



ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus

Mécanique

Géry de Saxcé and Charles-Michel Marle

Presentation of Jean-Marie Souriau's book *Structure des systèmes dynamiques*

Volume 354 (2026), p. 735-759

Online since: 13 August 2026

<https://doi.org/10.5802/crmeca.368>



This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



*The Comptes Rendus. Mécanique are a member of the
Mersenne Center for open scientific publishing*
www.centre-mersenne.org — e-ISSN : 1873-7234



Book review

Presentation of Jean-Marie Souriau's book *Structure des systèmes dynamiques*

Géry de Saxcé ^{*,,^a} and Charles-Michel Marle ^{,^b}

^a Univ. Lille, CNRS, Centrale Lille, UMR 9013 — LaMcube — Laboratoire de mécanique multiphysique multiéchelle, Lille, France

^b Sorbonne Université, Paris, France

E-mail: gery.de-saxce@univ-lille.fr (G. de Saxcé)

Abstract. Jean-Marie Souriau's book *Structure des systèmes dynamiques*, published in 1970, republished recently by Gabay, translated in English and published under the title *Structure of Dynamical Systems, a Symplectic View of Physics*, is a work with an exceptional wealth which, fifty years after its publication, is still topical. In this paper, we give a rather detailed description of its content and we intend to highlight the ideas that to us, are the most creative and promising.

Keywords. Dynamical systems, symplectic mechanics, moment map, statistical mechanics, geometric quantization.

Note. Article submitted by invitation.

Manuscript received 21 January 2026, accepted 11 May 2026, online since 13 August 2026.

Jean-Marie Souriau's book *Structure des systèmes dynamiques* [1] was published in 1970 in a book collection for students in the first year of master's degree in mathematics. It is directed in fact to mathematicians, beginners or experienced, wishing to know the applications of mathematics to physical sciences, and to physicists concerned with knowing certain mathematical tools useful for their researches. The author was very aware of this since, in his Introduction, he gives reading recommendations adapted to both reader categories. It is a work with an exceptional wealth which, fifty years after its publication, is still topical. In the first part, we shall be describing its content. In the second one, we shall discuss the most original aspects. As conclusion, we shall indicate why this book seem to us still today a valuable source for the students and researchers in mathematics, mechanics and physics.

* Corresponding author

1. Book content

1.1. *A quick survey*

This book comprises a rather large introduction (20 pages) and five chapters. The first two chapters, entitled *Differential geometry* and *Symplectic geometry*, are purely mathematical. The third one, entitled *mechanics*, begins with a classical presentation of the mechanics of material point systems. Very soon, the author introduces the concept of *manifold of motions* of a system. Then he applies the methods of symplectic geometry presented in the previous chapter both to classical and relativistic mechanical systems. An important paragraph deals with isolated systems admitting a symmetry group, acting transitively on the manifold of motions, that the author considers as models of *elementary particles*. Next the author studies the dynamics of systems of elementary particles, taking into account their interactions. The fourth chapter, entitled *statistical mechanics*, contains two sections. The first one, essentially mathematical, presents measure and integration theories, together with notions of probability theory. The author defines the *statistical states* of a dynamical system, the *entropy* of a statistical state and proposes a generalisation of the notion of *Gibbs state*. In the second paragraph, he uses these notions to treat certain systems seen in physics: classical and relativistic ideal gas, systems of null mass particles, to name a few. He clarifies the interpretation of the parameters which the Gibbs state depends on in terms of thermodynamical quantities, and proposes an interesting generalisation of the concept of thermodynamical equilibrium. The fifth and last chapter, entitled *Geometric quantization*, presents a construction which allows to associate to a symplectic manifold satisfying certain conditions another manifold, of larger dimension, called by the author a *quantum manifold*. This construction is due to the author. A slightly different but equivalent construction was proposed independently by the American mathematician B. Kostant [2]. In the second and last section of the fifth chapter, the author applies this construction to the quantization of physical systems.

The book is illustrated by many figures which mostly are very meaningful schematic representations of the geometric constructions used by the author. The references to other books or papers, fairly limited in number, are not gathered in a bibliographic list, but indicated in footnotes. An index, very detailed and easy to use, and the list of main notations completes the book.

1.2. *Detailed presentation*

The introduction

The author evokes the book by Joseph-Louis Lagrange (1736–1813) *Mécanique analytique* [3], written at the end of the 18th century. This famous work is at the origin of the *mécanique analytique classique* which was, during the 19th century and the first half of the 20th century, an essential part of the scientific learning in the French universities and high schools. According to the author, it is an unfinished work in which some chapters are only sketched. For him, the form used to present this theory and the concepts it uses (instead of concepts, the author writes the *categories* in the epistemological, Aristotelian or Kantian sense) were fixed by Lagrange's successors such as Siméon Denis Poisson (1781–1840), William Rowan Hamilton (1805–1865) and Carl Gustav Jakob Jacobi (1804–1851). The author considers that the form thus given to analytical mechanics, although it gives to this theory a formal mathematical perfection, has lost an important part of Lagrange's thought. The great discoveries of the first quarter of the 20th century (special

and general relativity, quantum theory) taught us that the words *time*, *space* and *matter* not necessarily have the obvious meaning ascribed thereto. For the author, classical analytical mechanics, which remains an essential ingredient of current physical theories, is not outdated, although certain concepts it uses are so because they do not have the required covariance, in other words because they are in contradiction with Galilean relativity. He wishes to show in his book that a better consideration of Lagrange's thought allows a formulation of this theory compatible with the most recent discoveries in physical sciences.

Using the very simple example of the motion of a material point, the author explains which concepts he is going to use: evolution space, space of motions of a system, Lagrange's form. The concept of space of motions of a dynamical system seems to us the most important: it is the set of all possible motions of the considered system. The author discusses in depth its usefulness, often underestimated by scholars mostly interested in the study of one particular motion of the considered system. He presents also all the mathematical tools that he will use: differential forms, Lie groups, symplectic forms, etc. Hence he reviews fairly accurately every chapter of his book and completes his introduction with advises to the readers.

Chapter I: Differential Geometry

This chapter presents in less than 70 pages numerous delicate notions: differential manifolds, tangent and cotangent fiber bundles, submanifolds, covering spaces, vector fields and differential equations, Lie bracket of two vector fields, exterior derivative, foliations, Lie groups, calculus of variation. The author presents the concept of differential manifold in a rather original way which does not use the previous presentation of topological manifolds, undoubtedly to make this notion easily accessible to beginning students. Differential manifolds are not assumed to be Hausdorff. The *space of motions* of certain mechanical systems encountered in Chapter III are indeed non-Hausdorff manifolds. The author's language sometimes slightly differs from that generally used: for example, he calls an *embedding* what most of geometers call an *injective immersion*. However, readers can easily avoid misunderstandings, since the author scrupulously defines all the terms he uses.

The paragraph devoted to Lie groups contains a detailed presentation of the actions of a Lie group on a differential manifold and of its adjoint representation. The author will define the coadjoint representation in the next chapter, with the study of the moment map of the action of a dynamical group on a presymplectic or symplectic manifold. The main classical Lie groups (linear, orthogonal, unitary, symplectic) are described in a very original manner. The section about calculus of variations presents, besides the classical Euler–Lagrange equations, the extremality criterion using Cartan's form, often called *Euler–Cartan theorem*, that establishes a link between calculus of variation and symplectic geometry, together with Noether's theorem.

Chapter II: Symplectic Geometry

It is also essentially mathematical. Clearly shorter than the previous one (46 pages), it begins with the study of a finite dimensional vector space E equipped with a bilinear skew-symmetric form σ . The author defines the concepts of *orthogonality* with respect to σ , of *isotropic co-isotropic* and *self-orthogonal* vector subspaces, also called by other authors *Lagrangian vector subspaces* when σ is nondegenerate. The author proves that the rank of σ is always even and that given a coisotropic vector subspace, one can always build a basis of the kernel of σ , then complete it to obtain a basis of E , called *canonical*, in which the expression of σ is very simple. When σ is non degenerate, it is called a *symplectic form*, the dimension of E is even and the couple (E, σ) is called a *symplectic vector space*. Hence the author defines the *symplectic group* of (E, σ)

and studies its natural action on E , its features, together with those of its Lie algebra. He shows in particular that E has a complex vector space structure adapted, in a precise mathematical sense, to the symplectic form σ , and that it can be endowed with a Hermitian form of which σ is the imaginary part.

The author defines next symplectic and presymplectic manifolds and studies their properties. He shows that under certain conditions, the quotient of a presymplectic manifold by the kernel of its presymplectic form is a symplectic manifold, a result that he will use in the following chapter to define the *space of motions* of a dynamical system. He shows that to each differentiable function defined on a symplectic manifold, there is an associated vector field that he calls its *symplectic gradient*. The flow of this vector field leaves unchanged the symplectic form. He proves that the set of the differentiable functions defined on a symplectic manifold is endowed with a binary operation, the *Poisson bracket*, that makes it a Lie algebra of finite dimension. He defines and studies some remarkable submanifolds of a symplectic manifold (isotropic, co-isotropic and self-orthogonal submanifolds), studies their properties and gives several examples thereof. He proves *Darboux theorem*, whereby every point of a symplectic manifold is an element of the domain of a chart, called *canonical*, in which the symplectic form is expressed in a simple manner. Next he defines and studies the *symplectomorphisms*, also called *canonical transformations*, and their generalisations, the *canonical similarities*, together with the *infinitesimal canonical transformations* (called by other authors *locally Hamiltonian vector fields*).

