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Research article

A pseudo-continuous media approach

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Abstract. This paper introduces a novel mathematical framework, termed the pseudo-continuous media approach, for treating strong discontinuities (such as cracks, interfaces, and contact surfaces) within continuum mechanics. Traditional methods typically require explicit geometric representation of discontinuities, leading to complex meshing and algorithmic challenges. In contrast, our method leverages distribution theory to embed the discontinuity conditions directly into the governing equations, allowing the problem to be posed on a simpler, uncracked domain. Our distributional approach is conceptually aligned with the rigorous framework of Special Functions of Bounded Variation (SBV) which provides a natural setting for functions with jump discontinuities. We present the theoretical foundation and the corresponding finite element formulation. The method demonstrates significant advantages in geometric simplification, mathematical elegance, and numerical robustness compared to traditional techniques like the eXtended Finite Element Method (XFEM), positioning itself as a novel numerical methodology that implements the SBV philosophy using a standard finite element framework.

Keywords. Continuous media, discontinuities, distributions, variational formulation.

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Introduction

Within the standard framework of continuum mechanics, the treatment of strong discontinuities (e.g., cracks) presents a significant challenge. The conventional approach involves isolating the discontinuous surface γ. The domain is split into $\Omega = \overline{\Omega} \backslash \gamma$, where $\overline{\Omega}$ is the uncracked domain, and the solution is sought separately in the subdomains. These solutions are then coupled through additional conditions imposed on γ, such as traction-free boundary conditions for a crack.

This work presents a paradigm shift. Instead of adapting the geometry to the discontinuity, we adapt the mathematical formulation. By employing the theory of distributions, we can represent the discontinuous solution as a generalized function (a distribution) defined over the entire, simpler domain $\overline{\Omega}$. This goal is conceptually aligned with the framework of Special Functions of Bounded Variation (SBV) [1], which was developed precisely to provide a rigorous functional-analytic setting for functions with jump discontinuities.

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While the SBV framework provides the theoretical justification, our contribution is a novel numerical implementation for a predetermined discontinuity. We develop a mixed finite element strategy that uses:

- A regular part approximated by C^1-continuous elements on a fixed, simple mesh of $\overline{\Omega}$.
- An independent jump field defined on the crack surface γ.

This approach avoids the need for complex, conforming discretizations or specialized enrichment functions used in methods like XFEM.

The paper is structured as follows: Section 1 introduces a canonical model problem. Section 2 develops the core theoretical framework of pseudo-continuous representations, including a detailed explanation of the SBV derivative and regularity considerations.

Section 3 details the mixed finite element discretization, including the derivation of the traction-free condition and the explanation of the variational notation.

Section 4 discusses the advantages, limitations, extensions to interfaces and contact, and future work. Section 5 concludes the paper.

1. A model problem

To illustrate the core ideas without the algebraic burden of elasticity, we use the Poisson equation, a prototypical elliptic problem. Consider a domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ containing an internal crack, modeled as a hypersurface $\gamma \subset \mathbb{R}^n$ of dimension $n-1$. The cracked domain is $\Omega = \overline{\Omega} \setminus \gamma$, and its external boundary is $\Gamma = \partial\overline{\Omega}$.

We consider the following mixed boundary value problem. Find $u(\mathbf{x})$, for $\mathbf{x} \in \Omega$, such that:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \gamma^+ \quad \text{and} \quad \gamma^-. \end{cases} \tag{1}$$

Here, the homogeneous Neumann condition on γ models a traction-free crack surface in a simplified setting.

