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Research article

The Double Generator Boundary Augmented bracket structure: a structure-preserving space-time integration framework for coupled thermo-visco-elastodynamics

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Abstract. The Double Generator Boundary Augmented bracket structure is a double generator bracket formulation, tailored to model continuum thermodynamics. Based on the idea of bracket generated formulations, this framework encompasses balance principles and thermodynamics laws within a unique expression. The present paper develops the methodology to derive this structure from classical equations of continuum thermodynamics for two examples. We consider first a unidimensional small strains generalized standard material, with a general quadratic dissipation potential. Then, we consider the example of large strain thermo-visco-elastodynamics, within the multisymplectic framework. We derive, for the first time, a multisymplectic Poisson bracket for thermo- (visco)-elastodynamics. Eventually, both formulations are shown to recover exactly balance principles and thermodynamics laws.

This paper sets grounds necessary to develop variational integrators from the Double Generator Boundary Augmented bracket structure.

Keywords. Variational principle, generalized standard materials, thermo-visco-elastodynamics, large strains.

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1. Introduction

Variational approaches for the thermodynamics of irreversible processes are an increasingly important area of research. Along with the development of variational and geometric integrators, these approaches elucidate the underlying mathematical structure of the problem at hand, which needs to be preserved by the integrator. The symplectic [1,2] or multisymplectic structure as defined in [3,4] is acknowledged to be the underlying structure of reversible mechanics. However, formulating a unified variational principle for dissipative phenomena, is a problem that has not yet been solved [5].

In the context of continuum mechanics, dissipative phenomena are best described thanks to the theory of thermodynamics of irreversible processes [6,7]. One main element of the description of such processes is the definition of a dissipation potential, taking into account

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dissipative internal variables, as introduced by Biot [8]. Its mathematical characteristics are directly deduced from the properties of the considered problem. In mechanics of continua, the local state method, and the generalized standard materials framework [9–11] are used to define precisely the dissipation potential. Nevertheless, state laws and the internal variables do not provide a unified framework, as they are particular to each problem, and come as additional information.

Variational principles for thermodynamics of irreversible processes have been introduced in the literature since the work of Rayleigh [12], extended by the works of Biot [8,13–15], Bateman [16]. Incremental variational principles have also been developed by Stainier, Ortiz et al. [17,18]. In addition, the symplectic Brézis–Eckeland–Nayrolles principle [19,20] is a variational formulation of dissipative mechanics using both the formulation of a dissipation potential for generalized standard materials and the decomposition of the time rate of the evolution curve of the problem into an irreversible and a reversible part. Gay-Balmaz et al. [21,22] extend Euler–Lagrange variational principle to dissipative phenomena, setting thermodynamics into the Dirac structure [23]. Additionally, port-Hamiltonian theory is also based on the geometry of Dirac structure to develop general, geometric models, for dissipative systems [24], where the framework has been extended to nonlinear elastodynamics [25]. Along with these variational formulations, bracket formulations, extending the Poisson bracket (hence Hamiltonian mechanics of continua) to dissipative processes, have been subsequently developed over the past decades, with promising results. The idea of generalizing the Poisson bracket shall be acknowledged to Dirac [23], which adapts the Poisson bracket to constrained problems. Three major contributions introduce the idea of a dissipative additional bracket: Kaufman, Morrison and Grmela [26–28], from which many bracket formulations have emerged. Furthermore, the works of Vallis, Carnevale and Young develop a double bracket formulation as an extension of the Poisson bracket, suited for fluid dynamics [29–32]. Consequently, two main categories of bracket formulations are pointed out: single and double generator bracket formulations [33]. Single bracket formulations state that any functional is generated by the Hamiltonian function of the system, with a split between reversible and irreversible contributions. This formulation has been used in the context of fluid mechanics [34,35]. The double generator bracket formulation is mainly known through the metriplectic [36] and the GENERIC (General Equation for the Non Equilibrium Reversible-Irreversible Coupling) [37–39] frameworks. Romero [40] was the first one to bring the GENERIC framework into the solid mechanics community, applying it to a thermoelastic continuum. One important contribution is from [41], which makes the link between the GENERIC framework and standard generalized materials [9]. Furthermore, Grmela, Esen and coworkers proposed a geometric interpretation of the GENERIC equations, expressing it by means of a dissipation potential rather than a dissipative bracket [42,43]. GENERIC has also been derived in the context of large strain thermo-visco-elasticity [44]. Considering the metriplectic community, a recent study by Zaidni and Morrison has provided a methodology to construct metriplectic brackets for materials that fulfill Onsager reciprocal relations [45]. Nonetheless, one main difficulty lies in adapting these preexisting structures to specific problems.

The present work shows how to obtain, from the equations of continuum thermodynamics, the general mathematical structure, called the "Double Generator Boundary Augmented bracket structure" denoted as the DGBA bracket structure in the rest of the document, for the examples of a unidimensional small strains generalized standard material, and for large strain thermo-visco-elastodynamics. The proposed bracket structure is innovative for two reasons. First, its methodology : it is built from the evolution equation from continuum thermodynamics, meaning the fields are not adapted to a preexisting bracket formulation. Second, the presence of boundary terms that emerge during the construction of the structure, essential to recover the balance principles without additional hypotheses and to build variational integrators where boundary conditions must be taken into account. In both cases, the DGBA bracket structure emerges from the Hamilton variational principle of elastodynamics for the reversible part of the structure, and from the generalized standard materials framework for the dissipative part. The large strains example has the originality of being built within the multisymplectic framework. Crucially, the DGBA bracket structure is constructed such that the balance principles of linear momentum, angular momentum (encoded within the variational principle), as well as the first and second principle of thermodynamics, are exactly verified.

The paper is organized as follows. Section 2 introduces the main assumptions of the DGBA bracket structure. Then, Section 3 presents the two applications of the DGBA bracket structure, with the one-dimensional small strains generalized standard material in Section 3.1 and the example of tridimensional large strain multisymplectic thermo-visco-elasticity in Section 3.2.

2. General formulation of the Double Generator Boundary Augmented bracket structure

The DGBA bracket structure is grounded in the assumptions of thermodynamics with internal variables [6], where the global evolution of the system is assumed to be dependent on internal variables, encompassed in a state vector $\boldsymbol{z}$, supposed to be, without precision for the introduction, a set of M smooth functions, $\boldsymbol{z} \in \left((C^\infty(\Omega_X \times I)\right)^M$, where $\Omega_X \subset \mathbb{R}^3$ is a subset of the Euclidean space, representing the Lagrangian configuration of the continuum, and $I \subset \mathbb{R}$ is a time interval. Moreover, we consider that the evolution of $\boldsymbol{z}$ can be divided into a reversible and an irreversible part, such that

$$\dot{\boldsymbol{z}} = \dot{\boldsymbol{z}}_{\text{rev}} + \dot{\boldsymbol{z}}_{\text{irr}}. \tag{1}$$

We derive the DGBA bracket structure for continuum thermodynamics. In this setting, we define the functional $\mathscr{F}$ as the integration of a density f over an open subset $\omega_X \subset \Omega_X$:

$$\mathscr{F} : M \longrightarrow \mathbb{R}, \quad \boldsymbol{z} \longmapsto \mathscr{F}[\boldsymbol{z}] = \int_{\omega_X} f(\boldsymbol{z})\, \mathrm{d}\omega_X, \tag{2}$$

where $f : \mathbb{R}^M \to \mathbb{R}$ is a smooth function, and $\mathrm{d}\omega_X$ is the Lebesgue measure on ω_X.

Remark 1. The construction of the structure is on the subdomain ω_X, such that we can build the most general model, i.e. without particular boundary forces.

The two studied problems in the applications are within the first-gradient theory, where the state vector lies within the first dual jet bundle.

Remark 2. For instance, for discrete dynamics, one might have $f : T^*Q \to \mathbb{R}$ where Q is the configuration manifold with T^*Q its cotangent bundle, $q \in Q$ is a degree of freedom, $p \in T^*Q$ is the linear momentum [1].

The DGBA bracket structure is derived by taking the time derivative of the functional $\mathscr{F}$

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_{\omega_X} \left(\frac{\partial f}{\partial \boldsymbol{z}} \cdot \left(\frac{\partial \boldsymbol{z}}{\partial t}\right)_{\text{rev}} + \frac{\partial f}{\partial \boldsymbol{z}} \cdot \left(\frac{\partial \boldsymbol{z}}{\partial t}\right)_{\text{irr}} \right) \mathrm{d}\omega_X \tag{3}$$

$$= \underbrace{\int_{\omega_X} \frac{\partial f}{\partial \boldsymbol{z}} \cdot \boldsymbol{L}(\boldsymbol{z}) \cdot \frac{\partial \widehat{E}_{\text{tot}}}{\partial \boldsymbol{z}}\, \mathrm{d}\omega_X}_{=:\{\mathscr{F}, \widehat{\mathscr{E}}_{\text{tot}}\}} + \underbrace{\int_{\omega_X} \frac{\partial f}{\partial \boldsymbol{z}} \cdot \boldsymbol{M}(\boldsymbol{z}) \cdot \frac{\partial s}{\partial \boldsymbol{z}}\, \mathrm{d}\omega_X}_{=:(\mathscr{F}, \mathscr{S})} + \text{boundary terms} \tag{4}$$

$$= \{\mathscr{F}, \widehat{\mathscr{E}}_{\text{tot}}\} + (\mathscr{F}, \mathscr{S}) + \text{boundary terms} \tag{5}$$

where $\widehat{\mathscr{E}}_{\text{tot}}$ is the total energy functional

$$\widehat{\mathscr{E}}_{\text{tot}} = \int_{\omega_X} \widehat{E}_{\text{tot}}\, \mathrm{d}\omega_X$$

with $\widehat{E}_{\text{tot}}\colon \mathbb{R}^M \to \mathbb{R}$ a smooth function, representing the density of total energy, and $\mathscr{S}$ is the total entropy functional

$$\mathscr{S} = \int_{\omega_X} s \, \mathrm{d}\omega_X$$

with $s\colon \mathbb{R}^M \to \mathbb{R}$ a smooth function, representing the density of total entropy. Eq. (4) defines the skew-symmetric matrix

$$\boldsymbol{L}(\boldsymbol{z}) \in \mathscr{M}_M(\mathbb{R}), \quad \boldsymbol{L}(\boldsymbol{z}) = -\boldsymbol{L}(\boldsymbol{z})^\top,$$

and the symmetric positive matrix

$$\boldsymbol{M}(\boldsymbol{z}) \in \mathscr{M}_M(\mathbb{R}), \quad \boldsymbol{M}(\boldsymbol{z}) = \boldsymbol{M}(\boldsymbol{z})^\top, \quad \boldsymbol{a}^\top \boldsymbol{M}(\boldsymbol{z})\boldsymbol{a} \geq 0 \quad \forall\, \boldsymbol{a} \in \mathbb{R}^M,$$

respectively representative of the reversible and the dissipative bracket.

The development of the framework involves the time derivatives of the state vector, decoupled into its reversible and irreversible components. The state laws, in this proposed methodology, are expressed thanks to the classical tools of continuum thermodynamics. As noted in the introduction, properly selecting the state vector — which will lead to the expression of the structure through the application of the state laws — is of paramount importance. Hence, the choice of the state vector can only be rooted in a deep understanding of the physics at stake.

The right-hand side of Eq. (5) consists of the sum of two brackets $\{\cdot,\cdot\}$ and $(\cdot,\cdot)$, and boundary terms. The reversible evolution of the system lies in $\{\cdot,\cdot\}$, a bilinear skew-symmetric bracket

$$\{\cdot,\cdot\}\colon C^\infty(\mathbb{R}^M) \times C^\infty(\mathbb{R}^M) \longrightarrow C^\infty(\mathbb{R}^M).$$

It emerges from the Hamilton variational principle. Only skew-symmetry is required such that the conservation principles are verified:

$$\text{(skew-symmetry)} \qquad \{\mathscr{F},\mathscr{G}\} = -\{\mathscr{G},\mathscr{F}\} \quad \forall\, \mathscr{F},\mathscr{G}. \tag{6}$$

Remark 3. If the skew-symmetric bracket fulfills Jacobi identity, it could be seen as a Poisson bracket [1,2]:

$$\text{(Jacobi identity)} \qquad \big\{\{\mathscr{F},\mathscr{G}\},\mathscr{H}\big\} + \big\{\{\mathscr{G},\mathscr{H}\},\mathscr{F}\big\} + \big\{\{\mathscr{H},\mathscr{F}\},\mathscr{G}\big\} = 0 \quad \forall\, \mathscr{F},\mathscr{G},\mathscr{H}. \tag{7}$$

While the current manuscript only provides the verification of this identity (throughout a systematic application of the defined bracket), its precise geometrical structure description is left as a perspective of our work.

The second bracket $(\cdot,\cdot)$ is the dissipative bracket, describing the irreversible i.e. dissipative evolution of the system

$$(\cdot,\cdot)\colon C^\infty(\mathbb{R}^M) \times C^\infty(\mathbb{R}^M) \longrightarrow C^\infty(\mathbb{R}^M).$$

It is a bilinear, symmetric positive operator:

$$(\mathscr{F},\mathscr{G}) = (\mathscr{G},\mathscr{F}) \quad \forall\, \mathscr{F},\mathscr{G}, \tag{8}$$

$$(\mathscr{F},\mathscr{F}) \geq 0 \qquad \forall\, \mathscr{F}. \tag{9}$$

This formulation is part of the "two generators structure" family, described in [33], as the functional $\mathscr{F}$ is generated by the total energy and the total entropy. Finally, deriving the structure from the state laws involves boundary terms that must be present so that the structure can recover the balance principles (unlike GENERIC or metriplectic formulations). The question of taking into account boundary terms for bracket formulation in fluid dynamics has for instance been tackled by Eldred and co-authors [46], using the Lagrange–d'Alembert variational principle to take into account boundary conditions. This methodology slightly differs from the one presented in the following lines, as the boundary terms of the present bracket structure emerge from the building of the structure.

Balance principles are obtained for different choices of f, the general function. For instance, choosing f equal to the linear momentum leads to the conservation of linear momentum.

Then, in the same way, when f is equal to the angular momentum, we obtain the conservation of angular momentum. Additionally, choosing f to be equal respectively to the density of energy and to the density of entropy helps us to recover the first and the second principles of thermodynamics. We emphasize that the properties of the brackets, and the expressions of the boundary terms will be crucial to enforce the balance principles.

Remark 4. Unlike GENERIC models, the DGBA bracket structure do not require prescribing non-interaction conditions ($\{\mathscr{S},\mathscr{F}\} = 0$, and $(\widehat{\mathscr{E}}_{\text{tot}},\mathscr{F}) = 0$, $\forall\,\mathscr{F}$). These conditions are redundant with the expressions of the boundary terms stated in the DGBA bracket structure. By nullifying boundary integrals, we implicitly assume strong boundary conditions. Finally, boundary terms are in fact important to derive balance principles from the structure.

3. Applications

3.1. *Deriving the structure for small strains unidimensional generalized standard material*

The first example is a unidimensional continuum under the small strain assumption. The dissipative behavior of the system is guided by the evolution of the temperature $T\colon \Omega_X \times I \to \mathbb{R}^+$, and of N internal variables $\boldsymbol{\alpha} = \begin{pmatrix}\alpha_1 & \cdots & \alpha_N\end{pmatrix}^{\mathsf{T}}$, where $\alpha_i\colon \Omega_X \times I \to \mathbb{R}$ for each $i \in \{1,\ldots,N\}$. This formulation is fundamental to continuum thermodynamics, where the pair T and $\boldsymbol{\alpha}$ is referred to as the standard set of normal variables [10,11]. The objective of this example is to set the DGBA bracket structure for this example which has a rather general behavior, while restraining ourselves within the small strain example, and in the unidimensional framework, to make the calculus more convenient.

The state vector describing the problem is composed of the displacement $u\colon \Omega_X \times I \to \mathbb{R}$, the linear momentum $p\colon \Omega_X \times I \to \mathbb{R}$, the strain tensor $\varepsilon\colon \Omega_X \times I \to \mathbb{R}$, the temperature T and internal variables $\boldsymbol{\alpha} = \begin{pmatrix}\alpha_1 & \cdots & \alpha_N\end{pmatrix}^{\mathsf{T}}$

$$\boldsymbol{z} = \begin{pmatrix} u & p & \varepsilon & T & \boldsymbol{\alpha}\end{pmatrix}. \tag{10}$$

Remark 5. The linearized strain tensor, in one dimension, is exactly the space derivative of the displacement $\varepsilon = (\partial u/\partial X)$.

Remark 6. For instance, for thermo-visco-elasticity, we have the internal dissipative variables that are $\boldsymbol{\alpha} = \alpha_0 = \varepsilon_{\text{i}}$, and the strain component is only the elastic part of the strain $\varepsilon_{\text{e}} = \varepsilon - \varepsilon_{\text{i}}$ and $\boldsymbol{z} = \begin{pmatrix} u & p & \varepsilon_{\text{e}} & T & \varepsilon_{\text{i}}\end{pmatrix}$.

