extension and, consequently, $\widehat{G}_{\mathrm{II}}(\Delta a)$ and $\widehat{G}_{\mathrm{II}}^{\mathrm{FE}}(\Delta a)$ represent different estimations of the incremental ERR.

The following interpretation to the expression of the incremental ERR in (36) can be given: $\sigma_{xy}^{\mathrm{A}+} + f\sigma_{yy}^{\mathrm{A}+}$ represents the elastic restoring shear traction before crack extension Δa, which decreases to zero during this crack extension. □

3.3. *Comparison of different approaches for large interface cracks and small crack advances*

For a sufficiently large crack in absence of friction, that is, when $f \to 0$ and $\Delta a_0 \ll a$, both $\widetilde{G}_{\mathrm{II}}(\Delta a_0)$ and $\widehat{G}_{\mathrm{II}}(\Delta a_0)$ tend to the Irwin classical expression of the incremental ERR [38]

$$G_{\mathrm{II}}(\Delta a_0) = \frac{1}{2\Delta a_0}\int_0^{\Delta a_0} \sigma_{xy}^{+}\Delta u_x \,\mathrm{d}\rho. \tag{38}$$

Notwithstanding, it must be noticed that the most significant effect of friction is not associated to the appearance of the frictional stresses σ_{xy}^{-} in (23) or (35), but to the fact that the asymptotic behavior of the crack tip solution changes, according to the expressions given in Section 2.

In the frictionless case the square root stress singularity with $\lambda = 0.5$ appears at the crack tip, and consequently $G_{\mathrm{II}}(\Delta a_0)$ tends to a constant positive value when crack extension vanishes (i.e., when $\Delta a_0 \to 0$). In the presence of friction, it holds that $0.5 < \lambda < 1$ and, consequently, both $\widetilde{G}_{\mathrm{II}}(\Delta a_0)$ and $\widehat{G}_{\mathrm{II}}(\Delta a_0)$ vanish when the crack extension vanishes.

Introducing the first term of Comninou's asymptotic solution (3)–(6) into the definitions of $\widetilde{G}_{\mathrm{II}}(\Delta a_0)$, $\widehat{G}_{\mathrm{II}}(\Delta a_0)$ and $G_{\mathrm{II}}(\Delta a_0)$ leads to the following asymptotic power-law expressions

$$\widetilde{G}_{\mathrm{II}}(\Delta a_0) = \tilde{c}(\lambda)\frac{K_{\mathrm{II}}^2}{\widehat{E}}(\Delta a_0)^{(2\lambda-1)}. \tag{39}$$

$$\widehat{G}_{\mathrm{II}}(\Delta a_0) = \hat{c}(\lambda)\frac{K_{\mathrm{II}}^2}{\widehat{E}}(\Delta a_0)^{(2\lambda-1)}, \tag{40}$$

$$G_{\mathrm{II}}(\Delta a_0) = \bar{c}(\lambda)\frac{K_{\mathrm{II}}^2}{\widehat{E}}(\Delta a_0)^{(2\lambda-1)}, \tag{41}$$

where

$$\widetilde{c}(\lambda) = \bar{c}(\lambda) + c_{\mathrm{B}}(\lambda), \tag{42}$$

$$\widehat{c}(\lambda) = \bar{c}(\lambda) - c_{\mathrm{B}}(\lambda), \tag{43}$$

and

$$\bar{c}(\lambda) = \frac{\sin[(1-\lambda)\pi]}{2\lambda(2\pi)^{2(1-\lambda)}}\left[\frac{\Gamma(\lambda)\Gamma(1+\lambda)}{\Gamma(1+2\lambda)}\right], \tag{44}$$

$$c_{\mathrm{B}}(\lambda) = \frac{\sin[(1-\lambda)\pi]}{2\lambda(2\pi)^{2(1-\lambda)}}\left[\frac{\cos[(1-\lambda)\pi]}{2\lambda}\right], \tag{45}$$

where $\Gamma(\cdot)$ is the gamma function. Notice that $(\Delta a_0)^{(2\lambda-1)} \to 0$ when $\Delta a_0 \to 0$ and $0.5 < \lambda$. The dimensionless auxiliary functions $\bar{c}(\lambda)$, $\widetilde{c}(\lambda)$, $\widehat{c}(\lambda)$, and $c_{\mathrm{B}}(\lambda)$ are represented in Figure 6.

