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Tribute to an exemplary man: Yves Couder
Bouncing drops, memory

Hydrodynamic quantum field theory: the free particle

Yuval Dagan[*, a] **and John W. M. Bush**[b]

[a] Faculty of Aerospace Engineering, Technion - Israel Institute of Technology, Haifa, Israel

[b] Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA

E-mails: yuvalda@technion.ac.il (Y. Dagan), bush@mit.edu (J. W. M. Bush)

Abstract. We revisit de Broglie's double-solution pilot-wave theory in light of insights gained from the hydrodynamic pilot-wave system discovered by Couder and Fort [1]. de Broglie proposed that quantum particles are characterized by an internal oscillation at the Compton frequency, at which rest mass energy is exchanged with field energy. He further proposed that the resulting pilot-wave field satisfies the Klein–Gordon equation. While he developed a guidance equation for the particle, he did not specify how the particle generates the wave. Informed by the hydrodynamic pilot-wave system, we explore a variant of de Broglie's mechanics in which the form of the Compton-scale dynamic interaction between particle and pilot wave is specified. The particle is modeled as a localized periodic disturbance of the Klein–Gordon field at twice the Compton frequency. We simulate the evolution of the particle position by assuming that the particle is propelled by the local gradient of its pilot wave field. Resonance is achieved between the particle and its pilot wave, leading to self-excited motion of the particle. The particle locks into quasi-steady motion characterized by a mean momentum $\bar{p} = \hbar k$, where k is the wavenumber of the surrounding matter waves. Speed modulations along the particle path arise with the de Broglie wavelength and frequency ck. The emergent dynamics is strongly reminiscent of that arising in the hydrodynamic pilot-wave system, on the basis of which we anticipate the emergence of quantum statistics in various settings. Our results suggest the potential value of a new hydrodynamically-inspired pilot-wave theory for the motion of quantum particles.

Keywords. Klein–Gordon equation, Matter waves, Hydrodynamic quantum analogs, Pilot-wave theory, Free particle.

1. Introduction

> "*We believe that the debate on hidden variables is not closed.*"
> – Yves Couder, DTU Fluids Summer School, Krogerup Denmark, August 12, 2011.

[*] Corresponding author.

The de Broglie relation, $\mathbf{p} = \hbar\mathbf{k}$, relates the momentum of a microscopic quantum particle $\mathbf{p}$ to an associated wavenumber $\mathbf{k}$ through the reduced Planck's constant $\hbar$. The relation serves as a cornerstone of quantum mechanics; moreover, it remains the totality of the dynamical description of the free particle in the standard quantum formalism [2], where the notion of particle trajectories has been largely abandoned. Nevertheless, the physical origins of the de Broglie relation remain nebulous: it survives as a vestigial element of de Broglie's discarded physical picture of quantum dynamics [3, 4]. We here demonstrate the manner in which $\mathbf{p} = \hbar\mathbf{k}$ emerges naturally from a trajectory-based physical picture of the free quantum particle. On the basis of the physical analogy between our model system and pilot-wave hydrodynamics, our results further suggest a dynamical basis for the emergence of a statistical signature with the de Broglie wavelength, specifically, the dynamic constraint imposed on the particle by its quasi-monochromatic pilot-wave field. As it is rooted in de Broglie's original double-solution pilot-wave theory of quantum dynamics [3–5] but informed by the hydrodynamic pilot-wave system discovered by Couder and Fort [1], we refer to our fledgling theoretical model as "hydrodynamic quantum field theory".

1.1. *Hydrodynamic pilot-wave theory*

In 2005, Couder and Fort discovered a hydrodynamic pilot-wave system consisting of a millimetric oil droplet self-propelling across a vibrating bath of the same liquid [1, 6]. The system has extended the range of classical physics to include many features previously thought to be exclusively quantum [7–9]. Hydrodynamic quantum analogs achieved with this system now include tunneling [10–12], Landau levels [13, 14], Zeeman splitting [15], and Friedel oscillations [16, 17]. Quantized orbits arise for droplets walking in either a rotating frame [13, 14, 18] or a simple harmonic potential [19–21]. In the chaotic regime of these orbital pilot-wave systems, the drop switches intermittently between weakly unstable quantized orbits, giving rise to quantum-like statistics [14, 18–21]. When the droplet walks in a confined cavity, quantum-like statistics also emerge [22], along with effects similar to quantum superposition and the quantum mirage [23]. Other analogs have been more elusive, including diffraction from single and double slits [24], as has been contested by Bohr and coworkers [25, 26] but revisited by Pucci *et al.* [27], who confirmed a wave-like diffraction pattern in addition to the influence of the second slit on a particle passing through the first, a form of single-particle interference.

The walking-droplet system suggests a more general theoretical framework for exploring classical pilot-wave dynamics not accessible in the laboratory [7]. Doing so has led to the discovery of hydrodynamic spin states [28, 29], and rich two-particle dynamics [30]. Durey *et al.* [31] examined the stability of the self-propelling state in this general classical pilot-wave framework, showing the propensity for in-line oscillations and emergent statistical behavior with a wavelength corresponding to that of the pilot wave [17]. More adventurous still was Fort and Couder's [32] theoretical abstraction of inertial walkers, which exhibited an analog of the Bohr–Sommerfeld quantization rule.

The hydrodynamic quantum field theory introduced here represents an attempt to develop a theoretical model of quantum dynamics based on insights gained from the walking-droplet system. As such, it has a number of recent precursors. Andersen *et al.* [25] examined a dynamical system in which a particle locally excites a waveform satisfying Schrödinger's equation, then moves in response to gradients of that field. While orbital quantization consistent with the Bohr–Sommerfeld quantization rule was shown to emerge, they concluded that their model was not capable of giving rise to behavior analogous to the quantum double-slit experiment. Borghesi [33] proposed an elastic pilot-wave model wherein a point-particle is allowed to move within a non-dissipative elastic substrate. Shinbrot [34] examined the influence of periodic driving of the

Klein–Gordon equation, and found bound-state solutions with half-integer spin composed of spin-less particles. A number of investigators have taken inspiration from the walking droplets to inform and advance their theoretical modeling of quantum stochastic dynamics [35–38].

The hydrodynamic pilot-wave system discovered by Couder and Fort is the first realization of a macroscopic particle self-propelling through a resonant interaction with its self-generated quasi-monochromatic wave field. As such, it is markedly different from Bohmian mechanics [39, 40], closer in spirit to the original double-solution pilot-wave theory proposed in the 1920s by Louis de Broglie [7, 8]. de Broglie envisaged quantum particles as having an internal clock that interacts with a background field, exchanging rest mass energy and field energy at the Compton frequency [3–5]. While he developed a guidance equation that prescribes how the particle moves in response to the local wave, he did not specify the mechanism for wave generation [41]. In the hydrodynamic system, the droplet's bouncing plays the role of the particle's internal vibration in de Broglie's mechanics, the bath surface the role of the field, and the mechanism of wave generation is well understood [7, 8]. We shall explore here a dynamical system of the form originally proposed by Louis de Broglie, but informed by the walking-droplet system. The novel feature of our model is that we treat explicitly the production of the pilot wave by particle vibration at twice the Compton frequency.

A vertically vibrating bath becomes unstable at a critical vibrational acceleration known as the Faraday threshold, above which a standing field of subharmonic Faraday waves arises, with a wavelength prescribed by the standard water-wave dispersion relation [42]. In the parameter regime of interest in the hydrodynamic pilot-wave system, the characteristic wavelength of the Faraday waves is approximately 5 mm. The waves are dominated by surface tension rather than gravity. Thus, roughly speaking, the role of $\hbar$ in de Broglie's mechanics, as determines both the particle's natural frequency and wave energy, is played by surface tension in the hydrodynamic system [7]. Notably, the walking-droplet experiments are performed below the Faraday threshold: no waves would be present in the absence of the drop, which thus responds only to its own field. Nevertheless, the bath vibration is critical in predisposing the system to quasi-monochromatic waves with the Faraday wavelength [43], as arise when the droplet bounces at the Faraday frequency so that resonance is achieved between droplet and bath.