The author calls *dynamical group* of a symplectic or presymplectic manifold a Lie group acting on it by canonical transformations. He calls *moment map* of the dynamical group G of a presymplectic or symplectic manifold M a smooth map Ψ from M into the vector space $\mathcal{G}^*$, dual of the Lie algebra $\mathcal{G}$ of G , such that for every $Z \in \mathcal{G}$, the infinitesimal generator of the action on M of the one-parameter subgroup generated by Z is the symplectic gradient of the function linking, at every $x \in M$, the real $\langle \Psi(x), Z \rangle$. In the terminology used by most of geometers, this infinitesimal generator is called the *Hamiltonian vector field* whose Hamiltonian is the function $x \mapsto \langle \Psi(x), Z \rangle$.

The author gives several examples of dynamical groups, indicates sufficient conditions for the existence of a moment map and studies its properties. This leads him to propose a generalisation of Noether's theorem encountered in the part of the previous chapter concerning the calculus of variations. After a quick presentation of the cohomology of Lie groups and algebras, the author shows that to any moment map of the action of a dynamical group G on a connected presymplectic or symplectic manifold, there always exist an associated cocycle θ of G valued in the dual $\mathcal{G}^*$ of its Lie algebra. The differential of θ at the neutral element, which is the cocycle of the Lie algebra $\mathcal{G}$ associated to θ , is a skew-symmetric bilinear form on $\mathcal{G}$. The author calls *symplectic cocycle* a cocycle of G valued in $\mathcal{G}^*$ satisfying this property. He proves that the addition to the moment map of a constant element of $\mathcal{G}^*$ modifies the cocycle θ by addition of a coboundary, therefore does not change its cohomology class, which depends only on the G -action, not on the choice of the moment map. Moreover the cocycle θ allows to define an affine action of G on the dual $\mathcal{G}^*$ of its Lie algebra whose linear part is the coadjoint representation. The author then proves that the moment map is equivariant with respect to this affine action of G on $\mathcal{G}^*$ and its action on M as a dynamical group of this manifold. The orbits of this affine action are submanifolds embedded in $\mathcal{G}^*$ (in author's meaning, most of the geometers would say rather *immersed*) and are endowed with a symplectic form whose expression involves the cocycle of $\mathcal{G}$, that is the differential of θ at the neutral element. Nowadays this important result is expressed by saying that $\mathcal{G}^*$ possesses a *Poisson structure* whose *symplectic leaves* are the orbits of the affine action for which the moment map is equivariant, that this Poisson structure remains unchanged under this affine action and that the moment map is a *Poisson map*. In the special case in

which the cocycle θ vanishes, this Poisson structure on $\mathcal{G}^*$ was discovered by the Norwegian mathematician Sophus Lie (1842–1899). In the general case, it was rediscovered independently by Alexander Kirillov, Bertram Kostant and the author, Jean-Marie Souriau. In his book *Structure des systèmes dynamiques*, he does not use the concept of Poisson structure, rather he uses the fact that the orbits of the affine action of G on $\mathcal{G}^*$ are symplectic manifolds, which he calls *symplectic manifolds defined by a Lie group*. He shows that when the dynamical group G of a symplectic manifold (M, σ) is connected, and when its action is transitive and possesses a moment map Ψ , this map is a local symplectomorphism of M onto an orbit of the affine action of G on $\mathcal{G}^*$ for which Ψ is equivariant. The map Ψ is a symplectomorphism if and only if, moreover, the isotropy group of a point of M is connected. He presents some examples of this important result, often called nowadays *Kostant–Souriau theorem*.

Chapter III: Mechanics

Relatively long (105 pages), this chapter begins with the study of a mechanical system composed of material points in a fixed Galilean frame. The author writes *Newton's equation* expressing the equality of the force acting on each of those points and the product of the mass by its acceleration. He treats briefly the case of a unique material point placed in a Coulomb field (Kepler's problem), next the *N-body problem* of celestial mechanics. He introduces then the notion of *constraint*, presents the *principle of virtual work* and the conditions in which a constraint is called *ideal*. He uses, to study the motion of a rigid body, the *group of Euclidean displacements* and establishes the equations of motion.

Returning to the case of a system of N material points without constraints, subjected to forces expressed by differentiable functions of the time and of the positions and velocities of these points, the author shows that the equations of motion are expressed as the differential equation associated to a vector field depending on time, defined on the *evolution space* of the system (a set, isomorphic to $\mathbb{R}^{6N+1}$, made of multiplets composed of the time and of the positions and velocities of the N material points). These equations determine a *foliation in curves* of the evolution space. Each of these curves is the mathematical expression of a possible motion of the system. It is why the author calls *space of motions* of the system the set made of these curves. He proves that the space of motions is a differential manifold (not always Hausdorff) of dimension $6N$ and there exists a natural projection, smooth and anywhere of rank $6N$, of the evolution space on the space of motions. Then he proves the existence, on the evolution space of the system, of a remarkable 2-form σ , which he calls *Lagrange's form*, because it was used in 1808 by Lagrange in his works in celestial mechanics. As pointed out by the author, this form was used in mechanics around 1950 by the French mathematician François Gallissot [4]. For a unique material point of mass m whose position and velocity vectors are, respectively, $\mathbf{r}$ and $\mathbf{v}$, on which acts a force $\mathbf{F}$, the components of these three vectors in a fixed orthonormal frame of the space being, respectively, (r_1, r_2, r_3) , (v_1, v_2, v_3) and (F_1, F_2, F_3) , this form reads

$$\sigma = \sum_{i=1}^3 (m dv_i - F_i dt) \wedge (dr_i - v_i dt), \text{ that can be written } \sigma = (m d\mathbf{v} - \mathbf{F} dt) \wedge (d\mathbf{r} - \mathbf{v} dt),$$

where the symbol $\wedge$ denotes the operator combining the dot product and the exterior product.

The Lagrange form of a system of N material points is the sum of the Lagrange forms of all points of the system. It determines the vector field of which the system's motions are the integral curves, since its kernel is the sub-bundle that determines the foliation in curves of the evolution space. The author remarks that it is always possible to choose N differentiable vector fields $\mathbf{B}_j$, defined on the evolution space, and to define N other vector fields $\mathbf{E}_j$ so that the force $\mathbf{F}_j$ acting on the j -th material point is $\mathbf{F}_j = \mathbf{E}_j - \mathbf{B}_j \times \mathbf{v}_j$, where $\mathbf{v}_j$ is the velocity of this material point.

When changing the reference frame, the parameterization of the evolution space and the expression of the Lagrange form are modified, particularly because inertial forces must be included in the forces acting on the material points (centrifugal force and Coriolis' force). The author states the *principe of Galilean relativity* which claims the existence of preferential reference frames, called *inertial reference frames*, such that the expression of the Lagrange form of an *isolated* system is the same in all inertial reference frames. He shows that the relative motion of an inertial reference frame with respect to another one is a translation motion at a constant velocity. The set of all changes of inertial reference frames is a Lie group, called the *Galilei group*. The author gives a matrix expression of this group, which is of dimension 10. It acts on the evolution space of an unconstrained system of material points by an action which preserves the Lagrange form. Using the principle of virtual works, the author extends the notions of evolution space, Lagrange form and space of motions, for a system of N material points involving ideal constraints, which can be either holonomic or non holonomic with a linear dependence on the velocities. The results obtained for the unconstrained systems of material points remain valid, provided that the constraints remain the same in all inertial reference frames.

The author calls *Maxwell's principle* the hypothesis that the exterior derivative of the Lagrange 2-form of a general dynamical system (not necessarily made of material points) vanishes. For a system of material points, since the vector fields $\mathbf{B}_j$ which appear in the forces $\mathbf{F}_j = \mathbf{E}_j - \mathbf{B}_j \times \mathbf{v}_j$ can be freely chosen, the author formulates Maxwell's principle in the following form: the vector fields $\mathbf{B}_j$ can be chosen in such a way that the Lagrange form is closed. This condition determines these fields in a unique manner. The author proves that, as a consequence of this principle, the vector fields $\mathbf{E}_j$ and $\mathbf{B}_j$ must not depend on the velocities of the material points and must verify both *Maxwell equations*

$$\text{rot} \mathbf{E}_j + \frac{\partial \mathbf{B}_j}{\partial t} = 0, \quad \text{div} \mathbf{B}_j = 0, \quad 1 \leq j \leq N.$$

The author shows that Maxwell's principle is well verified in a lot of cases: the N -body problem, a material point in the gravity field, an electrically charged particle in an external electromagnetic field. In this latter case, the above equations are the first two Maxwell's equations (Maxwell–Faraday equation and Maxwell–Thomson equation) that must be verified by the external electromagnetic field. The author deduces from them the well-known formula giving the expression of the *Laplace force*, and concludes that this force *is not* a relativistic effect since it results from the application of Maxwell's principle in the framework of classical mechanics.

By contrast, the creation of an electromagnetic field by electric charges in motion, mathematically described by the last two Maxwell equations (Maxwell–Ampère equation and Maxwell–Gauss equation) are relativistic effects which do not appear in the framework of classical mechanics. The author adopts Maxwell's principle as a new *law of the mechanics*. This allows him the study, in the framework of classical mechanics, systems more general than those made of material points. For the systems of material points, Maxwell's principle allows, under certain conditions, to define a Lagrangian and to show that the Lagrange form is nothing else than the exterior derivative of the Cartan form defined in the first chapter, in the study of calculus of variations. We can then apply the *principle of least action*, often considered as an essential piece of analytical mechanics. Moreover, when the Lagrangian is hyper-regular, one can associate to it a Hamiltonian and use for the study of motions the Lagrangian and Hamiltonian formalisms. Without denying the importance of the principle of least action nor the usefulness of these formalisms, the author declares that these concepts seem to him less fundamental than Maxwell's principle. His viewpoint seems to him justified because the existence of a Lagrangian is ensured only locally, and because there exist important systems, such as those made of particles with spin, to which Maxwell's principle applies while they have not a globally defined Lagrangian. In the sequel, the

author will not use the principle of least action except to present, very briefly, the *method of variation of constants* introduced by Lagrange in 1809 during his works on the slow variations of the orbital elements of planets.