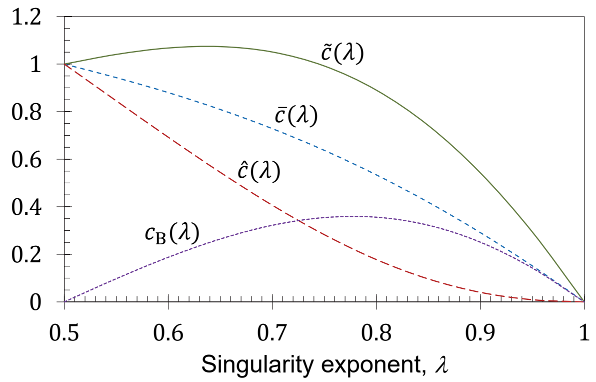

Figure 6. Coefficients $\bar{c}(\lambda)$, $\widehat{c}(\lambda)$, $\widetilde{c}(\lambda)$, and $c_{\mathrm{B}}(\lambda)$ used in the asymptotic expressions of $G_{\mathrm{II}}(\Delta a_0)$, $\widehat{G}_{\mathrm{II}}(\Delta a_0)$ and $\widetilde{G}_{\mathrm{II}}(\Delta a_0)$.

4. Evaluation of the incremental ERR for a finite crack extension

Due to the weak stress singularity at the frictional interface crack tip, when the crack extension vanishes, $\Delta a_0 \to 0$, the three estimations of the incremental ERR, $\widetilde{G}_{\mathrm{II}}(\Delta a_0)$, $\widehat{G}_{\mathrm{II}}(\Delta a_0)$ and $G_{\mathrm{II}}(\Delta a_0)$, presented in the previous sections vanish as well.

Since the incremental ERR is dependent on the crack extension, a characteristic finite crack extension Δa_{c} must be employed as discussed in Section 1. Depending on the approach employed for estimation of the incremental ERR, the following criteria may be used for assessing crack propagation

$$\widetilde{G}_{\mathrm{II}}(\Delta\widetilde{a}_{\mathrm{c}}) = \widetilde{G}_{\mathrm{c}}, \tag{46}$$

$$\widehat{G}_{\mathrm{II}}(\Delta\widehat{a}_{\mathrm{c}}) = \widehat{G}_{\mathrm{c}}, \tag{47}$$

$$G_{\mathrm{II}}\left(\Delta\bar{a}_{\mathrm{c}}\right) = \bar{G}_{\mathrm{c}}. \tag{48}$$

If Δa_{c} is sufficiently small in comparison with the crack length a, in view of the relations shown in (41)–(45), certain relations may be established between the different characteristic finite crack extensions $\Delta\widetilde{a}_{\mathrm{c}}$, $\Delta\widehat{a}_{\mathrm{c}}$ and $\Delta\bar{a}_{\mathrm{c}}$, if the critical ERRs (fracture energies) $\widetilde{G}_{\mathrm{c}}$, $\widehat{G}_{\mathrm{c}}$ and $\bar{G}_{\mathrm{c}}$, are considered equal to the critical ERR of the material G_{IIc}.

Conversely, if a characteristic finite crack extension is considered in the material and $\Delta\widetilde{a}_c$, $\Delta\widehat{a}_c$ and $\Delta\bar{a}_c$ are defined equal to this characteristic finite crack extension, the expressions in (39)–(45) allow relationships to be established between $\widetilde{G}_c$, $\widehat{G}_c$ and $\bar{G}_c$ which allow criteria defined in (46)–(48) to provide identical predictions for crack propagation.