Following the original theoretical model of Couder and Fort [24], a hierarchy of models of increasing sophistication have been developed to describe the walking droplets (see [44] for a recent review). We here follow Molácek and Bush's [45, 46] theoretical description of the vertical and horizontal dynamics of droplets walking on the surface of a vibrating bath in resonance with their guiding wave. Time-averaging over the bouncing period eliminates the vertical dynamics from consideration, and provides a description of the horizontal motion revealed by strobing the droplet at the Faraday frequency. In this strobed frame, the drop appears to glide along a line of constant wave amplitude, dressed by a wave form that is stationary in the drop's frame of Refs [47, 48]. The stroboscopic model of Oza *et al.* [49] is based on the assumption that the vertical motion is fast relative to the horizontal, so the drop may be treated as a continuous source of waves along its path. The resulting trajectory equation describing the drop's horizontal displacement $\mathbf{x_p}$ takes the form

$$m\ddot{\mathbf{x}}_\mathbf{p} = -D\dot{\mathbf{x}}_\mathbf{p} - mg\nabla h(\mathbf{x_p}, t). \tag{1}$$

The drop is propelled by a wave force proportional to the slope of the local wave field $h(\mathbf{x}, t)$, and resisted by a linear drag induced during flight and impact, characterized by a constant drag coefficient D. The wave field is that produced by prior bounces, and may be expressed as an integral along the droplet path of the form

$$h(\mathbf{x}, t) = A\int_{-\infty}^{t} J_0(k_F|\mathbf{x}-\mathbf{x_p}(s)|)\mathrm{e}^{-(t-s)/(T_F M_e)}\,\mathrm{d}s, \tag{2}$$

where T_F is the Faraday period, k_F is the Faraday wave number, A is a constant that depends on the drop's impact phase, and J_0 is the Bessel function of the first kind. The memory parameter

$$M_e = \frac{T_d}{T_F(1-\gamma/\gamma_F)}, \tag{3}$$

depends on the decay time of the waves in the absence of vibrational forcing T_d, as well as the proximity of the vibrational acceleration γ to the Faraday threshold, γ_F [50]. Notably, M_e prescribes the rate of decay of the wave field, and increases as the Faraday threshold is approached. The quantum-like features of the hydrodynamic pilot-wave system emerge in the high-memory (large M_e) limit.

The richness of the hydrodynamic pilot-wave system arises due to the temporal nonlocality of the wave force, whose form renders the drop dynamics hereditary: predicting the drop motion requires knowledge of its past [9, 13, 50]. It is also worth noting that the walker motion may be described in terms of three distinct speeds, the phase and group speeds of its piloting Faraday wave field, and the walker speed. For the special case of free, steady motion, the self-propelling particle is dressed by a stationary wave form: when strobed at the Faraday frequency, this wave form is unchanging (see Figure 1(a–c) and [47, 48]).

A key component in the dynamics of the walking droplets is resonance between the walker's bouncing motion and its wave field. Owing to the bath vibration, the result of this resonance is a highly structured, quasi-monochromatic wave field that imposes a dynamic constraint on the bouncing droplet. The quasi-monochromatic wave field is critical for the three established paradigms for the emergence of quantum-like statistics from pilot-wave hydrodynamics. The first such paradigm has emerged from studies of orbital pilot-wave dynamics, where the droplet carves out a wave field that promotes motion along a finite number of quantized periodic orbital states of relatively simple geometric form [13, 14, 18–20, 51]. When the system becomes chaotic, the droplet switches intermittently between these weakly unstable periodic orbits, giving rise to a multimodal statistical signature whose precise form reflects the relative instability of the unstable quantized orbits [14, 19, 21]. In the second paradigm, the droplet speed oscillates with the wavelength of its pilot wave [52, 53], leading to a statistical signature with this same wavelength [16, 17]. This mechanism is most clearly demonstrated when a walker interacts with a submerged well: the drop is first drawn inwards at uniform speed along a spiral, then exits the well radially with speed oscillations induced by its interaction with the well [17]. The result of these speed oscillations on an ensemble of trajectories is a statistical signature reminiscent of Friedel oscillations [54, 55], with the Faraday wavelength in place of the de Broglie wavelength [16, 17]. The third paradigm arises from the droplet interacting with its pilot-wave field in such a way as to execute a random walk with a characteristic step length comparable to λ_F, mean speed U and resulting diffusivity $U\lambda_F$ [51,56–58]. We shall see here that both in-line oscillations and structured random walks of the free particle are also a feature of the variant of de Broglie's mechanics developed here.

1.2. *de Broglie's matter waves*

de Broglie's pilot-wave theory was an attempt to reconcile quantum mechanics and relativity [3–5]. de Broglie suggested that a particle at rest has an associated internal clock with the Compton frequency $\omega_c = m_0c^2/\hbar$, as defined by the de Broglie–Einstein relation, $E = m_0c^2 = \hbar\omega_c$, where m_0 is the particle rest mass and c is the speed of light. He proposed that this clock generates a standing wave through the exchange between rest-mass energy and wave energy. Schrödinger proposed that the particle vibration, or so-called Zitterbewegung, takes place at twice the Compton frequency [59–61]. Notably, such fast oscillations are not yet observable experimentally for

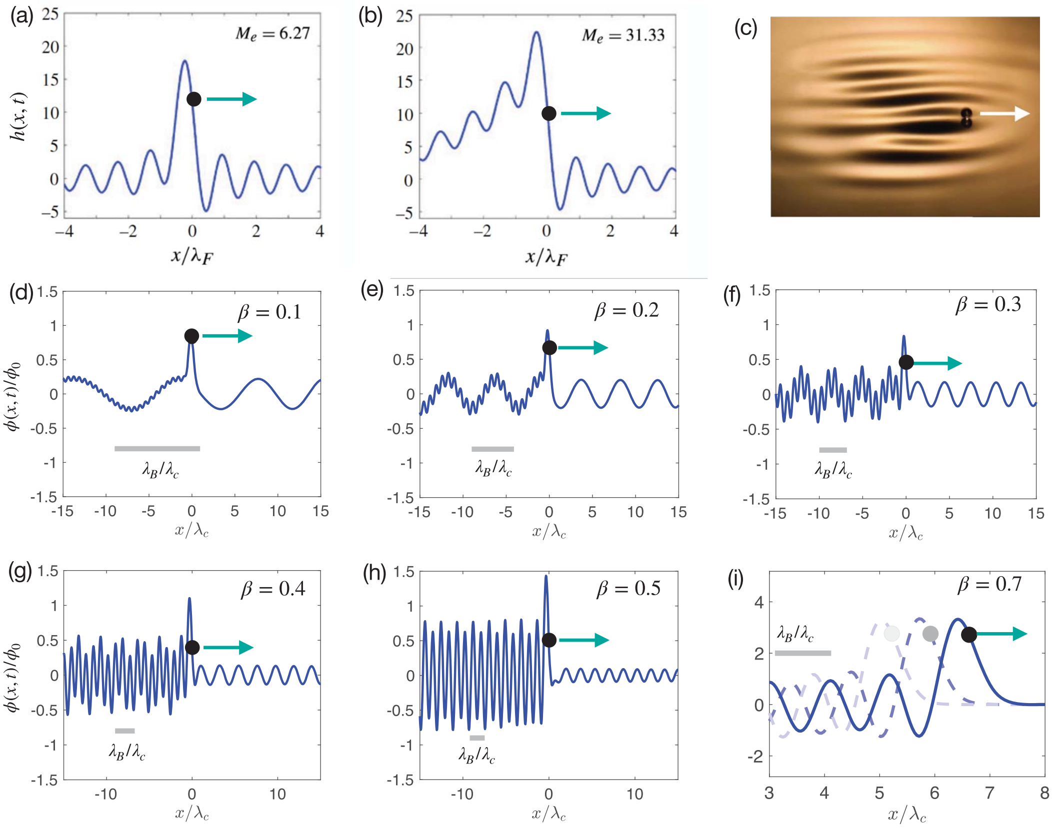

Figure 1. Comparison of the pilot wave fields in the walking droplet system (a–c) and our HQFT (d–f). Simulated wave profiles along the direction of motion of the walker [49] for memories (a) $M_e = 6.27$ and (b) 31.33. Wave amplitudes are shown in μm. (c) Snapshot of the strobed motion of a walking droplet reveals its quasi-steady pilot-wave field translating uniformly with the drop [47, 48]. (d–h) The predicted pilot wave fields for particles moving at different constant speeds, $v = \beta c$, with β varying from 0.1 to 0.7. The corresponding de Broglie wavelength $\lambda_B = \lambda_c/(\beta\gamma)$ is shown for reference. (i) Strobing the motion at $\beta = 0.7$ at the Compton frequency reveals three successive, indistinguishable waveforms separated by the time interval $\tau_c = 2\pi/\omega_c$. In both the hydrodynamic and quantum systems, the particle moves along a point of constant amplitude of its strobed pilot-wave field.

most particles with current experimental capabilities; for example, the circular Zitter frequency of the electron is 1.6×10^{21} s^{-1}. Nevertheless, some evidence of the Zitterbewegung has recently been reported in Bose–Einstein condensates [62] and trapped ion systems [63].