The Lagrange form projects onto the manifold of motions, and its projection is a closed form, which is symplectic since it is automatically non degenerate. When the system is isolated, the Galilei group acts on the evolution space and on the space of motions. The moment map of this action on the evolution space is a first integral of the motion. The author details its ten components, which can be regrouped in three vectors $\mathbf{p}$, $\mathbf{l}$, $\mathbf{g}$ and a scalar E . He gives their physical meaning: $\mathbf{p}$ is the total linear momentum; $\mathbf{l}$ is the total angular momentum; the equality $\mathbf{g} = \text{Constant}$ conveys the fact that the center of mass moves on a straight line at constant velocity; the scalar E , defined modulo an additive constant, is the total energy.

Using a result due to V. Bargmann [5], who proved that the symplectic cohomology of the Galilei group is of dimension 1, the author shows that the class of symplectic cohomology of the action of the Galilei group on the space of motions of an isolated system made of material points in interaction can be interpreted as the *total mass*. The writing of the equations of motion gives two noteworthy results: the vector fields $\mathbf{B}_j$ necessarily all vanish; the total force and the total torque of the interaction forces are necessarily zero. The first result shows that in classical mechanics, a system made of material points cannot describe moving magnets. The second one expresses the *principle of equality of action and reaction*, which appears as a consequence of Maxwell's principle and of the principle of Galilean relativity.

Next the author presents several properties of the dynamical groups of a dynamical system and gives numerous examples. For the Kepler problem (motion of a material point in a Coulomb field) he shows how to *regularize* the manifold of motions and explains the origin of the exceptional first integral, often called the *Lenz vector* or the *Laplace vector*, which should rather be called the *eccentricity vector*. Although the author does not say it, it should be stressed that this first integral was discovered by the Swiss mathematician Jakob Hermann (1678–1833) [6].

In the paragraph *The Principles of symplectic mechanics*, the author first works in the framework of non relativistic, classical mechanics. He no longer is limited to systems of material points and adopts the three following assertions as new axioms of mechanics:

- (1) The space of the motions of a dynamical system is a *connected symplectic manifold*.
- (2) If several dynamical systems evolve independently, the manifold of motions of the composite system is the *symplectic direct product* of the spaces of motions of the component systems.
- (3) If a dynamical system is isolated, its manifold of motions admits the *Galilei group* as a dynamical group.

It is an *extension* of the principles generally admitted in classical mechanics, which will allow the author to consider new dynamical systems having a physical interest.

Since, for an isolated system, the author identifies the *mass* of the system with the number m which marks the class of cohomology of the Galilei group action on the space of motions, he can now consider systems of positive, null or negative mass. He keeps, for the more general systems that he will consider, the physical interpretation of the components of the Galilean moment which he has previously given: the vectors $\mathbf{l}$ and $\mathbf{p}$ are, respectively, the *angular momentum* and the *linear momentum* of the system, the scalar E is its *energy*. As for $\mathbf{g}$, it allows, for a system of non vanishing mass m , to define the *center of mass* of the system that the author calls the *center of gravity* or the *barycenter*: it is the point whose position vector, at each instant t , is $\mathbf{R} = (\mathbf{p}t + \mathbf{g})/m$. Hence this point moves on a straight line at constant velocity $\mathbf{p}/m$. The author

chooses as fundamental quantities the length L , the time T and the action A , and indicates the dimensionnal equations of the encountered quantities in the mathematical description of the dynamical system: coordinates of an element of the evolution space, Lagrange form, components of the matrices of an element of the Galilei group, of an element of its Lie algebra and of an element of its dual.

Next, the author proves two theorems concerning the action of a dynamical system on a symplectic manifold, that he might have placed in Chapter II. The first provides, by means of the action of a dynamical group G on a connected symplectic manifold V and the associated symplectic cocycle, the expressions of a moment and that of the associated symplectic cocycle of the restriction of this operation to an invariant subgroup G' of G . The second applies to a moment ψ of the action of a connected Abelian dynamical group G on a connected symplectic manifold V when the differential f at the identity element of the symplectic cocycle θ associated to this moment is a non degenerate bilinear form on the Lie algebra $\mathcal{G}$ of G .

The author points out that there exist two Abelian subgroups of dimension 3 of the Galilei group of changes of inertial reference frame of space-time: the group of translations of the frame of space, the frame of time remaining unchanged, and the group of changes of inertial reference frame of space-time in which a first inertial frame is replaced by a second one whose relative motion with respect to the first one is a translation at a constant velocity, the time frame remaining once again unchanged. The direct product of these two groups is a normal subgroup of the Galilei group, isomorphic to the additive group $\mathbb{R}^6$.

Applying the above-mentioned theorems, the author shows that the space of motions of an isolated system of mass $m \neq 0$ is the direct product of two spaces of motions: the *space of motions of the center of mass*, isomorphic to the space of motions of a material point of mass m , and the *space of motions around the center of mass*, in which the center of mass remains always at the origin of the reference frame of space. This important result, called the *barycentric decomposition* of the motions of an isolated system, is well known in classical mechanics. The Galilei group is a dynamical group both of the space of motions of the center of mass and of the space of motions around the center of mass, but acts on the last one only through the quotient by its normal Abelian subgroup isomorphic to $\mathbb{R}^6$. This quotient is isomorphic to $\text{SO}(3) \times \mathbb{R}$. Hence the direct product of the Galilei group and of $\text{SO}(3) \times \mathbb{R}$ is a dynamical group of any isolated system of non vanishing mass. The author gives several examples including the one, non classical, of a particle with spin.

Next the author tackles the study of the relativistic systems. He calls *Lorentz frame* an inertial reference frame of space-time in which the time unit has been chosen in such a way that the velocity of light is equal to 1. He presents the essential mathematical concepts used in the special theory of relativity: the Minkowski space, the Lorentz group and the Poincaré group. He gives the expressions of a diffeomorphism of $\text{SO}(3) \times \mathbb{R}^3$ onto the restricted Lorentz group (the connected component of this group containing the identity) and of matrix representations of the Lie algebras of these groups. He points out, without proof, several important properties: the Lie algebra of the Poincaré group is equal to its derived algebra; the symplectic cohomology of the Poincaré group, and more generally its cohomology valued in the dual of its Lie algebra, are trivial.

In special relativity, the passage from a Lorentz frame to another one is made by the action of an element of the restricted Poincaré group. The author establishes the formula of change of Lorentz frames and gives its matrix expression. The first two axioms of symplectic mechanics remain unchanged, and in the third one, the Galilei group is replaced by the restricted Poincaré group. Therefore these axioms are:

- (1) The space of motions of a dynamical system is a *connected symplectic manifold*.
- (2) If several dynamical systems evolve independently, the manifold of motions of the composite system is the *direct product* of the spaces of motions of the component systems.
- (3) If a dynamical system is isolated, its manifold of motions admits the *restricted Poincaré group* as a dynamical group.

It is always possible to choose the moment map of the action of the restricted Poincaré group on the space of motions of an isolated relativistic dynamical system so that the cocycle associated to the moment map is null. This condition determines the moment map in an unique way, while in classical mechanics, the moment map of the action of the Galilei group on the space of motions of an isolated system depends on an arbitrary additive constant. Moreover, in relativistic mechanics, the barycentric decomposition of motions of an isolated system no longer exists.

The intersection of the Galilei and Poincaré groups, considered as subgroups of the group of affine transformations of space-time, is a dynamical group of dimension 7 of isolated dynamical systems, both classical and relativistic. The moment map of its action on the space of motions of an isolated system is composed of the vectors $\mathbf{l}$, $\mathbf{p}$ and the scalar E , that the author interpreted in classical mechanics as being the angular momentum, the linear momentum and the energy. For a relativistic system, the author chooses to conserve for these quantities the same interpretation as in classical mechanics. The choice of a Lorentz frame allows to associate to the couple $(\mathbf{p}, E)$ a vector P of the Minkowski space-time, called the *4-momentum* or the *energy-momentum vector* of the system. Next the author indicates some formulae useful in geometry of oriented Minkowski space, referring for more details to his book [7]. The action of the Poincaré group on the space of motions of an isolated relativistic system reveals a second vector W of the Minkowski space-time, called the *polarization vector*, orthogonal to the energy-momentum vector P .

In a long section, the author proposes a mechanistic description of elementary particles. In the framework of relativistic mechanics, an isolated dynamical system is said to be *elementary* when the Poincaré group acts transitively on the space of its motions. The moment map of its action is then a symplectic diffeomorphism of this space onto a coadjoint orbit of the Poincaré group. For the author, the so defined *elementary systems* are mathematical models for *elementary particles* of physicists. He uses the type (timelike, spacelike or lightlike, or in other words isotropic) of the quadrivectors P and W , defined in the previous section, for a classification of elementary systems. By these means, he obtains a large part of the physicists' classification of elementary particles. Below, briefly summarized, his results are presented.

