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Research article

Revisiting intrinsic curve type strength criteria for yield design analyses

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Abstract. Intrinsic curve type strength criteria for isotropic materials rely on the assumption that failure of a material element only depends on the values of the major and minor principal stresses exerted on it. The intrinsic curve is classically defined as the envelope of limit Mohr circles corresponding to yielding. It can also be considered as the envelope of a family of Coulomb's criteria depending on one scalar parameter. This alternative definition is exploited for providing the expressions for the maximum resisting work in the general case of a material with an intrinsic curve type yield criterion, with or without tension cutoff, both for strain rate tensors and velocity discontinuities. The application of these results to yield design analysis problems is discussed from various viewpoints, with one additional contribution to determining the critical height of a vertical cut.

Keywords. Intrinsic curve, Yield criterion, Maximum resisting work, Yield design analyses, Vertical cut.

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1. Yield criteria of the intrinsic curve type

1.1. *Definition and general results*

After the experiments reported by Tresca in a series of Memoirs to the French Academy of Sciences (1864–1870) (cf. [1]), where the Tresca yield criterion was introduced together with a first concept of a plastic flow rule, "*Mohr* [2] *proposed a criterion of yielding to the effect that permanent deformation occurred only when a stress circle touched a certain curve in the* (σ,τ) *plane*" [3] (Figure 1).

The original French terminology "*courbe intrinsèque*" is referred to in Caquot and Kerisel's famous textbook [4] as having been introduced more than 20 years earlier (than 1949) by one of the Authors[1], and Figure 2 shows how the concept of an *intrinsic curve* appears in the lecture notes of the course delivered by Caquot at the French *École nationale supérieure des mines* [6].

As a matter of fact, the basic concept of an intrinsic curve originates from experimental results obtained with granular materials and other materials commonly used in civil engineering and

[1] "*C'est cette courbe à laquelle l'un de nous a donné il y a plus de 20 ans le nom de courbe intrinsèque du matériau dans les conditions des expériences*". This is the curve which, one of us, more than 20 years ago, named as the intrinsic curve of the material within the considered experimental conditions. Viz. [5].

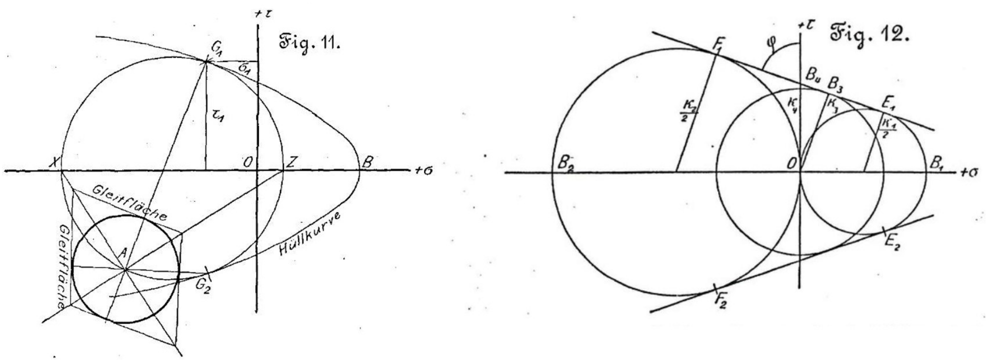

Figure 1. The *Hüllkurve*, (Hull curve; Ref. [2]). (By courtesy of Professor Samuel Forest.)

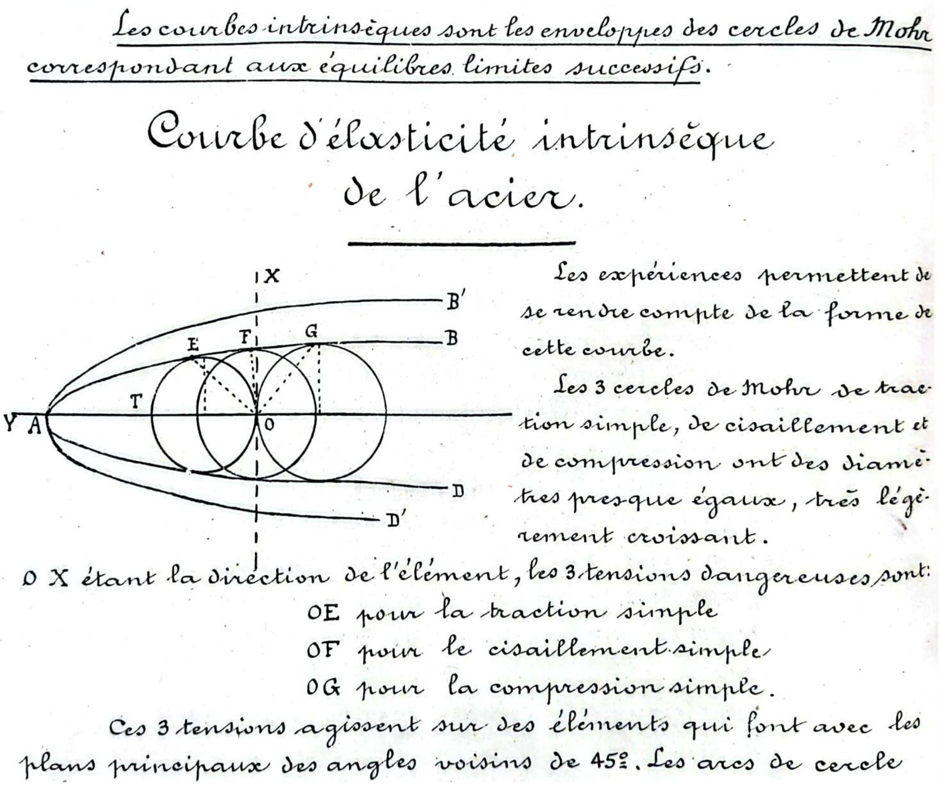

Figure 2. The intrinsic curve as it appears in Caquot [6]. (By courtesy of Professor Samuel Forest.)

considered as isotropic, which indicate that the Mohr circles[2] corresponding to yielding admit an envelope in the (σ, τ) plane of the Mohr stress representation.