According to de Broglie's original double-solution pilot-wave theory [5], there are two distinct waves, the statistical wave of standard quantum mechanics and a real particle-centered pilot wave responsible for guiding the particle. A feature of de Broglie's theory that he stressed is the "harmony of phases" that ensures the resonance between the particle's vibration and its pilot wave in an arbitrary frame of reference. Let ω be the particle oscillation frequency in a stationary frame of reference. When set into motion at speed v, due to the relativistic Doppler shift, this frequency is reduced to ω/γ, where $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor [61]. However, a concurrent increase of the particle frequency $mc^2/\hbar$ arises due to the relativistically boosted mass, $m = \gamma m_0$, thus cancelling the frequency reduction. The particle vibration thus maintains resonance with its pilot wave in any frame of reference, a resonance described by de Broglie as "une grande loi de la Nature" [3].

In the original derivation of his particle guidance equation, de Broglie assumed a monochromatic pilot wave characterized by a single wave number k [3]. For the sake of simplicity, we here consider a one-dimensional setting, for which the phase Φ of such a wave may be expressed as

$$\Phi = kx - \omega t. \tag{4}$$

The assumption underlying the harmony of phases is that the phase is a scalar and hence invariant to Lorentz transformation. As such, the phase measured from any inertial frame of reference is equal to that viewed from the particle's frame of reference:

$$\frac{\omega}{\gamma}\frac{x}{v} = \gamma\omega\left(\frac{x}{v} - \frac{x}{v_\phi}\right), \tag{5}$$

from which it follows that the phase velocity of the pilot wave is $v_\phi = c^2/v$. Next, taking the spatial derivative of the pilot-wave phase and substituting for the phase velocity $v_\phi = \omega/k$ and frequency $\omega = \gamma\omega_c$, we deduce

$$\frac{\partial\Phi}{\partial x} = k = \frac{\omega}{v_\phi} = \frac{\omega v}{c^2} = \frac{\gamma m_0 v}{\hbar}, \tag{6}$$

from which follows the de Broglie relation:

$$p = \gamma m_0 v = \hbar k. \tag{7}$$

This relation serves to define the de Broglie wavelength $\lambda_B = 2\pi/k$ in terms of the particle momentum. The generalization to three dimensions is straightforward and yields $\mathbf{p} = \gamma m_0 \mathbf{v} = \hbar\nabla\Phi = \hbar\mathbf{k}$.

Subsequently, de Broglie proposed that the pilot wave evolves according to the Klein–Gordon equation [5, 64], and so deduced an alternative derivation of his guidance equation. Assume a quasi-monochromatic one-dimensional wave solution, $\phi(x,t) = A(x,t)e^{i\Phi(x,t)}$, where $\Phi = kx - \omega t$ and A is a real amplitude that varies over a lengthscale large relative to $2\pi/k$. By substituting into the Klein–Gordon equation,

$$\phi_{tt} - c^2\phi_{xx} + \omega_c^2\phi = 0, \tag{8}$$

one finds that the imaginary part of the equation takes the form:

$$\Phi_t A_t - c^2\Phi_x A_x = 0, \tag{9}$$

where subscripts denote partial derivatives with respect to x and t. If the particle moves along a point of constant amplitude $A(x,t)$ of its pilot wave field, then the Lagrangian derivative of this amplitude in the particle's frame of reference must vanish:

$$\frac{DA}{Dt} = A_t + vA_x = 0, \tag{10}$$

yielding the particle speed $v = -A_t/A_x$. Substitution from (10) then yields a relationship known as de Broglie's guidance equation:

$$v = -c^2\frac{\Phi_x}{\Phi_t}. \tag{11}$$

Substituting for $\Phi_x = k$ and $\Phi_t = -\omega$ yields $v = c^2k/\omega$. Eliminating ω using the expression for the relativistic energy of a particle $E = \hbar\omega = \gamma m_0 c^2$ yields (7) directly. Notably, this derivation of the de Broglie relation follows from the assumption that the particle moves along lines of constant wave amplitude, as is the case in pilot-wave hydrodynamics (see Figure 1a–c).

The form of the pilot-wave field ϕ was not clearly defined by de Broglie, and his inability to do so was the principal impediment to advancing his theory. Notably, both derivations of the de Broglie relation (7) reviewed above rely on the assumption of a monochromatic or quasi-monochromatic wave (4). Following the work of Bohm [39,40], de Broglie suggested that the pilot wave takes the form of the solution of Schrödinger's equation, with the only distinction between these two waves being that the pilot wave has a singularity in the vicinity of the particle. Unsure of the form of the pilot wave, de Broglie invested considerable effort in demonstrating that his guidance equation (11) has broader generality [65], including when the pilot wave has a quasi-monochromatic form.

De Broglie's uncertainty in the form of the pilot wave was rooted in his inability to specify the coupling between particle and wave. To be precise, while his theory indicates how the particle moves in response to its pilot wave, it does not specify the manner in which this wave is generated by the particle. This shortcoming of de Broglie's theory (and likewise, incidentally, of Bohmian mechanics [39, 40]) was pointed out explicitly by Holland [41]: "*... we can envisage a more active role for the particle, something which is not even admitted as conceivable in the conventional view. This may, for instance, enter as a 'source' of the* Ψ *(pilot-wave) field through an inhomogeneous term in the wave equation ...*" It is precisely this approach that we take in developing our hydrodynamic quantum field theory, where the mechanism for wave production has been suggested by the walking droplet system.

In Section 2, we describe our mathematical model, an extension of de Broglie's mechanics infused with intuition gained from the walking-droplet system. The model captures the essential features common to the hydrodynamic system and de Broglie's mechanics, including the resonance between the particle and its quasi-monochromatic guiding wave, but goes beyond de Broglie's theory in specifying the mechanism of wave generation. In Section 3, we present model simulations of, in turn, particle kinematics (Section 3.1) and dynamics (Section 3.2). The latter illustrates the natural emergence of a mean motion consistent with $p = \hbar k$, and in-line speed oscillations with the de Broglie wavelength. The implications of our results are discussed in Section 4.

2. Mathematical model

We proceed by extending de Broglie's double-solution pilot-wave program through inclusion of a mechanism for particle-induced wave generation. We model relativistic particles as localized disturbances in a scalar field ϕ that evolves according to the Klein–Gordon wave equation. Traditionally, this equation describes the evolution of the wave function of spin-0 particles, from which particle statistics is derived. Notably, it also describes the evolution of the Higgs field [66]. Here, we consider the Klein–Gordon equation to describe the real pilot wave generated through a local interaction with the vibrating particle. Using this conceptual framework, we examine the self-propulsion of a microscopic particle by its pilot-wave field.

We consider the forced Klein–Gordon equation in one dimension,

$$\phi_{tt} - c^2\phi_{xx} + \omega_c^2\phi = -\epsilon_p f(t)\delta_a(x - x_p), \tag{12}$$

where ϕ is the real, scalar pilot-wave field. We seek solutions of this equation forced by a localized vibration: $f(t) = \sin(2\omega_c t)$, ϵ_p is a constant and $\delta_a(x) = (1/|a|\sqrt{\pi})\mathrm{e}^{-(x/a)^2}$ is a modified delta function that serves to localize the driving oscillation to the vicinity of the particle location x_p. The parameter a defines the width of the modified delta function, and so the extent of the particle's influence on the wave-field. In our analysis, we set $a = (1/2)\lambda_c$ so that the particle influence arises on the scale of the Compton wavelength. Notably, the form chosen for δ_a is consistent with de Broglie's postulate of a localized disturbance creating a wave form that extends to infinity while decaying spatially [3]. Our mechanism for wave generation is similar in spirit to that in the hydrodynamic pilot-wave model of Milewski *et al.* [43], wherein the droplet is treated as a localized exciter of its pilot-wave field. The resulting wave form can then not be simply expressed in a simple form equivalent to (2); rather, it is solved for directly from the governing partial differential equation. While the hereditary nature of the system is thus less evident in the mathematical formulation, the critical feature of path-memory is present in both models.