Next, we derive the DGBA bracket structure for this problem, by expressing each state law for the variables of the state vector, i.e. expressing their time derivatives within

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \int_{\omega_X} \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial f}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial f}{\partial T}\frac{\partial T}{\partial t} + \left(\frac{\partial f}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \frac{\partial \boldsymbol{\alpha}}{\partial t}\right) \mathrm{d}\omega_X. \tag{11}$$

3.1.1. *Evolution laws*

Evolution law of internal variables. First, we set the evolution law for the internal variables. Consider a generalized standard material [9] with a quadratic dissipation potential

$$\phi\colon \mathbb{R}^N \longrightarrow \mathbb{R}, \quad \left(\frac{\partial \alpha_1}{\partial t},\ldots,\frac{\partial \alpha_N}{\partial t}\right) \longmapsto \phi\left(\frac{\partial \alpha_1}{\partial t},\ldots,\frac{\partial \alpha_N}{\partial t}\right) = \frac{1}{2}\sum_{ij} \frac{\partial \alpha_i}{\partial t} V_{ij} \frac{\partial \alpha_j}{\partial t} = \frac{1}{2}\left(\frac{\partial \boldsymbol{\alpha}}{\partial t}\right)^{\mathsf{T}} \boldsymbol{V} \frac{\partial \boldsymbol{\alpha}}{\partial t},$$

where $V_{ij} \in \mathbb{R}$ is a coefficient shown later to be positive and $\boldsymbol{V} \in \mathscr{M}_N(\mathbb{R})$ is the matrix composed of the latter coefficients. The Onsager reciprocity relationships [47–49] yield the symmetry of the operator $V_{ij} = V_{ji} \Leftrightarrow \boldsymbol{V}^{\mathsf{T}} = \boldsymbol{V}$. Furthermore, the Biot relationships give, for each of the dissipative state variables, in the absence of thermodynamic forces

$$\rho \frac{\partial w}{\partial \alpha_i} + \rho \frac{\partial \phi}{\partial\left(\frac{\partial \alpha_i}{\partial t}\right)} = 0 \quad \forall\, i \in \{1, \ldots, N\}$$

where

$$w \colon \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^N \longrightarrow \mathbb{R}, \quad (\varepsilon, T, \boldsymbol{\alpha}) \longmapsto w(\varepsilon, T, \boldsymbol{\alpha})$$

is the Helmholtz free energy, and $\rho \in \mathbb{R}^+$ is the density of the material. The second principle of thermodynamics [11] yields

$$\sigma \frac{\partial \varepsilon}{\partial t} - \rho\left(\frac{\mathrm{d}w}{\mathrm{d}t} + \frac{\partial T}{\partial t} s\right) - \frac{1}{T} q \frac{\partial T}{\partial X} \geq 0$$

where $\sigma \colon \Omega_X \times I \to \mathbb{R}$ is the unidimensional Cauchy stress tensor, and $q \colon \Omega_X \times I \to \mathbb{R}$ is the heat flux vector. The dissipation can be divided into mechanical and thermal contributions, each of which must remain positive. Therefore, with further calculus on the mechanical part of the dissipation, and taking into account the state laws [11] $\rho(\partial w/\partial \varepsilon) = \sigma$ and $(\partial w/\partial T) = -s$, where $s \colon \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$, $(\varepsilon, T, \boldsymbol{\alpha}) \mapsto s(\varepsilon, T, \boldsymbol{\alpha})$ is the density of entropy, one obtains

$$\begin{aligned}
-\rho \sum_i \frac{\partial w}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial t} \geq 0 \quad &\Longleftrightarrow \quad \rho \sum_i \frac{\partial \phi}{\partial\left(\frac{\partial \alpha_i}{\partial t}\right)} \frac{\partial \alpha_i}{\partial t} \geq 0 \\
&\Longleftrightarrow \quad \sum_{ij} \frac{\partial \alpha_j}{\partial t} V_{ij} \frac{\partial \alpha_i}{\partial t} \geq 0 \\
&\Longleftrightarrow \quad \left(\frac{\partial \boldsymbol{\alpha}}{\partial t}\right)^{\mathsf{T}} \boldsymbol{V} \frac{\partial \boldsymbol{\alpha}}{\partial t} \geq 0.
\end{aligned}$$

From this expression, it is clear that $\boldsymbol{V}$ is positive, semi-definite. Furthermore, in this context $\boldsymbol{V}$ is definite, since a situation where the double product equals zero implies no evolution of the dissipation, or $(\partial \boldsymbol{\alpha}/\partial t) = 0$, which is not relevant here. Therefore, $\boldsymbol{V}$ is symmetric definite positive, hence invertible. Its inverse matrix is also symmetric definite positive by definition. Then, the Biot relationships yield

$$\frac{\partial w}{\partial \alpha_i} + \frac{\partial \phi}{\partial\left(\frac{\partial \alpha_i}{\partial t}\right)} = 0 \quad \forall\, i \quad \Longleftrightarrow \quad \frac{\partial \boldsymbol{\alpha}}{\partial t} = -\boldsymbol{V}^{-1} \frac{\partial w}{\partial \boldsymbol{\alpha}}.$$

Finally, to align with the DGBA bracket structure, derivatives of the entropy with respect to the state vector must be incorporated. Hence, we introduce the specific heat capacity $c \in \mathbb{R}^+$ and define the Helmholtz free energy w in terms of the internal energy $\widehat{e}_{\mathrm{int}} \colon \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$, $(\varepsilon, T, \boldsymbol{\alpha}) \mapsto \widehat{e}_{\mathrm{int}}(\varepsilon, T, \boldsymbol{\alpha})$, entropy s, and temperature T:

$$c = T \frac{\partial s}{\partial T} = \frac{\partial \widehat{e}_{\mathrm{int}}}{\partial T}, \tag{12}$$

$$w(\varepsilon, T, \boldsymbol{\alpha}) = \widehat{e}_{\mathrm{int}}(\varepsilon, T, \boldsymbol{\alpha}) - T s(\varepsilon, T, \boldsymbol{\alpha}). \tag{13}$$

Remark 7. Equation (12) is of paramount importance to derive the dissipative bracket as a linear function of $\partial s/\partial \boldsymbol{z}$, which can be a difficult computational point in such bracket formulations.

Remark 8. The Helmholtz free energy is defined as the Legendre transform of the internal energy with respect to the entropy. Then, the "natural" variable for the internal energy is the entropy. However, given the definition of the specific heat capacity, we can demonstrate that the relationships between the entropy, internal energy, and temperature are all bijective, thus invertible. Consequently, a transition from an entropy representation to a temperature, or an internal energy representation [41,50] is possible. Finally, as we aim for the system to be governed by the state vector defined in Eq. (10), the temperature representation is adopted.

Therefore, we obtain the evolution law of the internal dissipative variables:

$$\frac{\partial \boldsymbol{\alpha}}{\partial t} = -\frac{1}{c} T \boldsymbol{V}^{-1} \frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}} \frac{\partial s}{\partial T} + T \boldsymbol{V}^{-1} \frac{\partial s}{\partial \boldsymbol{\alpha}}. \tag{14}$$

Evolution law of temperature. Second, we derive the evolution law of the temperature. By considering the time derivative of Gibbs' equation, one obtains (for details, see Section 1.1 of the supplementary document)

$$\rho \frac{\mathrm{d}\widehat{e}_{\text{int}}}{\mathrm{d}t} = \rho \frac{\mathrm{d}w}{\mathrm{d}t} - \rho \frac{\partial T}{\partial t} \frac{\partial w}{\partial T} - \rho T \frac{\partial}{\partial t} \frac{\partial w}{\partial T} \quad \Longleftrightarrow \quad \frac{\partial T}{\partial t} = \underbrace{-\frac{1}{\rho c} \frac{\partial q}{\partial X} - \frac{1}{c} \left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}} \right)^{\mathsf{T}} \frac{\partial \boldsymbol{\alpha}}{\partial t}}_{=:(\partial T/\partial t)_{\text{irr}}} \underbrace{- \frac{1}{c} T \frac{\partial s}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial t}}_{=:(\partial T/\partial t)_{\text{rev}}}. \tag{15}$$

The equation indicates that the evolution of temperature can be decomposed into irreversible components, encompassing the first two terms that represent conduction and internal variables, and a reversible component, which corresponds to the time derivative of the strain. This observation has already been noted by the community of GENERIC in the context of thermomechanics [44,51]. Moreover, models of reversible evolution of temperature, which solve the paradox of infinite velocity heat wave propagation within the Fourier law of conduction, have been extensively proposed by Green and Naghdi [52], Cattaneo [53]. Additionally, Maugin developed a formulation of the Hamilton variational principle that recovers the Green and Naghdi law [54]. Finally, this partition of the time derivative of the temperature motivates the partition of the time derivative of the state vector, as defined in Eq. (1), into two distinct parts.

Therefore, the evolution of the irreversible part of the derivative of the temperature yields

$$\left(\frac{\partial T}{\partial t} \right)_{\text{irr}} = \frac{K}{\rho c} \frac{\partial}{\partial X} \left(\frac{T^2}{c} \frac{\partial}{\partial X} \left(\frac{\partial s}{\partial T} \right) \right) - \frac{1}{c^2} T \left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}} \right)^{\mathsf{T}} \boldsymbol{V}^{-1} \frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}} \frac{\partial s}{\partial T} - \frac{1}{c} T \left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}} \right)^{\mathsf{T}} \boldsymbol{V}^{-1} \frac{\partial s}{\partial \boldsymbol{\alpha}}, \tag{16}$$

where we used the Fourier law of conduction $q = -K(\partial T/\partial X)$ (with $K \in \mathbb{R}^+$ the material's conductivity) along with the definition of the specific heat capacity (12) and the evolution of internal dissipative variables (14).

Evolution law of displacement, linear momentum — the Hamilton variational principle. Finally, we describe the evolution of the displacement field, and of the linear momentum. These fields are related to the reversible bracket, rooted in the Hamilton variational principle. We begin by deriving the Hamilton equations for continuum mechanics. Consider the action integral over a time interval $\omega_t \subset \mathbb{R}$

$$\mathscr{A}[u] = \int_{\omega_t} \int_{\omega_X} l\left(u, \frac{\partial u}{\partial t}, \varepsilon, T, \boldsymbol{\alpha} \right) \mathrm{d}\omega_X \, \mathrm{d}\omega_t$$

where $l \colon \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$, $(u, \partial u/\partial t, \varepsilon, T, \boldsymbol{\alpha}) \mapsto l(u, \partial u/\partial t, \varepsilon, T, \boldsymbol{\alpha})$ is the Lagrangian density function, taking for variables displacement and its derivatives, temperature and internal variables. Consider the Legendre transformation of the Lagrangian density to obtain the Hamiltonian density, defined as

$$l^*(u, p, \varepsilon, T, \boldsymbol{\alpha}) = p \frac{\partial u}{\partial t} - l\left(u, \frac{\partial u}{\partial t}, \varepsilon, T, \boldsymbol{\alpha} \right), \quad \frac{\partial l}{\partial \left(\frac{\partial u}{\partial t} \right)} = p$$

defining the conjugate momentum p. Therefore, one can define the dual action for small strains elastodynamics

$$\mathscr{A}^*[u, p] = \int_{\omega_t} \int_{\omega_X} \left(p \frac{\partial u}{\partial t} - l^*(u, p, \varepsilon, T, \boldsymbol{\alpha}) \right) \mathrm{d}\omega_X \, \mathrm{d}\omega_t$$

on which we apply the Hamilton variational principle for variations with respect to the displacement and the linear momentum only

$$\delta \mathscr{A}^*[u, p] + \int_{\omega_t} \int_{(\partial \omega_X)^{\text{Neumann}}} b \delta u \, \mathrm{d}(\partial \omega_X) \, \mathrm{d}\omega_t = 0, \quad \forall \, \delta u(X, t) = 0 \text{ on } \partial \omega_t, \ \forall \, \delta p$$

where b are boundary Neumann conditions on $(\partial\omega_X)^{\text{Neumann}} \subseteq \partial\omega_X$, $b\colon (\partial\omega_X)^{\text{Neumann}} \to \mathbb{R}$. We obtain the classical equations of elastodynamics:

$$\frac{\partial u}{\partial t} = \frac{\partial l^*}{\partial p} \qquad \forall\, X \in \omega_X, \tag{17}$$

$$\frac{\partial p}{\partial t} = -\frac{\partial l^*}{\partial u} + \frac{\partial}{\partial X}\left(\frac{\partial l^*}{\partial \varepsilon}\right) \qquad \forall\, X \in \omega_X, \tag{18}$$

$$\frac{\partial l^*}{\partial \varepsilon} = b \qquad \forall\, X \in (\partial\omega_X)^{\text{Neumann}}. \tag{19}$$

Remark 9. For a functional $\mathscr{F} = \mathscr{F}[u,p] = \int_{\omega_X} f(u,p,\varepsilon)\,\mathrm{d}\omega_X$ depending only on displacement u and linear momentum p one obtains the following canonical Poisson bracket (here in small strain, with links to the Poisson bracket developed by Simo et al. [55] in the context of nonlinear elasticity):

$$\begin{aligned}\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} &= \int_{\omega_X}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial t}\right)\mathrm{d}\omega_X \\ &= \underbrace{\int_{\omega_X}\left(\left(\frac{\partial f}{\partial u} - \frac{\partial}{\partial X}\left(\frac{\partial f}{\partial \varepsilon}\right)\right)\frac{\partial l^*}{\partial p} - \frac{\partial f}{\partial p}\left(\frac{\partial l^*}{\partial u} - \frac{\partial}{\partial X}\left(\frac{\partial l^*}{\partial \varepsilon}\right)\right)\right)\mathrm{d}\omega_X}_{=:\,\{\mathscr{F},\mathscr{L}^*\}\text{: canonical Poisson bracket}} + \int_{\partial\omega_X}\frac{\partial f}{\partial \varepsilon}\frac{\partial u}{\partial t}\,\mathrm{d}(\partial\omega_X) \\ &= \{\mathscr{F},\mathscr{L}^*\} + \int_{\partial\omega_X}\frac{\partial f}{\partial \varepsilon}\frac{\partial u}{\partial t}\,\mathrm{d}(\partial\omega_X),\end{aligned}$$

where

$$\mathscr{L}^* = \int_{\omega_X} l^*\,\mathrm{d}\omega_X.$$

The additional boundary term, often considered as equal to zero [35,56–58], will be of high importance when deriving the equations of the DGBA bracket structure, applied to recover balance principles.

We express the latter equations in terms of the total energy $\widehat{E}_{\text{tot}}$. The Hamilton density function of continuum mechanics is considered as the sum of the kinetic energy and the density of Helmholtz free energy, leading to

$$l^*(u,p,\varepsilon,T,\boldsymbol{\alpha}) = \frac{1}{2}p\frac{1}{\rho}p + \rho w(\varepsilon,T,\boldsymbol{\alpha}) = \underbrace{\frac{1}{2}p\frac{1}{\rho}p + \rho\widehat{e}_{\text{int}}(\varepsilon,s,\boldsymbol{\alpha})}_{\widehat{E}_{\text{tot}}(u,p,\varepsilon,T,\boldsymbol{\alpha})} - \rho T s(\varepsilon,T,\boldsymbol{\alpha}).$$

Remark 10. We notice that $\widehat{E}_{\text{tot}}$ is not a specific energy, unlike the internal energy and the Helmholtz free energy, as its unit is $\mathrm{J\,m^{-3}}$, whereas the entropy and internal energy are expressed in $\mathrm{J\,kg^{-1}}$. This distinction will be significant in subsequent calculations to ensure dimensional homogeneity in the equations.

Furthermore, the temperature can be defined as the derivative of the total energy with respect to the entropy

$$T = \frac{\partial E_{\text{tot}}}{\partial s} \quad\Longleftrightarrow\quad T = \frac{1}{\rho}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\right)\left(\frac{\partial s}{\partial T}\right)^{-1}.$$

The definition of the temperature through the Legendre transformation from an entropy to a temperature representation of the total energy can be related to the work of Gay-Balmaz and co-authors, which adapted their dissipative Lagrangian formalism to a free-energy Lagrangian, that is naturally dependent on the temperature [59]. Therefore, when deriving the Hamilton equations in terms of the total energy (giving the state laws required to obtain the reversible bracket), we obtain

$$\frac{\partial u}{\partial t} = \frac{\partial \widehat{E}_{\text{tot}}}{\partial p}, \tag{20}$$

$$\frac{\partial p}{\partial t} = -\frac{\partial \widehat{E}_{\text{tot}}}{\partial u} + \frac{\partial}{\partial X}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial \varepsilon} - \left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\right)\left(\frac{\partial s}{\partial T}\right)^{-1}\frac{\partial s}{\partial \varepsilon}\right), \tag{21}$$

$$\left(\frac{\partial T}{\partial t}\right)_{\text{rev}} = -\left(\frac{\partial s}{\partial T}\right)^{-1}\frac{\partial s}{\partial \varepsilon}\frac{\partial}{\partial X}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial p}\right). \tag{22}$$

This completes the set of equations required to derive the DGBA bracket structure.

3.1.2. *Expression of the DGBA bracket structure*

In the preceding section we have shown how to obtain each of the evolution laws for the state vector displayed in Eq. (10). Then, the DGBA bracket structure yields

$$\begin{aligned}
\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} &= \int_{\omega_X}\Bigg(\frac{\partial f}{\partial u}\underbrace{\frac{\partial u}{\partial t}}_{\text{Eq (20)}} + \frac{\partial f}{\partial p}\underbrace{\frac{\partial p}{\partial t}}_{\text{Eq (21)}} + \frac{\partial f}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial f}{\partial T}\Bigg(\underbrace{\left(\frac{\partial T}{\partial t}\right)_{\text{rev}}}_{\text{Eq (22)}} + \underbrace{\left(\frac{\partial T}{\partial t}\right)_{\text{irr}}}_{\text{Eq (16)}}\Bigg) + \left(\frac{\partial f}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}}\underbrace{\frac{\partial \boldsymbol{\alpha}}{\partial t}}_{\text{Eq (14)}}\Bigg)\mathrm{d}\omega_X \\
&= \{\mathscr{F},\widehat{\mathscr{E}}_{\text{tot}}\} + (\mathscr{F},\mathscr{S}) + \int_{\partial\omega_X}\frac{\partial f}{\partial \varepsilon}\frac{\partial u}{\partial t}\,\mathrm{d}(\partial\omega_X) - \int_{\partial\omega_X}\frac{\partial f}{\partial p}\left(\frac{\partial s}{\partial T}\right)^{-1}\frac{\partial s}{\partial \varepsilon}\frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\,\mathrm{d}(\partial\omega_X) \\
&\qquad - \int_{\partial\omega_X}\frac{1}{\rho}\frac{\partial f}{\partial T}K\frac{T^2}{c^2}\frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right)\mathrm{d}(\partial\omega_X)
\end{aligned}$$

where

$$\begin{aligned}
\{\mathscr{F},\widehat{\mathscr{E}}_{\text{tot}}\} &= \int_{\omega_X}\Bigg(\left(\frac{\partial f}{\partial u} - \frac{\partial}{\partial X}\left(\frac{\partial f}{\partial \varepsilon}\right)\right)\frac{\partial \widehat{E}_{\text{tot}}}{\partial p} - \frac{\partial f}{\partial p}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial u} - \frac{\partial}{\partial X}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial \varepsilon}\right)\right) \\
&\qquad + \left(\frac{\partial s}{\partial T}\right)^{-1}\frac{\partial s}{\partial \varepsilon}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial f}{\partial p}\right) - \frac{\partial f}{\partial T}\frac{\partial}{\partial X}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial p}\right)\right)\Bigg)\mathrm{d}\omega_X \\
&= \int_{\omega_X}\frac{\partial f}{\partial \boldsymbol{z}}\cdot\boldsymbol{L}(\boldsymbol{z})\cdot\frac{\partial \widehat{E}_{\text{tot}}}{\partial \boldsymbol{z}}\,\mathrm{d}\omega_X \\
(\mathscr{F},\mathscr{S}) &= \int_{\omega_X}\Bigg(\frac{1}{\rho}\frac{\partial}{\partial X}\left(\frac{\partial f}{\partial T}\right)K\frac{T^2}{c^2}\frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right) + \frac{\partial f}{\partial T}\frac{1}{c}\frac{T}{c}\left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}}\boldsymbol{V}^{-1}\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\frac{\partial s}{\partial T} \\
&\qquad - \frac{\partial f}{\partial T}\frac{T}{c}\left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}}\boldsymbol{V}^{-1}\frac{\partial s}{\partial \boldsymbol{\alpha}} - \frac{T}{c}\left(\frac{\partial f}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}}\boldsymbol{V}^{-1}\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\frac{\partial s}{\partial T} + T\left(\frac{\partial f}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}}\boldsymbol{V}^{-1}\frac{\partial s}{\partial \boldsymbol{\alpha}}\Bigg)\mathrm{d}\omega_X \\
&= \int_{\omega_X}\frac{\partial f}{\partial \boldsymbol{z}}\cdot\boldsymbol{M}(\boldsymbol{z})\cdot\frac{\partial s}{\partial \boldsymbol{z}}\,\mathrm{d}\omega_X
\end{aligned} \tag{23}$$

with the following matrix representations of the reversible bracket $\boldsymbol{L}(\boldsymbol{z})$ and of the dissipative bracket $\boldsymbol{M}(\boldsymbol{z})$:

$$\boldsymbol{L}(\boldsymbol{z}) = \begin{pmatrix}
0 & 1 & 0 & 0 \cdots 0 \\
-1 & 0 & \left(\frac{\partial s}{\partial T}\right)^{-1}\frac{\partial s}{\partial \varepsilon}\frac{\partial \square_{\text{left}}}{\partial X} & \vdots \quad \vdots \\
0 & -\left(\frac{\partial s}{\partial T}\right)^{-1}\frac{\partial s}{\partial \varepsilon}\frac{\partial \square_{\text{right}}}{\partial X} & 0 & \vdots \quad \vdots \\
0 & \cdots & \cdots & 0 \cdots 0 \\
\vdots & & & \vdots \quad \vdots \\
0 & \cdots & \cdots & 0 \cdots 0
\end{pmatrix},$$

$$\boldsymbol{M}(\boldsymbol{z}) = \begin{pmatrix} 0\ 0\ 0 & 0 & \boldsymbol{0}^\top \\ 0\ 0\ 0 & 0 & \boldsymbol{0}^\top \\ 0\ 0\ 0 & 0 & \boldsymbol{0}^\top \\ 0\ 0\ 0 & \frac{1}{c^2}\left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^\top \boldsymbol{V}^{-1}\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}} T + \frac{1}{\rho}\frac{\partial \Box_{\text{left}}}{\partial X} K\left(\frac{T}{c}\right)^2 \frac{\partial \Box_{\text{right}}}{\partial X} & -\frac{1}{c} T \left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^\top \boldsymbol{V}^{-1} \\ \boldsymbol{0\ 0\ 0} & -\frac{1}{c} T \boldsymbol{V}^{-1}\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}} & T\boldsymbol{V}^{-1} \end{pmatrix}.$$

Integration by parts has been applied to the conduction part of the dissipative bracket (see Section 1.2 of the supplementary document).