Assuming this result to be valid for all types of experiments performed on the material under consideration implies that its domain of resistance is defined by a criterion that only involves the normal and tangential components, σ and τ, of the stress vector acting on any facet, whatever its orientation.

[2]The Mohr circle with maximum diameter is classically called *the* Mohr circle.

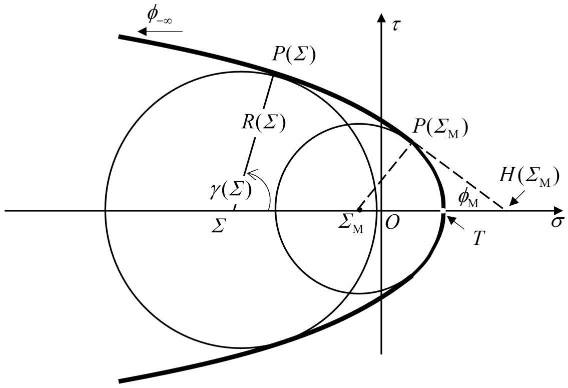

Figure 3. Intrinsic curve and ultimate limit Mohr circle.

From a mathematical viewpoint, the yield criterion is governed by the values of the major and minor principal stresses, independently of the value of the intermediate one. It follows that, the principal stresses being ordered according to $\sigma_3 \leq \sigma_2 \leq \sigma_1$ [3], the yield criterion can be written as a relationship between σ_1 and σ_3 only

$$(\sigma_1 - \sigma_3) - g(\sigma_1 + \sigma_3) \leq 0 \tag{1.1}$$

or, equivalently, between $R = (\sigma_1 - \sigma_3)/2$ and $\Sigma = (\sigma_1 + \sigma_3)/2$, which are, respectively, the radius of *the* Mohr circle and the abscissa of its centre in the plane of the Mohr stress representation, in the form

$$R \leq R(\Sigma), \tag{1.2}$$

where $R(\Sigma) = g(2\Sigma)/2$ can be determined experimentally by means of classical triaxial tests for instance.

Equality in (1.2) defines any *limit Mohr circle* through the positive scalar function R that relates its radius to its centre abscissa

$$R = R(\Sigma), \tag{1.3}$$

with $-\infty < \Sigma \leq \Sigma_M$, where Σ_M stands for the abscissa of the centre of the *ultimate* limit Mohr circle in the positive direction of the σ axis. It is worth noting that this ultimate limit Mohr circle defines T, the maximum tensile normal stress that can be sustained by the material (Figure 3), in the form

$$T = \Sigma_M + R(\Sigma_M) \geq 0. \tag{1.4}$$

From its very definition, the intrinsic curve is the envelope of the limit Mohr circles defined by (1.3); it consists of two arcs that are symmetric about the σ axis. In the particular case when $R(\Sigma_M) = 0$ these arcs meet on the σ axis, which corresponds to an intrinsic curve "with a sharp summit" in the same way as the Coulomb yield criterion.

The existence of such a real non-degenerated envelope implies that $R(\Sigma)$ must be a positive, decreasing function of Σ, $R'(\Sigma) \leq 0$, with the additional condition [3, pp. 294–300], [7, pp. 23–30].

$$-R'(\Sigma) = \left|\frac{\mathrm{d}R}{\mathrm{d}\Sigma}\right| < 1. \tag{1.5}$$

The contact point of a limit Mohr circle with the intrinsic curve is the point $P(\Sigma)$ such that (Figure 3)

$$\cos\gamma(\Sigma) = -\mathrm{d}R/\mathrm{d}\Sigma = -R'(\Sigma). \tag{1.6}$$

[3]Tensile stresses are counted positive.

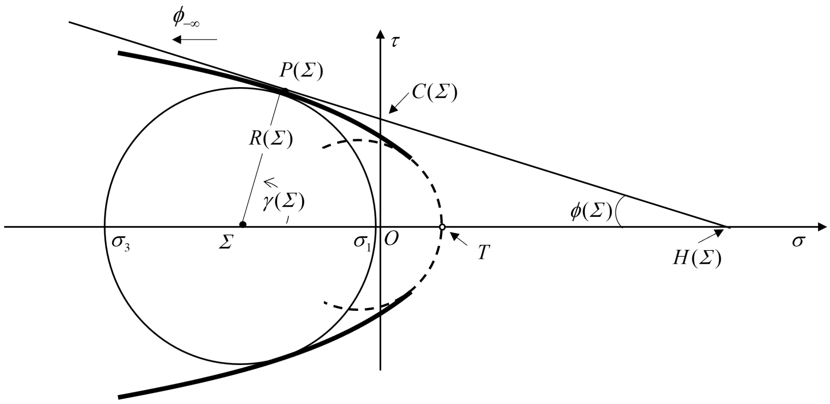

Figure 4. Intrinsic curve and Coulomb intrinsic curve tangent at point $P(\Sigma)$.

Assuming $R(\Sigma)$ to be a concave function of its argument, i.e., $R''(\Sigma) \leq 0$, implies the convexity of the domain delimited by (both arcs of) the intrinsic curve *and the ultimate limit Mohr circle* in the (σ, τ) plane, and vice versa. This domain comes out as the *domain of resistance* of the isotropic material under concern in the (σ, τ) plane of the Mohr stress representation. It is usually modelled as unbounded for $\Sigma \to -\infty$, as a description of the behaviour of the material within the range of stress states that can practically be exerted.