Here, informed by the hydrodynamic system, we couple the wave equation (12) to a guidance equation in which the particle velocity is proportional to the gradient of the wave amplitude

$$\gamma \dot{x}_p = -\alpha \frac{\partial \phi}{\partial x}, \tag{13}$$

in a stationary frame. The wave amplitude coefficient α serves as the only free parameter in our system. We note that (13) resembles the guidance equation (1) in pilot-wave hydrodynamics in the limit of negligible drop inertia: the particle simply moves in response to gradients of its pilot-wave field. In what follows, we shall see that it yields the salient result of de Broglie's physical picture, specifically $p = \hbar k$.

Relativistic considerations are worth noting. For relativistic particles, the numerical scheme employed here includes a proper Lorentz contraction for an observer that is not moving with the particle. The wave field ϕ is a scalar function, necessarily invariant to Lorentz transformation. Conversely, the spatial derivative appearing in the trajectory equation (13) is not invariant to Lorentz transformation due to the contraction of length and dilation of time in a moving frame of reference. The numerical implementation is thus carried out as follows. First, the wave equation is solved in the stationary frame of reference. After each time-step, the wave gradient at the particle location is updated and translated to the particle frame of Ref. [67]. In this particle frame, we use (13) to update the added speed due to the change in the wave gradient over one time-step. Subsequently, we use the Lorentz transformation to update the added speed in the stationary frame according to:

$$\dot{x}'_p = \frac{\mathrm{d}x'_p}{\mathrm{d}t'} = \frac{\mathrm{d}x_p - v\,\mathrm{d}t}{\mathrm{d}t - \frac{v\,\mathrm{d}x_p}{c^2}}. \tag{14}$$

In the non-relativistic limit, $\gamma \to 1$, this procedure for updating the particle speed is not necessary and (13) can be computed directly. As noted in Section 1.2, the phase of oscillation is the same in any frame of reference due to the harmony of phases. The particle oscillations and coupled particle-wave system are thus consistent with relativity [67].

The coupled Klein–Gordon equation (12) and guidance equation (13) are discretized using finite differences. An explicit finite-difference method was derived to solve the Klein–Gordon wave equation, and a Runge–Kutta scheme is employed to advance in time the guidance equation (13), which is nonlinear due to the dependence of γ on the particle speed [68].

The numerical scheme is second-order accurate in space and first-order in time. A time step of $\Delta t = 10^{-3}\tau_c$, where $\tau_c = 2\pi/\omega_c$, and a constant grid cell size of $\Delta x = 2 \times 10^{-3}\lambda_c$ were used to resolve the smallest time and length scales of the system. The simulation domain varied from 200 to 600 Compton wavelengths, according to the case solved. Zero-wave-amplitude boundary conditions were applied, effectively imposing infinite-wall potentials at the boundaries of the computational domain. However, boundaries were sufficiently distant that they did not affect the simulated particle dynamics. A maximum integration time of $1.2 \times 10^4 \tau_c$ was used to collect statistics, corresponding to approximately 10^6 time steps.

Finally, we note that our model has a single free parameter, the coupling constant α appearing in the guidance equation (13), as relates the particle velocity to the gradient of the pilot-wave field. Choosing a different α changes both the mean particle momentum p and the wavenumber k; however, these always change in concert such that $p = \hbar k$. Choosing the coupling constant α in our model is thus analogous to choosing the particle energy in quantum mechanics.

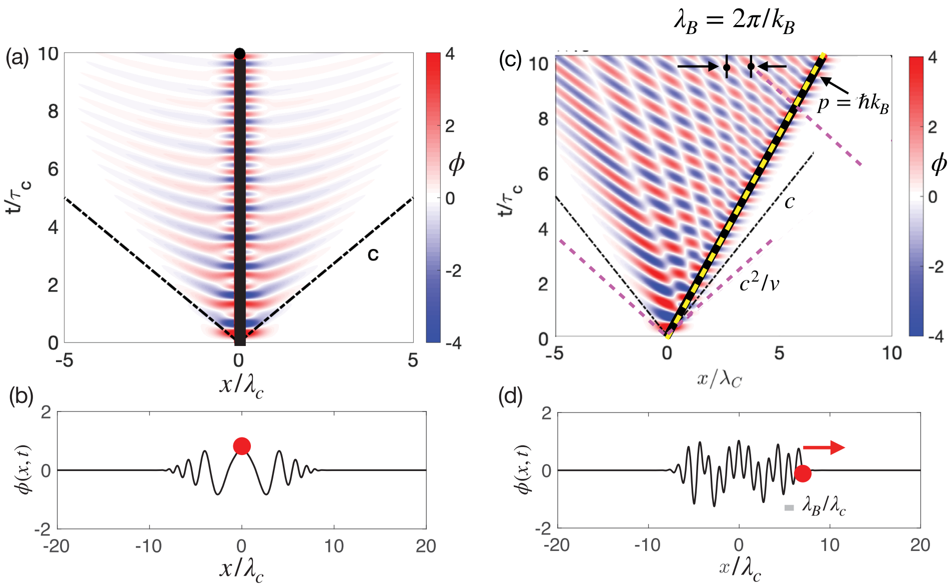

Figure 2. Pilot-wave kinematics. The pilot waves generated by (a,b) stationary and (c,d) steadily translating particles. (a) Spatio-temporal map indicates the evolution of the normalized wave form, ϕ (color map), generated by a stationary particle (indicated in black). (b) The normalized wave field in (a) at a time $t/\tau_c = 10$. Far from the particle location, the wave form has a characteristic wavelength λ_c. (c) Spatio-temporal map indicates the wave form, ϕ (color map), generated by a particle in uniform motion at speed $v = 0.7c$. The slope of the (dashed pink) line of constant phase indicates a phase speed c^2/v. The light cone is indicated by the dashed black line. Indicated in yellow is the speed calculated from the measured wavelength $\lambda = 2\pi/k$ according to $v = \hbar k/(\gamma m_0)$, from which we infer that $\lambda = \lambda_B$ in the vicinity of the particle. (d) Normalized wave field of the particle in (c) at a time $t = 10\tau_c$ indicates the limited upstream extent of the pilot wave. The de Broglie wavelegnth $\lambda_B = \lambda_c/(\gamma\beta) = 1.02\lambda_c$ is shown for reference.

3. Results and discussion

The results of our theoretical model are presented as follows. In Section 3.1, we explore the effect of particle vibration and translation on the pilot-wave field using *kinematic* simulations; specifically, we prescribe a constant particle speed, so the trajectory equation (13) need not be solved. This constraint is relaxed in Section 3.2, where we investigate the fully dynamic coupling between the particle and its pilot-wave expressed in (12) and (13), as results in the particle's self-propulsion.