Remark 11. The operators $\partial\Box_{\text{left}}/\partial X$ and $\partial\Box_{\text{right}}/\partial X$ are defined as [60]

$$\begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} 0 & \frac{\partial \Box_{\text{left}}}{\partial X} \\ \frac{\partial \Box_{\text{right}}}{\partial X} & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{\partial a_1}{\partial X} b_2 + a_2 \frac{\partial b_1}{\partial X}.$$

Remark 12. For the coupled thermo-visco-elastodynamics problem, consider the strain tensor to be split into a reversible (elastic) and an irreversible (anelastic) part [11]:

$$\varepsilon = \varepsilon_{\text{e}} + \varepsilon_{\text{i}},$$

where $\varepsilon_{\text{e}} : \Omega_X \times I \to \mathbb{R}$ and $\varepsilon_{\text{i}} : \Omega_X \times I \to \mathbb{R}$. We consider the viscoelastic part to answer the standard linear solid model (which will be the basis on which the large strain viscoelastodynamics state law will be built upon [61]), where $E_\infty \in \mathbb{R}$ is the elasticity modulus of the elastic branch, and E_{an} is the elasticity modulus of the inelastic (viscous) branch. $V^{-1} \in \mathbb{R}$ will be the inverse of the viscoelastic modulus. The equations of motion for the thermo-visco-elastodynamics coupled problem can then be obtained through the space Galerkin expansion of f with respect to its variables

$$\mathscr{F} = \int_{\omega_X} \left(\delta u(X) u(X,t) + \delta p(X) p(X,t) + \delta\varepsilon(X)\varepsilon(X,t) + \delta T(X) T(X,t) + \delta\varepsilon_{\text{an}}(X)\varepsilon_{\text{an}}(X,t)\right) \mathrm{d}\omega_X\big)$$

and the definitions of the potentials of thermo-visco-elastodynamics at small strains (where the viscous thermo-mechanical coupling is on the reversible part of the strain tensor)

$$\rho s(\varepsilon, T, \varepsilon_{\text{an}}) = k(\varepsilon - \varepsilon_{\text{an}}) + \rho c\left(\frac{T}{T_0} - 1\right),$$
$$\widehat{E}_{\text{tot}}(u, p, \varepsilon, T, \varepsilon_{\text{an}}) = \frac{1}{2\rho} pp + \frac{1}{2} E_\infty \varepsilon^2 + \frac{1}{2} E_{\text{an}} (\varepsilon - \varepsilon_{\text{an}})^2 + kT_0(\varepsilon - \varepsilon_{\text{an}}) + \frac{1}{2}\frac{\rho c}{T_0}(T^2 - T_0^2) + f_{\text{d}} u.$$

The equations of motion are then:

$$\frac{\partial u}{\partial t} = \frac{1}{\rho} p,$$
$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial X},$$
$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial X}\big((E_\infty + E_{\text{an}})\varepsilon - E_{\text{an}}\varepsilon_{\text{an}} - k(T - T_0)\big),$$
$$\rho c \frac{\partial T}{\partial t} + kT_0 \frac{\partial}{\partial X}\left(\frac{1}{\rho} p\right) = \frac{\partial}{\partial X}\left(K \frac{\partial T}{\partial X}\right) + \big(-E_{\text{an}}(\varepsilon - \varepsilon_{\text{an}}) - kT_0\big) V^{-1} \big(-E_{\text{an}}(\varepsilon - \varepsilon_{\text{an}}) + k(T - T_0)\big),$$
$$\frac{\partial \varepsilon_{\text{an}}}{\partial t} = V^{-1} E_{\text{an}}(\varepsilon - \varepsilon_{\text{an}}) - V^{-1} k (T - T_0).$$

3.1.3. *Properties of the brackets*

Properties of the reversible bracket. First, we verify that the reversible bracket satisfies the skew-symmetry property required in the preamble (6), confirmed in Section 1.3 of the supplementary document.

Remark 13. The reversible bracket can be considered as a Poisson bracket if the Jacobi identity is verified. However, verifying this identity (7) is not obvious, as the representative matrix $\boldsymbol{L}(\boldsymbol{z})$ of the reversible bracket explicitly depends on the state vector $\boldsymbol{z}$. The calculus is developed in Section 1.4 of the supplementary document, considering the dependence of the entropy and the temperature on the fields of the problems as defined in equation (13). The method followed is mainly computational, as we have straightforwardly applied the defined bracket two times, as required by the identity. Furthermore, as noted in the supplementary document, the proof is lead on the densities of the functions (it indeed implies the verification of the integrated Jacobi identity). The Jacobi identity is verified provided the following conditions on the entropy hold:

$$\begin{aligned} \frac{\partial^3 s}{\partial \varepsilon^2 \partial X} &= \frac{\partial}{\partial X}\left(\frac{\partial^2 s}{\partial \varepsilon^2}\right) = 0, \\ \frac{\partial^3 s}{\partial T \partial \varepsilon \partial X} &= \frac{\partial}{\partial X}\left(\frac{\partial^2 s}{\partial T \partial \varepsilon}\right) = 0. \end{aligned} \tag{24}$$

These conditions, involving third partial derivatives of the entropy, are physically reasonable. Consider, for instance, the example of linear thermo-elasticity, where the entropy is a linear function of strain and temperature, without coupling:

$$s(\varepsilon, T) = s_0 + \frac{k}{\rho}\varepsilon + c\frac{T - T_0}{T_0}.$$

Consequently, one should bear in mind that such hypothesis must be verified by the material at stake before using the DGBA bracket structure as proposed in this communication.

Properties of the dissipative bracket. Moreover, both the symmetry and the positivity of the dissipative bracket are verified respectively in Section 1.5 of the supplementary document and Section A.1 of the present document. The proof of the positivity relies on the positivity of the material's conductivity K, and the matrix $\boldsymbol{V}^{-1}$.

3.1.4. *Conservation laws*

As it is the main idea behind the DGBA bracket structure, we now prove how to derive the classical conservation laws of continuum mechanics using the obtained structure.

Conservation of linear momentum. Consider f to be the linear momentum

$$f = p, \quad \mathscr{F} = \int_{\omega_X} p \, \mathrm{d}\omega_X.$$

Therefore, setting this expression within the obtained structure yields (noting that only $(\partial f / \partial p)$ is not equal to zero)

$$\begin{aligned} \int_{\omega_X} \frac{\partial p}{\partial t}\,\mathrm{d}\omega_X &= \int_{\omega_X}\left(\left(-\frac{\partial \widehat{E}_{\mathrm{tot}}}{\partial u} + \frac{\partial}{\partial X}\frac{\partial \widehat{E}_{\mathrm{tot}}}{\partial \varepsilon}\right) - \left(\frac{\partial s}{\partial T}\right)^{-1}\frac{\partial s}{\partial \varepsilon}\frac{\partial \widehat{E}_{\mathrm{tot}}}{\partial T}\right)\mathrm{d}(\partial\omega_X) \\ &= \int_{\omega_X}\left(-\frac{\partial \widehat{E}_{\mathrm{tot}}}{\partial u} + \frac{\partial}{\partial X}\left(\frac{\partial \widehat{E}_{\mathrm{tot}} - \rho T s}{\partial \varepsilon}\right)\right)\mathrm{d}\omega_X \\ &= \int_{\omega_X}\left(-\frac{\partial \widehat{E}_{\mathrm{tot}}}{\partial u} + \frac{\partial}{\partial X}\left(\frac{\partial l^*}{\partial \varepsilon}\right)\right)\mathrm{d}\omega_X \\ \iff \quad 0 &= \int_{\omega_X}\left(-\frac{\partial p}{\partial t} - \frac{\partial \widehat{E}_{\mathrm{tot}}}{\partial u} + \frac{\partial}{\partial X}\left(\rho\frac{\partial w}{\partial \varepsilon}\right)\right)\mathrm{d}\omega_X \quad \forall\, \omega_X. \end{aligned}$$

Since this equality is defined for all subset $\omega_X \subset \Omega_X$, we obtain the conservation of linear momentum (considering that $\rho(\partial w / \partial \varepsilon) = \sigma$):

$$\frac{\partial p}{\partial t} = -\frac{\partial \widehat{E}_{\mathrm{tot}}}{\partial u} + \frac{\partial \sigma}{\partial X}.$$

Conservation of angular momentum. As we have chosen to present a one-dimensional problem to simplify the notations and enhance understanding, the conservation of angular momentum is not addressed here.

First law of thermodynamics. We consider f to be equal to the total energy, considered to be the sum of the kinetic and the internal energy

$$f = \widehat{E}_{\text{tot}}, \quad \mathscr{F} = \widehat{\mathscr{E}}_{\text{tot}} = \int_{\omega_X} \underbrace{\left(\frac{1}{2} p \frac{1}{\rho} p + \rho \widehat{e}_{\text{int}}(\varepsilon, T, \boldsymbol{\alpha})\right)}_{=\widehat{E}_{\text{tot}}} \mathrm{d}\omega_X$$

and insert it within the DGBA bracket structure

$$\frac{\mathrm{d}\widehat{\mathscr{E}}_{\text{tot}}}{\mathrm{d}t} = (\widehat{\mathscr{E}}_{\text{tot}}, \mathscr{S}) - \int_{\partial\omega_X} \frac{1}{\rho} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T} K \frac{T^2}{c^2} \frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right) \mathrm{d}(\partial\omega_X) + \int_{\partial\omega_X} \frac{\partial \widehat{E}_{\text{tot}}}{\partial \varepsilon} \frac{\partial u}{\partial t} \mathrm{d}(\partial\omega_X) - \int_{\partial\omega_X} \frac{\partial \widehat{E}_{\text{tot}}}{\partial p} \left(\frac{\partial s}{\partial T}\right)^{-1} \frac{\partial s}{\partial \varepsilon} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T} \mathrm{d}(\partial\omega_X) \tag{25}$$

giving

$$\begin{aligned}
\int_{\omega_X} \left(\frac{1}{\rho} p \frac{\partial p}{\partial t} + \rho \frac{\partial \widehat{e}_{\text{int}}}{\partial t}\right) \mathrm{d}\omega_X
&= \underbrace{\int_{\omega_X} \frac{1}{\rho} \frac{\partial}{\partial X}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\right) K \frac{T^2}{c^2} \frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right) \mathrm{d}\omega_X - \int_{\partial\omega_X} \frac{1}{\rho} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T} K \frac{T^2}{c^2} \frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right) \mathrm{d}(\partial\omega_X)}_{=(\mathrm{I})} \\
&\quad + \underbrace{\int_{\omega_X} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T} \left(\left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \boldsymbol{V}^{-1} \frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}} \frac{1}{c} \frac{T}{c} \frac{\partial s}{\partial T} - \left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \boldsymbol{V}^{-1} \frac{\partial s}{\partial \boldsymbol{\alpha}} \frac{T}{c}\right) \mathrm{d}\omega_X}_{=(\mathrm{II})} \\
&\quad + \underbrace{\int_{\omega_X} \left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \left(-\boldsymbol{V}^{-1} \frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}} \frac{T}{c} \frac{\partial s}{\partial T} + T \boldsymbol{V}^{-1} \frac{\partial s}{\partial \boldsymbol{\alpha}}\right) \mathrm{d}\omega_X}_{=(\mathrm{III})} \\
&\quad + \underbrace{\int_{\partial\omega_X} \frac{\partial \widehat{E}_{\text{tot}}}{\partial \varepsilon} \frac{\partial u}{\partial t} \mathrm{d}(\partial\omega_X) - \int_{\partial\omega_X} \frac{\partial \widehat{E}_{\text{tot}}}{\partial p} \left(\frac{\partial s}{\partial T}\right)^{-1} \frac{\partial s}{\partial \varepsilon} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T} \mathrm{d}(\partial\omega_X)}_{=(\mathrm{IV})}.
\end{aligned}$$

Each term on the right-hand side of the equal sign is analysed separately. First, the conduction term (I) becomes

$$\begin{aligned}
&\int_{\omega_X} \frac{1}{\rho} \frac{\partial}{\partial X}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\right) K \frac{T^2}{c^2} \frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right) \mathrm{d}\omega_X - \int_{\partial\omega_X} \frac{1}{\rho} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T} K \frac{T^2}{c^2} \frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right) \mathrm{d}(\partial\omega_X) \\
&\quad = -\int_{\omega_X} \frac{1}{\rho} \frac{\partial}{\partial X}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\right) K \frac{1}{c} \frac{\partial T}{\partial X} \mathrm{d}\omega_X + \int_{\partial\omega_X} \frac{1}{\rho} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T} K \frac{1}{c} \frac{\partial T}{\partial X} \mathrm{d}(\partial\omega_X) \\
&\quad = \int_{\omega_X} \underbrace{\frac{1}{\rho} \frac{1}{c} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T}}_{=1} \frac{\partial}{\partial X}\left(K \frac{\partial T}{\partial X}\right) \mathrm{d}\omega_X \\
&\qquad \underbrace{- \int_{\partial\omega_X} \frac{1}{\rho} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T} K \frac{1}{c} \frac{\partial T}{\partial X} \mathrm{d}(\partial\omega_X) + \int_{\partial\omega_X} \frac{1}{\rho} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T} K \frac{1}{c} \frac{\partial T}{\partial X} \mathrm{d}(\partial\omega_X)}_{=0} \\
&\quad = \int_{\omega_X} \frac{\partial}{\partial X}(-q) \mathrm{d}\omega_X \\
&\quad = -\int_{\omega_X} \frac{\partial q}{\partial X} \mathrm{d}\omega_X.
\end{aligned}$$

Next, the second term (II) is

$$\begin{aligned}\int_{\omega_X} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\left(\left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \boldsymbol{V}^{-1}\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\frac{1}{c}\frac{T}{c}\frac{\partial s}{\partial T} - \left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \boldsymbol{V}^{-1}\frac{\partial s}{\partial \boldsymbol{\alpha}}\frac{T}{c}\right)\mathrm{d}\omega_X \\
&= \int_{\omega_X} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\frac{1}{c}\left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \boldsymbol{V}^{-1}\left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}} - T\frac{\partial s}{\partial \boldsymbol{\alpha}}\right)\mathrm{d}\omega_X \\
&= \int_{\omega_X} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\frac{1}{c}\left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \boldsymbol{V}^{-1}\frac{\partial (\widehat{e}_{\text{int}} - Ts)}{\partial \boldsymbol{\alpha}}\mathrm{d}\omega_X \\
&= \int_{\omega_X} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\frac{1}{c}\left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \boldsymbol{V}^{-1}\frac{\partial w}{\partial \boldsymbol{\alpha}}\mathrm{d}\omega_X \\
&= -\int_{\omega_X} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\frac{1}{c}\left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \frac{\partial \boldsymbol{\alpha}}{\partial t}\mathrm{d}\omega_X \\
&= -\int_{\omega_X} \left(\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \frac{\partial \boldsymbol{\alpha}}{\partial t}\mathrm{d}\omega_X\end{aligned}$$

and the term (III) leads to

$$\begin{aligned}\int_{\omega_X}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}}\left(-\boldsymbol{V}^{-1}\frac{\partial \widehat{e}_{\text{int}}}{\partial \boldsymbol{\alpha}}\frac{T}{c}\frac{\partial s}{\partial T} + T\boldsymbol{V}^{-1}\frac{\partial s}{\partial \boldsymbol{\alpha}}\right)\mathrm{d}\omega_X &= \int_{\omega_X}\sum_i -\frac{\partial \widehat{E}_{\text{tot}}}{\partial \boldsymbol{\alpha}}^{\mathsf{T}}\boldsymbol{V}^{-1}\frac{\partial w}{\partial \boldsymbol{\alpha}}\mathrm{d}\omega_X \\
&= \int_{\omega_X}\sum_i \left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}}\frac{\partial \boldsymbol{\alpha}}{\partial t}\mathrm{d}\omega_X \\
&= -(\text{II}).\end{aligned}$$