With the tangential and normal stresses corresponding to point $P(\Sigma)$ being denoted by $\tau(\Sigma)$ and $\sigma(\Sigma)$ respectively, both arcs of the intrinsic curve are described in a parametric form by (1.6) and

$$\begin{cases} |\tau(\Sigma)| = R(\Sigma)\sin\gamma(\Sigma) \\ \sigma(\Sigma) = \Sigma + R(\Sigma)\cos\gamma(\Sigma) = \Sigma - R(\Sigma)R'(\Sigma). \end{cases} \tag{1.7}$$

At point $P(\Sigma)$, the tangent to the intrinsic curve is defined by

$$\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}(\Sigma) = -\cot\gamma(\Sigma). \tag{1.8}$$

It follows that, at point $P(\Sigma)$, the intrinsic curve is tangent to the Coulomb intrinsic curve defined by parameters $\phi(\Sigma) = \pi/2 - \gamma(\Sigma) > 0$ and $C(\Sigma)$, as indicated in Figure 4, with

$$\begin{cases} \sin\phi(\Sigma) = \cos\gamma(\Sigma) = -R'(\Sigma) \\ C(\Sigma) = \dfrac{R(\Sigma)}{\cos\phi(\Sigma)} + \Sigma\tan\phi(\Sigma), \end{cases} \tag{1.9}$$

which corresponds to a cohesion pressure $H(\Sigma)$

$$H(\Sigma) = C(\Sigma)\cot\phi(\Sigma) = \frac{R(\Sigma)}{\sin\phi(\Sigma)} + \Sigma = -\frac{R(\Sigma)}{R'(\Sigma)} + \Sigma \tag{1.10}$$

and Coulomb's criterion being written as

$$|\tau(\Sigma)| - C(\Sigma) + \sigma(\Sigma)\tan\phi(\Sigma) \leq 0 \tag{1.11}$$

or, equivalently,

$$|\tau(\Sigma)| - (H(\Sigma) - \sigma(\Sigma))\tan\phi(\Sigma) \leq 0. \tag{1.12}$$

Positivity of $C(\Sigma)$ and $H(\Sigma)$ implies

$$R(\Sigma) - \Sigma R'(\Sigma) > 0. \tag{1.13}$$

This condition being fulfilled over the range $-\infty < \Sigma \le \Sigma_{\rm M}$ ensures that the zero-stress state lies inside the domain of resistance[4].

1.2. *Parametric description*

The arcs of the intrinsic curve described in (1.7) constitute the envelope of the family of Coulomb's intrinsic curves, the straight lines defined by (1.11) or (1.12), depending on Σ as a parameter, under the condition that

$$\begin{gathered} R(\Sigma) = (H(\Sigma) - \Sigma)\sin\phi(\Sigma) \text{ shall be a positive, decreasing, concave function,} \\ \text{with } R(\Sigma_M) - \Sigma_M R'(\Sigma_M) > 0. \end{gathered} \tag{1.14}$$

Let $\phi_{-\infty} > 0$ denote the value of $\phi(\Sigma)$ when $\Sigma \to -\infty$, describing the asymptotic direction of the intrinsic curve. Hence, as a consequence of (1.14), the parametric definition

$$\begin{cases} \Sigma \mapsto H(\Sigma) \\ \Sigma \mapsto \phi(\Sigma) \end{cases} \tag{1.15}$$

can be solved as a monotonous decreasing function

$$H = H(\phi) \tag{1.16}$$

over the range $0 < \phi_{-\infty} \le \phi \le \phi_{\rm M}$. It follows that the arcs of the intrinsic curve can also be defined as the *envelope of the straight lines*

$$0 < \phi_{-\infty} \le \phi \le \phi_{\rm M}, \qquad |\tau| - (H(\phi) - \sigma)\tan\phi = 0. \tag{1.17}$$

The contact point of the ultimate limit Mohr circle with the intrinsic curve is $P(\Sigma_M)$, as shown in Figure 3, with centre Σ_M such that (1.4) can now be written as

$$T - \Sigma_{\rm M} = R(\Sigma_M) = (H(\Sigma_{\rm M}) - \Sigma_{\rm M})\sin\phi(\Sigma_{\rm M}). \tag{1.18}$$

As a result, the domain of resistance in the (σ, τ) plane is defined by

$$\begin{cases} \displaystyle\operatorname*{Max}_{\phi_{-\infty} < \phi \le \phi(\Sigma_{\rm M})} \{|\tau| - (H(\phi) - \sigma)\tan\phi\} \le 0 \\ \displaystyle\operatorname*{Max}_{\phi(\Sigma_{\rm M}) < \phi \le \pi/2} \{|\tau| - (T - \Sigma_{\rm M})\cos\phi\} \le 0 \end{cases} \tag{1.19}$$

which can be interpreted as follows:

- the first line in (1.19) describes the domain delimited by both arcs of the intrinsic curve as the intersection of the family of Coulomb's domains depending on parameter ϕ, with $\phi_{-\infty} < \phi \le \phi(\Sigma_{\rm M})$, these arcs being generated as the envelope of the corresponding Coulomb intrinsic curves;
- the second line, where $\phi(\Sigma_{\rm M}) \le \phi \le \pi/2$, describes the tangential generation of the circular cap of the domain of resistance in the (σ, τ) plane with the limitation imposed by the maximum tensile normal stress.

This domain is obviously convex.

In the Haigh–Westergaard stress space (cf. [8]), with coordinates the non-ordered principal stresses $\sigma_i, \sigma_j, \sigma_k$, the domain of resistance characterised by (1.1) and (1.4) is defined by

$$f(\underline{\underline{\sigma}}) = \operatorname{Max}\{(\sigma_i - \sigma_j) - g(\sigma_i + \sigma_j), \sigma_i - T \mid i, j = 1, 2, 3\} \le 0. \tag{1.20}$$

Consistently with (1.19), it is delimited by the envelope of the corresponding Coulomb's domains depending on parameter ϕ, with $\phi_{-\infty} < \phi \le \phi(\Sigma_{\rm M})$, and the limitation set on the

[4] Since $R(\Sigma)$ is a concave function, it is sufficient that $R(\Sigma_M) - \Sigma_M R'(\Sigma_M) > 0$.

principal stresses by the maximum tensile normal stress $T = \Sigma_{\mathrm{M}} + R(\Sigma_{\mathrm{M}})$. Its definition can be written as

$$\begin{cases} T = \Sigma_{\mathrm{M}} + R(\Sigma_{\mathrm{M}}), \\ \phi_{-\infty} < \phi \leq \phi(\Sigma_{\mathrm{M}}), \\ \mathrm{Max}\{\sigma_i(1+\sin\phi) - \sigma_j(1-\sin\phi) - 2H(\phi)\sin\phi \mid i,j = 1,2,3\} \leq 0 \\ \mathrm{Max}\{\sigma_i - T \mid i = 1,2,3\} \leq 0. \end{cases} \tag{1.21}$$

Incidentally, it is worth noting that, from this definition, any Mohr circle such that

$$\sigma_1 = T, \quad T - 2g(\Sigma_M) < \sigma_3 < \sigma_1 \tag{1.22}$$

comes out as a *limit* Mohr circle, which is physically consistent.