5. Determination of the critical finite crack extension and the critical SIF using CC for large cracks

Although the use of a characteristic finite crack extension as a material parameter controlling crack propagation may have physical sense in some materials, cf. [40], the use of a critical interface strength τ_c and a critical Mode II strain energy release rate G_{IIc} are commonly employed in crack propagation criteria of closed cracks. This is the case, e.g., of CZMs [42–50].

The use of the Coupled stress and energy Criterion (CC) of the Finite Fracture Mechanics (FFM) proposed by Leguillon [53] and later by Cornetti et al. [54], allows establishing an unambiguous definition of the critical finite crack extension Δa_c as a structural parameter for a frictional interface crack.

If a pointwise stress criterion is used in the considered finite crack extension Δa_0, cf. [53], and the incremental ERR proposed in this paper, $\widetilde{G}_{II}(\Delta a_0)$, defined in (23), is used for the energy criterion, the CC establishes that crack propagation will take place if the following inequalities are simultaneously fulfilled

$$\sigma^A_{xy}(\rho,0) \geq \tau_c \quad \text{for} \quad 0<\rho<\Delta a_0 \tag{49}$$

and

$$\widetilde{G}_{II}(\Delta a_0) \geq G_{IIc}. \tag{50}$$

If the first singular term of the near-tip asymptotic solution is dominant along the crack extension, both criteria can be written in terms of K_{II}, yielding

$$\sigma^A_{xy}(\Delta a_0,0) = K_{II}(2\pi\Delta a_0)^{(\lambda-1)} \geq \tau_c \tag{51}$$

and

$$\widetilde{G}_{II}(\Delta a_0) = \widetilde{c}(\lambda)\frac{K_{II}^2}{\widehat{E}}(\Delta a_0)^{(2\lambda-1)} \geq G_{IIc}, \tag{52}$$

where it has been considered that $\sigma^A_{xy}(\rho,0)$ decreases when ρ increases.

The critical values of K_{II} and Δa_0, denoted as K_{IIc} and Δa_c, can be obtained from the solution of the system of equations

$$\sigma^A_{xy}(\Delta\widetilde{a}_c,0) = \tau_c \quad \text{and} \quad \widetilde{G}_{II}(\Delta\widetilde{a}_c) = G_{IIc}, \tag{53}$$

which yields

$$\Delta\widetilde{a}_c = \frac{(2\pi)^{2(\lambda-1)}}{\widetilde{c}(\lambda)}\frac{G_{IIc}\widehat{E}}{\tau_c^2} \tag{54}$$

and

$$\widetilde{K}_{IIc} = (2\pi\Delta\widetilde{a}_c)^{(1-\lambda)}\tau_c. \tag{55}$$

Thus, $\Delta\widetilde{a}_c$ is a multiple of the Irwin length for interface cracks in shear $G_{IIc}\widehat{E}/\tau_c^2$.

The following non-dimensional variables can be defined using these critical values $\widetilde{K}_{IIc}$ and $\Delta\widetilde{a}_c$

$$\widetilde{K}^n_{II} = \frac{K_{II}}{\widetilde{K}_{IIc}} \quad \text{and} \quad \Delta\widetilde{a}^n_0 = \frac{\Delta a_0}{\Delta\widetilde{a}_c}, \tag{56}$$

which allows rewriting the CC in the following non-dimensional form, cf. [57]:

$$\widetilde{K}^n_{II} \geq \left(\Delta\widetilde{a}^n_0\right)^{(1-\lambda)} \tag{57}$$

and

$$\widetilde{K}^n_{II} \geq \left(\Delta\widetilde{a}^n_0\right)^{\left(\frac{1}{2}-\lambda\right)}. \tag{58}$$

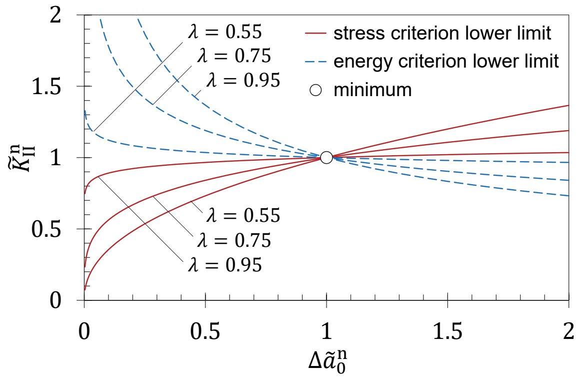

Figure 7. Representation of the non-dimensional form of the coupled stress and energy criterion (CC) for $0.5 < \lambda < 1$.