Hence the two terms (II) and (III) cancel each other: (II) + (III) = 0. Finally, the fourth term (IV) yields

$$\begin{aligned}\int_{\partial\omega_X}\frac{\partial \widehat{E}_{\text{tot}}}{\partial \varepsilon}\frac{\partial u}{\partial t}\mathrm{d}(\partial\omega_X) - \int_{\partial\omega_X}\frac{\partial \widehat{E}_{\text{tot}}}{\partial p}\left(\frac{\partial s}{\partial T}\right)^{-1}\frac{\partial s}{\partial \varepsilon}\frac{\partial \widehat{E}_{\text{tot}}}{\partial T}\mathrm{d}(\partial\omega_X) &= \int_{\partial\omega_X}\left(\frac{\partial(\widehat{E}_{\text{tot}} - \rho Ts)}{\partial \varepsilon}\frac{\partial \widehat{E}_{\text{tot}}}{\partial p}\right)\mathrm{d}(\partial\omega_X) \\
&= \int_{\partial\omega_X}\rho\frac{\partial w}{\partial \varepsilon}\frac{\partial \widehat{E}_{\text{tot}}}{\partial p}\mathrm{d}(\partial\omega_X) \\
&= \int_{\partial\omega_X}\sigma\frac{\partial u}{\partial t}\mathrm{d}(\partial\omega_X).\end{aligned}$$

Therefore, we have

$$\int_{\omega_X}\left(\frac{1}{\rho}p\frac{\partial p}{\partial t} + \rho\frac{\partial \widehat{e}_{\text{int}}}{\partial t}\right)\mathrm{d}\omega_X = -\int_{\omega_X}\frac{\partial q}{\partial X}\mathrm{d}\omega_X + \int_{\partial\omega_X}\sigma\frac{\partial u}{\partial t}\mathrm{d}(\partial\omega_X).$$

Finally, we insert the definition of the linear momentum of Eq. (20) and the time derivative of the linear momentum of Eq. (21), to obtain

$$\begin{aligned}&\int_{\omega_X}\left(\frac{\partial u}{\partial t}\frac{\partial}{\partial X}\left(\overbrace{\rho\frac{\partial w}{\partial \varepsilon}}^{=\sigma}\right) + \rho\frac{\partial \widehat{e}_{\text{int}}}{\partial t}\right)\mathrm{d}\omega_X = -\int_{\omega_X}\frac{\partial q}{\partial X}\mathrm{d}\omega_X + \int_{\partial\omega_X}\sigma\frac{\partial u}{\partial t}\mathrm{d}(\partial\omega_X) \\
\implies\quad &\int_{\omega_X}\left(\frac{\partial u}{\partial t}\frac{\partial \sigma}{\partial X} + \rho\frac{\partial \widehat{e}_{\text{int}}}{\partial t}\right)\mathrm{d}\omega_X = -\int_{\omega_X}\frac{\partial q}{\partial X}\mathrm{d}\omega_X + \int_{\partial\omega_X}\sigma\frac{\partial u}{\partial t}\mathrm{d}(\partial\omega_X) \\
\implies\quad &\int_{\omega_X}\left(-\frac{\partial^2 u}{\partial X\partial t}\sigma + \rho\frac{\partial \widehat{e}_{\text{int}}}{\partial t}\right)\mathrm{d}\omega_X + \int_{\partial\omega_X}\frac{\partial u}{\partial t}\sigma\,\mathrm{d}(\partial\omega_X) + \int_{\omega_X}\frac{\partial q}{\partial X}\mathrm{d}\omega_X = \int_{\partial\omega_X}\sigma\frac{\partial u}{\partial t}\mathrm{d}(\partial\omega_X) \\
\implies\quad &\int_{\omega_X}\left(\rho\frac{\partial \widehat{e}_{\text{int}}}{\partial t} - \frac{\partial \varepsilon}{\partial t}\sigma + \frac{\partial q}{\partial X}\right)\mathrm{d}\omega_X = 0,\end{aligned}$$

thus giving the first law of thermodynamics

$$\rho \frac{\partial \widehat{e}_{\text{int}}}{\partial t} = \sigma \dot{\varepsilon} - \operatorname{div} q.$$

Second law of thermodynamics. Assume f to be equal to the total entropy $\mathscr{F} = \mathscr{S}$

$$f = s, \quad \mathscr{F} = \int_{\omega_X} s \, \mathrm{d}\omega_X.$$

We consider the dependencies of the entropy as described in (13), such that $(\partial s/\partial p) = 0$ and $(\partial s/\partial u) = 0$. Then, one has

$$\begin{aligned}
\frac{\mathrm{d}\mathscr{S}}{\mathrm{d}t} &= \{\mathscr{S}, \widehat{\mathscr{E}}_{\text{tot}}\} + (\mathscr{S}, \mathscr{S}) + \int_{\partial\omega_X} \frac{\partial s}{\partial \varepsilon} \frac{\partial u}{\partial t} \mathrm{d}(\partial\omega_X) \\
&\quad - \int_{\partial\omega_X} \frac{\partial s}{\partial p} \left(\frac{\partial s}{\partial T}\right)^{-1} \frac{\partial s}{\partial \varepsilon} \frac{\partial \widehat{E}_{\text{tot}}}{\partial T} \mathrm{d}(\partial\omega_X) - \int_{\partial\omega_X} \frac{1}{\rho} \frac{\partial s}{\partial T} K \frac{T^2}{c^2} \frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right) \mathrm{d}(\partial\omega_X) \\
\iff \quad \frac{\mathrm{d}\mathscr{S}}{\mathrm{d}t} &+ \int_{\partial\omega_X} \frac{1}{\rho} \frac{\partial s}{\partial T} K \frac{T^2}{c^2} \frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right) \mathrm{d}(\partial\omega_X) = (\mathscr{S}, \mathscr{S}) + \{\mathscr{S}, \widehat{\mathscr{E}}_{\text{tot}}\} + \int_{\partial\omega_X} \frac{\partial s}{\partial \varepsilon} \frac{\partial u}{\partial t} \mathrm{d}(\partial\omega_X)
\end{aligned}$$

where

$$\begin{aligned}
\{\mathscr{S}, \widehat{\mathscr{E}}_{\text{tot}}\} &+ \int_{\partial\omega_X} \frac{\partial s}{\partial \varepsilon} \frac{\partial u}{\partial t} \mathrm{d}(\partial\omega_X) \\
&= \int_{\omega_X} \left(-\frac{\partial}{\partial X}\left(\frac{\partial s}{\partial \varepsilon}\right) \frac{\partial \widehat{E}_{\text{tot}}}{\partial p} - \left(\frac{\partial s}{\partial T}\right)^{-1} \frac{\partial s}{\partial \varepsilon} \frac{\partial s}{\partial T} \frac{\partial}{\partial X}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial p}\right)\right) \mathrm{d}\omega_X + \int_{\partial\omega_X} \frac{\partial s}{\partial \varepsilon} \frac{\partial u}{\partial t} \mathrm{d}(\partial\omega_X) \\
&= -\int_{\omega_X} \left(\frac{\partial}{\partial X}\left(\frac{\partial s}{\partial \varepsilon}\right) \frac{\partial \widehat{E}_{\text{tot}}}{\partial p} + \frac{\partial s}{\partial \varepsilon} \frac{\partial}{\partial X}\left(\frac{\partial \widehat{E}_{\text{tot}}}{\partial p}\right)\right) \mathrm{d}\omega_X + \int_{\partial\omega_X} \frac{\partial s}{\partial \varepsilon} \frac{\partial u}{\partial t} \mathrm{d}(\partial\omega_X) \\
&= -\int_{\omega_X} \frac{\partial}{\partial X}\left(\frac{\partial s}{\partial \varepsilon} \frac{\partial \widehat{E}_{\text{tot}}}{\partial p}\right) \mathrm{d}\omega_X + \int_{\partial\omega_X} \frac{\partial s}{\partial \varepsilon} \frac{\partial u}{\partial t} \mathrm{d}(\partial\omega_X) \\
&= \int_{\partial\omega_X} -\frac{\partial s}{\partial \varepsilon} \frac{\partial \widehat{E}_{\text{tot}}}{\partial p} \mathrm{d}(\partial\omega_X) + \int_{\partial\omega_X} \frac{\partial s}{\partial \varepsilon} \frac{\partial u}{\partial t} \mathrm{d}(\partial\omega_X) \\
&= 0.
\end{aligned}$$

Referring to Remark 4, we obtain, without any assumption, a non-interaction condition that is necessary to preserve the second principle of thermodynamics. The second term of the left-hand side of the equation is equal to

$$\int_{\partial\omega_X} \frac{1}{\rho} \frac{\partial s}{\partial T} K \frac{T^2}{c^2} \frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right) \mathrm{d}(\partial\omega_X) = \int_{\partial\omega_X} \frac{1}{\rho} \frac{\partial}{\partial X}\left(\frac{q}{T}\right) \mathrm{d}\omega_X$$

while the right-hand side of the equation leads to

$$(\mathscr{S}, \mathscr{S}) = \int_{\omega_X} \left[\frac{1}{\rho} \frac{1}{T^2} K \left(\frac{\partial T}{\partial X}\right)^2 + \left(\frac{\partial w}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \boldsymbol{V}^{-1} \frac{1}{T} \frac{\partial w}{\partial \boldsymbol{\alpha}} \right] \mathrm{d}\omega_X.$$

Then, as we have shown that the dissipative bracket is positive for any functional $\mathscr{F}$ (see Section A.1), we obtain the second principle of thermodynamics, and its expression as the Clausius–Duhem identity

$$\int_{\omega_X} \left(\rho \frac{\partial s}{\partial t} + \frac{\partial}{\partial X}\left(\frac{q}{T}\right)\right) \mathrm{d}\omega_X \geq 0 \quad \iff \quad \int_{\omega_X} \frac{1}{\rho T} \left[-\frac{1}{T} q \frac{\partial T}{\partial X} - \sum_i \left(\frac{\partial w}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \frac{\partial \boldsymbol{\alpha}}{\partial t} \right] \mathrm{d}\omega_X \geq 0$$

or, in local form

$$\rho \frac{\partial s}{\partial t} + \operatorname{div}\left(\frac{q}{T}\right) \geq 0 \quad \iff \quad -\frac{1}{T} q \frac{\partial T}{\partial X} - \sum_i \left(\frac{\partial w}{\partial \boldsymbol{\alpha}}\right)^{\mathsf{T}} \frac{\partial \boldsymbol{\alpha}}{\partial t} \geq 0.$$

3.2. *Deriving the DGBA bracket structure for tridimensional large strain thermo-visco-elastodynamics*

We consider a tridimensional thermo-visco-elastodynamics problem at large strains. The reference (Lagrangian, or material) configuration is denoted by $\Omega_X \subset \mathbb{R}^3$, while the deformed (Eulerian, or spatial) configuration is denoted by $\Omega_x \subset \mathbb{R}^3$, linked through the smooth mapping $\underline{\chi}\colon \Omega_X \times I \to \Omega_x$. The basis vectors of the Lagrangian configuration will have uppercase indexes, such as $\underline{e}_A$, while basis vectors of the Eulerian configuration will have lowercase indexes $\underline{e}_a$ (this notation will also be followed by the components of vectors, tensors, when expressed in index notation). In the same manner, the position in the Lagrangian configuration is denoted by $\underline{X}$, and $\underline{x} = \underline{\chi}(\underline{X}, t)$ in the Eulerian configuration. The state vector is built upon the learnings of the small strain example. First, we consider the configuration mapping $\underline{\chi}$, and the linear momentum $\underline{p}\colon \Omega_X \to \Omega_x$ as components of the state vector. Additionally, we introduce the first Piola–Kirchhoff stress tensor $\underline{\underline{\Pi}}$ a two-point material-spatial tensor, such that $\underline{\underline{\Pi}} = \Pi_{aA}\underline{e}_a \otimes \underline{e}_A$, where $\Pi_{aA}\colon \Omega_X \times I \to \mathbb{R}$. Indeed, as we are in field theories, the multisymplectic framework arising from the Hamilton variational principle is more suitable [3,4,62]. This introduces a major distinction from the small strain example: the state vector does not include the derivative of the configuration $\underline{\underline{F}}\colon T\Omega_X \to T\Omega_x$ where

$$\underline{\underline{F}} = \frac{\partial \chi_a}{\partial X_A}\underline{e}_a \otimes \underline{e}_A, \quad \frac{\partial \chi_a}{\partial X_A} = F_{aA}, \quad F_{aA}\colon \Omega_X \times I \longrightarrow \mathbb{R}$$

(which, in the small strain one-dimensional setting, would be equivalent to using the small strain tensor $\varepsilon = (\partial u/\partial X)$). Finally, the temperature $T\colon \Omega_X \times I \to \mathbb{R}^+$ is considered, and the inverse inelastic Cauchy–Green purely Lagrangian second order tensor $\underline{\underline{C}}_{\mathrm{i}}^{-1} = (C_i^{-1})_{AB}\underline{e}_A \otimes \underline{e}_B$, $(C_i^{-1})_{AB}\colon \Omega_X \times I \to \mathbb{R}$ is used, as it is an objective tensor that describes finite viscoelasticity [44]. The state vector is then defined as

$$\boldsymbol{z} = \begin{pmatrix} \underline{\chi} & \underline{p} & \underline{\underline{\Pi}} & T & \underline{\underline{C}}_{\mathrm{i}}^{-1} \end{pmatrix}$$

and the density of the functional $\mathscr{F}$ will be a smooth function of these variables

$$f\colon \Omega_x \times \Omega_x \times T\Omega_x \times \mathbb{R}^+ \times T\Omega_X \longrightarrow \mathbb{R}, \quad (\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}, T, \underline{\underline{C}}_{\mathrm{i}}^{-1}) \longmapsto f(\underline{\chi}, \underline{p}, \underline{\underline{\Pi}}, T, \underline{\underline{C}}_{\mathrm{i}}^{-1}).$$

The total energy density $\widehat{E}_{\mathrm{tot}}$, the Hamiltonian density l^*, the Lagrangian density l will be defined as such. As in the small strain setting, we will demonstrate how to express the physics of both the reversible and irreversible parts of the problem.

Remark 14. One could notice that the first Piola–Kirchhoff stress tensor is not a Lagrangian tensor. Hence, the chain rule on the time derivative that applies to the other Lagrangian quantities will not apply to this tensor. Therefore, the development is not as straightforward as before, and we will need to express carefully our time derivatives using the Lie derivative, which is an objective time rate [63]. The objective will finally be to have an expression of the DGBA bracket structure for large strain thermo-visco-elastodynamics where the potentials (the total and internal energy, the entropy) will depend on the state vector.

3.2.1. *Evolution laws*

Evolution law of the inverse inelastic Cauchy–Green tensor. First, we start with the description of the evolution law for the inverse of the inelastic part of the Cauchy–Green tensor, derived from finite viscoelasticity models. A key assumption of finite viscoelasticity is the decomposition of the gradient of transformations into an elastic and an inelastic part, due to [64]:

$$\underline{\underline{F}} = \underline{\underline{F}}_{\mathrm{e}} \cdot \underline{\underline{F}}_{\mathrm{i}},$$

where we define the elastic part of the gradient of transformation a two-point material-spatial second order tensor $\underline{\underline{F}}_{\mathrm{e}}\colon T\Omega_X \to T\Omega_x$, and the inelastic part of the gradient of transformation, a fully material second order tensor $\underline{\underline{F}}_{\mathrm{i}}\colon T\Omega_X \to T\Omega_X$ [65]. From this partition, we define the following expressions of the Cauchy–Green tensor:

$$\underline{\underline{C}} = \underline{\underline{F}}^{\mathsf{T}}\underline{\underline{F}}, \quad \underline{\underline{C}}_{\mathrm{e}} = \underline{\underline{F}}_{\mathrm{e}}^{\mathsf{T}}\underline{\underline{F}}_{\mathrm{e}}, \quad \underline{\underline{C}}_{\mathrm{i}}^{-1} = \underline{\underline{F}}_{\mathrm{i}}^{-1}\underline{\underline{F}}_{\mathrm{i}}^{-\mathsf{T}}.$$

To establish the constitutive equation of viscoelasticity, we refer to the fundamental work of Reese and Govindjee [61]. They introduced the splitting of the Helmholtz free energy into an equilibrium and an out-of-equilibrium part

$$w\colon \mathbb{R}\times T\Omega_x \times T\Omega_X \longrightarrow \mathbb{R}, \quad w(T,\underline{\underline{F}},\underline{\underline{C}}_{\mathrm{e}}) = w_{\mathrm{EQ}}(T,\underline{\underline{F}}) + w_{\mathrm{NEQ}}(T,\underline{\underline{C}}_{\mathrm{e}})$$

using the Helmholtz free energy to build their behavior law, a standard process for phenomenological continuum thermodynamics [11].

Remark 15. Examples of detailed expressions of these free energies can be found in [44].

Other models of finite linear viscoelasticity exist: Kaliske [66] presents simple examples of viscoelastic stress-strain laws for unidimensional examples, in small strains, and for the generalized Maxwell model; Simo [67] presents a model of finite viscoelasticity which does not satisfy the second law of thermodynamics, as pointed out by Reese and Govindjee [68]. The developments of Reese and Govindjee yield the following viscoelastic evolution law:

$$\mathscr{L}_v(\underline{\underline{b}}_{\mathrm{e}}) = -2(\underline{\underline{\underline{\underline{V}}}}^{-1} : \underline{\underline{\tau}}_{\mathrm{NEQ}})\cdot\underline{\underline{b}}_{\mathrm{e}} \iff \frac{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}{\partial t} = -4\underline{\underline{F}}^{-1}\left[\underline{\underline{\underline{\underline{V}}}}^{-1} : \left(\underline{\underline{F}}^{-\mathsf{T}}\cdot\frac{\partial w_{\mathrm{NEQ}}}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}\cdot\underline{\underline{C}}_{\mathrm{i}}^{-1}\cdot\underline{\underline{F}}^{\mathsf{T}}\right)\right]\cdot\underline{\underline{F}}\cdot\underline{\underline{C}}_{\mathrm{i}}^{-1},$$

where $\underline{\underline{\underline{\underline{V}}}}^{-1}$ is a fourth-order Eulerian tensor of inelastic flow, $\underline{\underline{\underline{\underline{V}}}}^{-1} = (V^{-1})_{abcd}\underline{e}_a\otimes\underline{e}_b\otimes\underline{e}_c\otimes\underline{e}_d$, $(V^{-1})_{abcd}\in\mathbb{R}$, $\underline{\underline{b}}_{\mathrm{e}}$ is the Finger deformation tensor, a second order Eulerian second order tensor $\underline{\underline{b}}_{\mathrm{e}} = \underline{\underline{F}}_{\mathrm{e}}\cdot\underline{\underline{F}}_{\mathrm{e}}^{\mathsf{T}}$, and $\underline{\underline{\tau}}_{\mathrm{NEQ}}$ is the out-of-equilibrium Piola stress tensor (the Piola stress tensor is split into an equilibrium and an out-of-equilibrium part, as for the Helmholtz free energy $\underline{\underline{\tau}} = \underline{\underline{\tau}}_{\mathrm{EQ}} + \underline{\underline{\tau}}_{\mathrm{NEQ}}$, $\underline{\underline{\tau}} = \tau_{AB}\underline{e}_A\times\underline{e}_B$, $\tau_{AB}\colon\Omega_X\times I\to\mathbb{R}$, with the same definitions for the equilibrium and the out-of-equilibrium parts), and $\mathscr{L}_v$ is the Lie derivative. The precise expression of this tensor can be found in [61]. Details of calculus to obtain the viscoelastic law are in Section 2.1 of the supplementary document. Furthermore, we set

$$\underline{\underline{\underline{\underline{N}}}} = [\underline{\underline{F}}^{-1}\cdot(\underline{\underline{F}}^{-1}\cdot\underline{\underline{\underline{\underline{V}}}}^{-1}\cdot\underline{\underline{F}})^{\mathsf{T}}\cdot\underline{\underline{F}}]$$

as the fourth order Lagrangian tensor of inelastic flow [44], such that

$$\underline{\underline{\underline{\underline{N}}}} = \underline{\underline{\underline{\underline{N}}}}^{\mathsf{T}}, \quad N_{QACR} = N_{AQRC}\in\mathbb{R}$$

as shown in Section 2.2 of the supplementary document, for which we have

$$\frac{\partial f}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \frac{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}{\partial t} = -4\left[\frac{\partial \widehat{e}_{\mathrm{int}}}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}\cdot\underline{\underline{C}}_{\mathrm{i}}^{-1}\right] : \underline{\underline{\underline{\underline{N}}}} : \left[\frac{\partial f}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}\cdot\underline{\underline{C}}_{\mathrm{i}}^{-1}\right] + 4\left[\frac{\partial s}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}\cdot\underline{\underline{C}}_{\mathrm{i}}^{-1}\right] : T\underline{\underline{\underline{\underline{N}}}} : \left[\frac{\partial f}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}\cdot\underline{\underline{C}}_{\mathrm{i}}^{-1}\right]. \tag{26}$$

Remark 16. One could be surprised that, unlike the first example, there is no dissipation potential that is presented. Nonetheless, the evolution law given for the inelastic part of the Cauchy–Green tensor comes from the application of the Clausius–Duhem identity, implying the existence of a dissipation potential. Its developed expression is not required to derive the evolution law.