2. Intrinsic curve type strength criteria: maximum resisting work

Except for the case of Tresca's criterion, intrinsic curve type strength criteria are not associated with any concept of a constitutive law, or plastic flow rule, regarding yielding when the criterion is saturated. Consistently, they are only considered in yield design analyses, where the mathematical compatibility between the equilibrium equations for the structure under concern and the resistance of its constituent material shall be checked, within the framework of the theory of Yield design [9,10], through the implementation of static approaches and, more usually, kinematic approaches. The latter ones call for the definition of the concept of *maximum resisting work*, whose mathematical expression is only derived from the yield criterion itself.

2.1. *Definition*

For a material whose domain of resistance is defined by a convex function f of the stress tensor $\underline{\underline{\sigma}}$ in the form

$$f(\underline{\underline{\sigma}}) \leq 0, \tag{2.1}$$

given a strain rate tensor $\underline{\underline{d}}$, it is recalled that the *maximum resisting work* that can be developed in $\underline{\underline{d}}$ by a stress tensor $\underline{\underline{\sigma}}$ abiding by (2.1) is the volume density $\pi(\underline{\underline{d}})$ defined through

$$\pi(\underline{\underline{d}}) = \mathrm{Sup}\{\underline{\underline{\sigma}} : \underline{\underline{d}} \mid f(\underline{\underline{\sigma}}) \leq 0\}, \tag{2.2}$$

where the notation " : " stands for the doubly contracted product of two 2nd Rank tensors i.e., $\underline{\underline{\sigma}} : \underline{\underline{d}} = \sigma_{ij} d_{ji}$ (with summation on repeated indices).

In the same way, in the case of a discontinuity $\underline{V}$, of the velocity field $\underline{U}$, across a jump surface with outward normal $\underline{n}$, the *maximum resisting work* that can be developed by a stress tensor $\underline{\underline{\sigma}}$ abiding by (2.1) is the surface density $\pi(\underline{V}, \underline{n})$ defined through

$$\pi(\underline{V}, \underline{n}) = \mathrm{Sup}\{\underline{V} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \mid f(\underline{\underline{\sigma}}) \leq 0\}. \tag{2.3}$$

From a mathematical viewpoint, within the context of convex analysis, function $\pi(\underline{\underline{d}})$ is the *support function* of the convex domain defined by $f(\underline{\underline{\sigma}}) \leq 0$. It is obtained as the value of the product $\underline{\underline{\sigma}} : \underline{\underline{d}}$ at any point $\underline{\underline{\sigma}} * (\underline{\underline{d}})$ on the convex yield surface defined by $f(\underline{\underline{\sigma}}) = 0$, where $\underline{\underline{d}}$ is oriented along an outward normal:

$$\pi(\underline{\underline{d}}) = \underline{\underline{\sigma}} * (\underline{\underline{d}}) : \underline{\underline{d}}. \tag{2.4}$$

As a result, $\pi(\underline{\underline{d}})$ is finite, except when $\underline{\underline{d}}$ is oriented in a direction along which the domain $f(\underline{\underline{\sigma}}) \leq 0$ is not bounded, in which case $\pi(\underline{\underline{d}})$ is infinite.

2.2. *Maximum resisting work for a material with a Coulomb yield criterion with tension cutoff*

Coulomb criterion with tension cutoff is the particular case of a criterion evoked in Section 1, when the classical Coulomb's criterion is capped by an ultimate limit Mohr circle with

$$\sigma_1 = T \geq 0, \Sigma_{\mathrm{M}} = (T - C\cos\phi)/(1-\sin\phi). \tag{2.5}$$

In terms of non-ordered principal stresses, this criterion can be written as

$$f(\underline{\underline{\sigma}}) = \mathrm{Max}\{\sigma_i(1+\sin\phi) - \sigma_j(1-\sin\phi) - 2C\cos\phi, \sigma_i - T \mid i,j = 1,2,3\} \leq 0. \tag{2.6}$$

Referring to Section 1, the corresponding domain of resistance in the (σ, τ) plane can also be described through (1.19) as

$$\begin{cases} \mathrm{Max}\{|\tau| - C + \sigma\tan\phi\} \leq 0 \\ \underset{\phi<\alpha\leq\pi/2}{\mathrm{Max}}\{|\tau| - (T - \Sigma_{\mathrm{M}})\cos\alpha\} \leq 0. \end{cases} \tag{2.7}$$

The expressions of the maximum resisting work for this criterion can be found in Salençon [10] in the forms

$$\begin{cases} \underset{\phi,H,T}{\pi}(\underline{\underline{d}}) = +\infty \quad \text{if } \mathrm{tr}\,\underline{\underline{d}} < (|d_1| + |d_2| + |d_3|)\sin\phi \\ \underset{\phi,H,T}{\pi}(\underline{\underline{d}}) = C(|d_1| + |d_2| + |d_3| - \mathrm{tr}\,\underline{\underline{d}})\tan(\pi/4 + \phi/2) \\ \qquad + \dfrac{T}{1-\sin\phi}(\mathrm{tr}\,\underline{\underline{d}} - (|d_1| + |d_2| + |d_3|)\sin\phi) \quad \text{if } \mathrm{tr}\,\underline{\underline{d}} \geq (|d_1| + |d_2| + |d_3|)\sin\phi \end{cases} \tag{2.8}$$

and

$$\begin{cases} \underset{\phi,H,T}{\pi}(\underline{V}, \underline{n}) = +\infty \quad \text{if } \underline{V}\cdot\underline{n} < |\underline{V}|\sin\phi \\ \underset{\phi,H,T}{\pi}(\underline{V}, \underline{n}) = C(|\underline{V}| - \underline{V}\cdot\underline{n})\tan(\pi/4 + \phi/2) \\ \qquad + \dfrac{T}{1-\sin\phi}(\underline{V}\cdot\underline{n} - |\underline{V}|\sin\phi) \quad \text{if } \underline{V}\cdot\underline{n} \geq |\underline{V}|\sin\phi. \end{cases} \tag{2.9}$$