Case 1: a particle with spin. It is when P is timelike and when W (which, being orthogonal to P , is spacelike) is non-zero. The author proves that two real numbers m and s , whose expressions are given in terms of P and W , can be interpreted as the *mass* and the *spin* of the particle. These numbers are constant on the space of motions of the system. The mass m is non-zero, but can be either positive or negative, while the spin s is always strictly positive. For each given pair (m, s) , with $m \neq 0$ and $s > 0$, there exists only one model of particle with mass m and spin s . Its space of motions is 8-dimensional. For each motion of the system, there exists an affine straight line of the Minkowski space-time, parallel to the timelike quadrivector P , interpreted as the *trajectory* of the particle. By expressing the energy-momentum quadrivector P in any Lorentz reference frame, the author observes that the norm of the velocity v of the particle in that reference frame is always smaller than 1 (which is, with the chosen units, the norm of the velocity of light) and obtains the famous Einstein's formula

$$E = \frac{m}{\sqrt{1 - \|v\|^2}}.$$

Case 2: a particle without spin. It is when P is timelike and $W = 0$. There still exists a real number $m \neq 0$, expressed in terms of P , interpreted as the *mass* of the particle, constant on the space of motions of the system. For each given real $m \neq 0$, there exists only one model of particle without spin with mass m , and its space of motions is 6-dimensional. Still in that case, for each motion of the system, there exists an affine straight line of the Minkowski space-time, parallel to the timelike quadrivector P , interpreted as the *trajectory* of the particle. In any Lorentz reference frame the norm of the velocity of the particle is always smaller than 1, the energy E and the velocity v of the particle are related by the above written Einstein's formula.

Case 3: a massless particle. It is when both P and W are non-zero and lightlike. The author defines three real numbers $\eta = \pm 1$, $\chi = \pm 1$ and $s > 0$, interpreted, respectively, as the *sign of the energy*, the *helicity* and the *spin* of the particle, expressed in terms of P and W , constant on the space of motions of the system. For each triple (η, χ, s) of real numbers satisfying $\eta = \pm 1$, $\chi = \pm 1$ and $s > 0$, there exists only one model of massless particle with sign of the energy η , helicity χ and spin s , and its manifold of motions is 6-dimensional. For each motion of the system, there exists a three-dimensional affine subspace of the Minkowski space-time which, interpreted in any Lorentz reference frame, can be described as a two-dimensional spacelike plane moving at the velocity of light, called the *wavefront* of the particle. There is no more an affine straight line of the Minkowski space-time which can be considered as the trajectory of the particle. More precisely, for each Lorentz reference frame, there is such an affine straight line, which is lightlike and contained in the wavefront of the particle. However, this straight line depends on the chosen Lorentz reference frame and sweeps the whole wavefront when the considered Lorentz reference frame takes all the possible values.

The author briefly indicates the existence of other elementary systems, for example *tachyons*, which do not correspond to known elementary particles. Then he looks at *non relativistic elementary particles*, beginning by particles without spin. He obtains a mathematical model of such particles by means of a suitable change of variables in which the velocity of light c appears, and then by letting $c \mapsto +\infty$. The model so obtained corresponds to material points considered in classical non-relativistic mechanics. The same procedure starting with relativistic particles with spin leads to a model of non-relativistic material point with spin, interpreted as a *proper angular momentum*. He briefly indicates another way in which mathematical models of non-relativistic particles could be obtained, using as spaces of motions orbits of affine actions of the Galilei group on the dual of its Lie algebra, which may involve a symplectic cocycle. By this means he obtains models of non-relativistic massless particles moving at an infinite velocity, each model being characterized by three real numbers $\chi = \pm 1$, $s > 0$ and $k > 0$, called, respectively, the *helicity*, the *spin* and the *color* of the considered particle.

At the end of this section, the author explains that the theory of general relativity argues in favour of physical elementary dynamical systems whose space of motions admits the full Poincaré group G' as a dynamical group. Such a system's space of motions may have several connected components. The full Poincaré group G' has four connected components, and all its elements are obtained by composition of an element of the restricted Poincaré group G (the connected component of the neutral element) with elements of two discrete subgroups, each with two elements: the group of *space inversions* (exchange of right and left), and the group of *time reversals* (exchange of past and future). The author fully discusses geometric properties of the moment map of a Hamiltonian action of G' on a (maybe non-connected) symplectic manifold, as well as geometric properties of its coadjoint orbits, and presents their consequences for physical isolated elementary systems whose space of motions admits G' as a dynamical group. He considers elementary particles first with a non-zero mass, then with zero mass. Known massless particles (photons and neutrinos) exist with two opposite helicities, and the author

considers this fact as an argument in favour of the admission of the group of space inversions as an invariance group of mechanics.

Chapter III ends with a study of particles dynamics first in the framework of classical mechanics in a fixed inertial reference frame. Starting from the dynamics of a free material point, the author explains how to describe the dynamics of an electrically charged particle submitted to an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$. He uses a modification of the Lagrange form to account for the effects of $\mathbf{E}$, $\mathbf{B}$ and the electric charge of the particle. Using the fact that $\mathbf{E}$ and $\mathbf{B}$ satisfy the Maxwell equations, he proves that this modified Lagrange form still is closed, and that its kernel, of dimension 1, still determines a foliation in curves of the evolution space, and derives the equations of motion. The same procedure, in which the symplectic form of the space of free motions of a relativistic particle is used (instead of that of the space of free motions of a classical particle), leads to the equations of motion of a relativistic electrically charged particle submitted to an electromagnetic field. In these equations, in good agreement with experimental results, appears the relativistic correction of the linear momentum well known by physicists.

For a particle with spin, either classical or relativistic, submitted to an electromagnetic field, the author explains that experiments show that still another term must be added to the Lagrange form. This term is the product by a constant μ (later interpreted as the module of the magnetic moment of the particle) of the exterior derivative of a 1-form ϖ which, at the non-relativistic limit, is the product of dt by a function. The equations of motion so obtained in the non-relativistic approximation are in good agreement with the Stern and Gerlach experiments, as well as with the precession of spin and the magnetic resonance phenomena. For a relativistic particle these equations become very complicated. When the electromagnetic field is constant, they are used for the measurement of the *anomalous magnetic moment* of particles.

The author now considers n non-interacting particles, which may be either free, or subjected to a field. Let U_i be the space of motions of the system made by the i -th particle alone. The space of motions U of the system is the direct product of the spaces of motions U_i , $1 \leq i \leq n$. The author describes two different ways in which an evolution space can be built for the system, which lead to two different evolution spaces, respectively called the *synchronous evolution space* and the *asynchronous evolution space*. The synchronous evolution space is obtained by adding to the space of motions U one dimension for the time, and by taking the initial conditions of all particles at the same time. Its dimension is $\dim U + 1$. The asynchronous evolution space, built by taking the initial conditions of the particles at different times, is of dimension $\dim U + n$, and involves n different times, one for each particle. Both the synchronous and the asynchronous evolution spaces are presymplectic manifolds which project onto the space of motions U . The synchronous evolution space should be used only for non-relativistic particles since it involves a notion of simultaneity, which in relativistic physics depends on the choice of a Lorentz reference frame. When all the particles of the system are identical and submitted to the same field, the spaces of motions U_i , for all $i \in \{1, \dots, n\}$, are equal to U_1 and one could think that the space of motions of the system is $(U_1)^n$. However, experiments have shown that motions which only differ by the labelling of particles should be identified. The true space of motions of the system is the set of equivalence classes of n -uples $(x_1, \dots, x_n) \in (U_1)^n$ which satisfy $x_i \neq x_j$ for $i \neq j$, $1 \leq i, j \leq n$, two n -uples $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ being equivalent if there exists a permutation σ of $\{1, \dots, n\}$ such that $x'_i = x_{\sigma(i)}$ for all i , $1 \leq i \leq n$. The author indicates the expression of the Lagrange form and remarks that the use of a set of equivalence classes as a set of motions of the system is a consequence of the indiscernability of particles which does not involve quantum mechanics.

A “classical” method for obtaining the equations of motion of a system of interacting particles uses the synchronous evolution space of the system when the particles do not interact, with a

Lagrange form modified by addition of a suitable term involving an *interaction potential*. This method works successfully for celestial mechanics. The author uses it for a system of non-relativistic particles with spin, indicates the equations of motion and, when the system is isolated, writes the formulae which express the constancy of the Galilean moment map. The equations so obtained take into account the electrostatic and magnetostatic forces exerted by each moving particle on the other particles. However, they do not account for other small, but measurable relativistic effects, such as the Laplace force. To account for these effects, the author considers the use of the asynchronous evolution space. He proves that the use of that space would prohibit the existence of interactions between the particles, and concludes that a way to get around this difficulty could be to abandon the idea of localized particles, the distinction between particles and fields being a non-relativistic approximation.

The *scattering theory* is an approximate mathematical description of a system of interacting particles in which it is assumed that when each particle is far enough from all other particles, the interaction forces can be neglected. The author presents a mathematically rigorous version of this theory. He assumes that unscattered and scattered dynamical systems share the same evolution space with two different Lagrange forms, equal of an open subset Ω of the evolution space. The complement of Ω , on which the Lagrange forms of the scattered and unscattered systems are not equal, is called the *scattering source*. A scattered motion is said to be *constrained* when it is wholly contained in the scattering source. The author looks at unconstrained scattered motions contained in the scattering source only for a bounded time interval. He states, without a complete proof, a theorem according to which such a motion coincides with two different unscattered motions, one before it enters the scattering source and another one after it has finally left the scattering source. The so established correspondence between two unscattered motions is a symplectomorphism between two open subsets of the set of motions of the unscattered system. The author defines the *symmetry group* of the scattering source and proves that it is a dynamical group for both systems, scattered as well as unscattered. For *bounded, static or conservative* scattering sources, some properties of this group can be deduced. The author briefly presents the dynamical system made by photons in a refracting telescope. Due to a term containing a length (later interpreted as the wavelength of the light) in the formulae he obtains, the image of a distant star can never be a point. When this term is neglected, one obtains the *geometrical optics* approximation. Using the scattering theory, the author discusses, in relativistic physics, the reflection of light on a moving mirror and obtains formulae for the *Doppler effect due to reflection*.