Evolution law of temperature. Second, we derive the evolution law of the temperature, with the particularity that potentials (as the total energy, the entropy, or the Helmholtz free energy) must be defined in terms of the first Piola–Kirchhoff stress tensor. Consider as a starting point the Legendre transformation of the Helmholtz free energy with respect to the transformation gradient to obtain

$$\begin{aligned}\rho w^*(T, \underline{\underline{\Pi}}, \underline{\underline{C}}_{\mathrm{i}}^{-1}) &= \underline{\underline{\Pi}} : \underline{\underline{F}} - \rho w(T, \underline{\underline{F}}, \underline{\underline{C}}_{\mathrm{i}}^{-1}) \\ &= \underbrace{\underline{\underline{\Pi}} : \underline{\underline{F}} + \rho T s(T, \underline{\underline{F}}, \underline{\underline{C}}_{\mathrm{i}}^{-1})}_{\rho T s^*(T, \underline{\underline{\Pi}}, \underline{\underline{C}}_{\mathrm{i}}^{-1})} - \rho \widehat{e}_{\mathrm{int}}(T, \underline{\underline{F}}, \underline{\underline{C}}_{\mathrm{i}}^{-1})\end{aligned} \tag{27}$$

where we define the density of entropy $s\colon \mathbb{R}^+ \times T\Omega_x \times T\Omega_X \to \mathbb{R}$, $(T, \underline{\underline{F}}, \underline{\underline{C}}_{\mathrm{i}}^{-1}) \mapsto s(T, \underline{\underline{F}}, \underline{\underline{C}}_{\mathrm{i}}^{-1})$ and the internal energy $\widehat{e}_{\mathrm{int}}\colon \mathbb{R}^+ \times T\Omega_x \times T\Omega_X \to \mathbb{R}$, $(T, \underline{\underline{F}}, \underline{\underline{C}}_{\mathrm{i}}^{-1}) \mapsto \widehat{e}_{\mathrm{int}}(T, \underline{\underline{F}}, \underline{\underline{C}}_{\mathrm{i}}^{-1})$. We define the Legendre transformation of the entropy with respect to the gradient of transformation as follows

$$s^*\colon \mathbb{R}^+ \times T\Omega_x \times T\Omega_X \longrightarrow \mathbb{R}, \quad (T, \underline{\underline{\Pi}}, \underline{\underline{C}}_{\mathrm{i}}^{-1}) \longmapsto s^*(T, \underline{\underline{\Pi}}, \underline{\underline{C}}_{\mathrm{i}}^{-1})$$

where

$$\rho T s^*(T, \underline{\underline{\Pi}}, \underline{\underline{C}}_{\mathrm{i}}^{-1}) = \underline{\underline{\Pi}} : \underline{\underline{F}} + \rho T s(T, \underline{\underline{F}}, \underline{\underline{C}}_{\mathrm{i}}^{-1}), \quad \frac{\partial s^*}{\partial \underline{\underline{\Pi}}} = \underline{\underline{F}}.$$

Remark 17. We observe in Eq. (27) that the Legendre transformation from the gradient of transformations to the first Piola–Kirchhoff stress tensor can be applied to either the entropy or the internal energy. Calculus displayed in the appendix (see Section B.3) show that we need to apply the Legendre transformation on the entropy. Then, the internal energy, through a change of variables (that does not take into account a Legendre transformation) is represented in terms of the first Piola–Kirchhoff stress tensor, defining $\widehat{e}^*_{\mathrm{int}}\colon \mathbb{R}^+ \times T\Omega_x \times T\Omega_X \to \mathbb{R}$, $(T, \underline{\underline{\Pi}}, \underline{\underline{C}}_{\mathrm{i}}^{-1}) \mapsto \widehat{e}^*_{\mathrm{int}}(T, \underline{\underline{\Pi}}, \underline{\underline{C}}_{\mathrm{i}}^{-1})$.

Furthermore, we need to be extremely cautious when deriving each term and applying the chain rule to the potentials. The rule cannot be used straightforward with mixed (or Eulerian) tensors as variables of the potentials. The key to solve the problem is to move from the representation of the potentials in terms of the first Piola–Kirchhoff stress tensor $\underline{\underline{\Pi}}$ to a representation in terms of the second Piola–Kirchhoff stress tensor $\underline{\underline{S}}$, and then applying the rules of derivation. This change of representation lies in the invertible relationship

$$\underline{\underline{\Pi}} = \underline{\underline{F}} \cdot \underline{\underline{S}}. \tag{28}$$

Each of these steps is clearly depicted in Section B.3. Therefore, we obtain the following evolution law of temperature:

$$\frac{\partial T}{\partial t} = \underbrace{-\frac{1}{\rho c}\,\mathrm{DIV}_{\underline{X}}(\underline{Q}) - \frac{1}{c}\frac{\partial \widehat{e}^*_{\mathrm{int}}}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} : \frac{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}{\partial t}}_{=:(\partial T/\partial t)_{\mathrm{irr}}} + \underbrace{\frac{T}{c}\frac{\partial s^*}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}}}_{=:(\partial T/\partial t)_{\mathrm{rev}}}, \tag{29}$$

where $c \in \mathbb{R}^+$ is the thermal capacity, $\rho \in \mathbb{R}^+$ is the material density, and $\mathrm{DIV}_{\underline{X}}(\underline{\cdot})$ is the Lagrangian divergence. As within the small strain setting, the temperature evolution can be split into a reversible and an irreversible part. We apply the Fourier law of conduction $\underline{Q} = -\underline{\underline{K}} \cdot (\partial T/\partial \underline{X})$ (where $\underline{\underline{K}}$ is the second order tensor of conduction, purely Lagrangian $\underline{\underline{K}} = K_{AB}\underline{e}_A \otimes \underline{e}_B$, $K_{AB}\colon \Omega_X \to \mathbb{R}$, symmetric (proved through Onsager reciprocity relationships), and becomes a constant for an isotropic material [11]) as well as the visco-elastic stress-strain law (26) to obtain the irreversible evolution of temperature

$$\left(\frac{\partial T}{\partial t}\right)_{\text{irr}} = -\frac{1}{\rho}\frac{\partial}{\partial \underline{X}}\cdot\left(\underline{\underline{K}}\frac{T^2}{c^2}\frac{\partial}{\partial \underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\right) + \frac{1}{c^2}4\left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right) : T\underline{\underline{\underline{\underline{N}}}} : \left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right)\frac{\partial s^*}{\partial T} - \frac{1}{c}4\left(\frac{\partial s^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right) : T\underline{\underline{\underline{\underline{N}}}} : \left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right). \tag{30}$$

Evolution law of the configuration, linear momentum and first Piola–Kirchhoff stress tensor — multisymplectic Poisson bracket. Third, we derive the evolution laws for the configuration, the linear momentum, and the first Piola–Kirchhoff stress tensor, from the expression of the Hamilton variational principle. First, we define the dual action

$$\mathscr{A}^*[\underline{\chi},\underline{p},\underline{\underline{\Pi}}] = \int_{\omega_t}\int_{\omega_X}\left(\underline{p}\cdot\frac{\partial \underline{\chi}}{\partial t} - \underline{\underline{\Pi}} : \frac{\partial \underline{\chi}}{\partial \underline{X}} - l^*(\underline{\chi},\underline{p},\underline{\underline{\Pi}},T,\underline{\underline{C}}_{\text{i}}^{-1})\right)\mathrm{d}\omega_X\,\mathrm{d}\omega_t \tag{31}$$

where we obtained the Hamiltonian density function by performing two partial Legendre transformations on the derivatives of the configuration

$$l^*(\underline{\chi},\underline{p},\underline{\underline{\Pi}},T,\underline{\underline{C}}_{\text{i}}^{-1}) = \underline{p}\cdot\frac{\partial \underline{\chi}}{\partial t} - \underline{\underline{\Pi}} : \frac{\partial \underline{\chi}}{\partial \underline{X}} - l\left(\underline{\chi},\frac{\partial \underline{\chi}}{\partial t},\frac{\partial \underline{\chi}}{\partial \underline{X}},T,\underline{\underline{C}}_{\text{i}}^{-1}\right),\quad \underline{p} = \frac{\partial l}{\partial\left(\frac{\partial \underline{\chi}}{\partial t}\right)},\quad \underline{\underline{\Pi}} = -\frac{\partial l}{\partial\left(\frac{\partial \underline{\chi}}{\partial \underline{X}}\right)}.$$

Then, Hamilton's principle states that

$$\delta\mathscr{A}^* + \int_{\omega_t}\int_{(\partial\omega_X)^{\text{Neumann}}} \underline{b}\cdot\delta\underline{\chi}\,\mathrm{d}(\partial\omega_X)\,\mathrm{d}\omega_t = 0 \qquad \forall\,\delta\underline{\chi} = \underline{0} \text{ on } \partial\omega_t, \delta\underline{p}, \delta\underline{\underline{\Pi}}$$

where $\underline{b}$ are prescribed forces on a part of the boundary $(\partial\omega_X)^{\text{Neumann}} \subseteq \partial\omega_X$. Variations are computed with respect to the configuration, the linear momentum, and the Piola–Kirchhoff stress tensors, as it would have been outside the thermo-visco-elastodynamics context. Then, after deriving each of the variations of the action in Section B.1, we obtain the following system of equations (within the body)

$$-\frac{\partial \underline{p}}{\partial t} + \mathrm{DIV}_{\underline{X}}(\underline{\underline{\Pi}}) - \frac{\partial l^*}{\partial \underline{\chi}} = \underline{0}, \tag{32}$$

$$\frac{\partial \underline{\chi}}{\partial t} - \frac{\partial l^*}{\partial \underline{p}} = \underline{0}, \tag{33}$$

$$-\frac{\partial \underline{\chi}}{\partial \underline{X}} - \frac{\partial l^*}{\partial \underline{\underline{\Pi}}} = \underline{0}, \tag{34}$$

in which we recognize the multisymplectic formulation of linear elastodynamics [3,69]

$$\begin{pmatrix}\frac{\partial l^*}{\partial \underline{\chi}}\\ \frac{\partial l^*}{\partial \underline{p}}\\ \frac{\partial l^*}{\partial \underline{\underline{\Pi}}}\end{pmatrix} = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}\underline{\chi}\\ \underline{p}\\ \underline{\underline{\Pi}}\end{pmatrix} + \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}\frac{\partial}{\partial \underline{X}}\begin{pmatrix}\underline{\chi}\\ \underline{p}\\ \underline{\underline{\Pi}}\end{pmatrix}. \tag{35}$$

Remark 18. The differential operator $\partial/\partial\underline{X}$ in Eq. (35) is a simplified notation used for convenience. It is equal to

$$\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}\frac{\partial}{\partial \underline{X}}\begin{pmatrix}\underline{\chi}\\ \underline{p}\\ \underline{\underline{\Pi}}\end{pmatrix} := \begin{pmatrix}\mathrm{DIV}_{\underline{X}}(\underline{\underline{\Pi}})\\ \underline{0}\\ -\mathrm{GRAD}_{\underline{X}}(\underline{\chi}) = -\underline{\underline{F}}\end{pmatrix}$$

where $\underline{\underline{F}}$ is the deformation gradient.

Remark 19. It is possible to obtain a reversible skew-symmetric bracket for large strain elastodynamics, as depicted in Section B.2. The whole proof lies in the use of the Lie derivative to obtain finally

$$\begin{aligned}
&\int_{\omega_X}\left(\frac{\partial f}{\partial \underline{\chi}}\cdot\frac{\partial \underline{\chi}}{\partial t}+\frac{\partial f}{\partial \underline{p}}\cdot\frac{\partial \underline{p}}{\partial t}-\frac{\partial f}{\partial \underline{\underline{F}}}:\dot{\underline{\underline{F}}}\right)\mathrm{d}\omega_X\\
&=\int_{\omega_X}\left(\frac{\partial f}{\partial \underline{\chi}}\cdot\frac{\partial l^*}{\partial \underline{p}}-\frac{\partial f}{\partial \underline{p}}\cdot\frac{\partial l^*}{\partial \underline{\chi}}+\underline{\underline{\nabla}}_X\frac{\partial f}{\partial \underline{p}}:\frac{\partial l^*}{\partial \underline{\underline{\Pi}}}\,\underline{\underline{S}}-\frac{\partial f}{\partial \underline{\underline{\Pi}}}:\underline{\underline{\nabla}}_X\frac{\partial l^*}{\partial \underline{p}}\,\underline{\underline{S}}\right)\mathrm{d}\omega_X+\int_{\partial\omega_X}\frac{\partial f}{\partial \underline{p}}\cdot(\underline{\underline{\Pi}}\cdot\underline{N})\,\mathrm{d}(\partial\omega_X)\\
&=:\{\mathscr{F},\mathscr{L}^*\}+\int_{\partial\omega_X}\frac{\partial f}{\partial \underline{p}}\cdot(\underline{\underline{\Pi}}\cdot\underline{N})\,\mathrm{d}(\partial\omega_X)
\end{aligned}$$

with

$$\mathscr{L}^*=\int_{\omega_X} l^*\,\mathrm{d}\omega_X.$$

This bracket is skew-symmetric provided that $\underline{\underline{S}}=\underline{\underline{S}}^{\mathsf{T}}$, which is a property of the second Piola–Kirchhoff stress tensor encoding the conservation of angular momentum. Furthermore, for a test function $f=\delta\underline{p}(\underline{X})\cdot u(\underline{X},t)+\delta\underline{\chi}(\underline{X})\cdot\underline{p}(\underline{X},t)+\delta\underline{\underline{\Pi}}(\underline{X}):\underline{\underline{F}}(\underline{X},t)$, one can recover the expression of the equations in terms of the multisymplectic skew-symmetric matrix [69]. Finally, this bracket differs from the one developed by Simo and co-authors, as the phase space of this bracket takes into account the first Piola–Kirchhoff stress tensor instead of the Cauchy–Green tensor, making the Legendre transformation one step further, as noted by Bridges [3].

As in the small strain setting, we extend the problem to reversible thermo- (visco)-elasticity by taking into account the total energy into the definition of the Hamiltonian density function

$$l^*(\underline{\chi},\underline{p},\underline{\underline{\Pi}},T,\underline{\underline{C}}_{\mathrm{i}}^{-1})=\widehat{E}^*_{\mathrm{tot}}(\underline{\chi},\underline{p},\underline{\underline{\Pi}},T,\underline{\underline{C}}_{\mathrm{i}}^{-1})-\rho T s^*(\underline{\underline{\Pi}},T,\underline{\underline{C}}_{\mathrm{i}}^{-1})$$

where the dual entropy is obtained through a Legendre transformation, while the total energy in terms of the Piola–Kirchhoff stress tensor is obtained through a simple change of variable (with the same method as for the internal energy).