The non-dimensional form of the CC is depicted in Figure 7 for three different values of λ in the range $0.5 < \lambda < 1$. The lower limit curves of the stress criterion and the energy criterion represent the values of $\widetilde{K}_{\mathrm{II}}^{\mathrm{n}}$ and $\Delta\tilde{a}_0^{\mathrm{n}}$ that respectively fulfil the equalities in (57) and (58). Consequently, for each value of λ, crack propagation would take place for $\widetilde{K}_{\mathrm{II}}^{\mathrm{n}}$ and $\Delta\tilde{a}_0^{\mathrm{n}}$ pairs located above the corresponding curves of the stress and energy criteria.

Since, for all λ in the range $0.5 < \lambda < 1$, the stress criterion lower limit increases with $\Delta\tilde{a}_0^{\mathrm{n}}$ and the energy criterion lower limit decreases with $\Delta\tilde{a}_0^{\mathrm{n}}$, the minimum value of $\widetilde{K}_{\mathrm{II}}^{\mathrm{n}}$ for which crack propagation can take place is $\widetilde{K}_{\mathrm{II}}^{\mathrm{n}} = 1$ and the corresponding crack extension would be $\Delta\tilde{a}_0^{\mathrm{n}} = 1$.

In other words, the minimum value of $\widetilde{K}_{\mathrm{II}}$ for which crack propagation can take place is $\widetilde{K}_{\mathrm{IIc}}$ and the corresponding crack extension would be $\Delta\tilde{a}_{\mathrm{c}}$.

Notice that, in view of the evolution of the lower limit curves of the stress criterion and the energy criterion shown in Figure 7, for all λ in the range $0.5 < \lambda < 1$ the CC for a frictional crack can be written in the following pointwise stress criterion form

$$\sigma_{xy}^{\mathrm{A}}(\Delta\tilde{a}_{\mathrm{c}}, 0) \geq \tau_{\mathrm{c}} \quad \text{with} \quad \Delta\tilde{a}_{\mathrm{c}} = \frac{(2\pi)^{2(\lambda-1)}}{\tilde{c}(\lambda)} \frac{G_{\mathrm{IIc}}\widehat{E}}{\tau_{\mathrm{c}}^2}. \tag{59}$$

Notice that, when $\lambda \to 1$, corresponding to $f \to +\infty$, it holds that $\widetilde{G}_{\mathrm{II}}(\Delta a_0) \to 0$, for a fixed Δa_0, and $\Delta\tilde{a}_{\mathrm{c}} \to \infty$, and, consequently, crack propagation is not possible in that case.

For the opposite limit case, that is, for $\lambda = 0.5$, which corresponds either to a frictionless crack or to the case of similar materials with the Dundurs parameter $\beta = 0$, the lower limit curve of the energy criterion becomes a horizontal line, $\widetilde{K}_{\mathrm{II}}^{\mathrm{n}} = 1$, see Figure 8. Consequently, the minimum value of $\widetilde{K}_{\mathrm{II}}^{\mathrm{n}}$ for which crack propagation can take place is $\widetilde{K}_{\mathrm{II}}^{\mathrm{n}} = 1$ and the corresponding crack extension would be any $\Delta\tilde{a}_0^{\mathrm{n}}$ value in the range $0 \leq \Delta\tilde{a}_0^{\mathrm{n}} \leq 1$. In other words, if the energy criterion is fulfilled, infinitesimal or finite crack extensions can take place. The maximum extent of the crack extension is given by

$$\lambda = 0.5 \quad \Longrightarrow \quad \Delta\tilde{a}_{\mathrm{c}} = \Delta a_{\mathrm{ch}} = \frac{1}{2\pi} \frac{G_{\mathrm{IIc}}\widehat{E}}{\tau_{\mathrm{c}}^2}. \tag{60}$$