It is worth noting that Equations (2.9) illustrate the result that, from definition (2.3) and the characteristic property of *the* Mohr circle, $\underset{\phi,H,T}{\pi}(\underline{V}, \underline{n})$ is the *support function* of the convex domain delimited, in the Mohr stress representation plane (σ, τ), by *the Coulomb intrinsic curve and the ultimate limit Mohr circle.*

2.3. *Maximum resisting work in the case of an intrinsic curve type yield criterion*

As recalled in Section 2.1, for any given value of $\underline{\underline{d}}$, determining the maximum resisting work $\underset{\mathrm{IC}}{\pi}(\underline{\underline{d}})$ amounts to looking for a point $\underline{\underline{\sigma}}*(\underline{\underline{d}})$ on the boundary of the domain of resistance defined by (1.21), where $\underline{\underline{d}}$ is oriented along an outward normal. If such a point exists, then

$$\underset{\mathrm{IC}}{\pi}(\underline{\underline{d}}) = \underline{\underline{\sigma}}*(\underline{\underline{d}}) : \underline{\underline{d}}. \tag{2.10}$$

Since $\underline{\underline{\sigma}}*(\underline{\underline{d}})$ lies on the boundary of the domain of resistance, it determines the Coulomb yield criterion with tension cutoff defined by ϕ and $H(\phi)$, that is tangent to the domain of resistance at this point. As a result, $\underset{\mathrm{IC}}{\pi}(\underline{\underline{d}})$, the value of the maximum resisting work for strain rate tensor $\underline{\underline{d}}$, is equal to the value of $\underset{\phi,H}{\pi}(\underline{\underline{d}})$ for the tangent Coulomb criterion. Then, from (2.8), we derive that the corresponding expression of $\underset{\mathrm{IC}}{\pi}(\underline{\underline{d}})$ is governed by the value of $\mathrm{tr}\,\underline{\underline{d}}/\sum_i |d_i|$ in the following way:

$$\begin{cases} \underset{\mathrm{IC}}{\pi}(\underline{\underline{d}}) = +\infty \quad \text{if } \phi = \sin^{-1}\left(\mathrm{tr}\,\underline{\underline{d}}/\sum_i |d_i|\right) < \phi_{-\infty} \\ \underset{\mathrm{IC}}{\pi}(\underline{\underline{d}}) = \underset{\phi,H(\phi)}{\pi}(\underline{\underline{d}}) = H(\phi)\,\mathrm{tr}\,\underline{\underline{d}} \quad \text{if } \phi_{-\infty} \leq \phi = \sin^{-1}\left(\mathrm{tr}\,\underline{\underline{d}}/\sum_i |d_i|\right) \leq \phi_{\mathrm{M}}. \end{cases} \tag{2.11}$$

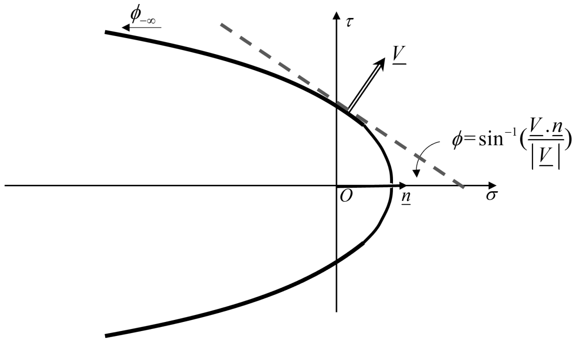

Figure 5. Maximum resisting work for a velocity jump.

In the same way, for a velocity jump $\underline{V}$, $\underset{\mathrm{IC}}{\pi}(\underline{V},\underline{n})$ is governed by the value of $\underline{V}\cdot\underline{n}/|\underline{V}|$:

$$\begin{cases}\underset{\mathrm{IC}}{\pi}(\underline{V},\underline{n}) = +\infty \quad \text{if } \phi = \sin^{-1}(\underline{V}\cdot\underline{n}/|\underline{V}|) < \phi_{-\infty}\\ \underset{\mathrm{IC}}{\pi}(\underline{V},\underline{n}) = \underset{\phi,H(\phi)}{\pi}(\underline{V},\underline{n}) = H(\phi)\underline{V}\cdot\underline{n} \quad \text{if } \phi_{-\infty} \le \phi = \sin^{-1}(\underline{V}\cdot\underline{n}/|\underline{V}|) \le \phi_{\mathrm{M}}.\end{cases} \tag{2.12}$$

When $\underline{\underline{d}}$ is oriented along an outward normal to the ultimate limit Mohr circle, i.e., when

$$\phi_{\mathrm{M}} \le \sin^{-1}\left(\mathrm{tr}\,\underline{\underline{d}}/\sum_i |d_i|\right) \le \pi/2, \tag{2.13}$$

the expressions of $\underset{\mathrm{IC}}{\pi}(\underline{\underline{d}})$ are given by (2.8)–(2.9) with $\phi = \phi_{\mathrm{M}}$.