It is well known that this problem has a unique solution in the Sobolev space $\widetilde{H_0^1}(\Omega)$ [2], defined as:

$$\widetilde{H_0^1}(\Omega) = \left\{ v \in H^1(\Omega) \,\middle|\, v|_\Gamma = 0 \right\},$$

where $\widetilde{H_0^1}(\Omega)$ is the space of functions which are not required to be continuous across γ and not required to vanish on the entire boundary $\partial\Omega$ since they are not required to vanish on γ, defined as:

$$\widetilde{H_0^1}(\Omega) = \left\{ v \in H^1(\Omega) \,\middle|\, v|_\Gamma = 0 \right\}.$$

The solution satisfies the standard variational formulation:

$$\int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} = \int_\Omega f v \, d\mathbf{x} \quad \forall\, v \in \widetilde{H_0^1}(\Omega). \tag{2}$$

A key feature of the solution is its *jump* across the crack:

$$[u]_\gamma := u|_{\gamma^+} - u|_{\gamma^-} \neq 0.$$

Our goal is to capture this behavior without explicitly cutting the domain.

2. Pseudo-continuous representation using distribution theory

2.1. *Extension by distributions*

The objective is to define an extension $\bar{u}$ of the solution u from the cracked domain Ω to the entire, uncracked domain $\overline{\Omega}$. This extension must encapsulate the discontinuous behavior of u across γ. Since $\bar{u}$ is discontinuous, its classical derivatives do not exist on γ; we must therefore define them in the sense of distributions.

2.2. *Deriving the distributional formulation*

Starting from the solution $u \in H_0^1(\Omega)$ of Problem (1), we consider its action on a test function $v \in H^2(\overline{\Omega})$. For u sufficiently regular away from γ, integration by parts yields:

$$-\int_{\Omega} \Delta u \cdot v \, d\mathbf{x} = \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{x} - \int_{\gamma^+} \frac{\partial u}{\partial n^+} v \, d\mathcal{H}^{n-1} - \int_{\gamma^-} \frac{\partial u}{\partial n^-} v \, d\mathcal{H}^{n-1}.$$

Noting that the unit normal vectors on opposite crack faces satisfy $\mathbf{n}^+ = -\mathbf{n}^- = \mathbf{n}$, and using the boundary conditions $\partial u / \partial n = 0$ on $\gamma^{\pm}$, we obtain the fundamental relation:

$$-\int_{\Omega} \Delta u \cdot v \, d\mathbf{x} = -\int_{\Omega} u \Delta v \, d\mathbf{x} - \int_{\gamma} [u] \frac{\partial v}{\partial n} \, d\mathcal{H}^{n-1} \quad \forall \, v \in H^2(\overline{\Omega}). \tag{3}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (surface measure on γ).

2.3. *The SBV framework and distributional derivatives*

The SBV (Special functions of Bounded Variation) framework provides the rigorous mathematical foundation for our approach. For a function $u \in SBV(\Omega)$, the distributional derivative Du is a measure that decomposes into two parts:

$$Du = \nabla u \, \mathcal{L}^n + [u] \, \nu_u \, \mathcal{H}^{n-1} \llcorner J_u \tag{4}$$

where:

- ∇u is the *approximate gradient* defined $\mathcal{L}^n$-almost everywhere (the classical gradient where the function is smooth). This represents the regular, continuous part of the deformation.
- $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure (volume measure in $\mathbb{R}^n$). The term $\nabla u \mathcal{L}^n$ is the part of the derivative that is absolutely continuous with respect to the Lebesgue measure.
- $[u] = u^+ - u^-$ is the *jump* of u across the discontinuity set J_u (the crack surface γ). It represents the magnitude of the discontinuity.
- ν_u is the *unit normal vector* to the jump set J_u, oriented from the minus side to the plus side.
- The notation $\mathcal{H}^{n-1} \llcorner J_u$ indicates the restriction of this measure to the jump set.
- The product $[u] \, \nu_u \, \mathcal{H}^{n-1} \llcorner J_u$ is a *singular measure* concentrated on the discontinuity surface. It captures the effect of the jump.