3.1. *Kinematics: the pilot wave form*

In Figure 2, we present the wave forms generated by both stationary and uniformly translating particles. For the particle at rest (Figure 2a), after an initial transient, a standing wave form with a characteristic wavelength λ_c emerges. Figure 2(b) shows a snapshot of the transient wave form ten Compton periods after the initiation of particle vibration. Figure 2(c) shows the relatively complex wave structure that emerges when the particle is set into motion at a constant speed of $v = 0.7c$. The form of the accompanying wave form ten Compton periods after the initiation

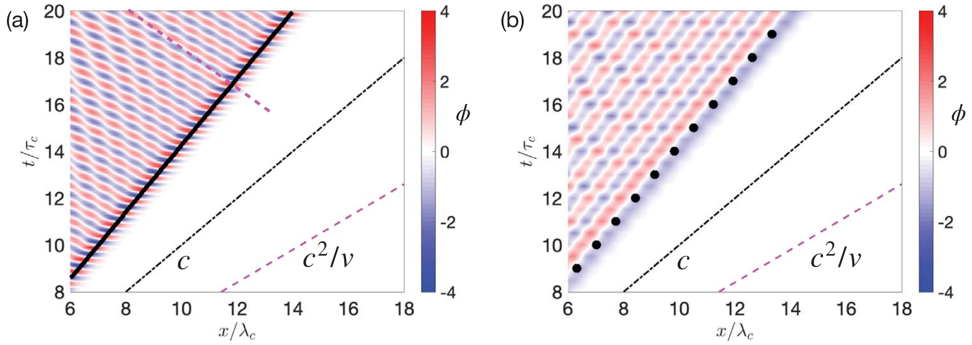

Figure 3. Pilot-wave kinematics. Spatio-temporal map indicates the evolution of the normalized wave field, ϕ (color map), generated by a particle (indicated in black) moving at a uniform speed $v = 0.7c$. (a) In the full, unstrobed solution, the phase speed is comparable to de Broglie's superluminal phase speed, c^2/v. (b) Strobed solution, presented at integer multiples of the Compton period $\tau_c = 2\pi/\omega_c$. Note that de Broglie's phase speed, c^2/v, is not apparent in the strobed framework. The light cone is indicated by the dashed black line with slope c.

of particle motion in shown in Figure 2(d). The dependence of the form of the pilot wave form on $\beta = v/c$ is shown in Figure 1(d–h), where the de Broglie wavelength $\lambda_B = \lambda_c/(\gamma\beta)$ is indicated for reference. Notably, signatures of both the de Broglie and Compton wavelengths are evident in all cases. As $\beta \to 1$, the pilot wave has only a weak signature in advance of the particle, and necessarily resides within the particle's light cone.

We proceed by measuring the wavelength of the waves in the vicinity of the translating particle towards the end of the simulation shown in Figure 2(c). If we identify this wavelength with the de Broglie wavelength, λ_B, then the corresponding speed $v_p = \hbar k_B/\gamma m_0$ is shown as a dashed yellow line, and is virtually indistinguishable from the prescribed particle speed indicated in black. Note that the phase velocity of these waves is comparable but not precisely equal to de Broglie's phase velocity, c^2/v. When the waves in the vicinity of the particle are strobed at ω_c, the constant speed motion is apparent through the wave field (Figure 3). We conclude that these pilot waves, generated by the particle vibrating at $2\omega_c$ and translating at uniform speed, have wavelength λ_B as measured in a stationary frame, and so are consistent with the proposed form of de Broglie's matter waves. The form of the waves in this kinematic case has recently been deduced analytically by Durey & Bush [69], who solved the appropriate initial-value problem.

3.2. *Dynamics: self-propulsion and in-line oscillations of free particles*

We proceed by simulating a particle free to move through interaction with its pilot-wave field, as described by the coupled equations (12) and (13), with coupling constant $\alpha = 0.045$. We consider a closed computational domain that extends between $x = -600\lambda_c$ and $x = 600\lambda_c$. A zero-wave-amplitude condition is applied at the boundaries. Our focus is on the initial trajectory of particles, and we investigate only the range $x = -20\lambda_c$ to $x = 20\lambda_c$, in which the boundaries have a negligible influence. The resulting pilot-wave dynamics are shown in Figure 4.

The particle vibrates at twice the Compton frequency, exciting localized waves similar to the standing waves shown in Figure 2(a,b). In order to break symmetry and so initiate motion, a small initial random wave perturbation is applied, approximately four orders of magnitude smaller than the maximum wave amplitude generated by the particle. This perturbation initially causes

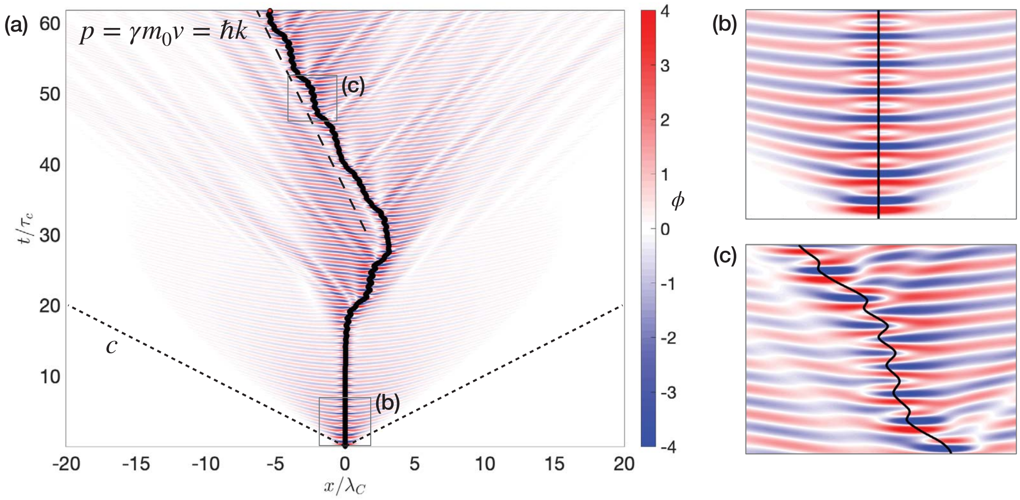

Figure 4. Pilot-wave dynamics of a free particle. (a) Spatio-temporal map illustrates the evolution of the normalized pilot-wave field, ϕ (color map), generated by the trajectory of a free particle (indicated in black). The coupling constant in (12), $\alpha = 0.045$. Dashed black lines correspond to $\bar{p} = \hbar k$, as calculated from the wavelength in window (c), where the particle has a quasi-steady speed of approximately 0.25c. The particle's light cone is indicated by the dotted black line with slope c. Closeup views of (b) the initial quasi-static state and (c) the subsequent quasi-steady, self-propelling state.

the particle to oscillate about its initial position ($x = 0$) with small amplitude, on a scale not perceptible in Figure 4. As time proceeds, the particle vibration causes the local wave amplitude to increase. At $t \approx 20\tau_c$, symmetry is broken and the particle is deflected, in this case, to the right. The subsequent motion is highly sensitive to the initial perturbation. Thereafter, the particle locks into a quasi-steady motion, in this case, propelling itself to the left. Measuring the wavelength in the vicinity of the particle as we did for the kinematic case, we find that the emergent mean motion is again consistent with $\bar{p} = \hbar k$ (dashed black lines). Moreover, a fine-scale particle dynamics is now apparent: the particle speed is modulated by an oscillation about the mean, with the de Broglie wavelength.

By strobing the free particle solution at ω_c, as we did in the kinematic case (Figure 3), we reveal that the particle trajectory may be traced through the propagation of its accompanying pilot-wave packet, despite the fact that its trajectory is relatively complex (Figure 5). Thus, as in the walker system, the microscopic particle is "dressed" with an accompanying quasi-monochromatic wave form that is effectively stationary in the particle frame of reference. The range of influence of the particle is thus extended through its pilot-wave field, a feature that is key to the emergent quantum-like behavior in several hydrodynamic quantum analogs [7, 9].

Figure 6 illustrates 50 particle trajectories, each corresponding to a separate simulation of the same configuration, but with a different random initial wave perturbation. Each trajectory is initiated from the same location ($x = 0$), but then deflected in a random manner by the initial perturbation. The particle trajectories all reach a quasi-steady state in which they oscillate about a constant mean speed such that $\bar{p} = \hbar k$. Most particles undergo sporadic changes in direction and then settle into a new quasi-steady motion.

The power spectral density of the time series describing the speed evolution of a particle moving with mean speed $v = 0.32c$ ($\alpha = 0.05$) is shown in Figure 7. The spectrum reveals three

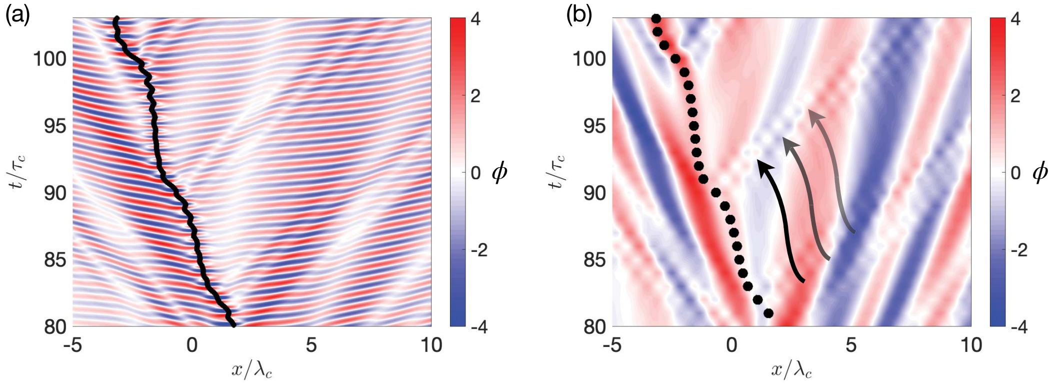

Figure 5. Spatio-temporal wave-field map of the free particle (indicated in black) showing: (a) the full unstrobed solution, and (b) the strobed wave solution, presented at integer multiples of the Compton period $\tau_c = 2\pi/\omega_c$. The conditions are the same as in Figure 4.