3.2.2. *Expression of the DGBA bracket structure*

Using the previous developments, we have the expression of the DGBA bracket structure for large strain thermo-visco-elastodynamics

$$\begin{aligned}
\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t}&=\int_{\omega_X}\left(\frac{\partial f}{\partial \underline{\chi}}\cdot\frac{\partial \underline{\chi}}{\partial t}+\frac{\partial f}{\partial \underline{p}}\cdot\frac{\partial \underline{p}}{\partial t}-\frac{\partial f}{\partial \underline{\underline{F}}}:\dot{\underline{\underline{F}}}+\frac{\partial f}{\partial T}\left(\left(\frac{\partial T}{\partial t}\right)_{\mathrm{rev}}+\left(\frac{\partial T}{\partial t}\right)_{\mathrm{irr}}\right)+\frac{\partial f}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}:\frac{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}{\partial t}\right)\\
&=\{\mathscr{F},\widehat{\mathscr{E}}_{\mathrm{tot}}\}+(\mathscr{F},\mathscr{S}^*)+\int_{\partial\omega_X}\frac{\partial f}{\partial \underline{p}}\cdot(\underline{\underline{\Pi}}\cdot\underline{N})\,\mathrm{d}(\partial\omega_X)-\int_{\partial\omega_X}\frac{1}{\rho}\frac{\partial f}{\partial T}\underline{\underline{K}}\left(\frac{T}{c}\right)^2\cdot\frac{\partial}{\partial \underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\cdot\underline{N}\,\mathrm{d}(\partial\omega_X)
\end{aligned}$$

where

$$\begin{aligned}
\{\mathscr{F},\widehat{\mathscr{E}}_{\mathrm{tot}}\}=\int_{\omega_X}\Bigg(&\frac{\partial f}{\partial \underline{\chi}}\cdot\frac{\partial \widehat{E}^*_{\mathrm{tot}}}{\partial \underline{p}}-\frac{\partial f}{\partial \underline{p}}\cdot\frac{\partial \widehat{E}^*_{\mathrm{tot}}}{\partial \underline{\chi}}+\underline{\underline{\nabla}}_X\frac{\partial f}{\partial \underline{p}}:\frac{\partial \widehat{E}^*_{\mathrm{tot}}}{\partial \underline{\underline{\Pi}}}\,\underline{\underline{S}}-\frac{\partial f}{\partial \underline{\underline{\Pi}}}:\underline{\underline{\nabla}}_X\frac{\partial \widehat{E}^*_{\mathrm{tot}}}{\partial \underline{p}}\,\underline{\underline{S}}\\
&-\frac{\partial \widehat{E}^*_{\mathrm{tot}}}{\partial T}\left(\frac{\partial s^*}{\partial T}\right)^{-1}\underline{\underline{\nabla}}_X\frac{\partial f}{\partial \underline{p}}:\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}\,\underline{\underline{S}}+\frac{\partial f}{\partial T}\left(\frac{\partial s^*}{\partial T}\right)^{-1}\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}:\underline{\underline{\nabla}}_X\frac{\partial \widehat{E}^*_{\mathrm{tot}}}{\partial \underline{p}}\,\underline{\underline{S}}\Bigg)\mathrm{d}\omega_X
\end{aligned}$$

$$(\mathscr{F},\mathscr{S}^*)=\int_{\omega_X}\left[\frac{1}{\rho}\frac{\partial}{\partial \underline{X}}\left(\frac{\partial f}{\partial T}\right)\cdot \underline{\underline{K}}\left(\frac{T}{c}\right)^2\cdot\frac{\partial}{\partial \underline{X}}\left(\frac{\partial s^*}{\partial T}\right)+\frac{\partial f}{\partial T}\frac{4}{c^2}\left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right):T\underline{\underline{\underline{\underline{N}}}}:\left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right)\frac{\partial s^*}{\partial T}\right.$$
$$-\frac{\partial f}{\partial T}\frac{4}{c}\left(\frac{\partial s^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right):T\underline{\underline{\underline{\underline{N}}}}:\left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right)-\frac{\partial s^*}{\partial T}\frac{4}{c}\left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right):T\underline{\underline{\underline{\underline{N}}}}:\left(\frac{\partial f}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right)$$
$$\left.+4\left(\frac{\partial s^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right):T\underline{\underline{\underline{\underline{N}}}}:\left(\frac{\partial f}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right)\right]\mathrm{d}\omega_X.$$

Remark 20. As in the small strain setting, the equations of motion are obtained through the Galerkin expansion of the function f, as

$$f=\delta \underline{p}(\underline{X})\cdot\underline{\chi}(\underline{X},t)+\delta\underline{\chi}(\underline{X})\cdot\underline{p}(\underline{X},t)+\delta\underline{\underline{\Pi}}(\underline{X}):\underline{\underline{F}}(\underline{X},t)+\delta T(\underline{X})T(\underline{X},t)+\delta \underline{\underline{C}}_{\text{i}}^{-1}:\underline{\underline{C}}_{\text{i}}^{-1}(\underline{X},t)$$

which, after calculus, gives the following equations of motion:

$$\frac{\partial\underline{\chi}}{\partial t}=\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{p}},$$
$$\dot{\underline{\underline{F}}}=\underline{\underline{\nabla}}_X\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{p}},$$
$$\frac{\partial\underline{p}}{\partial t}=-\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{\chi}}-\frac{\partial}{\partial \underline{X}}\left(\frac{\partial l^*}{\partial \underline{\underline{\Pi}}}\underline{\underline{S}}\right),$$
$$\frac{\partial T}{\partial t}=\left(\frac{\partial s^*}{\partial T}\right)^{-1}\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}:\underline{\underline{\nabla}}_X\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{p}}\underline{\underline{S}}+\frac{1}{\rho}\frac{\partial}{\partial \underline{X}}\cdot\left(\underline{\underline{K}}\cdot\frac{\partial T}{\partial \underline{X}}\right)+4\left(\frac{\partial w^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right):\underline{\underline{\underline{\underline{N}}}}:\left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right),$$
$$\frac{\partial \underline{\underline{C}}_{\text{i}}^{-1}}{\partial t}=-4\left(\left(\frac{\partial w^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right):\underline{\underline{\underline{\underline{N}}}}\right)\cdot \underline{\underline{C}}_{\text{i}}^{-1}.$$

The equations are not expanded in terms of the expressions of the potentials, as the technicality of these might infer with the understanding of the meaning of the equations. We can draw conclusions from the expressions obtained above. The first equation corresponds to the definition of the first Legendre transformation on the Hamiltonian density function, while the second is exactly the space derivative of the first one. The third equation is the conservation of linear momentum, observing that $(\partial l^*/\partial\underline{\underline{\Pi}})\underline{\underline{S}}=-\underline{\underline{\Pi}}$. Then, the fourth equation is the evolution of temperature, where the first term (right to the equal sign) corresponds to the thermo-mechanical coupling, the second one to the conduction effects, and the third one to the thermal-viscous coupling. Finally, the last equation is the evolution equation of the inverse of the inelastic Cauchy–Green tensor, expressed in terms of the fourth order Lagrangian tensor of inelastic flow. It can be expressed again in terms of the Eulerian fourth order tensor of inelastic flow, hence as in the first expression of the evolution of the inelastic Cauchy–Green tensor using index calculus as follows. First, the expression of the fourth order, Lagrangian tensor of inelastic flow is given, in index notation

$$N_{QACR}=\left[\underline{\underline{F}}^{-1}\cdot(\underline{\underline{F}}^{-1}\cdot\underline{\underline{\underline{\underline{V}}}}^{-1}\cdot\underline{\underline{F}})^{\mathsf{T}}\cdot\underline{\underline{F}}\right]_{QACR}=[F^{-\mathsf{T}}]_{pQ}[F^{-1}]_{Aa}[V^{-1}]_{apqb}[F]_{bC}[F^{\mathsf{T}}]_{Rq}.$$

Then, the expression of the evolution law obtained through the Galerkin expansion of the DGBA bracket structure becomes

$$\begin{aligned}\frac{\partial \underline{\underline{C}}_{\text{i}}^{-1}}{\partial t}&=-4\left(\left(\frac{\partial w^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right):\underline{\underline{\underline{\underline{N}}}}\right)\cdot \underline{\underline{C}}_{\text{i}}^{-1}\\&=-4\left(\frac{\partial w^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right)_{QR}[F^{-\mathsf{T}}]_{pQ}[F^{-1}]_{Aa}[V^{-1}]_{apqb}[F]_{bC}[F^{\mathsf{T}}]_{Rq}[C_{\text{i}}^{-1}]_{CK}\\&=-4[F^{-1}]_{Aa}[V^{-1}]_{apqb}[F^{-\mathsf{T}}]_{pQ}\left(\frac{\partial w^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot \underline{\underline{C}}_{\text{i}}^{-1}\right)_{QR}[F^{\mathsf{T}}]_{Rq}[F]_{bC}[C_{\text{i}}^{-1}]_{CK}\end{aligned}$$

$$= -4[F^{-1}]_{Aa}\left[\underline{\underline{\underline{\underline{V}}}}^{-1} : \left(\underline{\underline{F}}^{-\mathsf{T}} \cdot \underline{\underline{C}} \cdot \frac{\partial w_{\mathrm{NEQ}}}{\partial \underline{\underline{C}}} \cdot \underline{\underline{F}}^{\mathsf{T}}\right)\right]_{ab}[F]_{bC}[C_{\mathrm{i}}^{-1}]_{CK}$$

$$= -4\underline{\underline{F}}^{-1}\left[\underline{\underline{\underline{\underline{V}}}}^{-1} : \left(\underline{\underline{F}}^{-\mathsf{T}} \cdot \underline{\underline{C}} \cdot \frac{\partial w_{\mathrm{NEQ}}}{\partial \underline{\underline{C}}} \cdot \underline{\underline{F}}^{\mathsf{T}}\right)\right] \cdot \underline{\underline{F}} \cdot \underline{\underline{C}}_{\mathrm{i}}^{-1}.$$

Properties of the reversible bracket. The skew-symmetry is verified in Section 2.5 of the supplementary document.

Remark 21. The verification of the Jacobi identity is proved for functionals which successive partial derivatives are consistent with the particular expression of the total energy for large strain thermo-visco-elastodynamics. It is carefully displayed within Section 2.8 of the supplementary document. The proof develops the same methodology as for the small strain example, i.e. consecutively applying the skew-symmetric operator to the quantities considered in the Jacobi identity. The general proof without these assumptions is left out for future investigations.

Properties of the dissipative bracket. Second, the symmetry of the dissipative bracket is verified in Section 2.6 of the supplementary document, relying on the symmetry of both the conduction and Lagrangian viscous flow tensors (where the symmetry of the latter is proved in Section 2.3 of the supplementary document). The positivity of the dissipative bracket is verified in Section 2.7 of the supplementary document, based on the positivity of these tensors.

3.2.3. *Conservation laws*

Finally, we verify that conservation laws are indeed ensured by the DGBA bracket structure.

Conservation of linear momentum. It is demonstrated by taking the general functional $\mathscr{F}$ to be equal to the total linear momentum

$$\mathscr{F} = \int_{\omega_X} \underline{p}\,\mathrm{d}\omega_X.$$

Therefore, the proof of conservation of linear momentum is

$$\int_{\omega_X} \frac{\partial \underline{p}}{\partial t}\,\mathrm{d}\omega_X = \int_{\omega_X} -\frac{\partial \widehat{E}^*_{\mathrm{tot}}}{\partial \underline{\chi}}\,\mathrm{d}\omega_X + \int_{\partial\omega_X} \underline{\underline{\Pi}} \cdot \underline{N}\,\mathrm{d}(\partial\omega_X) \quad \Longrightarrow \quad \int_{\omega_X} \left(\frac{\partial \underline{p}}{\partial t} + \frac{\partial \widehat{E}^*_{\mathrm{tot}}}{\partial \underline{\chi}} - \mathrm{DIV}_{\underline{X}}\,\underline{\underline{\Pi}}\right)\mathrm{d}\omega_X = 0$$

which is true for all open subset $\omega_X \subset \Omega_X$. We obtain the Lagrangian expression of the conservation of linear momentum

$$\frac{\partial \underline{p}}{\partial t} + \frac{\partial \widehat{E}^*_{\mathrm{tot}}}{\partial \underline{\chi}} - \mathrm{DIV}_{\underline{X}}\,\underline{\underline{\Pi}} = \underline{0}.$$

Conservation of angular momentum. We set the test function to be equal to the angular momentum

$$f = (\underline{\chi} \times \underline{p}), \quad \mathscr{F} = \int_{\omega_X} (\underline{\chi} \times \underline{p})\,\mathrm{d}\omega_X.$$

This expression of f leads to the following expression of the DGBA bracket structure:

$$0 = \int_{\omega_X} \underline{p} \times \left(\frac{\partial \underline{\chi}}{\partial t} - \frac{\partial \widehat{E}_{\mathrm{tot}}}{\partial \underline{p}}\right)\mathrm{d}\omega_X - \int_{\omega_X} \underline{\chi} \times \left(\frac{\partial \underline{p}}{\partial t} + \frac{\partial \widehat{E}_{\mathrm{tot}}}{\partial \underline{\chi}} - \mathrm{DIV}_{\underline{X}}\,\underline{\underline{\Pi}}\right)\mathrm{d}\omega_X$$
$$+ \int_{\omega_X} \underline{\nabla}_{\underline{X}}\left(\frac{\partial \underline{\chi} \times \underline{p}}{\partial \underline{p}}\right) : \left(\frac{\partial l^*}{\partial \underline{\underline{\Pi}}} \cdot \underline{\underline{S}} + \underline{\underline{\Pi}}\right)\mathrm{d}\omega_X.$$

This equation can be solved for fields such that the first Legendre transformation (33), the conservation of linear momentum (32), and eventually both the second Legendre transformation (34) and the relationship between the first and second Piola–Kirchhoff stress tensors (28) hold true.

$$\underline{\underline{\Pi}} = \underline{\underline{F}} \cdot \underline{\underline{S}} \quad \text{where } \underline{\underline{S}}^{\mathsf{T}} = \underline{\underline{S}} \quad (\Leftrightarrow\ \underline{\underline{\Pi}} \cdot \underline{\underline{F}}^{\mathsf{T}} = \underline{\underline{F}} \cdot \underline{\underline{\Pi}}^{\mathsf{T}}).$$

Therefore, as noticed in Remark 19, the conservation of angular momentum is already encoded within the skew-symmetry of the reversible bracket.

First law of thermodynamics. It is derived taking the general functional to be equal to the total energy, that is the sum of the kinetic energy and the internal energy.

$$f = \widehat{E}^*_{\text{tot}}, \quad \mathscr{F} = \widehat{\mathscr{E}}^*_{\text{tot}} = \int_{\omega_X} \underbrace{\left(\frac{1}{2}\frac{1}{\rho}\underline{p}\cdot\underline{p} + \rho\widehat{e}^*_{\text{int}}\right)}_{=\hat{E}^*_{\text{tot}}} \mathrm{d}\omega_X.$$

Its time derivative is given by

$$\begin{aligned}\frac{\mathrm{d}\widehat{\mathscr{E}}^*_{\text{tot}}}{\mathrm{d}t} &= \{\widehat{\mathscr{E}}^*_{\text{tot}}, \widehat{\mathscr{E}}^*_{\text{tot}}\} + (\widehat{\mathscr{E}}^*_{\text{tot}}, \mathscr{S}^*) \\ &\quad + \int_{\partial\omega_X} \frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{p}} \cdot (\underline{\underline{\Pi}}\cdot\underline{N})\,\mathrm{d}(\partial\omega_X) - \int_{\partial\omega_X} \frac{1}{\rho}\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial T}\underline{\underline{K}}\left(\frac{T}{c}\right)^2 \cdot \frac{\partial}{\partial \underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\cdot\underline{N}\,\mathrm{d}(\partial\omega_X) \end{aligned} \tag{36}$$

where the skew-symmetric bracket vanishes by skew-symmetry. The symmetric bracket yields

$$\begin{aligned}(\widehat{\mathscr{E}}^*_{\text{tot}}, \mathscr{S}^*) &= \int_{\omega_X} \frac{1}{\rho}\frac{\partial}{\partial \underline{X}}\left(\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial T}\right)\cdot\underline{\underline{K}}\left(\frac{T}{c}\right)^2\cdot\frac{\partial}{\partial \underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\mathrm{d}\omega_X \\ &\quad + \int_{\omega_X}\Bigg[\underbrace{\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial T}\frac{1}{c}\frac{4}{c}\left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right):T\underline{\underline{\underline{\underline{N}}}}:\left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right)\frac{\partial s^*}{\partial T}}_{=(\text{a})} \\ &\quad \underbrace{-\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial T}\frac{1}{c}4\left(\frac{\partial s^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right):T\underline{\underline{\underline{\underline{N}}}}:\left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right)}_{=-(\text{b})} \\ &\quad \underbrace{-\frac{\partial s^*}{\partial T}\frac{4}{c}\left(\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right):T\underline{\underline{\underline{\underline{N}}}}:\left(\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right)}_{=-(\text{a})} \\ &\quad \underbrace{+4\left(\frac{\partial s^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right):T\underline{\underline{\underline{\underline{N}}}}:\left(\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right)}_{=(\text{b})}\Bigg]\mathrm{d}\omega_X\end{aligned}$$

leaving only the conduction term which becomes, when summed with the boundary conduction term,

$$\begin{aligned}&\int_{\omega_X}\frac{1}{\rho}\frac{\partial}{\partial \underline{X}}\left(\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial T}\right)\cdot\underline{\underline{K}}\left(\frac{T}{c}\right)^2\cdot\frac{\partial}{\partial \underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\mathrm{d}\omega_X - \int_{\partial\omega_X}\frac{1}{\rho}\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial T}\underline{\underline{K}}\left(\frac{T}{c}\right)^2\cdot\frac{\partial}{\partial \underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\cdot\underline{N}\,\mathrm{d}(\partial\omega_X) \\ &\qquad = \int_{\omega_X}\frac{1}{\rho}\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial T}\frac{1}{c}\cdot\frac{\partial}{\partial \underline{X}}\left(\underline{\underline{K}}\cdot\frac{\partial T}{\partial \underline{X}}\right)\mathrm{d}\omega_X - \int_{\partial\omega_X}\frac{1}{\rho}\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial T}\cdot\underline{\underline{K}}\frac{1}{c}\cdot\frac{\partial T}{\partial \underline{X}}\mathrm{d}(\partial\omega_X) \\ &\qquad\qquad - \int_{\partial\omega_X}\frac{1}{\rho}\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial T}\underline{\underline{K}}\left(\frac{T}{c}\right)^2\cdot\frac{\partial}{\partial \underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\cdot\underline{N}\,\mathrm{d}(\partial\omega_X) \\ &\qquad = \int_{\omega_X} -\mathrm{DIV}_{\underline{X}}\,\underline{Q}\,\mathrm{d}\omega_X.\end{aligned}$$

Furthermore, the time derivative of the total energy becomes, when taking into account the conservation of linear momentum, as well as the definition of the first Legendre transformation (from velocity to the linear momentum)

$$\begin{aligned}\frac{\mathrm{d}\widehat{\mathscr{E}}^*_{\text{tot}}}{\mathrm{d}t} &= \int_{\omega_X}\left(\frac{1}{\rho}\underline{p}\cdot\frac{\partial \underline{p}}{\partial t}+\rho\frac{\partial \widehat{e}^*_{\text{int}}}{\partial t}\right)\mathrm{d}\omega_X\\ &= \int_{\omega_X}\left(\frac{\partial \underline{\chi}}{\partial t}\cdot\left(-\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{\chi}}+\mathrm{DIV}_{\underline{X}}(\underline{\underline{\Pi}})\right)+\rho\frac{\partial \widehat{e}^*_{\text{int}}}{\partial t}\right)\mathrm{d}\omega_X\\ &= \int_{\omega_X}\left(-\frac{\partial^2\underline{\chi}}{\partial \underline{X}\partial t}:\underline{\underline{\Pi}}+\rho\frac{\partial \widehat{e}^*_{\text{int}}}{\partial t}\right)\mathrm{d}\omega_X+\int_{\partial\omega_X}\frac{\partial \underline{\chi}}{\partial t}\cdot\underline{\underline{\Pi}}\cdot\underline{N}\,\mathrm{d}(\partial\omega_X)\end{aligned}$$

where the boundary term vanishes thanks to the remaining boundary term within the structure (first term of Eq. (36)). Therefore, we obtain the first law of thermodynamics for large strains, in the Lagrangian framework

$$\rho\frac{\partial \widehat{e}^*_{\text{int}}}{\partial t} = \frac{\partial \underline{\underline{F}}}{\partial t}:\underline{\underline{\Pi}}-\mathrm{DIV}_{\underline{X}}\,\underline{Q}.$$