This decomposition allows us to define the extended solution $\bar{u}$ as a distribution on $\overline{\Omega}$:

Definition 1 (Distributional Gradient). *The gradient of $\bar{u}$, denoted $\nabla \bar{u}$, is defined as the functional:*

$$\langle \nabla \bar{u}, v \rangle = -\int_{\overline{\Omega}} \bar{u} \nabla \cdot v \, d\mathbf{x} + \int_{\gamma} [u] (v \cdot \mathbf{n}) \, d\mathcal{H}^{n-1} \quad \forall \, v \in H^1(\overline{\Omega}). \tag{5}$$

Definition 2 (Distributional Laplacian). *The Laplacian of $\bar{u}$, denoted $\Delta\bar{u}$, is defined as the functional:*

$$\langle \Delta\bar{u}, v\rangle = \int_{\overline{\Omega}} \bar{u}\,\Delta v\, d\mathbf{x} + \int_{\gamma} \left(\left[\frac{\partial u}{\partial n} \right] v - [u] \frac{\partial v}{\partial n} \right) d\mathscr{H}^{n-1} \quad \forall\, v \in H^2(\overline{\Omega}). \tag{6}$$

2.4. *Justification of the second-Order formulation*

In classical finite elements for the Poisson equation, one uses the symmetric form $\int \nabla u \cdot \nabla v$. That requires $u \in H^1$ and leads to continuity across elements. Here, u is discontinuous across γ, and its gradient is not square-integrable on γ. To distribute the discontinuity into a surface term, we integrate by parts twice, transferring *two derivatives* onto the test function v. This yields Δv and a boundary term involving $[u]\partial v/\partial n$. This is not an arbitrary choice but a direct consequence of the distributional definition of $\Delta\bar{u}$ in the dual of H^2. It allows the jump to appear naturally as a measure, without modifying the mesh.

2.5. *Regularity considerations*

It is essential to clarify the regularity of the different fields involved:

- The *physical solution u* on the cracked domain Ω belongs to $H^1(\Omega)$.
- The *extended field $\bar{u}$* is a *distribution* on $\overline{\Omega}$. It belongs to $H^{-2}(\overline{\Omega})$, the dual space of $H_0^2(\overline{\Omega})$.
- The *test functions v* are required to be in $H_0^2(\overline{\Omega})$ because the weak form involves Δv.

This is standard in distribution theory: we test the equation $\Delta\bar{u} = f$ against smooth functions v, yielding:

$$\langle \Delta\bar{u}, v\rangle = \langle f, v\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between H^{-2} and H_0^2.

2.6. *The pseudo-continuous weak form*

Incorporating the boundary condition $\partial u/\partial n|_\gamma = 0$ (which implies $[\partial u/\partial n] = 0$), and starting from the definition of the distributional Laplacian, we obtain the pseudo-continuous weak form: find $\bar{u}$ such that:

$$-\int_{\overline{\Omega}} \bar{u}\,\Delta v\, d\mathbf{x} = \int_{\overline{\Omega}} f\, v\, d\mathbf{x} + \int_{\gamma} [\bar{u}] \frac{\partial v}{\partial n}\, d\mathscr{H}^{n-1} \quad \forall\, v \in H_0^2(\overline{\Omega}). \tag{7}$$

3. Mixed finite element discretization

3.1. *The mixed variational formulation*

The key insight of our numerical approach is to recognize that the jump $[u]$ is not a property of the regular part $\bar{u}_h$ itself, but an independent kinematic variable. We therefore introduce a mixed two-field variational scheme:

- A regular part $\bar{u}_h \in V_h \subset H_0^2(\overline{\Omega})$ approximated with C^1-continuous elements on the uncracked domain.
- An independent jump field j_γ defined only on the crack surface γ, representing the displacement discontinuity $[u]$.

The physical displacement u is then reconstructed from the regular part and the independent jump field as:

$$u(\mathbf{x}) = \underbrace{\bar{u}_h(\mathbf{x})}_{\text{regular part}} + \underbrace{j_\gamma(\mathbf{x}_\gamma)\, H_\gamma(\mathbf{x})}_{\text{jump part}},$$

where H_γ is a Heaviside function aligned with γ (1 on the + side and 0 on the − side).