Collisions of relativistic free particles with non-zero masses are finally discussed by the author, with the help of the scattering theory. Though no valid model of relativistic interacting particles is available, the author obtains some properties of the symplectomorphism which relates the free motions of the particles before and after their collision. He proves that this symplectomorphism commutes with the action of elements of the Poincaré group, which implies that the total momentum and the total energy of the system of particles are conserved by collisions. As a conclusion, he states that knowing this symplectomorphism could allow to study the *constrained motions* by the technique of analytic continuation.

Chapter IV: Statistical mechanics

It contains two sections. The first one, of about 50 pages, is essentially mathematical. The second one, of about 35 pages, presents the principles of statistical mechanics.

The first section of this chapter begins by introducing various concepts related to smooth manifolds, topological spaces, ordered vector spaces (called *Riesz spaces*), normed vector spaces,

specially Banach and Hilbert spaces. A very condensed course (about 30 pages) in Measure Theory and Integration follows, with some notions in Probability. The author uses the presentation, privileged by the Boubaki school, in which a measure on a smooth Hausdorff manifold V is an element of the topological dual vector space of the space of continuous, compactly supported functions on V , instead of defining first measurable parts of V , and then a measure as a denombrably additive function defined on the set of measurable parts. Then the author defines *probability measures*. A measure λ on V is said to be *defined by an everywhere positive continuous field of densities* on V , when its expression in every chart is the product, by an everywhere strictly positive continuous function, of the Lebesgue measure. The measure λ being fixed, the author considers a probability measure whose density with respect to λ is a continuous function $\rho \geq 0$. He defines the λ -entropy of this probability measure by setting

$$s_\lambda(\rho) = \begin{cases} \int_V -\rho(x) \log(\rho(x)) \lambda(dx) & \text{if this integral converges,} \\ -\infty & \text{if the above integral does not converge.} \end{cases}$$

By convention, at points $x \in V$ where $\rho(x) = 0$, the value taken by the function $x \mapsto -\rho(x) \log(\rho(x))$ is 0.

A continuous map Ψ , defined on V , with values in a finite-dimensional vector space E being given, the author calls *generalized Gibbs probability law* any completely continuous probability law for which the map Ψ is integrable, whose density ρ with respect to λ is expressed as

$$\rho(x) = \exp\left(-\left(z + \langle Z, \Psi(x) \rangle\right)\right), \quad \text{with } x \in V.$$

In the above equality, Z is an element of the dual vector space E^* of E , which must be such that the integrals below, appearing on the right hand sides of the equalities defining I_0 and I_1 , are convergent. The real z must be chosen in such a way that the integral of ρ on V with respect to the measure λ is 1.

The *normal distribution* on $\mathbb{R}^n$ is a generalized Gibbs probability law for a suitable choice of the map Ψ .

The mean value of Ψ , for the probability law of density ρ with respect to λ , being denoted by M , and setting

$$I_0 = \int_V \exp\left(-\langle Z, \Psi(x) \rangle\right) \lambda(dx), \quad I_1 = \int_V \Psi(x) \exp\left(-\langle Z, \Psi(x) \rangle\right) \lambda(dx),$$

the author can write

$$z = \log(I_0), \quad M = \int_V \Psi(x) \rho(x) \lambda(dx) = \frac{I_1}{I_0}.$$

The λ -entropy of the generalized Gibbs law of density ρ with respect to λ is

$$s(\rho) = z + \langle Z, M \rangle.$$

The author proves that, on the set of completely continuous probability laws for which the mean value of Ψ is M , the λ -entropy functional has a strict maximum at the generalized Gibbs probability law of density ρ with respect to λ . Moreover, when the set of values of Ψ is not contained in an affine subspace of E of dimension strictly smaller than $\dim E$, he proves that the map $Z \mapsto M$ is injective. He derives conditions for the differentiability with respect to Z , under the sign $\int$, of the integrals I_0 and I_1 . He will later improve these results in his paper [8], published a few years after his book [1].

The author then assumes that the Hausdorff manifold V is endowed with a symplectic form and that a connected Lie group G acts on it by a Hamiltonian action. For λ , he chooses the Liouville measure, and for the map Ψ a moment map of the action of G . The vector space E is therefore now the dual vector space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G , and its dual E^* is $\mathfrak{g}$. He denotes

by Ω the largest open subset of $\mathfrak{g}$ on which the integrals I_0 and I_1 are convergent and define differentiable functions of the variable Z whose differentials are continuous and can be obtained by differentiation under the $\int$ sign. A generalized Gibbs probability law can then be associated to each $Z \in \Omega$. The author indicates the corresponding expressions of $z = \log(I_0)$, of the entropy s and of the mean value M of the moment map, considered as functions of the variable $Z \in \mathfrak{g}$. He proves that when G acts effectively on V , the map $Z \mapsto M$ is injective and open, therefore is a diffeomorphism of Ω onto an open subset Ω^* of $\mathfrak{g}^*$. By using the inverse map, z , s and Z can be considered as functions of the variable M , which spans the open subset Ω^* of $\mathfrak{g}^*$. The author proves that these functions can be differentiated and indicates the expressions of their differentials. Adding a constant to the moment map does not change the generalized Gibbs laws on the manifold V . Their set is called by the author the *Gibbs set* of the considered Lie group action. Moreover the author proves that the open subset Ω of $\mathfrak{g}$ is a union of orbits of the adjoint action of G and is endowed with a definite negative Euclidean metric.

Results obtained by the author in the first section of Chapter IV are, in the second section, applied to dynamical systems encountered in physics. A *statistical state* of such a system is a probability law on the space of its motions. The author explains that in the *kinetic theory of gases*, a gas in a container at rest in a Galilean reference frame is considered as an assembly of very small material particles whose motions are governed by the laws of classical mechanics, interacting by instantaneous elastic collisions between themselves and with the walls of the container. With these assumptions, the gas can be modelled by a conservative Hamiltonian system. The one-dimensional group of time translations acts on the symplectic manifold of motions of the system by a Hamiltonian action, with the energy as a moment map. The author explains that the entropy of the system increases with time, so assuming that the *natural equilibria* of the gas are elements of the Gibbs set of the group of time translations is a very reasonable assumption. Each Gibbs state is determined by an element Z of the one-dimensional Lie algebra of this group, which is a way of measuring the *gas temperature*. Each Gibbs state is unaffected by the action of the group of time translations, since the adjoint action of this group on its Lie algebra is trivial. The author indicates the dimension equations of quantities which determine a Gibbs state, or appear in its description. He explains that when a compound system is the union of several subsystems, the tensor product of natural equilibria of the subsystems is a natural equilibrium of the compound system only when the element Z of the Lie algebra of the group of time translations which determine the natural equilibria is the same for all subsystems, in other words only when the temperatures of all the subsystems are equal. If this condition is not satisfied, the compound system is in a state of natural equilibrium only when the subsystems cannot exchange energy between them. As soon exchanges of energy can occur, even when they are very tiny, the compound system is no more in a natural state of equilibrium.

For the determination of the Gibbs states of a gas, one has to calculate the integral I_0 . The *ideal gas approximation* is when, for this calculation, the only motions taken into account are those for which, at the considered time, no collisions occur between particles or between a particle and the walls of the container, and when the total volume of the particles is considered as negligible in comparison with the volume of the container. The author successively considers monoatomic and polyatomic ideal gases and derives, for a Gibbs state, the probability distribution of velocities of the particles, which was determined by Maxwell in 1860. He explains the principle of an *ideal gas thermometer* used for precise measurements of the temperature. He is led to the formula $Z = 1/(kT)$, which expresses the absolute temperature T in terms of the element Z of the Lie algebra of the group of time translations, and of *Boltzmann's constant* k . He obtains the Mariotte, Gay-Lussac and Avogadro laws for perfect gases well known by physicists. He proves that for an ideal gas thermometer, the temperature is a random variable whose probability law is very

concentrated around its mean value, and converges (for the weak topology) towards the Dirac measure when the number of particles tends to infinity.

For a conservative system, the entropy s of a Gibbs state is a smooth function of the energy E whose derivative is the element Z of the Lie algebra which determines the Gibbs state. Units being chosen, Z becomes a real number. Experimental results show that this number is always strictly positive. The author can therefore consider the energy E of a Gibbs state as a function of its entropy s and of other variables which describe the system. For example, when the considered system is a gas, these additional variables are the volume of the container, the number of particles, etc. The author assumes that these variables can be globally described by an element u in a smooth Hausdorff manifold. Infinitesimal variations of a Gibbs state are therefore described by the differential of E ,

$$dE = \frac{ds}{Z} + \varpi du = TdS + \varpi du,$$

where ϖ denotes the partial differential of E with respect to u , and $S = ks$ (k being Boltzmann's constant). The author calls S and u the *position variables*, T and ϖ the associated *tension variables*. For example, when the considered system is a gas in a container whose volume may vary, the above expression becomes

$$dE = TdS - pdV,$$

where p is the *pressure*.

The author then explains that the infinitesimal variation of the energy E is the sum of the infinitesimal variations of two quantities Q and W , called respectively the *heat* and the *work*, which are not functions, but rather *action integrals* in the sense of calculus of variations. Variations of heat and work depend on the path followed while the adjustable variables which describe the state of the system changed. Different types of evolution of the state of a system should be separately considered, for example *adiabatic evolution*, *isothermal evolution*, etc.

The author defines the *heat capacities* of a gas at *constant volume* and at *constant pressure*, and indicates the expressions of the thermodynamic functions of an *ideal fluid*. These expressions depend on the chosen model for the particles. A table indicates their expressions and values for various models (material point, particle with spin, etc). Agreement with experimental results is good for a monoatomic gases, not so good for polyatomic gases and for solids at low temperatures.