Although the incremental ERR proposed in this paper, $\widetilde{G}_{\mathrm{II}}(\Delta a_0)$, has been used for the derivation of the CC for a frictional interface crack, analogous formulations can be obtained evaluating the incremental ERR using either $G_{\mathrm{II}}(\Delta a_0)$ or $\widehat{G}_{\mathrm{II}}(\Delta a_0)$, just by replacing coefficient $\hat{c}(\lambda)$ by either $\bar{c}(\lambda)$ or $\tilde{c}(\lambda)$, respectively. Conclusions obtained using $\widetilde{G}_{\mathrm{II}}(\Delta a_0)$ hold for the other approaches but yielding different critical values of K_{II} and Δa_0, namely $\overline{K}_{\mathrm{IIc}}$ and $\Delta\bar{a}_{\mathrm{c}}$, or $\widehat{K}_{\mathrm{IIc}}$ and $\Delta\hat{a}_{\mathrm{c}}$, respectively.

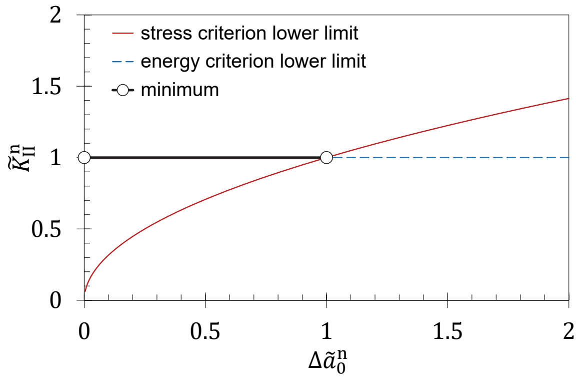

Figure 8. Representation of the non-dimensional form of the coupled stress and energy criterion (CC) for $\lambda = 0.5$.

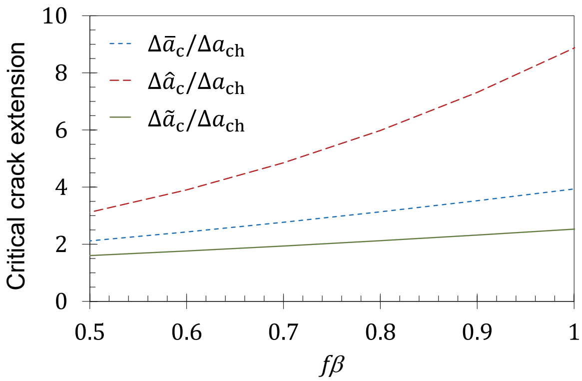

Figure 9. Critical crack extensions given by the different approaches for evaluating incremental ERR in the coupled stress and energy criterion (CC).

Critical crack extensions $\Delta\bar{a}_c$, $\Delta\hat{a}_c$ and $\Delta\tilde{a}_c$ are compared in Figure 9 assuming the same value of G_{IIc}, $\widehat{E}$ and τ_c for all three approaches. When $\lambda = 0.5$ ($f\beta = 0$) the three different critical crack extensions yield the same characteristic length Δa_{ch}.

Considering that the critical interface strength τ_c and the critical Mode II energy release rate G_{IIc} are material parameters, for sufficiently large cracks, criterion in (59) allows predicting crack growth, considering that $\widetilde{G}_{II}(\Delta\tilde{a}_c)$ is the incremental ERR associated with crack propagation. Analogous expressions can be easily obtained for predicting crack growth considering that either $G_{II}(\Delta\bar{a}_0)$ or $\widehat{G}_{II}(\Delta\hat{a}_c)$ are the incremental ERRs associated with crack propagation. A detailed comparison of the aforementioned approaches for determining the incremental ERR associated with a frictional interfacial crack propagation with experimental results or with numerical predictions obtained with different approaches lies beyond the scope of this work.