Theoretical status

In the SBV framework, any function $u \in SBV(\Omega)$ decomposes uniquely into a regular H^1 component and a jump concentrated on J_u. Our approximation $\bar{u}_h \in H_0^2(\overline{\Omega})$ (continuous derivatives) approximates the regular part, and j_γ approximates the jump amplitude. The equality is understood *in the limit of mesh refinement* within a Galerkin approximation.

3.2. *The Discrete Variational Problem*

Let $V_h \subset H_0^2(\overline{\Omega})$ be a finite-dimensional subspace spanned by C^1-continuous basis functions: $V_h = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\}$. Let $J_h \subset L^2(\gamma, \mathcal{H}^{n-1})$ be a finite-dimensional space for the jump field, spanned by basis functions $\{\psi_1, \psi_2, \text{span}\ldots, \psi_M\}$ defined on γ. The associated norm is:

$$\| j_\gamma \|^2_{L^2(\gamma)} = \int_\gamma j_\gamma^2 \, d\mathcal{H}^{n-1}(\gamma).$$

The mixed discrete problem is: find $(\bar{u}_h, j_\gamma) \in V_h \times J_h$ such that:

$$-\int_{\overline{\Omega}} \bar{u}_h \Delta v_h \, d\mathbf{x} = \int_{\overline{\Omega}} f \, v_h \, d\mathbf{x} + \int_\gamma j_\gamma \frac{\partial v_h}{\partial n} \, d\mathcal{H}^{n-1}(\gamma) \quad \forall\, v_h \in V_h, \tag{8}$$

supplemented by an additional equation that links j_γ to $\bar{u}_h$ via the traction condition on γ. For the traction-free crack, this additional equation enforces that the variational derivative with respect to j_γ vanishes. The equation that enforces the traction-free condition is written as:

$$\int_\gamma \frac{\partial \bar{u}_h}{\partial n} \, \delta j_\gamma \, d\mathcal{H}^{n-1}(\gamma) = 0 \quad \forall\, \delta j_\gamma \in J_h, \tag{9}$$

where δj_γ represents a variation (or test function) for the jump field j_γ. This is the weak form of the condition $\partial u / \partial n = 0$ on $\gamma^\pm$.

3.3. *Linear system assembly*

Expressing $\bar{u}_h = \sum_{j=1}^N U_j \phi_j$ and $j_\gamma = \sum_{k=1}^M I_k \psi_k$, and using test functions $v_h = \phi_i$ and $\delta j_\gamma = \psi_l$, we obtain the coupled linear system:

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{uj} \\ \mathbf{K}_{ju} & \mathbf{K}_{jj} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{J} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}, \tag{10}$$

where:

$$
\begin{aligned}
(K_{uu})_{ij} &= -\int_{\overline{\Omega}} \phi_j \Delta\phi_i \, d\mathbf{x}, \\
(K_{uj})_{ik} &= \int_{\gamma} \psi_k \frac{\partial \phi_i}{\partial n} \, d\mathcal{H}^{n-1}(\gamma), \\
(K_{ju})_{lj} &= \int_{\gamma} \frac{\partial \phi_j}{\partial n} \psi_l \, d\mathcal{H}^{n-1}(\gamma), \\
(K_{jj})_{lk} &= \text{(terms from the traction condition)}, \\
F_i &= \int_{\overline{\Omega}} f \phi_i \, d\mathbf{x}.
\end{aligned}
\tag{11}
$$

3.4. *Implementation remarks*

C^1-continuity: The formulation necessitates C^1-continuous elements (e.g., Hermite, Argyris, or isogeometric analysis elements) because the weak form involves second-order derivatives of test functions.

Localized effect: The coupling terms $\int_\gamma \psi_k \frac{\partial\phi_i}{\partial n} \, d\mathcal{H}^{n-1}(\gamma)$ are non-zero only for basis functions ϕ_i whose supports intersect the crack γ. This leads to a highly localized modification of the stiffness matrix.