Since the group of time translations is a subgroup, but not a *normal* subgroup, of the Galilei group, a dynamical system conservative in some inertial reference frame is not conservative in a different inertial reference frame. This important remark leads the author to introduce the new concept of *covariant statistical mechanics* by proposing the following principle:

When a dynamical system is invariant by the action of some Lie subgroup G' of the Galilei group, its natural equilibria are the elements of the Gibbs set of the action of G' .

Each natural equilibrium of such a system is determined by an element Z of the Lie algebra $\mathfrak{g}'$ of G' , which is a Lie subalgebra of the Lie algebra of the Galilei group. The author observes that Z generalizes the inverse of the temperature, and discusses its physical interpretation. For this purpose he chooses an inertial reference frame $\mathcal{R}_0$ of the Galilei space-time and considers the reference frame $\mathcal{R} = \exp(tZ)\mathcal{R}_0$, which generally is not inertial. He looks at the indications given, in the reference frame $\mathcal{R}$, by an ideal gas thermometer in equilibrium with the considered system. It amounts to observe the system in a moving frame. The author proves that it is as if $\mathcal{R}$ were an inertial frame and if the particles of gas were submitted to additional forces (inertial forces, i.e., centrifugal and Coriolis forces). By this means he obtains general formulae for the

probability density of the Gibbs state associated to Z . He then discusses in greater detail several examples: the wind (the relative motion of $\mathcal{R}$ with respect to $\mathcal{R}_0$ is a translation at constant velocity), accelerating rocket (this relative motion is a uniformly accelerated translation), gas in a centrifuge (this relative motion is now a rotation around an axis at a constant angular velocity). In this last example he considers also a system made by particles with spin, and finds that the most probable orientation of the particle's spin is parallel to the rotation axis.

One may wish to apply the above principle for a system invariant by the whole Galilei group. However, the subset Ω of the Lie algebra of the Galilei group made by elements which determine a Gibbs state is then empty. Looking at the motions of the system around its center of mass, the author is led to consider the just discussed equilibria in a rotating frame. For the author, the rotation of celestial bodies observed in Astronomy confirms the validity of his principle of covariant statistical mechanics.

For a relativistic dynamical system, the Galilei group must be replaced with the Poincaré group. As above, the Gibbs state of the action of the whole Poincaré group is empty, and the author is led to consider systems invariant by the action of a subgroup G' of the Poincaré group. He presents in more detail several examples: an ideal gas in a container at rest in an inertial frame (in that example Maxwell's distribution law for the velocities of particles is slightly modified); relativistic wind; statistical equilibria of photons. In this last example, the number of photons cannot be fixed. The author explains how such a system can be described and even takes into account the fact that photons can have two opposite circular polarizations. However, his formula is not in agreement with the one obtained by Planck for the black-body radiation, which is in very good agreement with experimental observations. The author concludes that this last example must be dealt with in the framework of quantum mechanics.

Chapter V: Geometric quantization

It contains two long sections, each of around 40 pages. Before describing their contents, let us recall that a *contact form* on a smooth Hausdorff manifold Y is a differential 1-form ω , which nowhere vanishes on Y , whose exterior derivative $d\omega$ is non degenerate on $\ker \omega$. The existence of a contact form on the manifold Y has several important consequences: Y must be odd-dimensional, the 2-form $d\omega$ is a *presymplectic form* and there exists on Y a unique vector field R_Y , called the *Reeb vector field* (in honour of the French mathematician Georges Reeb), determined by the two equalities $i(R_Y)(d\omega) = 0$ and $i(R_Y)\omega = 1$.

The author defines a *quantum manifold* as a smooth manifold Y endowed with a contact form ω such that all integral curves of the Reeb vector field R_Y are 2π -periodic. These curves are then the orbits of an operation on Y of the one-dimensional torus. The set of these curves, in other words the quotient of Y by this operation, is a symplectic manifold (U, σ_U) , called by the author the *basis* of the quantum manifold Y . The quantum manifold projects on U , and the pull-back by the projection map of the symplectic form σ_U is the presymplectic form $d\omega$.

A *quantization* of a given Hausdorff symplectic manifold (U, σ_U) is defined by the author as the construction of a quantum manifold whose basis is (U, σ_U) . A symplectic manifold is said to be *quantizable* when its quantization is possible. The author indicates, without proof, a necessary and sufficient condition in which a given symplectic manifold is quantizable: the cohomology class of its symplectic form must be $2n\pi$, with n an integer. It is satisfied when σ_U is the exterior derivative of a 1-form. The manifolds of motions of many mechanical systems, for example harmonic oscillators, Kepler's problem, and the N -body problem of celestial mechanics, are therefore quantizable. The author proves that the manifold of motions of a non-relativistic particle with spin is quantizable if and only if the spin of the particle is integer or half integer,

when expressed with $\hbar$ as unit. The author indicates several examples of quantizable symplectic manifolds, together with the full description of the corresponding quantum manifolds: two-dimensional spheres of integer or half-integer radii, spaces of motions of a relativistic particle, first with a non-zero mass, then with zero mass, with spin $1/2$ or 1 .

Isomorphisms of quantum manifolds are called by the author *quantomorphisms*. Any quantomorphism between two quantum manifolds projects onto a symplectomorphism between their bases. Two quantizations Y and Y' of the same symplectic manifold are said to be *equivalent* when there exists between them a quantomorphism which projects onto the identity map of their common basis. A symplectic manifold is said to be *monoquantizable* when all its quantizations are equivalent.

A group Γ of quantomorphisms of a quantum manifold Y projects onto a group of symplectomorphisms of its basis U , and its projection is a group homomorphism. Conversely, a group G of symplectomorphisms of the basis U is said to be *liftable* if there exists a group Γ of quantomorphisms of Y which projects onto it. When in addition the projection of Γ onto G is a group isomorphism, Γ is said to be an *isomorphic lift* of G . The author proves that the set of isomorphic lifts of G is in one to one correspondence with the set of its characters (group homomorphisms of G into the one-dimensional torus).

A theorem due to the author states that a simply connected quantizable symplectic manifold is monoquantizable. Another theorem explains how to quantize a symplectic manifold U when a quantization of a covering manifold U' of U is known. Conversely, when a quantization Y of U is known and when the group of symplectomorphisms of Y determined by the covering manifold U' isomorphically lifts onto a group Γ of quantomorphisms of Y , Γ is a discrete group and can be used to build a covering manifold Y' of Y , which quantizes U' . Using these two theorems, it can be proven that there exist as many non equivalent quantizations of a quantizable connected symplectic manifold as its homotopy group has distinct characters. Therefore, being simply connected, the two-dimensional sphere, any symplectic vector space, the space of motions of a free particle, (with or without spin, non-relativistic or relativistic), when quantizable, are monoquantizable. The author then discusses the quantizability of the manifold of motions of a system of N particles without interactions. If each of these particles can be distinguished from the others and has an integer or half-integer spin, the space of motions of the system is quantizable and simply connected, therefore monoquantizable. But if each particle cannot be distinguished from the others, the space of motions of the system has exactly two non-equivalent quantizations, corresponding to the two distinct characters of the group of permutations of a set of N elements. These two quantifications are the Bose–Einstein and Fermi–Dirac quantizations, well known to physicists.

A smooth vector field, defined on a quantum manifold Y , whose flow acts on Y by quantomorphisms, is called by the author an *infinitesimal quantomorphism*. The Lie derivative of the contact form ω defined on Y with respect to an infinitesimal quantomorphism vanishes. The author proves that each infinitesimal quantomorphism is associated to a smooth function defined on the basis U of Y . For example, the Reeb vector field is an infinitesimal quantomorphism associated to the constant function whose value at any point in U is 1 . The Lie bracket of two infinitesimal quantomorphisms is an infinitesimal quantomorphism, whose associated function is the Poisson bracket of the functions associated to these two infinitesimal quantomorphisms.

The author then discusses the *quantization* of a dynamical group of a quantizable symplectic manifold U , with the quantum manifold Y as quantization. When a Lie group acts on Y by quantomorphisms, it acts also on the basis U by symplectomorphisms, the projection of Y onto U being equivariant with respect to these actions. Therefore G is a dynamical group of U , and is

said to be a *quantizable* dynamical group. A quantizable dynamical group of U is always liftable. The author proves that when a dynamical group of U is quantizable, its symplectic cohomology is zero. He gives examples which prove that this necessary condition is not sufficient.

A dynamical group of U which is liftable, but not quantizable, may have an extension which still is a dynamical group of U and is quantizable. As a first example, the author considers a symplectic vector space E which acts on itself by translations. It is a dynamical group of E . The product $Y = E \times T$, where T is the one-dimensional torus, is a quantization of E , which happens to be a Lie group, called the *Weyl group*, extension of the Abelian group E . It acts on itself by translations, which are quantomorphisms. The Weyl group is therefore a quantizable extension of the Abelian group E , which itself is not quantizable since its symplectic cohomology is not zero. As a second example, the author considers the group $SO(3)$ acting on a two-dimensional sphere S_2 centered on the origin of $\mathbb{R}^3$, endowed with its element of area form as symplectic form. This symplectic manifold is quantizable if its total area is an integer or an half-integer. In this latter case, the author proves that the dynamical group $SO(3)$, which is liftable since S_2 is simply connected, is not quantizable, though its symplectic cohomology is zero. He proves also that the group $SU(2)$ is a quantizable extension of $SO(3)$. A similar situation occurs for the restricted Poincaré group, which is a dynamical group of the manifold of motions of a relativistic particle. This group is quantizable if and only if the particle's spin is integer. If the particle's spin is half-integer, there exists a quantizable extension of the restricted Poincaré group described by the author in terms of *Dirac's spinors*.