Second law of thermodynamics. Consider

$$f = s^*,\quad \mathscr{F} = \mathscr{S}^* = \int_{\omega_X} s^*\,\mathrm{d}\omega_X$$

yielding, within the DGBA bracket structure expression

$$\frac{\mathrm{d}\mathscr{S}^*}{\mathrm{d}t}+\int_{\partial\omega_X}\frac{1}{\rho}\frac{\partial s^*}{\partial T}\underline{\underline{K}}\left(\frac{T}{c}\right)^2\cdot\frac{\partial}{\partial \underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\cdot\underline{N}\,\mathrm{d}(\partial\omega_X) = \{\mathscr{S}^*,\widehat{\mathscr{E}}^*_{\text{tot}}\}+(\mathscr{S}^*,\mathscr{S}^*) \tag{37}$$

where

$$\begin{aligned}\{\mathscr{S}^*,\widehat{\mathscr{E}}^*_{\text{tot}}\} &= \int_{\omega_X}\left(0-0-\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}:\underline{\nabla}_X\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{p}}\,\underline{\underline{S}}+0+\frac{\partial s}{\partial T}\left(\frac{\partial s^*}{\partial T}\right)^{-1}\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}:\underline{\nabla}_X\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{p}}\,\underline{\underline{S}}-0\right)\mathrm{d}\omega_X\\ &= \int_{\omega_X}\left(-\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}:\underline{\nabla}_X\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{p}}\,\underline{\underline{S}}+\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}:\underline{\nabla}_X\frac{\partial \widehat{E}^*_{\text{tot}}}{\partial \underline{p}}\,\underline{\underline{S}}\right)\mathrm{d}\omega_X\\ &= 0.\end{aligned}$$

Again, as in the small strain setting, we refer to Remark 4: our development fulfills *a posteriori* the non-interaction condition. The following developments demonstrate that the equation aligns precisely with the classical Clausius–Duhem identity [11]. First, we express the dissipative bracket as

$$\begin{aligned}(\mathscr{S}^*,\mathscr{S}^*) = \int_{\omega_X}\Bigg[&\frac{1}{\rho}\frac{\partial}{\partial \underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\cdot\underline{\underline{K}}\left(\frac{T}{c}\right)^2\cdot\frac{\partial}{\partial \underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\\ &+\left(\frac{\partial s^*}{\partial T}\frac{2}{c}\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}-2\frac{\partial s^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right):T\underline{\underline{\underline{\underline{N}}}}:\left(\frac{\partial s^*}{\partial T}\frac{2}{c}\frac{\partial \widehat{e}^*_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}-2\frac{\partial s^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right)\Bigg]\mathrm{d}\omega_X\end{aligned}$$

and we have

$$\frac{\partial s^*}{\partial T}\frac{2}{c}\frac{\partial \widehat{e}_{\text{int}}}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}-2\frac{\partial s^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1} = \frac{2}{T}\frac{\partial(\widehat{e}^*_{\text{int}}-Ts^*)}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1} = \frac{2}{T}\frac{\partial w^*}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1} = -\frac{2}{T}\frac{\partial w}{\partial \underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}.$$

Then

$$\int_{\omega_X}\left(\frac{\partial s^*}{\partial T}\frac{2}{c}\frac{\partial\widehat{e}^*_{\text{int}}}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}-2\frac{\partial s^*}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\underline{\underline{C}}_{\text{i}}^{-1}\right):T\underline{\underline{\underline{\underline{N}}}}:\left(\frac{\partial s^*}{\partial T}\frac{2}{c}\frac{\partial\widehat{e}^*_{\text{int}}}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}-2\frac{\partial s^*}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\right)\mathrm{d}\omega_X$$
$$=\int_{\omega_X}\frac{2}{T}\frac{\partial w}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}:T\underline{\underline{\underline{\underline{N}}}}:\frac{2}{T}\frac{\partial w}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\cdot\underline{\underline{C}}_{\text{i}}^{-1}\,\mathrm{d}\omega_X$$
$$=\int_{\omega_X}\frac{1}{T}\left(-\frac{\partial w}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}:\frac{\partial\underline{\underline{C}}_{\text{i}}^{-1}}{\partial t}\right)\mathrm{d}\omega_X.$$

Furthermore, we have

$$\int_{\omega_X}\frac{1}{\rho}\frac{\partial}{\partial\underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\cdot\underline{\underline{K}}\left(\frac{T}{c}\right)^2\cdot\frac{\partial}{\partial\underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\mathrm{d}\omega_X=\int_{\omega_X}\frac{1}{\rho}\frac{\partial T^{-1}}{\partial\underline{X}}c\cdot\underline{\underline{K}}\left(\frac{T}{c}\right)^2\cdot\frac{\partial T^{-1}}{\partial\underline{X}}c\,\mathrm{d}\omega_X$$
$$=\int_{\omega_X}\frac{1}{\rho}\frac{1}{T^2}\frac{\partial T}{\partial\underline{X}}\cdot\underline{\underline{K}}\cdot\frac{\partial T}{\partial\underline{X}}\mathrm{d}\omega_X.$$

Then, we have

$$(\mathscr{S}^*,\mathscr{S}^*)=\int_{\omega_X}\left(\frac{1}{\rho}\frac{1}{T^2}\frac{\partial T}{\partial\underline{X}}\cdot\underline{\underline{K}}\cdot\frac{\partial T}{\partial\underline{X}}+\frac{1}{T}\left(-\frac{\partial w}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}:\frac{\partial\underline{\underline{C}}_{\text{i}}^{-1}}{\partial t}\right)\right).$$

Moreover, as in the small strain setting, one can show that

$$\int_{\partial\omega_X}\frac{1}{\rho}\frac{\partial s^*}{\partial T}\underline{\underline{K}}\left(\frac{T}{c}\right)^2\cdot\frac{\partial}{\partial\underline{X}}\left(\frac{\partial s^*}{\partial T}\right)\cdot\underline{N}\,\mathrm{d}(\partial\omega_X)=\int_{\omega_X}\frac{1}{\rho}\mathrm{DIV}_{\underline{X}}\left(\frac{\underline{\underline{Q}}}{T}\right)\mathrm{d}\omega_X.$$

Thus, Eq. (37) finally leads to

$$\int_{\omega_X}\left(\rho\frac{\partial s^*}{\partial t}+\mathrm{DIV}_{\underline{X}}\left(\frac{\underline{\underline{Q}}}{T}\right)\right)\mathrm{d}\omega_X=(\mathscr{S}^*,\mathscr{S}^*)=\int_{\omega_X}\frac{1}{T}\left(-\rho\frac{\partial w}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}:\frac{\partial\underline{\underline{C}}_{\text{i}}^{-1}}{\partial t}+\frac{1}{T}\left(\underline{\underline{K}}\cdot\frac{\partial T}{\partial\underline{X}}\right)\cdot\frac{\partial T}{\partial\underline{X}}\right)\mathrm{d}\omega_X\geq 0$$

which is positive by the positivity of the dissipative bracket. Therefore, we recover both the second principle of thermodynamics and Clausius–Duhem identity for the large strain problem

$$\rho\frac{\partial s^*}{\partial t}+\mathrm{DIV}_{\underline{X}}\left(\frac{\underline{\underline{Q}}}{T}\right)\geq 0\quad\Longleftrightarrow\quad-\frac{1}{T}\underline{Q}\cdot\frac{\partial T}{\partial\underline{X}}-\rho\frac{\partial w}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}:\frac{\partial\underline{\underline{C}}_{\text{i}}^{-1}}{\partial t}\geq 0.$$

3.2.4. *Linearization of the dissipative bracket*

The large strains dissipation bracket can be particularised to derive the unidimensional small strain thermo-visco-elasticity bracket. One can show that the Sidoroff multiplicative decomposition leads to the partition of the strain tensor into an elastic and a viscoelastic part of small strain visco-elasticity

$$\left|\frac{\partial u}{\partial X}\right|\ll 1,\qquad\underline{\underline{F}}=\underline{\underline{F}}_{\text{e}}\cdot\underline{\underline{F}}_{\text{i}}\;\Rightarrow\;\varepsilon=\varepsilon_{\text{e}}+\varepsilon_{\text{i}}$$

where $\varepsilon_{\text{e}}=(\partial u/\partial X)_{\text{e}}$ and $\varepsilon_{\text{i}}=(\partial u/\partial X)_{\text{i}}$. Then, the expression of the large strain dissipative bracket yields

$$(\mathscr{F},\mathscr{S})=\int_{\omega_X}\left[\frac{1}{\rho}\frac{\partial}{\partial X}\left(\frac{\partial f}{\partial T}\right)K\left(\frac{T}{c}\right)^2\frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right)+\frac{\partial f}{\partial T}\frac{4}{c^2}\left(\frac{\partial\widehat{e}_{\text{int}}}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\underline{\underline{C}}_{\text{i}}^{-1}\right)V^{-1}\left(\frac{\partial\widehat{e}_{\text{int}}}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\underline{\underline{C}}_{\text{i}}^{-1}\right)\frac{\partial s}{\partial T}\right.$$
$$\left.-\frac{\partial f}{\partial T}\frac{4}{c}\left(\frac{\partial s}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\underline{\underline{C}}_{\text{i}}^{-1}\right)TV^{-1}\left(\frac{\partial\widehat{e}_{\text{int}}}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\underline{\underline{C}}_{\text{i}}^{-1}\right)-\frac{\partial s}{\partial T}\frac{4}{c}\left(\frac{\partial w}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\underline{\underline{C}}_{\text{i}}^{-1}\right)TV^{-1}\left(\frac{\partial f}{\partial\underline{\underline{C}}_{\text{i}}^{-1}}\underline{\underline{C}}_{\text{i}}^{-1}\right)\right]\mathrm{d}\omega_X.$$

Furthermore, a first order limited development of the Cauchy–Green tensor yields

$$\underline{\underline{C}}_{\text{i}}^{-1}=\left(\left(\frac{\partial u}{\partial X}\right)_{\text{i}}+1\right)^{-2}\approx 1-2\varepsilon_{\text{i}}.$$

Then, one may obtain the following expressions for the partial derivatives with respect of the inelastic Cauchy–Green inelastic tensor

$$\frac{\partial f}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}} \approx -\frac{1}{2}\frac{\partial f}{\partial \varepsilon_{\mathrm{i}}}$$

giving the following linearized expression of the dissipative bracket

$$\begin{aligned}
(\mathscr{F},\mathscr{S}) &= \int_{\omega_X} \left[\frac{1}{\rho}\frac{\partial}{\partial X}\left(\frac{\partial f}{\partial T}\right) K \left(\frac{T}{c}\right)^2 \frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right) + \frac{\partial f}{\partial T}\frac{1}{c^2}\left(\frac{\partial \widehat{e}_{\mathrm{int}}}{\partial \varepsilon_{\mathrm{i}}}(1-2\varepsilon_{\mathrm{i}})\right) V^{-1}\left(\frac{\partial \widehat{e}_{\mathrm{int}}}{\partial \varepsilon_{\mathrm{i}}}(1-2\varepsilon_{\mathrm{i}})\right)\frac{\partial s}{\partial T} \right.\\
&\qquad \left. -\frac{\partial f}{\partial T}\frac{1}{c}\left(\frac{\partial s}{\partial \varepsilon_{\mathrm{i}}}(1-2\varepsilon_{\mathrm{i}})\right) T V^{-1}\left(\frac{\partial \widehat{e}_{\mathrm{int}}}{\partial \varepsilon_{\mathrm{i}}}(1-2\varepsilon_{\mathrm{i}})\right) - \frac{\partial s}{\partial T}\frac{4}{c}\left(\frac{\partial w}{\partial \varepsilon_{\mathrm{i}}}(1-2\varepsilon_{\mathrm{i}})\right) T V^{-1}\left(\frac{\partial f}{\partial \varepsilon_{\mathrm{i}}}(1-2\varepsilon_{\mathrm{i}})\right)\right] \mathrm{d}\omega_X \\
&= \int_{\omega_X} \left[\frac{1}{\rho}\frac{\partial}{\partial X}\left(\frac{\partial f}{\partial T}\right) K \left(\frac{T}{c}\right)^2 \frac{\partial}{\partial X}\left(\frac{\partial s}{\partial T}\right) + \frac{\partial f}{\partial T}\frac{1}{c^2}\left(\frac{\partial \widehat{e}_{\mathrm{int}}}{\partial \varepsilon_{\mathrm{i}}}\right) V^{-1}\left(\frac{\partial \widehat{e}_{\mathrm{int}}}{\partial \varepsilon_{\mathrm{i}}}\right)\frac{\partial s}{\partial T} \right.\\
&\qquad \left. -\frac{\partial f}{\partial T}\frac{1}{c}\left(\frac{\partial s}{\partial \varepsilon_{\mathrm{i}}}\right) T V^{-1}\left(\frac{\partial \widehat{e}_{\mathrm{int}}}{\partial \varepsilon_{\mathrm{i}}}\right) - \frac{\partial s}{\partial T}\frac{4}{c}\left(\frac{\partial w}{\partial \varepsilon_{\mathrm{i}}}\right) T V^{-1}\left(\frac{\partial f}{\partial \varepsilon_{\mathrm{i}}}\right)\right] \mathrm{d}\omega_X \\
&= \int_{\omega_X} \frac{\partial f}{\partial \boldsymbol{z}} \cdot \boldsymbol{M}(\boldsymbol{z}) \cdot \frac{\partial s}{\partial \boldsymbol{z}} \, \mathrm{d}\omega_X
\end{aligned}$$

where

$$\boldsymbol{M}(\underline{z}) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\rho}\frac{\partial \square_{\mathrm{left}}}{\partial X} K \left(\frac{T}{c}\right)^2 \frac{\partial \square_{\mathrm{right}}}{\partial X} + \frac{1}{c^2}\frac{\partial \widehat{e}_{\mathrm{int}}}{\partial \varepsilon_{\mathrm{i}}} V^{-1} \frac{\partial \widehat{e}_{\mathrm{int}}}{\partial \varepsilon_{\mathrm{i}}} & -\frac{1}{c} T V^{-1}\frac{\partial \widehat{e}_{\mathrm{int}}}{\partial \varepsilon_{\mathrm{i}}} \\
0 & 0 & 0 & -\frac{1}{c} T V^{-1}\frac{\partial \widehat{e}_{\mathrm{int}}}{\partial \varepsilon_{\mathrm{i}}} & T V^{-1}
\end{pmatrix}$$

considering linear terms in ε_{i} and higher order terms to be neglected. This expression is exactly the expression of the DGBA bracket structure formulation of a viscoelastic material, i.e. for $\boldsymbol{V} = V$ a dissipation coefficient (23).

4. Conclusions and perspectives

The presented approach has successfully shown how to obtain the Double Generator Boundary Augmented bracket structure from classical continuum thermodynamics equations. Unlike other bracket structures, this methodology starts with the state laws. As a consequence, a key contribution, distinguishing it from other bracket frameworks (such as GENERIC, metriplectic...) is the inclusion of non-zero boundary operators, essential for ensuring conservation and thermodynamics laws.

The one-dimensional small strain example, with its simple scalar notation, familiarized us with the structure and methodology, providing many important results. We derived, for the first time, a bracket structure for a dissipative generalized standard material with a quadratic dissipation potential, using the Biot relationships and the Onsager reciprocity relationships. Embedding the physics within the reversible-irreversible structure clarifies the partition of the evolution of the temperature into reversible and irreversible contributions. While the reversible bracket has already been derived previously in the literature [60], the proof of the Jacobi identity reveals crucial assumptions on third order derivatives of the entropy. Finally, the obtained structure accurately recovers the principles of mechanics including conservation of linear momentum and the first and second laws of thermodynamics.

Second, the large strain tridimensional thermo-visco-elasticity example sets itself in the multisymplectic framework, natural for field theories. A multisymplectic reversible skew-symmetric bracket has been developed, verifying skew symmetry and the Jacobi identity under assumptions, for both elastodynamics and thermo-(visco)-elastodynamics. Finally, as in the small strain example, the structure recovers exactly the principles of mechanics: conservation of linear and angular momentum, first and second laws of thermodynamics.

This paper sets the framework to build variational integrators that preserve the DGBA bracket structure, i.e. that automatically enforce conservation and thermodynamic laws. Furthermore, the precise geometrical nature of the reversible bracket as a Poisson bracket, as well as the geometrical insight that is brought by the addition of the boundary terms are left out for future investigations.

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Declaration of interest

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

Underlying data

Supporting information for this article is available at https://hal-lara.archives-ouvertes.fr/hal-05223598v1 (see [70]).

Appendix A. Small strains unidimensional generalized standard material

A.1. *Positivity of the dissipative bracket*

We verify the positivity of the symmetric bracket in the following development

$$
\begin{aligned}
(\mathscr{F},\mathscr{F}) &= \int_{\omega_X}\left[\frac{1}{\rho}\frac{\partial}{\partial X}\left(\frac{\partial f}{\partial T}\right)K\frac{T^2}{c^2}\frac{\partial}{\partial X}\left(\frac{\partial f}{\partial T}\right)+\frac{\partial f}{\partial T}\rho\frac{1}{c}\frac{T}{c}\left(\frac{\partial\widehat{e}_{\text{int}}}{\partial\boldsymbol{\alpha}}\right)^{\top}\boldsymbol{V}^{-1}\frac{\partial\widehat{e}_{\text{int}}}{\partial\boldsymbol{\alpha}}\frac{\partial s}{\partial T}\right.\\
&\qquad\left.-\frac{\partial f}{\partial T}\rho\frac{T}{c}\left(\frac{\partial\widehat{e}_{\text{int}}}{\partial\boldsymbol{\alpha}}\right)^{\top}\boldsymbol{V}^{-1}\frac{\partial s}{\partial\boldsymbol{\alpha}}-\rho\frac{T}{c}\frac{\partial f}{\partial\boldsymbol{\alpha}}^{\top}\boldsymbol{V}^{-1}\frac{\partial\widehat{e}_{\text{int}}}{\partial\boldsymbol{\alpha}}\frac{\partial s}{\partial T}+\rho T\left(\frac{\partial f}{\partial\boldsymbol{\alpha}}\right)^{\top}\boldsymbol{V}^{-1}\frac{\partial s}{\partial\boldsymbol{\alpha}}\right]\mathrm{d}\omega_X\\
&= \int_{\omega_X}\left[\frac{1}{\rho}\left(\frac{\partial}{\partial X}\left(\frac{\partial f}{\partial T}\right)\frac{T}{c}\right)^2K+\rho T\left(\frac{1}{c}\frac{\partial f}{\partial T}\frac{\partial\widehat{e}_{\text{int}}}{\partial\boldsymbol{\alpha}}-\frac{\partial f}{\partial\boldsymbol{\alpha}}\right)^{\top}\boldsymbol{V}^{-1}\left(\frac{1}{c}\frac{\partial f}{\partial T}\frac{\partial\widehat{e}_{\text{int}}}{\partial\boldsymbol{\alpha}}-\frac{\partial f}{\partial\boldsymbol{\alpha}}\right)\right]\mathrm{d}\omega_X\\
&\geq 0
\end{aligned}
$$

since K, the material's conductivity, is positive, and $\boldsymbol{V}^{-1}$ is a symmetric positive definite matrix.