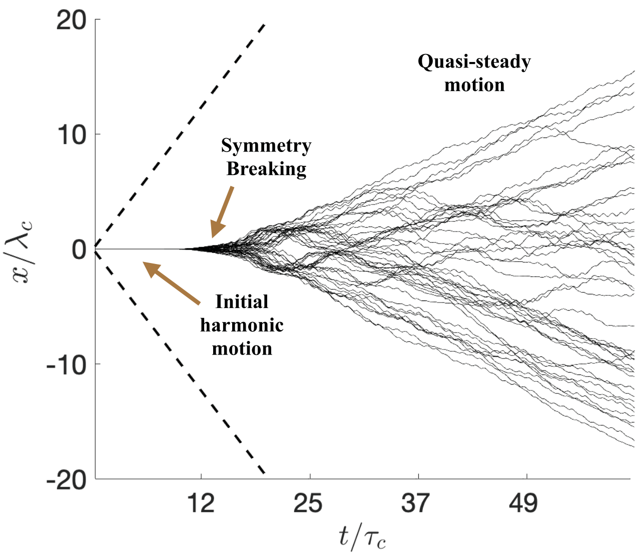

Figure 6. Symmetry breaking and the emergence of self-propulsion in an ensemble of 50 free-particle trajectories. The coupling constant $\alpha = 0.045$. An initial random perturbation, four orders of magnitude smaller than the maximum value of the wave-field, was applied to the wave-field in order to break symmetry and so initiate motion. The emergent self-propelling states are marked by a mean speed $\bar{v} = \hbar k/(\gamma m_0) = 0.25c$, in-line oscillations with frequency $\omega_{\text{mod}} = kc$ and sporadic reversals in direction.

significant frequencies, the first being the particle's vibration frequency $\omega = 2\omega_c$. The second and third frequencies, $\omega = \sqrt{(\omega_c^2 \pm k^2c^2)}$, can be understood as being shifted from the Compton frequency, ω_c, due to the particle's self-excited in-line oscillations at the modulation frequency $\omega_{\text{mod}} = kc$ (see also Figure 4c). By way of comparison, in the hydrodynamic pilot-wave system, the resonant walker bounces at the Faraday frequency, indicating a harmonic resonance between droplet and wave, and the in-line oscillations arise at a frequency of order $u_0 k_F$, where u_0 is the free walking speed [31].

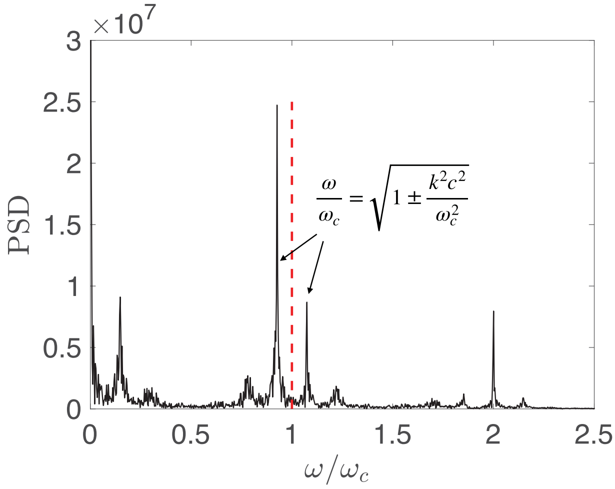

Figure 7. Temporal power spectral density of the velocity fluctuations of a free particle with mean speed $\bar{v} = 0.32c$, as computed over a time interval of 250 Compton periods. The coupling constant $\alpha = 0.05$. Note the prominence of the particle forcing frequency, $2\omega_c$, and the two dominant frequencies at $\omega = \sqrt{(\omega_c^2 \pm k^2c^2)}$, as arise due to the in-line oscillations at the frequency kc.

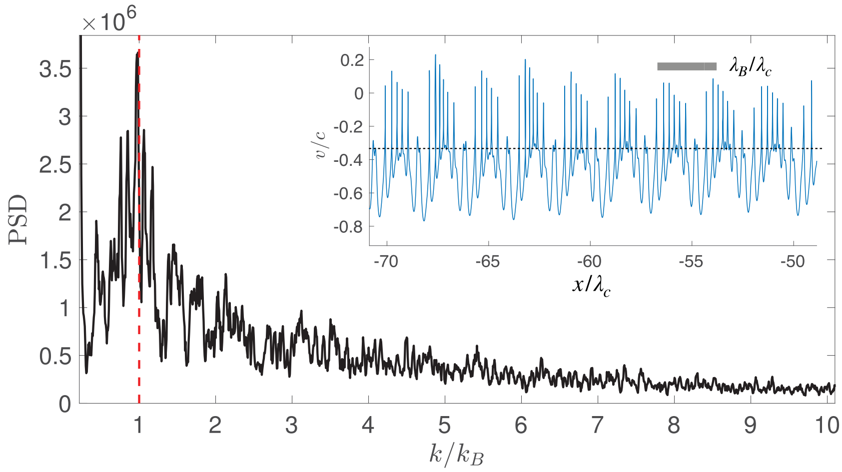

Figure 8. Spatial power spectral density of velocity fluctuations of a free particle with mean speed $\bar{v} = 0.32c$. The coupling constant $\alpha = 0.05$. The dashed line represents the de Broglie wavenumber, $k = \gamma m_0 \bar{v}/\hbar$. Inset: the deviations of the particle speed about the mean. The de Broglie wavelength $\lambda_B = \lambda_c/(\beta\gamma) = 2.96\lambda_c$ is shown for reference.

The speed modulations of a free particle with mean speed $\bar{v} = 0.32c$ are also characterized in Figure 8. A clear peak is apparent at the de Broglie wave number in the spatial power spectrum of these modulations. The in-line speed oscillations with the de Broglie wavelength are clearly apparent in the spatial dependence of the particle speed (inset of Figure 8). We conclude that the pilot-wave dynamics are characterized by in-line speed oscillations with the de Broglie wavelength and frequency ck. Our results thus suggest a dynamical interpretation of the

Table 1. Comparison of the hydrodynamic pilot-wave system and the current model

	Pilot-wave hydrodynamics	HQFT
Driving	Bath vibration	Zitterbewegung
Driving frequency	$2\,\omega_F$	$2\,\omega_c$
Particle vibration	Droplet bouncing	Zitterbewegung
Waves	Faraday waves	Matter waves
Wave natural frequency	ω_F	$\omega_c = \dfrac{m_0 c^2}{\hbar}$
Pilot wavelength	λ_F	$\lambda_B,\ \lambda_c$
Dispersion relation	$\omega_F^2 = g k_F + \dfrac{\sigma}{\rho} k_F^3$	$\omega^2 = \omega_c^2 + c^2 k^2$
Wave energetics	Drop GPE $\longleftrightarrow$ Surface energy	$m_0 c^2 \longleftrightarrow \hbar\omega$
Wave energy parameter	σ	$\hbar$
Mean velocity	Free walking speed: u_0	$\bar{v} = \hbar k/(\gamma m_0)$
Particle vibration length	Step size: u_0/ω_F	$\lambda_c = h/mc$
In-line oscillation frequency	$u_0 k_F$	$\omega_{\mathrm{mod}} = ck$
In-line oscillation length	λ_F	λ_B

relativistic quantum dispersion relation, $\omega^2 = \omega_c^2 + c^2 k^2$ [70]: the first term corresponds to the energy of internal vibration and wave generation, the second to the kinetic energy associated with in-line oscillations. Recall that in-line oscillations and structured random walks with the pilot wavelength are also prevalent in the hydrodynamic pilot-wave system [52, 53, 58], and have been shown to be responsible for the emergence of quantum-like statistics in a number of settings [16, 17, 22, 23, 51, 56, 57].