Finally, with $T = \Sigma_{\mathrm{M}} + g(\Sigma_{\mathrm{M}}) \ge 0$, we get

$$\begin{cases}\underset{\mathrm{IC},T}{\pi}(\underline{\underline{d}}) = \underset{\phi,H,T}{\pi}(\underline{\underline{d}}) + \infty \quad \text{if } \phi = \sin^{-1}\left(\mathrm{tr}\,\underline{\underline{d}}/\sum_i |d_i|\right) < \phi_{-\infty}\\ \underset{\mathrm{IC},T}{\pi}(\underline{\underline{d}}) = \underset{\phi,H(\phi)}{\pi}(\underline{\underline{d}}) = H(\phi)\mathrm{tr}\,\underline{\underline{d}} \quad \text{if } \phi_{-\infty} \le \phi = \sin^{-1}\left(\mathrm{tr}\,\underline{\underline{d}}/\sum_i |d_i|\right) \le \phi_{\mathrm{M}}\\ \underset{\mathrm{IC},T}{\pi}(\underline{\underline{d}}) = \underset{\phi_{\mathrm{M}},H_{\mathrm{M}},T}{\pi}(\underline{\underline{d}}) = C_{\mathrm{M}}(|d_1|+|d_2|+|d_3| - \mathrm{tr}\,\underline{\underline{d}})\tan(\pi/4+\phi_{\mathrm{M}}/2)\\ \qquad + \dfrac{T}{1-\sin\phi_{\mathrm{M}}}(\mathrm{tr}\,\underline{\underline{d}} - (|d_1|+|d_2|+|d_3|)\sin\phi_{\mathrm{M}}) \quad \text{if } \mathrm{tr}\,\underline{\underline{d}} \ge (|d_1|+|d_2|+|d_3|)\sin\phi_{\mathrm{M}}\end{cases} \tag{2.14}$$

and for a velocity jump,

$$\begin{cases}\underset{\mathrm{IC},T}{\pi}(\underline{V},\underline{n}) = +\infty \quad \text{if } \phi = \sin^{-1}(\underline{V}\cdot\underline{n}/|\underline{V}|) < \phi_{-\infty}\\ \underset{\mathrm{IC},T}{\pi}(\underline{V},\underline{n}) = \underset{\phi,H(\phi)}{\pi}(\underline{V},\underline{n}) = H(\phi)\underline{V}\cdot\underline{n} \quad \text{if } \phi_{-\infty} \le \phi = \sin^{-1}(\underline{V}\cdot\underline{n}/|\underline{V}|) \le \phi_{\mathrm{M}},\\ \underset{\mathrm{IC},T}{\pi}(\underline{V},\underline{n}) = \underset{\phi_{\mathrm{M}},H_{\mathrm{M}},T}{\pi}(\underline{V},\underline{n}) = C_{\mathrm{M}}(|\underline{V}| - \underline{V}\cdot\underline{n})\tan(\pi/4+\phi_{\mathrm{M}}/2)\\ \qquad + \dfrac{T}{1-\sin\phi_{\mathrm{M}}}(\underline{V}\cdot\underline{n} - |\underline{V}|\sin\phi_{\mathrm{M}}) \quad \text{if } \underline{V}\cdot\underline{n} \ge |\underline{V}|\sin\phi_{\mathrm{M}}.\end{cases} \tag{2.15}$$

Consistently with the final remark in Section 2.2, Equation (2.15) shows that $\underset{\mathrm{IC}}{\pi}(\underline{V},\underline{n})$ is the *support function of the domain delimited by the intrinsic curve and the ultimate limit Mohr circle in the* (σ,τ) *plane* (Figure 5)[5].

[5] To obtain a more comprehensive perspective on this result, refer to [9, pp. 53–55].

3. Applications to Yield design analysis

Yield design analyses for a material with an intrinsic curve criterion proceed from the implementation of the interior and exterior approaches of the theory.

Without any consideration for a tension cutoff, i.e., assuming that the intrinsic curve exhibits a sharp summit, an abundant literature has been devoted to plane limit equilibrium solutions, which are based upon Kötter's equations [11] for the stress field in the case of a material with a Coulomb or Tresca yield criterion.

These equations were completed by Mandel [12] in the general case. They can be used to build up complete statical solutions to be implemented within the interior approach framework, or partial statical solutions, without any conclusive relevance in that case. They can also be the basis for building up "incomplete solutions", following Bishop's terminology [13], where a plane velocity field is associated through the (outwards) normality rule with a partial limit statically admissible stress field. It may be worth recalling that these solutions fall within the exterior approach framework of the yield design theory: as first established by Bishop, the result of the corresponding exterior approach comes out directly from the partial stress field, without it being necessary to compute the maximum resisting work developed in the velocity field.

Within the framework of the Haar–Karman hypothesis [14], similar solutions have been proposed for axially symmetrical problems, which refer to the counterparts of Kötter's equations in the general case of a non-homogeneous material, with an intrinsic curve type yield criterion with a sharp summit [15–17].

The fact that the solutions here above only concern intrinsic curve type yield criteria with a sharp summit, essentially (if not uniquely) Coulomb or Tresca yield criteria, enhances the importance of the results displayed in Equations (2.14) and (2.15). They open the way to pure kinematical exterior approaches based upon the design of virtual velocity fields in the general case of a domain of resistance described by (1.19). Such virtual velocity fields $\underline{U}$ can be piecewise continuous and continuously differentiable (strain rate $\underline{\underline{d}}$) with velocity discontinuities $\underline{V}$ across jump surfaces with normal $\underline{n}$.

Since the exterior approach of the yield design theory states that the work by external forces exerted on the considered structure shall not be superior to the maximum resisting rate of work in any virtual velocity field, it follows that, for the exterior approach to yield significant results, only virtual velocity fields that comply with the condition that $\underset{\mathrm{IC}}{\pi}(\underline{\underline{d}})$ and $\underset{\mathrm{IC}}{\pi}(\underline{V},\underline{n})$, $\underset{\mathrm{IC},T}{\pi}(\underline{\underline{d}})$ and $\underset{\mathrm{IC},T}{\pi}(\underline{V},\underline{n})$ remain finite, shall be considered. This condition defines *relevant virtual velocity fields* for the problem, without any reference to any constitutive law whatsoever. It acts both as a constraint and a guide in devising relevant virtual velocity fields, for example, when considering rigid body velocity fields with velocity jump lines.