Appendix B. Large strain tridimensional thermo-visco-elastodynamics

B.1. *The Hamilton principle*

Considering the action defined in (31), its variation with respect to the configuration, the linear momentum, and the first Piola–Kirchhoff stress tensor is given by

$$\begin{aligned}
\delta\mathscr{A}^* &= \delta_{\underline{\chi}}\mathscr{A}^* + \delta_{\underline{p}}\mathscr{A}^* + \delta_{\underline{\underline{\Pi}}}\mathscr{A}^* \\
&= \int_{\omega_t}\int_{\omega_X}\Big[\underbrace{\underline{p}\cdot\delta\frac{\partial\underline{\chi}}{\partial t} - \underline{\underline{\Pi}}:\delta\frac{\partial\underline{\chi}}{\partial\underline{X}} - \frac{\partial l^*}{\partial\underline{\chi}}\delta\underline{\chi}}_{\delta_{\underline{\chi}}a^*} + \underbrace{\delta\underline{p}\cdot\underline{\chi} - \frac{\partial l^*}{\partial\underline{p}}\cdot\delta\underline{p}}_{\delta_{\underline{p}}a^*} \underbrace{-\delta\underline{\underline{\Pi}}:\frac{\partial\underline{\chi}}{\partial\underline{X}} - \frac{\partial l^*}{\partial\underline{\underline{\Pi}}}:\delta\underline{\underline{\Pi}}}_{\delta_{\underline{\underline{\Pi}}}a^*}\Big]\,\mathrm{d}\omega_X\,\mathrm{d}\omega_t \\
&= \int_{\omega_t}\int_{\omega_X}\left[-\frac{\partial\underline{p}}{\partial t} + \mathrm{DIV}_{\underline{X}}(\underline{\underline{\Pi}}) - \frac{\partial l^*}{\partial\underline{\chi}}\right]\cdot\delta\underline{\chi}\,\mathrm{d}\omega_X\,\mathrm{d}\omega_t - \int_{\omega_t}\int_{\partial\omega_X}(\underline{\underline{\Pi}}\cdot\underline{N})\cdot\delta\underline{\chi}\,\mathrm{d}(\partial\omega_X)\,\mathrm{d}\omega_t \\
&\quad + \underbrace{\int_{\omega_X}\int_{\partial\omega_t}\underline{p}\cdot\delta\underline{\chi}\,\mathrm{d}(\partial\omega_t)\,\mathrm{d}\omega_X}_{=0} + \int_{\omega_t}\int_{\omega_X}\left[\delta\underline{p}\cdot\frac{\partial\underline{\chi}}{\partial t} - \frac{\partial l^*}{\partial\underline{p}}\cdot\delta\underline{p} - \delta\underline{\underline{\Pi}}:\frac{\partial\underline{\chi}}{\partial\underline{X}} - \frac{\partial l^*}{\partial\underline{\underline{\Pi}}}:\delta\underline{\underline{\Pi}}\right]\mathrm{d}\omega_X\,\mathrm{d}\omega_t.
\end{aligned}$$

Thus, Hamilton's principle becomes, $\forall\,\delta\underline{\chi} = \underline{0}$ on $\partial\omega_t, \delta\underline{p}, \delta\underline{\underline{\Pi}}$

$$\begin{aligned}
&\delta\mathscr{A}^* + \int_{\omega_t}\int_{(\partial\omega_X)^{\mathrm{Neumann}}}\underline{b}\cdot\delta\underline{\chi}\,\mathrm{d}(\partial\omega_X)\,\mathrm{d}\omega_t = 0 \\
\iff\quad &\int_{\omega_t}\int_{\omega_X}\left[\left(-\frac{\partial\underline{p}}{\partial t} + \mathrm{DIV}_{\underline{X}}(\underline{\underline{\Pi}}) - \frac{\partial l^*}{\partial\underline{\chi}}\right)\cdot\delta\underline{\chi} + \left(\frac{\partial\underline{\chi}}{\partial t} - \frac{\partial l^*}{\partial\underline{p}}\right)\cdot\delta\underline{p} + \left(-\frac{\partial\underline{\chi}}{\partial\underline{X}} - \frac{\partial l^*}{\partial\underline{\underline{\Pi}}}\right):\delta\underline{\underline{\Pi}}\right]\mathrm{d}\omega_X\,\mathrm{d}\omega_t \\
&\qquad + \int_{\omega_t}\int_{(\partial\omega_X)^{\mathrm{Neumann}}}(\underline{b} - \underline{\underline{\Pi}}\cdot\underline{N})\cdot\delta\underline{\chi}\,\mathrm{d}(\partial\omega_X)\,\mathrm{d}\omega_t = 0
\end{aligned}$$

which gives the equations of elastodynamics

$$\frac{\partial\underline{p}}{\partial t} = \mathrm{DIV}_{\underline{X}}(\underline{\underline{\Pi}}) - \frac{\partial l^*}{\partial\underline{\chi}} \quad \text{within } \omega_X, \tag{38}$$

$$\frac{\partial\underline{\chi}}{\partial t} = \frac{\partial l^*}{\partial\underline{p}} \quad \text{within } \omega_X, \tag{39}$$

$$\frac{\partial\underline{\chi}}{\partial\underline{X}} = -\frac{\partial l^*}{\partial\underline{\underline{\Pi}}} \quad \text{within } \omega_X, \tag{40}$$

$$\underline{\underline{\Pi}}\cdot\underline{N} = \underline{b} \quad \text{on } (\partial\omega_X)^{\mathrm{Neumann}}. \tag{41}$$

B.2. *Skew-symmetric bracket for elastodynamics*

This appendix describes how to derive the antisymmetric multisymplectic bracket from the equations of the Hamilton variational principle. Consider a function f to be dependent on the configuration, the linear momentum, and the second Piola–Kirchhoff stress tensor (which is a Lagrangian stress tensor, thus its time derivative is objective by definition). Therefore, the time derivative of such a function yields

$$\int_{\omega_X}\frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}\omega_X = \int_{\omega_X}\left(\frac{\partial f}{\partial\underline{\chi}}\cdot\frac{\partial\underline{\chi}}{\partial t} + \frac{\partial f}{\partial\underline{p}}\cdot\frac{\partial\underline{p}}{\partial t} + \frac{\partial f}{\partial\underline{\underline{S}}}:\frac{\partial\underline{\underline{S}}}{\partial t}\right)\mathrm{d}\omega_X.$$

Furthermore, the Lie derivative of the first and second Piola–Kirchhoff stress tensors are expressed as

$$\begin{aligned}
&\mathscr{L}_v \underline{\underline{S}} = \dot{\underline{\underline{S}}} \ (= \overline{\underline{\underline{F}}^{-1}\underline{\underline{\Pi}}}^{\boldsymbol{\cdot}}),\\
&\mathscr{L}_v \underline{\underline{\Pi}} = \underline{\underline{F}}\mathscr{L}_v \underline{\underline{S}} = \underline{\underline{F}}\,\overline{\underline{\underline{F}}^{-1}\underline{\underline{\Pi}}}^{\boldsymbol{\cdot}} \Rightarrow \mathscr{L}_v \underline{\underline{\Pi}} = -\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}} + \dot{\underline{\underline{\Pi}}},\\
&\mathscr{L}_v \underline{\underline{\tau}} = \underline{\underline{F}}\frac{\mathrm{d}}{\mathrm{d}t}(\underline{\underline{F}}^{-1}\underline{\underline{\tau}}\,\underline{\underline{F}}^{-\mathsf{T}})\underline{\underline{F}}^{\mathsf{T}},
\end{aligned}$$

where we used the identity $\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1} = \underline{\underline{F}}\,\overline{\underline{\underline{F}}^{-1}}^{\boldsymbol{\cdot}}$. Then, we insert the strong laws obtained through the Hamilton variational principle within the time derivative of the function f, and we use the expanded expression of the time derivative of $\underline{\underline{S}}$ to obtain

$$\begin{aligned}
&\int_{\omega_X}\left(\frac{\partial f}{\partial \underline{\chi}}\cdot\frac{\partial \underline{\chi}}{\partial t} + \frac{\partial f}{\partial \underline{p}}\cdot\frac{\partial \underline{p}}{\partial t} + \frac{\partial f}{\partial \underline{\underline{S}}} : \frac{\partial \underline{\underline{S}}}{\partial t}\right)\mathrm{d}\omega_X\\
&\qquad = \int_{\omega_X}\left(\frac{\partial f}{\partial \underline{\chi}}\cdot\frac{\partial l^*}{\partial \underline{p}} - \frac{\partial f}{\partial \underline{p}}\cdot\frac{\partial l^*}{\partial \underline{\chi}} + \frac{\partial f}{\partial \underline{p}}\cdot \mathrm{DIV}_{\underline{X}}\,\underline{\underline{\Pi}} + \frac{\partial f}{\partial \underline{\underline{\Pi}}} : \underline{\underline{F}}\,\overline{\underline{\underline{F}}^{-1}}^{\boldsymbol{\cdot}}\underline{\underline{\Pi}} + \frac{\partial f}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}}\right)\mathrm{d}\omega_X\\
\iff\ &\int_{\omega_X}\left(\frac{\partial f}{\partial \underline{\chi}}\cdot\frac{\partial \underline{\chi}}{\partial t} + \frac{\partial f}{\partial \underline{p}}\cdot\frac{\partial \underline{p}}{\partial t} + \frac{\partial f}{\partial \underline{\underline{S}}} : \frac{\partial \underline{\underline{S}}}{\partial t} - \frac{\partial f}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}}\right)\mathrm{d}\omega_X\\
&\qquad = \int_{\omega_X}\left(\frac{\partial f}{\partial \underline{\chi}}\cdot\frac{\partial l^*}{\partial \underline{p}} - \frac{\partial f}{\partial \underline{p}}\cdot\frac{\partial l^*}{\partial \underline{\chi}} - \underline{\underline{\nabla}}_X\frac{\partial f}{\partial \underline{p}} : \underline{\underline{\Pi}} - \frac{\partial f}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}}\right)\mathrm{d}\omega_X + \int_{\partial\omega_X}\frac{\partial f}{\partial \underline{p}}\cdot(\underline{\underline{\Pi}}\cdot\underline{N})\,\mathrm{d}(\partial\omega_X).
\end{aligned}$$

Furthermore, we prove the equality between the derivatives with respect to the first and the second Piola–Kirchhoff stress tensors, and their contraction with their objective time derivatives. First we have the following identity

$$\frac{\partial f}{\partial \underline{\underline{S}}} = \frac{\partial f}{\partial \underline{\underline{\Pi}}} : \frac{\partial \underline{\underline{\Pi}}}{\partial \underline{\underline{S}}} = \underline{\underline{F}}^{\mathsf{T}}\frac{\partial f}{\partial \underline{\underline{\Pi}}}$$

which leads to

$$\begin{aligned}
\frac{\partial f}{\partial \underline{\underline{S}}} : \dot{\underline{\underline{S}}} &= \underline{\underline{F}}^{\mathsf{T}}\frac{\partial f}{\partial \underline{\underline{\Pi}}} : \overline{\underline{\underline{F}}^{-1}}^{\boldsymbol{\cdot}}\underline{\underline{\Pi}} + \underline{\underline{F}}^{\mathsf{T}}\frac{\partial f}{\partial \underline{\underline{\Pi}}} : \underline{\underline{F}}^{-1}\dot{\underline{\underline{\Pi}}}\\
&= \frac{\partial f}{\partial \underline{\underline{\Pi}}} : \underline{\underline{F}}\,\overline{\underline{\underline{F}}^{-1}}^{\boldsymbol{\cdot}}\underline{\underline{\Pi}} + \frac{\partial f}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}}\\
&= \frac{\partial f}{\partial \underline{\underline{\Pi}}} : \mathscr{L}_v\underline{\underline{\Pi}}
\end{aligned}$$

where $\mathscr{L}_v$ is the Lie derivative. Finally, we have

$$\begin{aligned}
\frac{\partial f}{\partial \underline{\underline{S}}} : \frac{\partial \underline{\underline{S}}}{\partial t} - \frac{\partial f}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}} &= \frac{\partial f}{\partial \underline{\underline{\Pi}}} : \mathscr{L}_v\underline{\underline{\Pi}} - \frac{\partial f}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{\Pi}}}\\
&= -\frac{\partial f}{\partial \underline{\underline{\Pi}}} : \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}}\\
&= -\frac{\partial f}{\partial \underline{\underline{F}}} : \left(\frac{\partial \underline{\underline{F}}}{\partial \underline{\underline{\Pi}}}\right) : \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}}\\
&= -\frac{\partial f}{\partial \underline{\underline{F}}} : \left(\frac{\partial(\underline{\underline{\Pi}}\,\underline{\underline{S}}^{-1})}{\partial \underline{\underline{\Pi}}}\right) : \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}}\\
&= -\frac{\partial f}{\partial \underline{\underline{F}}}\underline{\underline{S}}^{-\mathsf{T}} : \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}}\\
&= -\frac{\partial f}{\partial \underline{\underline{F}}} : \dot{\underline{\underline{F}}},
\end{aligned}$$

therefore leading to

$$\int_{\omega_X}\left(\frac{\partial f}{\partial \underline{\chi}}\cdot\frac{\partial \underline{\chi}}{\partial t}+\frac{\partial f}{\partial \underline{p}}\cdot\frac{\partial \underline{p}}{\partial t}-\frac{\partial f}{\partial \underline{\underline{F}}}:\underline{\underline{\dot{F}}}\right)\mathrm{d}\omega_X$$
$$=\int_{\omega_X}\left(\frac{\partial f}{\partial \underline{\chi}}\cdot\frac{\partial l^*}{\partial \underline{p}}-\frac{\partial f}{\partial \underline{p}}\cdot\frac{\partial l^*}{\partial \underline{\chi}}+\underline{\underline{\nabla}}_X\frac{\partial f}{\partial \underline{p}}:\frac{\partial l^*}{\partial \underline{\underline{\Pi}}}\,\underline{\underline{S}}-\frac{\partial f}{\partial \underline{\underline{\Pi}}}:\underline{\underline{\nabla}}_X\frac{\partial l^*}{\partial \underline{p}}\,\underline{\underline{S}}\right)\mathrm{d}\omega_X+\int_{\partial\omega_X}\frac{\partial f}{\partial \underline{p}}\cdot(\underline{\underline{\Pi}}\cdot\underline{N})\,\mathrm{d}(\partial\omega_X).$$

B.3. *Evolution of temperature*

This section proves Eq. (29) of evolution of the temperature. The main difficulty lies in defining the appropriate representations and transformations of the potentials so that the chain rule can be applied. First, start from Eq. (27). Then, since $\underline{\underline{\Pi}}=\underline{\underline{F}}\,\underline{\underline{S}}$, and $\underline{\underline{F}}^{-1}$ exists, potentials can be represented in terms of the second Piola–Kirchhoff stress tensor (where the dagger represents the representation in terms of the second Piola–Kirchhoff stress tensor $\cdot^\dagger = \cdot(\dots,\underline{\underline{S}})$). Furthermore, the internal energy is defined in terms of the first Piola–Kirchhoff stress tensor, not through a Legendre transformation, but through a change of variables (thanks to the same arguments used for moving from a first to a second Piola–Kirchhoff stress tensor representation). Therefore, we have the following equation

$$\rho w^\dagger(T,\underline{\underline{S}},\underline{\underline{C}}_{\mathrm{i}}^{-1})=\rho T s^\dagger(T,\underline{\underline{S}},\underline{\underline{C}}_{\mathrm{i}}^{-1})-\rho\widehat{e}^*_{\mathrm{int}}(T,\underline{\underline{\Pi}},\underline{\underline{C}}_{\mathrm{i}}^{-1}).$$

We then take the time derivative of this equation, to obtain (replacing the expression of the time derivative of the internal energy by the expression of the first law of thermodynamics)

$$\rho T\frac{\partial s^\dagger}{\partial T}\dot{T}=\rho\frac{\partial w^\dagger}{\partial T}\dot{T}-\rho\dot{T}s^\dagger+\rho\frac{\partial w^\dagger}{\partial \underline{\underline{S}}}:\underline{\underline{\dot{S}}}-\rho T\frac{\partial s^\dagger}{\partial \underline{\underline{S}}}:\underline{\underline{\dot{S}}}+\underline{\underline{\Pi}}:\underline{\underline{\dot{F}}}+\rho\frac{\partial(w^\dagger-Ts^\dagger)}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}:\dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}-\mathrm{DIV}_{\underline{X}}(\underline{Q}).$$

Then, as we have

$$\frac{\partial w}{\partial T}=-s\iff\frac{\partial(\underline{\underline{\Pi}}:\underline{\underline{F}}-w^*)}{\partial T}=-(s^*-\underline{\underline{\Pi}}:\underline{\underline{F}})\iff\frac{\partial w^*}{\partial T}=s^*.$$

Furthermore, one has

$$\begin{aligned}\rho\frac{\partial w^\dagger}{\partial \underline{\underline{S}}}:\underline{\underline{\dot{S}}}-\rho T\frac{\partial s^\dagger}{\partial \underline{\underline{S}}}:\underline{\underline{\dot{S}}}+\underline{\underline{\Pi}}:\underline{\underline{\dot{F}}}&=\rho\frac{\partial w^*}{\partial \underline{\underline{\Pi}}}:\mathscr{L}_v\underline{\underline{\Pi}}-\rho T\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}:\mathscr{L}_v\underline{\underline{\Pi}}+\underline{\underline{\Pi}}:\underline{\underline{\dot{F}}}\\
&=\rho\frac{\partial w^*}{\partial \underline{\underline{\Pi}}}:(\underline{\underline{\dot{\Pi}}}-\underline{\underline{\dot{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}})-\rho T\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}:(\underline{\underline{\dot{\Pi}}}-\underline{\underline{\dot{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}})+\underline{\underline{\Pi}}:\underline{\underline{\dot{F}}}\\
&=\underline{\underline{F}}:(\underline{\underline{\dot{\Pi}}}-\underline{\underline{\dot{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}})-\underline{\underline{F}}:\underline{\underline{\dot{\Pi}}}+\rho T\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}:\underline{\underline{\dot{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}}+\underline{\underline{\Pi}}:\underline{\underline{\dot{F}}}\\
&=\rho T\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}:\underline{\underline{\dot{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}}.\end{aligned}$$

Hence the expression of the evolution of the temperature is

$$\underbrace{\rho T\frac{\partial s^*}{\partial T}}_{=c}\dot{T}=\rho T\frac{\partial s^*}{\partial \underline{\underline{\Pi}}}:\underline{\underline{\dot{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{\Pi}}-\mathrm{DIV}_{\underline{X}}\,\underline{Q}-\rho\frac{\partial\widehat{e}^*_{\mathrm{int}}}{\partial \underline{\underline{C}}_{\mathrm{i}}^{-1}}:\dot{\overline{\underline{\underline{C}}_{\mathrm{i}}^{-1}}}.$$

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