Nevertheless, Graciani et al. [33] used $\widetilde{G}_{II}(\Delta a_0)$, with a prescribed critical extension $\Delta a_0 = 1\,\mu\mathrm{m}$, to obtain the friction coefficient and the critical Mode II ERR of glass fibre/matrix interface by the best fit between numerical predictions with boundary elements and experimental results of the single fibre fragmentation test, yielding $G_{IIc} = 12.12\,\mathrm{J/m^2}$ and $f = 1$. Using these values, crack propagation was simulated using a FE model with bilinear cohesive elements (with

$\tau_c = 110\,\text{MPa}$) and frictional contact along the fibre/matrix interface, yielding a remarkably good agreement. Introducing these fibre/matrix interfacial properties (along with the elastic properties of the materials given in [33]) into (54) gives a value of $\Delta\tilde{a}_c = 1.16\,\mu\text{m}$. The good agreement between $\Delta\tilde{a}_c$ and the prescribed critical extension Δa_0 used in [33] supports the use of FFM with the CC to assess frictional interfacial crack growth, using $\widetilde{G}_{\text{II}}(\Delta\tilde{a}_c)$ to determine incremental ERR associated to crack propagation.

6. Concluding remarks

A few concluding remarks are given below to summarise and clarify the results presented and to offer new perspectives on the modelling of frictional interface crack propagation.

(1) A challenging problem of cracks with a relevant frictional contact zone near the crack tip propagating along a perfectly bonded interface between dissimilar linear elastic materials is addressed. The Coulomb friction law is considered in the contact zone between the cracks faces. Since the classical Griffith criterion, which assumes infinitesimal crack growth, cannot be applied to determine the load for which such crack will grow because a null ERR is always obtained, a novel approach using the Coupled Criterion (CC) of Finite Fracture Mechanics (FFM) [53–57] is developed to solve the problem and to determine the critical load associated with crack propagation. To this end, the incremental ERR $G_{\text{II}}(\Delta a)$ and the critical crack extension Δa_c associated with crack propagation are used.

Finite crack advances Δa are considered, resulting in nonzero incremental ERR values. To compute the incremental ERR, the main issue is to properly account for the energy dissipated by friction along the contact zone between the crack faces due to a finite crack advance Δa. Two approaches for computing the incremental ERR from the near-tip solution are considered. In the first one, proposed in [33], all the energy released along the segment Δa is taken into account in the evaluation of the incremental ERR denoted as $\widetilde{G}_{\text{II}}(\Delta a)$, while in the second one, previously proposed in [24], an estimate of the energy dissipated due to friction along Δa is excluded from the incremental ERR, thus resulting in a different estimation of the incremental ERR denoted as $\widehat{G}_{\text{II}}(\Delta a)$. A third option is also studied in which the incremental ERR $G_{\text{II}}(\Delta a)$ is calculated ignoring friction effects. The critical values of the incremental ERR and the associated critical crack extension are determined using the CC of FFM.

Special attention is paid to the case where the most singular term in the asymptotic series of the solution at the crack tip governs the elastic solution along the segment Δa before the crack advance. In this case, closed-form expressions for the incremental ERR, the critical crack advance Δa_c and the critical Stress Intensity Factor (SIF) K_{IIc} have been derived for the three aforementioned approaches, showing that the three approaches can be formulated in a unified form. A straightforward comparison of these three approaches is then given, leading to three different definitions of the apparent critical fracture energies $\widetilde{G}_c$, $\widehat{G}_c$ and $\overline{G}_c$, corresponding to the same critical fracture energy of the material G_{IIc}, which result in the same critical load associated with crack propagation obtained from the three approaches.

The limit cases of zero and infinite friction coefficient are briefly analyzed showing that, in the former case, the critical crack advance Δa_c is an upper limit for all possible crack advances, while, in the latter case, no crack advance is possible due to infinite frictional energy dissipation.