Tresca and Coulomb criteria with zero-tension cutoff, which means that T is set to zero in (2.5), were first considered by Drucker and Prager [18], Drucker [19] for the stability analysis of a vertical cut. Chatzigogos, Pecker and Salençon [20,21] also referred to a Tresca criterion with zero-tension cutoff for the determination of the ultimate bearing capacity of shallow foundations under inclined eccentric loads. From a general viewpoint, these criteria which include cohesionless Coulomb's criterion, illustrate the concept of strength criteria for which the zero-stress state lies on the boundary of the domain of resistance, as considered by Frémond and Friaà [22]. For such criteria, it turns out that, when implementing the exterior approach of the Yield design theory, the minimization process often results in "vanishing" virtual velocity fields, such as those implemented in Drucker [19] or Salençon [23].

Recently, in an attempt to assess the sensitivity of yield design analyses to the tensile resistance of the constituent material, the problem of the critical height of a vertical cut was revisited within the framework of Tresca's and Coulomb's criteria with a non-zero tension cutoff [24],

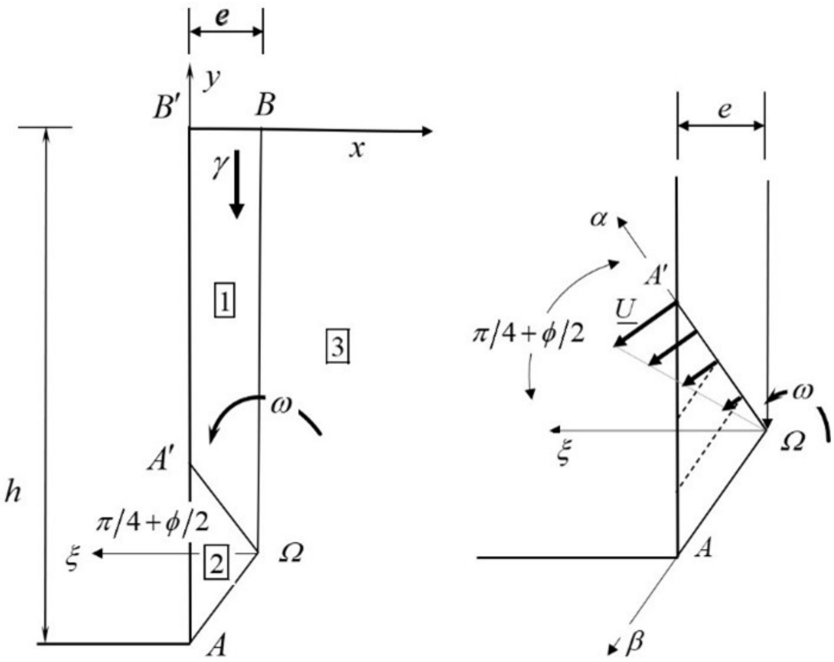

Figure 6. Drucker–Prager's virtual collapse mechanism.

implementing kinematical exterior approaches with the same virtual collapse mechanism as first devised by Drucker and Prager (Figure 6), with $\underset{\mathrm{IC},T}{\pi}(\underline{\underline{d}})$ and $\underset{\mathrm{IC},T}{\pi}(\underline{V},\underline{n})$ expressed by (2.14) and (2.15). With $\varepsilon = (e/h)$ as a non-dimensional geometrical parameter, the description of this virtual collapse mechanism can be briefly recalled as follows. Zone $\boxed{3}$ remaining motionless, the virtual velocity field $\underline{U}$ in zone $\boxed{1}$ defined as $\Omega A'B'B$ consists of an anticlockwise rigid body rotational motion, with angular velocity ω about point Ω. This implies that zone $\boxed{1}$ separates from zone $\boxed{3}$ with a velocity discontinuity $\underline{V}(y)$ along the ξ axis when crossing ΩB, whose magnitude is

$$V(y) = \omega[h(1-\varepsilon\tan(\pi/4+\phi/2)) + y]. \tag{3.1}$$

The velocity field is continuous across $\Omega A'$ and across ΩA. Complying with these boundary conditions, the velocity field in zone $\boxed{2}$, delimited by $\Omega AA'$, is defined as follows: referring to the α and β lines, $\underline{U}$ is constant along any β line and normal to $\Omega A'$ with magnitude $\omega\, x^{\alpha}$, where x^{α} denotes the abscissa of the considered β line along $\Omega\alpha$.

Using properly chosen non-dimensional factors, it turned out that the concept of a vanishing virtual collapse mechanism corresponds to the fact that the value $\varepsilon_{\mathrm{m}} = (e/h)_{\mathrm{m}}$ that defines the optimum virtual mechanism, tends to zero with T, as shown in Figure 7.

In addition, for any value of T, an upper bound to the critical height of the cut was obtained in the form

$$h_{\mathrm{cr}} \leq \frac{C}{\gamma}\tan(\pi/4+\phi/2)F\left(\frac{T}{2C}\tan(\pi/4+\phi/2)\right) \tag{3.2}$$

where function F is depicted in Figure 8.

Based upon this result and taking advantage of (1.16) and (1.19) as a definition of an intrinsic curve type domain of resistance with tension cutoff, we derive that, in the case when the vertical

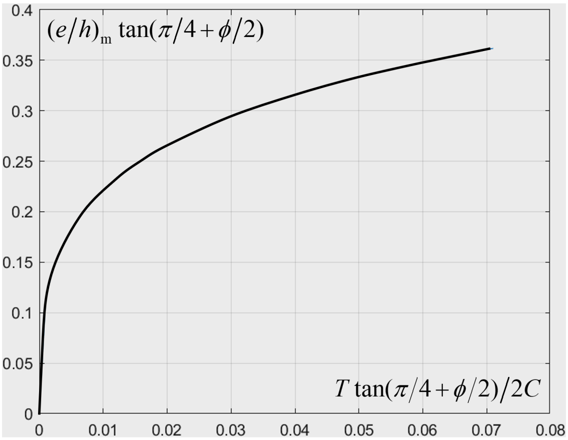

Figure 7. $(e/h)_{\mathrm{m}} \tan(\pi/4+\phi/2)$ as a function of $T\tan(\pi/4+\phi/2)/2C$.