Simple meshing: The mesh is generated for the uncracked domain $\overline{\Omega}$. The crack γ is simply an internal surface where these coupling integrals are computed; the mesh does not need to conform to it.

4. Discussion

4.1. *Comparison with XFEM and crack tip singularities*

XFEM's ability to capture the $1/\sqrt{r}$ stress singularity via asymptotic enrichment is a key advantage that we acknowledge [3]. Our current formulation does not model this singularity explicitly. It is better suited for situations where:

- The crack tip lies outside the region of interest (e.g., problems dominated by interface behavior).
- The far-field solution is the primary concern.
- Multiple or intersecting cracks make enrichment strategies cumbersome.

We note that our method captures singularities implicitly through the solution of the weak form with jump terms, but it does not guarantee the correct asymptotic behavior near the tip without sufficient mesh refinement. A promising direction is a hybrid approach: using our method for the jump away from the tip and XFEM-type enrichment near the tip, combining the strengths of both techniques.

4.2. *Extensions to interfaces and ontact*

The distributional framework is general and can be extended to other types of discontinuities:

Material interfaces: continuity of displacement $[u] = 0$ with possible jumps in flux. This requires modifying the surface integral terms to enforce displacement continuity while allowing flux jumps determined by material properties.

Contact problems: inequality constraints on the jump field (no interpenetration) and the introduction of contact pressure as a Lagrange multiplier. The weak form would be extended to include variational inequalities and additional Lagrange multiplier fields.

4.3. *Towards Evolving Discontinuities*

The extension to propagating cracks is natural within the same distributional setting:

- Represent the moving crack $\gamma(t)$ via a level-set function $\phi(\mathbf{x}, t)$.
- The jump field $j_{\gamma(t)}$ evolves according to a fracture criterion (e.g., energy release rate).

4.3.1. *Handling moving bases for $J_h(t)$*

When the crack $\gamma(t)$ evolves, the support of the jump field changes [4]. The space $J_h(t) \subset L^2(\gamma(t), \mathscr{H}^{n-1})$ is therefore time-dependent. We propose two practical approaches:

Explicit surface remeshing: At each time step, $\gamma(t)$ is discretized independently by a surface mesh, and a new basis $\{\psi_k(t)\}$ is constructed. This is analogous to standard moving-mesh methods but without re-meshing the bulk domain.

Level-set + fixed basis: Define a fixed set of shape functions $\{\psi_k(\mathbf{x})\}$ on the background mesh of $\overline{\Omega}$. At each time step, the integrals over $\gamma(t)$ are evaluated via a level-set function $\phi(\mathbf{x}, t)$ (e.g., $\gamma(t) = \{\phi = 0\}$). The restriction of ψ_k to $\gamma(t)$ provides a natural basis for $J_h(t)$. This avoids rebuilding the basis entirely.

A detailed implementation and comparison will be presented in a follow-up paper focused on numerical aspects.

5. Conclusion

The pseudo-continuous media framework represents a significant shift in computational mechanics. By leveraging distribution theory and SBV spaces, it transforms problems with complex geometric discontinuities into mathematically equivalent mixed formulations on simplified domains. The complexity is moved from the geometry (meshing) to the formulation (incorporating jumps via independent fields).

Our mixed finite element formulation, grounded in rigorous distribution theory and the SBV framework, provides a solid theoretical foundation for treating strong discontinuities without conforming meshes. The method:

- Uses a non-conforming mesh without geometric alignment to the discontinuity.
- Maintains mathematical consistency by treating the discontinuity intrinsically through an independent jump field.
- Avoids the need for additional enriched degrees of freedom required by XFEM, though with the trade-off of requiring C^1 continuity.
- Offers a clear pathway for extensions to interfaces, contact, and propagating cracks.

The theoretical developments presented here pave the way for a new class of numerical methods for problems with strong discontinuities. The numerical implementation and validation of the method will be the subject of a forthcoming paper, which will demonstrate its practical capabilities and confirm the theoretical predictions .

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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