The author denotes by $\mathcal{H}(Y)$ the vector space of smooth, complex-valued and compactly supported functions defined on a quantum manifold Y , which are equivariant with respect to the one-dimensional torus T actions on Y by the flow of the Reeb vector field and on $\mathbb{C}$ by multiplication (the torus T being identified with the set of complex numbers of modulus 1). After indicating the definitions and results in topology and functional analysis he is going to use, he proves that $\mathcal{H}(Y)$ is a pre-Hilbert space. He defines the *Hilbert space of the quantum manifold* Y as the completion of $\mathcal{H}(Y)$. He defines also C^* -algebras, indicates some of their properties and proves that the set of bounded linear endomorphisms of a Hilbert space is a C^* -algebra. He very shortly presents many concepts about operators on a Hilbert space: self-adjoint, normal, Hermitian, unitary operators, etc. For a given quantum manifold Y of basis U , the author then explains how to associate a Hermitian operator $\hat{u}$ on $\mathcal{H}(Y)$ to any smooth function u defined on the basis U . He proves that the map $u \mapsto \hat{u}$ is linear and injective, that for $u = 1$, $\hat{u}$ is the identity of $\mathcal{H}(Y)$, and that for any pair (u, u') of smooth functions on U ,

$$\hat{u} \circ \hat{u}' - \hat{u}' \circ \hat{u} = -i\widehat{\{u, u'\}}.$$

He illustrates this important result in the case when Y is the quantization of a symplectic vector space, the smooth functions on this vector space being canonical coordinates. The operators associated to smooth functions in this way often are not bounded, which makes their study and use very difficult. He proposes to build directly quantomorphisms acting on Y , without taking the detour of infinitesimal quantomorphisms and exponentiation. He proves a theorem which asserts the existence of an injective homomorphism of the group of quantomorphisms of Y into the group of unitary operators on the Hilbert space of Y .

The second section of Chapter V begins with the statement and explanation of the *Correspondence Principle*, well known by physicist. Quantum mechanics is not an autonomous theory: it cannot be formulated without using classical mechanics. According to the Correspondence Principle, for each physical phenomenon described in the framework of quantum mechanics, there should exist a corresponding "classical approximation" described in the framework of classical mechanics. For the author, this principle is an argument in favour of the extension of classical

mechanics he proposed in Chapter II, the *Maxwell Principle*. Without this extension, the use of classical mechanics would be limited to systems made of material points, excluding many phenomena encountered in physics such as particles with spin.

The *quantization* of a classical mechanical system is the construction of the quantum mechanical system of which this classical system is an approximation. In this section the author examines the possible application of the mathematical theory of Geometric Quantization developed in the previous section, to the quantization of real systems encountered in physics. His initial assumption is: the space U of classical motions of the system is a *quantizable symplectic manifold*. The first consequence of this assumption is: expressed with the constant $\hbar$ as unit, the value of the spin of particles must be either an integer or a half-integer. This consequence is in perfect agreement with experimental results.

Let the quantum manifold Y be a quantization of the classical manifold of motions U . A *state vector* of Y is any element Ψ with norm 1 of the pre-Hilbert space $\mathcal{H}(Y)$. The author explains its probabilistic interpretation: for each $\xi \in Y$, the real non-negative number $|\Psi(\xi)|^2 = \bar{\Psi}(\xi)\Psi(\xi)$ depends only on the projection x of ξ on the manifold of motions U ; the function so defined on U is the probability density, with respect to the Liouville measure, of a probability law on U which is the *statistical state* of the system (in the sense defined in Chapter IV) corresponding to the state vector Ψ .

An *observable* in classical mechanics is a smooth function u defined on the classical manifold of motions U . According to the British scientist P.A.M. Dirac, *observables* in quantum mechanics are Hermitian operators. In agreement with this view, the author has proven in the previous section that a Hermitian operator $\hat{u}$ could be associated to the classical observable u . This operator is the quantum observable of which u is the classical approximation. When applied to the phase space of a classical conservative system at a given time t , this idea leads to Dirac's *quantum equations of motion* and to the *commutation relations* chosen by the physicists W. Pauli and W. Heisenberg as fundamental ingredients of quantum mechanics.

In favorable cases, an isotropic foliation of the classical space of motions U can be lifted into a *Planck foliation* of the quantum manifold Y , that means a foliation such that the contact form defined on Y vanishes on all vectors tangent to the leaves. When a state vector Ψ is constant on each leaf of this Planck foliation, this vector state is said by the author to *satisfy Planck's condition* (relative to the considered Planck foliation). The *Planck space* is the set of state vectors which satisfy Planck's condition. For a conservative system, using as isotropic foliation of U the foliation whose leaves are the integral curves of the Hamiltonian vector field whose Hamiltonian is the energy, by writing explicitly Planck's condition, the author obtains the famous equality $E = h\nu$ which relates the energy E , the Planck constant h and the frequency ν .

The author then assumes that a dynamical group G acts on the classical manifold of motions U , with a moment map, and that there exists an Abelian normal subgroup $\tilde{G}$ of G whose symplectic cohomology is zero. He uses as isotropic foliation of U the foliation whose leaves are tangent to the vector sub-bundle of TU generated by Hamiltonian vector fields whose Hamiltonians are the components of the moment map of the action of $\tilde{G}$. He lifts this foliation into a Planck foliation of Y , writes explicitly the corresponding Planck's condition, and determines the associated Planck space. For an isolated system, G will be either the restricted Poincaré group or the Galilei group, according to whether the system is relativistic or non-relativistic. The Abelian subgroup $\tilde{G}$ will be the group of space-time translations. The author explains how Planck's condition leads to the quantum wave equations. For a non-relativistic material point, the author obtains by this means the *Schrödinger equation*, and for a relativistic material point, the *Klein–Gordon equation*. For a non-relativistic particle with spin $1/2$, using a $\mathbb{C}^2$ -valued variable as state vector, the author

proves that both its components satisfy a Schrödinger equation; it is the description of electrons proposed by Pauli. For a relativistic particle of spin $1/2$, he obtains *Dirac's equation* and proves that rather than the Poincaré group itself, it is a quantizable extension of this group which acts on the set of solutions of this equation. For him, the non-quantizability of the Poincaré group provides a natural explanation of this fact, well known to physicists. The author applies the same procedure for a massless particle, first with spin $1/2$, then with spin 1 .

Next the author considers an assembly of particles of a given type in indeterminate number. He denotes by U_Φ its manifold of motions and by U the manifold of motions of the system made by a single particle of the considered type. He calls U_Φ the *Fock's manifold* and explains that as a set, it is the set of all finite subsets of U . As a manifold, U_Φ is the sum of disjoint parts of different dimensions, each part U_n being the manifold of motions of the system made of a fixed number n of particles, with $0 \leq n < +\infty$. For U_0 , the author takes a singleton considered as a 0-dimensional manifold. As seen in the previous section, for each $n \geq 2$, the part U_n of U_Φ has two non-equivalent quantizations, one for each distinct character of the group of permutations of a set of n elements. The author explains that for some physical considerations, the same character should be chosen for all possible values of n , therefore he obtains two non-equivalent quantizations Y_Φ of U_Φ . On the pre-Hilbert space $\mathcal{H}(Y_\Phi)$, he defines *creation* and *annihilation* operators, which respectively increase or decrease the number of particles by one unit. Two different cases must be distinguished, depending on whether the particles are *fermions* or *bosons*.

At the end of Chapter V, the author discusses the notion of *average value* of an observable for a given state vector. Planck's condition appears as a sufficient condition so that the statistical and quantum mechanical averages of an observable coincide. Finally the author uses the notion of *function of positive type* defined on a group, with values in $\mathbb{C}$, to enlarge the definitions of a *quantum state* and of *average value* of an observable, to include states defined by a *density operator* encountered in quantum chemistry.

2. Comments

The great originality of the book *Structure des systèmes dynamiques* clearly appears in the detailed presentation of its contents given above. In this section, we will first try to identify the most innovative and promising ideas which can be found in it. Then we will write a few words about the terminology and the notations used by the author, which do not lack originality too.

Remarkable scientific concepts

Innovative ideas presented in the first two chapters are mainly related to the ways in which difficult subjects can be taught to beginner students. The first really remarkable scientific concept which appears in this book is, in our opinion, presented in Chapter III. The author begins with a short and rather classical account of the general principles of classical mechanics. Then he proves that on the evolution space of a dynamical system, there exists a remarkable presymplectic form, which he calls the *Lagrange form*. Its kernel determines the vector field whose integral curves are the *motions* of the system. This form projects onto the set of all possible motions, called the *space of motions*, and its projection is a *symplectic form*, i.e., a closed, non degenerate 2-form. The author chooses this property, called the *Maxwell principle*, as the founding principle of mechanics. For us, it is a very important innovation: the traditionally used concepts, such as the *configuration space*, the *evolution space*, the *phase space*, go into the background and leave

the first place to the *space of motions* and to the *Lagrange form* of which it is endowed. Thanks to this innovation, the author will be able to describe, in the framework of classical mechanics, with the dynamics of a particle with spin, though there is no Lagrangian for such a system, and with assemblies of an indeterminate number of such particles. Similarly, he will be able to describe systems made by massless relativistic particles, though there is no evolution space for such systems.

The space of motions of a dynamical system is not very often considered by modern authors, though it appeared as soon as 1808 in the works of Lagrange. This very natural concept has a nice mathematical property: the space of motions is always endowed with a smooth manifold structure. The *phase portrait* of a dynamical system, a frequently used closely related concept, very often has a much more complicated structure. One may wonder why the concept of space of motions is not used more by modern authors. Maybe it is because for some dynamical systems, the space of motions is a *non-Hausdorff* manifold. Another possible explanation is that some scientists are interested in the thorough description of particular motions of a system, rather than by the study of the set of all possible motions. By showing that many important results can be deduced from the symmetries of the space of motions of a system, the author proves that this reluctance is ungrounded. For example, in Chapter III, Section 12, he proves that the principle of equality of action and reaction appears as a consequence of Galilean relativity and Maxwell's principle.