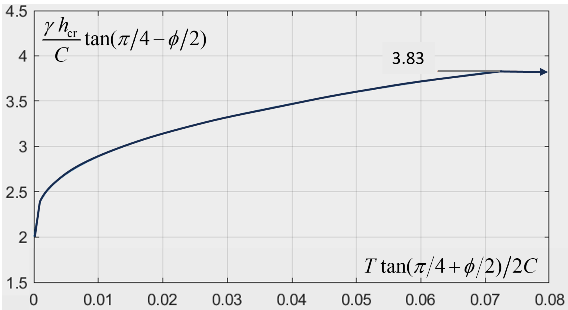

Figure 8. $(\gamma\, h_{\mathrm{cr}}/C)\tan(\pi/4-\phi/2)$ as a function F of $T\tan(\pi/4+\phi/2)/2C$.

cut is characterized by such a yield criterion, an upper bound to its critical height can be straightforwardly obtained in the form

$$h_{\mathrm{cr}} \leq \underset{\phi_\infty \leq \phi \leq \phi_{\mathrm{M}}}{\mathrm{Inf}} \left[\frac{H(\phi)\sin\phi}{\gamma(1-\sin\phi)} F\left(\frac{T(1+\sin\phi)}{2H(\phi)\sin\phi}\right)\right]. \tag{3.3}$$

In essence, this is the determination of the Coulomb criterion with maximum normal stress T, which most accurately fits the intrinsic curve as an upper bound, for the problem under consideration.

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Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than his research organization.

References

[1] J. Salençon, "About Tresca's memoirs on the fluidity of solids", *C. R. Méc.* **349** (2021), no. 1, pp. 1–7.
[2] O. Mohr, "Welche Umstände bedingen die Elastizitätsgrenze und den Bruch eines Materiales ?", *Z. Ver. Dtsch. Ing.* **44** (1900), pp. 1524–1530. 1572–1577.
[3] R. Hill, *The Mathematical Theory of Plasticity*, Clarendon Press: Oxford, 1950.
[4] A. Caquot and J. Kerisel, *Traité de Mécanique des sols*, Gauthier-Villars: Paris, 1949.
[5] A. Caquot, "Définition du domaine élastique dans les corps isotropes", in *Proceedings of the Fourth International Congress for Applied Mechanics*, vol. 24, Cambridge University Press: Cambridge, 1935.
[6] A. Caquot, *Cours de Résistance des Matériaux et Construction*, 1924. [Notes taken by students], année 1924–1925. By courtesy of Professor Samuel Forest.
[7] B. Halphen and J. Salençon, *Élastoplasticité*, Presses de l'École nationale des ponts et chaussées: Paris, 1987.
[8] G. E. Mase, *Theory and problems of Continuum Mechanics*, Schaum's Outline Series, McGraw-Hill: New York, 1970.
[9] J. Salençon, *Calcul à la rupture et analyse limite*, Presses de l'École nationale des ponts et chaussées: Paris, 1983.
[10] J. Salençon, *Yield Design*, ISTE Ltd, London and John Wiley & Sons: New York, 2013.
[11] F. Kötter, *Die Bestimmung des Druckes an gekrümmten Gleitflächen, eine Aufgabe aus der Lehre vom Erddruck*, Berlin Akad. Bericht: Berlin, 1903, pp. 229–233.
[12] J. Mandel, *Équilibre par tranches planes des solides à la limite d'écoulement*, L. Jean, Gap: France, 1943. *Travaux*, June-July-December, 1943.
[13] J. F. W. Bishop, "On the complete solution to problems of deformations of a plastic-rigid material", *J. Mech. Phys. Solids* **2** (1953), no. 1, pp. 43–53.
[14] A. Haar and T. von Karman, "Zur Theorie der Spannungszustände in plastischen und sandartigen Medien", *Nachr. Wiss. Göttingen Math. Phys. Kl.* **1909** (1909), pp. 204–218.
[15] B. G. Berezantzev, *Axial Symmetrical Problems of the Limit Equilibrium Theory of Earthy Medium*, Gostekhizdat: Moscow, 1952.
[16] J. Salençon, *Théorie de la plasticité pour les applications à la mécanique des sols*, Eyrolles: Paris, 1973.
[17] J. Salençon, *Applications of the Theory of Plasticity in Soil Mechanics*, John Wiley: Chichester, 1977.
[18] D. C. Drucker and W. Prager, "Soil mechanics and plastic analysis or limit design", *Quart. Appl. Math.* **10** (1952), pp. 157–165.
[19] D. C. Drucker, "Limit analysis of two- and three-dimensional soil mechanics problems", *J. Mech. Phys. Solids* **1** (1953), no. 4, pp. 217–226.
[20] J. Salençon and A. Pecker, "Ultimate bearing capacity of shallow foundation under inclined and eccentric loads. Part II: purely cohesive soil without tensile strength", *Eur. J. Mech. A* **14** (1995), no. 3, pp. 377–396.
[21] C. T. Chatzigogos, A. Pecker and J. Salençon, "Seismic bearing capacity of a circular footing on a heterogeneous soil", *Soils Found.* **47** (2007), no. 4, pp. 783–797.
[22] M. Frémond and A. Friaà, "Analyse limite. Comparaison des méthodes statique et cinématique", *C. R. Acad. Sci. Paris* **286A** (1978), pp. 107–110.
[23] J. Salençon, "Bearing capacity of a footing on a purely cohesive soil with linearly varying shear strength", *Géotechnique* **24** (1974), no. 3, pp. 443–446.
[24] J. Salençon, "About the critical height of a vertical cut", *C. R. Méc.* **352** (2024), pp. 269–275.