4. Conclusions

The similarities between the walker system and our extension of de Broglie's double-solution pilot-wave theory are summarized in Table 1. Matter waves play the role of capillary Faraday waves in the walker system. The Zitterbewegung of the quantum particle at twice the Compton frequency plays the role of the droplet bouncing at the Faraday frequency in generating the structured quasi-monochromatic pilot-wave field through a resonant interaction. Planck's constant $\hbar$ plays the role of surface tension in setting both the natural frequency of the quantum particle and its associated wave energy [7]. In both systems, a localized vibrating particle gives rise to the excitation of a quasi-monochromatic wave field centered on the particle, and resonance is achieved between the particle and wave. In both, the stationary state goes unstable to a dynamic state characterized by a particle surfing its quasi-monochromatic pilot wave. In both, the pilot-wave form depends explicitly on the particle's past. Both systems are characterized by two lengthscales, the scale of particle vibration and the wavelength of the guiding wave. Both systems are characterized by in-line speed oscillations with the wavelength of the pilot wave. Both are thus characterized by three timescales, the timescale of particle vibration, the timescale of particle translation and the timescale of in-line oscillations.

Our hydrodynamic quantum field theory has yielded a number of beguiling results in its depiction of the free quantum particle. We have shown that, just as a bouncing droplet destabilizes into a walking droplet at a critical vibrational acceleration, vibration of the free quantum particle gives rise to translation prescribed by $\bar{p} = \hbar k$, precisely as proposed by de Broglie. Through invocation of the physical analogy between our new hydrodynamic quantum field theory and the walking-droplet system, one can envisage how the former may give rise to statistical behavior consistent with that predicted by standard quantum mechanics in a number of settings. Specifically, the quasi-monochromatic pilot-wave will impose a dynamic constraint on the particle that will favor certain quantized states: chaotic motion may then prompt the intermittent switching between these states and the emergence of multimodal statistics. Moreover, modulations in the particle speed at the modulation frequency kc and with the de Broglie wavelength may provide a mechanism for generating quantum statistics in various quantum settings, including quantum corrals and Friedel oscillations [16, 17, 22, 23].

Our study has extended de Broglie's mechanics through inclusion of a mechanism for particle-induced pilot-wave generation based on that arising in pilot-wave hydrodynamics. Our study would seem to suggest the plausibility of a rational, trajectory-based theory for the dynamics of free quantum particles. The utility of our approach in rationalizing more complex quantum systems, including the motion of particles in response to applied forces and boundary interactions, is currently being examined. Particular attention will be given to coupling this dynamical picture to the Ensemble Interpretation of quantum mechanics [71], and assessing whether the statistical behavior of an ensemble of particles evolving according to the dynamics defined herein will be consistent with the standard formulation of quantum mechanics.

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References

[1] Y. Couder, S. Protier, E. Fort, A. Boudaoud, "Dynamical phenomena: walking and orbiting droplets", *Nature* **437** (2005), no. 7056, p. 208.

[2] D. J. Griffiths, D. F. Schroeter, *Introduction to Quantum Mechanics*, Cambridge University Press, 2018.

[3] L. De Broglie, "Recherches sur la théorie des quanta", PhD thesis, Migration-université en cours d'affectation, 1924.

[4] L. De Broglie, *Rapport au Veme Congres de Physique Solvay*, Gauthier-Villars, 1930.

[5] L. De Broglie, "The reinterpretation of wave mechanics", *Found. Phys.* **1** (1970), no. 1, p. 5-15.

[6] S. Protière, A. Boudaoud, Y. Couder, "Particle–wave association on a fluid interface", *J. Fluid Mech.* **554** (2006), p. 85-108.

[7] J. W. M. Bush, "Pilot-wave hydrodynamics", *Annu. Rev. Fluid Mech.* **47** (2015), p. 269-292.

[8] J. W. M. Bush, "The new wave of pilot-wave theory", *Phys. Today* **68** (2015), p. 47-53.

[9] J. W. M. Bush, Y. Couder, T. Gilet, P. A. Milewski, A. Nachbin, "Introduction to focus issue on hydrodynamic quantum analogs", *Chaos: An Interdiscip. J. Nonlinear Sci.* **28** (2018), no. 9, 096001.

[10] A. Eddi, E. Fort, F. Moisy, Y. Couder, "Unpredictable tunneling of a classical wave-particle association", *Phys. Rev. Lett.* **102** (2009), 240401.

[11] A. Nachbin, P. A. Milewski, J. W. M. Bush, "Tunneling with a hydrodynamic pilot-wave model", *Phys. Rev. Fluids* **2** (2017), no. 3, 034801.

[12] M. Hubert, M. Labousse, S. Perrard, "Self-propulsion and crossing statistics under random initial conditions", *Phys. Rev. E* **95** (2017), no. 6, 062607.

[13] E. Fort, A. Eddi, A. Boudaoud, J. Moukhtar, Y. Couder, "Path-memory induced quantization of classical orbits", *Proc. Natl Acad. Sci. USA* **107** (2010), no. 41, p. 17515-17520.

[14] D. M. Harris, J. W. M. Bush, "Droplets walking in a rotating frame: from quantized orbits to multimodal statistics", *J. Fluid Mech.* **739** (2014), p. 444-464.

[15] A. Eddi, J. Moukhtar, S. Perrard, E. Fort, Y. Couder, "Level splitting at macroscopic scale", *Phys. Rev. Lett.* **108** (2012), no. 26, 264503.

[16] T. Cristea-Platon, "Hydrodynamic analogues of quantum corrals and Friedel oscillations", PhD thesis, Massachusetts Institute of Technology, 2019.

[17] P. J. Sáenz, T. Cristea-Platon, J. W. Bush, "A hydrodynamic analog of Friedel oscillations", *Sci. Adv.* **6** (2020), no. 20, eaay9234.

[18] A. U. Oza, D. M. Harris, R. R. Rosales, J. W. M. Bush, "Pilot-wave dynamics in a rotating frame: on the emergence of orbital quantization", *J. Fluid Mech.* **744** (2014), p. 404-429.

[19] S. Perrard, M. Labousse, E. Fort, Y. Couder, "Chaos driven by interfering memory", *Phys. Rev. Lett.* **113** (2014), no. 10, 104101.

[20] M. Labousse, S. Perrard, Y. Couder, E. Fort, "Build-up of macroscopic eigenstates in a memory-based constrained system", *New J. Phys.* **16** (2014), no. 11, 113027.

[21] M. Durey, P. A. Milewski, "Faraday wave-droplet dynamics: discrete-time analysis", *J. Fluid Mech.* **821** (2017), p. 296-329.

[22] D. M. Harris, J. Moukhtar, E. Fort, Y. Couder, J. W. M. Bush, "Wavelike statistics from pilot-wave dynamics in a circular corral", *Phys. Rev. E* **88** (2013), no. 1, 011001.

[23] P. J. Sáenz, T. Cristea-Platon, J. W. M. Bush, "Statistical projection effects in a hydrodynamic pilot-wave system", *Nat. Phys.* **14** (2018), no. 3, p. 315.

[24] Y. Couder, E. Fort, "Single-particle diffraction and interference at a macroscopic scale", *Phys. Rev. Lett.* **97** (2006), no. 15, 154101.

[25] A. Andersen, J. Madsen, C. Reichelt, S. R. Ahl, B. Lautrup, C. Ellegaard, M. T. Levinsen, T. Bohr, "Double-slit experiment with single wave-driven particles and its relation to quantum mechanics", *Phys. Rev. E* **92** (2015), no. 1, 013006.

[26] M. Rode, J. Madsen, A. Andersen, "Wave fields in double-slit experiments with wave-driven droplets", *Phys. Rev. Fluids* **4** (2019), no. 10, 104801.

[27] G. Pucci, D. M. Harris, L. M. Faria, J. W. M. Bush, "Walking droplets interacting with single and double slits", *J. Fluid Mech.* **835** (2018), p. 1136-1156.

[28] M. Labousse, S. Perrard, Y. Couder, E. Fort, "Self-attraction into spinning eigenstates of a mobile wave source by its emission back-reaction", *Phys. Rev. E* **94** (2016), 042224.