(2) Note that the three approaches introduced for calculating the incremental ERR in Mode II provide three different ways of defining the fracture toughness for frictional interface cracks G_{IIc}, each leading to a different value of G_{IIc}. Of the two possible basic dissipation mechanisms ahead of the crack tip — the energy spent in forming a new crack segment Δa and the energy spent in frictional sliding along this new crack segment — these approaches

consider the latter differently. In the first approach, no energy is spent in frictional sliding along Δa_{c} resulting in $\widetilde{G}_{\mathrm{IIc}}$. In the second approach, the energy spent in frictional sliding remains constant as the relative tangential displacement increases, giving $\widehat{G}_{\mathrm{IIc}}$. In the third approach, the energy spent in frictional sliding increases linearly with the relative tangential displacement, resulting in $\overline{G}_{\mathrm{IIc}}$.

However, it has been shown that, under the assumption that the critical finite crack advance Δa_{c} is sufficiently small compared to other characteristic lengths in the problem, e.g., the crack length, so that the most singular term in the asymptotic expansion of the crack tip solution governs the solution in the neighbourhood of the crack tip of radius Δa_{c}, these three approaches are equivalent in the sense that, using the coherent values of G_{IIc} and incremental ERR, essentially the same predictions can be expected for the propagation of frictional interface crack.

Nevertheless, it would be worthwhile trying to discriminate between these approaches by comparing their predictions in situations where Δa_{c} is not small compared to other characteristic lengths of the problem and the friction coefficient can differ.

(3) The presented procedure is also well suited for computational modeling of stepwise propagation of frictional interface cracks as in each step of the computational process the present approach based on CC predicts a finite crack advance as a function of the overall problem configuration.

(4) The present approach could be applied to several experimental tests on both the macro and micro scales. Typical tests developed for the measurement of Mode II fracture toughness G_{IIc} of an interface in macroscopic specimens include Three-Point and Four-Point End-Notched-Flexure (3P- and 4P-ENF) [58,59]. For an analysis of the influence of friction on delamination propagation, see [60]. Single fibre tests developed to measure G_{IIc} of the fibre-matrix interface in microscopic specimens involving just one fibre. In the Single Fibre Fragmentation Test (SFFT), accounting for the effect of friction is crucial for an accurate measurement of G_{IIc}, as demonstrated in [33,61,62]. In the future, it would be interesting to monitor delamination propagation in 3P- or 4P-ENF tests using digital image correlation (DIC) to characterize the displacement field near the crack, detect the crack tip, and compute the critical SIF K_{IIc} at crack propagation.

(5) Although the present approach has been developed for isotropic materials, it could in principle also be used for anisotropic materials, by taking into account the results of the recent study of asymptotic solutions at the front of frictional interface cracks in anisotropic bimaterials under generalised plane strain in [35,36]. However, the present approach must be adapted to each specific configuration as described in [35,36]. This involves taking into account whether a plane of elastic symmetry in the bimaterial coincides with the interface plane or is perpendicular to the crack front.

Consider first the case of an orthotropic bimaterial with a plane of elastic symmetry coinciding with the interface plane, and another perpendicular to the crack front. In this case, the present approach can be applied quite straightforwardly by replacing the asymptotic solution in Section 2 by its orthotropic counterpart in [35,36].

However, adapting the present approach to more general configurations would be more involved, as it would need to consider the specific features of the asymptotic solution at the crack front, as described in [35,36]. For instance, in the case of a bimaterial with a single plane of elastic symmetry coinciding with the interface plane, the out-of-plane components of displacements and stresses must be considered as well. Considering the results in [35,36], we can conjecture that there will be a competition between two fracture modes: Mode *ii* the crack propagation by finite advances associated with a sliding direction corresponding to $\lambda > 0.5$, with zero ERR but positive incremental ERR (somewhat similar to Mode II in isotropic or orthotropic bimaterials, but with different angle of sliding), and Mode *iii* an infinitesimal crack propagation associated with a sliding direction corresponding to $\lambda = 0.5$, with a positive ERR (somewhat similar to Mode III in isotropic or orthotropic bimaterials, but with different angle of sliding). A generalisation of the present approach could be used to compute the incremental ERR in the Mode *ii*. The overall problem configuration, mainly determined by the specimen geometry, the (direction of) applied loads and the material properties, would govern the above described competition.

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