[29] A. U. Oza, R. R. Rosales, J. W. M. Bush, "Hydrodynamic spin states", *Chaos: An Interdiscip. J. Nonlinear Sci.* **28** (2018), no. 9, 096106.

[30] R. Valani, A. C. Slim, "Pilot-wave dynamics of two identical, in-phase bouncing droplets", *Chaos* **28** (2018), 096114.

[31] M. Durey, S. E. Turton, J. W. M. Bush, "Speed oscillations in classical pilot-wave dynamics", *Proc. R. Soc. Lond. A* **476** (2020), no. 2239, 20190884.

[32] E. Fort, Y. Couder, "Trajectory eigenmodes of an orbiting wave source", *Europhys. Lett.* **102** (2013), no. 1, 16005.

[33] C. Borghesi, "Equivalent quantum equations in a system inspired by bouncing droplets experiments", *Found. Phys.* **47** (2017), p. 933-958.

[34] T. Shinbrot, "Dynamic pilot wave bound states", *Chaos: An Interdiscip. J. Nonlinear Sci.* **29** (2019), no. 11, 113124.

[35] G. Grössing, S. Fussy, J. M. Pascasio, H. Schwabl, "Implications of a deeper level explanation of the de Broglie–Bohm version of quantum mechanics", *Quantum Stud. Math. Found.* **2** (2015), p. 133-140.

[36] A. M. Cetto, L. de la Peña, A. Valdés-Hernández, "Specificity of the Schrödinger equation", *Quantum Stud.: Math. Found.* **2** (2015), p. 275-287.

[37] M. Hatifi, R. Willox, S. Colin, T. Durt, "Bouncing oil droplets, de Broglie's quantum thermostat, and convergence to equilibrium", *Entropy* **20** (2018), no. 10, p. 780.

[38] J. Walleczek, G. Grössing, P. Pylkkänen, B. Hiley, "Emergent quantum mechanics: David Bohm Centennial Perspectives", *Entropy* **21** (2019), p. 113.

[39] D. Bohm, "A suggested interpretation of the quantum theory in terms of hidden variables, I", *Phys. Rev.* **85** (1952a), p. 66-179.

[40] D. Bohm, "A suggested interpretation of the quantum theory in terms of hidden variables, II", *Phys. Rev.* **85** (1952b), p. 180-193.

[41] P. R. Holland, *The Quantum Theory of Motion: An Account of the de Broglie-Bohm Causal Interpretation of Quantum Mechanics*, Cambridge University Press, 1995.

[42] J. Miles, D. Henderson, "Parametrically forced surface waves", *Annu. Rev. Fluid Mech.* **22** (1990), no. 1, p. 143-165.

[43] P. A. Milewski, C. A. Galeano-Rios, A. Nachbin, J. W. M. Bush, "Faraday pilot-wave dynamics: modelling and computation", *J. Fluid Mech.* **778** (2015), p. 361-388.

[44] S. Turton, M. Couchman, J. W. Bush, "A review of the theoretical modeling of walking droplets: toward a generalized pilot-wave framework", *Chaos: An Interdiscip. J. Nonlinear Sci.* **28** (2018), no. 9, 096111.

[45] J. Moláček, J. W. M. Bush, "Drops walking on a vibrating bath: towards a hydrodynamic pilot-wave theory", *J. Fluid Mech.* **727** (2013a), p. 612-647.

[46] J. Moláček, J. W. M. Bush, "Drops bouncing on a vibrating bath", *J. Fluid Mech.* **727** (2013b), p. 582-611.

[47] D. M. Harris, J. W. M. Bush, "The pilot-wave dynamics of walking droplets", *Phys. Fluids* **25** (2013), no. 9, 091112.

[48] P.-T. Brun, D. M. Harris, V. Prost, J. Quintela, J. W. M. Bush, "Shedding light on pilot-wave phenomena", *Phys. Rev. Fluids* **1** (2016), no. 5, 050510.

[49] A. U. Oza, R. R. Rosales, J. W. M. Bush, "A trajectory equation for walking droplets: hydrodynamic pilot-wave theory", *J. Fluid Mech.* **737** (2013), p. 552-570.

[50] A. Eddi, E. Sultan, J. Moukhtar, E. Fort, M. Rossi, Y. Couder, "Information stored in Faraday waves: the origin of a path memory", *J. Fluid Mech.* **674** (2011), p. 433-463.

[51] M. Durey, P. A. Milewski, J. W. M. Bush, "Dynamics, emergent statistics, and the mean-pilot-wave potential of walking droplets", *Chaos* **28** (2018), no. 9, 096108.

[52] Ø. Wind-Willassen, J. Moláček, D. M. Harris, J. W. M. Bush, "Exotic states of bouncing and walking droplets", *Phys. Fluids* **25** (2013), no. 8, 082002.

[53] V. Bacot, S. Perrard, M. Labousse, Y. Couder, E. Fort, "Multistable free states of an active particle from a coherent memory dynamics", *Phys. Rev. Lett.* **122** (2019), no. 10, 104303.

[54] J. Friedel, "Electronic structure of primary solid solutions in metals", *Adv. Phys.* **3** (1954), no. 12, p. 446-507.

[55] K. Kanisawa, M. Butcher, H. Yamaguchi, Y. Hirayama, "Imaging of Friedel oscillation patterns of two-dimensionally accumulated electrons at epitaxially grown InAs (111) A surfaces", *Phys. Rev. Lett.* **86** (2001), no. 15, p. 3384.

[56] T. Gilet, "Dynamics and statistics of wave-particle interactions in a confined geometry", *Phys. Rev. E* **90** (2014), no. 5, 052917.

[57] T. Gilet, "Quantumlike statistics of deterministic wave-particle interactions in a circular cavity", *Phys. Rev. E* **93** (2016), no. 4, 042202.

[58] M. Hubert, M. Labousse, S. Perrard, M. Labousse, N. Vandewalle, Y. Couder, "Tunable bimodal explorations of space from memory-driven deterministic dynamics", *Phys. Rev. E* **100** (2019), no. 032201.

[59] E. Schrödinger, "About the force-free motion in relativistic quantum mechanics", *Session Phys. Math.* **31** (1930), p. 418.

[60] D. Hestenes, "The zitterbewegung interpretation of quantum mechanics", *Found. Phys.* **20** (1990), no. 10, p. 1213-1232.

[61] L. De la Peña, A. M. Cetto, A. Valdés-Hernández, "The emerging quantum", in *The Physics behind Quantum Mechanics*, Springer International Publishing, Cham, 2015.

[62] C. Qu, C. Hamner, M. Gong, C. Zhang, P. Engels, "Observation of Zitterbewegung in a spin-orbit-coupled Bose-Einstein condensate", *Phys. Rev. A* **88** (2013), no. 2, 021604.

[63] R. Gerritsma, G. Kirchmair, F. Zähringer, E. Solano, R. Blatt, C. Roos, "Quantum simulation of the Dirac equation", *Nature* **463** (2010), no. 7277, p. 68.

[64] L. De Broglie, "La mécanique ondulatoire et la structure atomique de la matière et du rayonnement", *J. Phys. Radium* **8** (1927a), no. 5, p. 225-241.

[65] L. De Broglie, "L'univers à cinq dimensions et la mécanique ondulatoire", *J. Phys. Radium* **8** (1927b), no. 2, p. 65-73.

[66] P. W. Higgs, "Broken symmetries and the masses of gauge bosons", *Phys. Rev. Lett.* **13** (1964), no. 16, p. 508.

[67] D. Greenspan, *Particle Modeling*, Springer Science & Business Media, 2013.

[68] W. Greiner *et al.*, *Relativistic Quantum Mechanics, Vol. 3*, Springer, 1990.

[69] M. Durey, J. W. Bush, "Hydrodynamic quantum field theory: the onset of particle motion and the form of the pilot wave", *Front. Phys.* **8** (2020), p. 300.

[70] J. J. Sakurai, E. D. Commins, *Modern Quantum Mechanics*, revised ed., 1995.

[71] L. E. Ballentine, "Ensembles in quantum mechanics", in *Compendium of Quantum Physics* (D. Greenberger, K. Hentschel, F. Weinert, eds.), Springer, 2009, p. 199-201.