In classical (non-relativistic) mechanics, the cohomological interpretation of the *mass* of an isolated dynamical system is, in our opinion, another innovative idea worth mentioning. A cohomology class appears indeed in the mathematical expression of the action of the Galilei group on the space of motions of such a system. It is an element of a one-dimensional vector space, and the author has proven that it can be interpreted as the mass of the system. In classical mechanics, it is legitimate to consider isolated systems with a positive, null or negative mass. It will be done by the author in his study of the behaviour of elementary systems with respect to time or space reversals.

Another originality of this book is that it presents dynamical systems in the framework of relativistic physics as well as in the framework of classical, non-relativistic mechanics. This is made possible thanks to the use of the concept of state of motions. In classical, non-relativistic mechanics, the Kepler problem and the n -bodies problem are very good mathematical models for systems of gravitationally interacting material points. In relativistic physics, there is no such mathematical model for a system of interacting electrically charged particles. These particles interact by means of the electromagnetic field created by their motion. Once created, this electromagnetic field evolves according to its own laws. Up to now, no mathematical model is available to describe the motions of the particles with the integral curves of a vector field defined on some hypothetic evolution space, only depending on the positions and motions of the particles. The author manages to get around this difficulty by using the space of motions of the system and Maxwell's principle.

In classical mechanics, the symmetry group of the space of motions of an isolated dynamical system is the Galilei group. In relativistic physics, this group is the Poincaré group. This change of symmetry group has important consequences clearly described by the author. The status of *mass* is in relativistic physics very different from its status in classical mechanics, due to the fact that the symplectic cohomology of the Poincaré group is trivial. In relativistic physics, there is no more a *barycentric decomposition* of the state of motions of an isolated system, as in classical mechanics, because the Poincaré group has no privileged normal Abelian subgroup, as the Galilean group has.

The works of the German mathematician Emmy Noether [9] told us that *first integrals* of a Lagrangian or Hamiltonian system very often are linked with symmetries of the equations of motion. Most textbooks in classical mechanics published before the author's book only indicate how a real valued first integral is determined by each one-parameter symmetry group of these equations. We believe that Jean-Marie Souriau is, with Stephen Smale [10,11], among the first scientists who considered the geometric structure of the full set of these first integrals: they are the components of the *moment map* of the symmetry group's action, defined on the evolution space and valued in the dual vector space of the Lie algebra of this group.

An *elementary system* is a relativistic dynamical system whose space of motions is an homogeneous space of the restricted Poincaré group. This purely mathematical definition, due to the author, seems to us very important because the classification of elementary systems reveals many properties of physicists' *elementary particles*, especially their *mass*, their *spin* and, for massless particles, their *helicity* (whose possible values, 1 or -1 , correspond to the two different circular polarizations of photons). In the non-relativistic approximation, elementary systems are models of elementary particles in the framework of classical mechanics. By this means, Geometric Optics appears, when the particle's spin is negligible, as a part of classical mechanics. The author's works so appear in continuity with those of Hamilton [12], who introduced the *characteristic function* first in Optics before using it in mechanics. Interested readers will find in the long Introduction of the book [13] a very nice discussion of the symplectic aspects of geometric Optics and Electromagnetism.

Chapter IV also contains several worth mentioning innovations. For the author, a *statistical state* of a dynamical system is a probability measure on the space of motions of the system. This definition is more natural than that generally used, for example by George Mackey [14] who, instead of the state of motions, uses the phase space at a given time. However, the most important innovation contained in this chapter seems to us the generalization of the notion of a Gibbs state, in which the energy is replaced by the moment map of the action of a symmetry group. This very natural generalization (the energy being the moment map of the action of the one-dimensional group of time translations) involves remarkable new features when the symmetry group is not Abelian. This group acts on the dual space of its Lie algebra by an affine action involving a symplectic cocycle, with the coadjoint action as linear part, for which the moment map is equivariant. By using generalized Gibbs states, the author develops a kind of *Lie groups thermodynamics*, which seems to us very interesting, as well from the mathematician's viewpoint as for possible applications in physics. In this theory, the generalized temperature and the generalized quantity of heat are, respectively, elements of the Lie algebra of the symmetry group and of its dual vector space. The author proves that an open convex subset of the Lie algebra is endowed with a Riemannian metric which plays an important part in Statistics and Information theory.

The seeds of Geometric Quantization, presented in Chapter V, can be found in George Mackey's small book [14], published in 1963. Jean-Marie Souriau [1] and Bertram Kostant [2], its main creators, independently proposed two slightly different, but equivalent versions of this theory. On the basis made by a symplectic manifold, Kostant defines a bundle whose fibres are complex lines, endowed with a connection whose curvature is the symplectic form of its basis. Similarly, Souriau defines a bundle whose fibres are circles, endowed with a contact form ω , whose Reeb vector field is tangent to the fibres. Moreover, the exterior derivative of ω projects onto the symplectic form of the basis. Souriau's circle bundle is the *principal bundle* which is associated to Kostant's complex line bundle, and the connection form of Kostant's line bundle is the contact form ω of Souriau. While Souriau quantizes the *manifold of motions* of a classical dynamical system, Kostant quantizes its *phase space* at a given time. However, this difference in their choice

of the quantized symplectic manifold may not be as important as it seems, because Souriau often uses the local symplectomorphism which exists between the space of motions and the phase space, especially when he derives the quantized wave equations.

Many remarkable results are presented in Chapter V. For instance, the author proves that a quantizable system made by indistinguishable particles has exactly two non-equivalent quantizations, both experimentally observed: they are the Fermi–Dirac and Bose–Einstein quantizations well known by physicists. He proves too that wave functions of quantum mechanics can be described with functions defined on the quantum manifold, and that the usual spatio-temporal description is obtained by means of a Fourier transform. The quantization of the space of motions of an assembly of indistinguishable particles leads to the *second quantization formalism*: quantum vacuum, creation and annihilation operators. The author also shows that the geometric obstructions encountered when the action of a symmetry group on the classical space of motions of a system is lifted to the quantum manifold, explains some facts well known by physicists: the lift of the Abelian group of translations is the non-Abelian *Weyl group*; elements of the Galilean group can be separately lifted, but this full group does not act on the quantum manifold (fact due to a cohomological obstruction discovered by Valentine Bargmann [5]); the Poincaré group can be lifted for a particle of integer spin, but not for a particle of half-integer spin, in which case it is a two sheets covering space of the Poincaré group which can be lifted, which leads to the use of Dirac’s spinors.

Finally, we must quote the many conjectural ideas presented by the author in the last section of his book, for example about the behaviour of elementary systems with respect to space or time reversals, and about a generalization of the notion of quantum state.

Language and notations of the author

The author’s language, although slightly different from the one generally used in differential geometry, is perfectly logical and understandable. One of the very few terms which could cause misunderstandings is that of *embedding* (in French, *plongement*). For the author, an embedding is a smooth map defined on a smooth manifold V , or on an open subset of V , with values in another smooth manifold V' , which is injective (the author uses the term *regular* for *injective*) and whose rank is everywhere equal to the dimension of V . Most geometers call such a map *an injective immersion*, and use the term *embedding* for injective immersions which, in addition, are homeomorphisms of their domain of definition onto their image, endowed with the topology induced by that of V' . It seems to us that the choice made by the author is very reasonable because injective immersions are much more frequently encountered than embeddings. For example, orbits of a Lie group action, as well as leaves of a foliation, always are immersed in the manifold in which they are contained, and much more rarely embedded. This indication only concerns the original version of the book in French. In its English translation [15], the translators use the usual term *injective immersion* and, in a footnote, indicate that for the author a *submanifold* is an *immersed submanifold*.

Another particularity of the author’s language which seems to us worth mentioning, is about the notion of an *Euclidean vector space* of finite dimension. For the author, it is a real or complex finite-dimensional vector space E endowed with a symmetric, bilinear form g satisfying the following conditions.

- If E is a real vector space, g is assumed to be non degenerate, which means that for any $x \in E$, $x \neq 0$, there exists $y \in E$ such that $g(x, y) \neq 0$. The author does not impose

the condition g positive. He still uses the term *Euclidean vector space*, not the term *pseudo-Euclidean vector space* when g is definite without being positive, as done by most mathematicians.

- If E is a complex n -dimensional vector space, g is a symmetric, bilinear form *for the structure of $2n$ -dimensional real vector space of E* underlying its structure of n -dimensional complex vector space. The form g is assumed to be non degenerate and to satisfy, for all pair (x, y) of elements in E , $g(ix, iy) = g(x, y)$, where $i = \sqrt{-1}$.

This convention seems to us very logical and useful, because it unifies two different concepts often taught separately: the concept of real Euclidean finite-dimensional vector space and that of complex finite-dimensional Hermitian vector space.

Some notations of the author seem to us disconcerting. For example, he calls *variable* any symbol, such as the letter y , and associates it to a map. He denotes by $[\text{value } y](a)$ the value of the map associated to y at the point a of its domain of definition. He uses different symbols for the variable, the associated map, the point of its domain of definition and its value at that point. So it is sometimes difficult to understand some expressions he writes, because the reader must simultaneously have in mind the meaning of many symbols.

Given a vector field f defined on a smooth manifold V , the author calls *derivation* associated to f the operation which, to any smooth map A defined on V and valued in a smooth manifold V' , associates the map whose value, at each $x \in V$, is the image of the vector $f(x)$ by the linear map tangent to A at x . He denotes derivations by symbols such as d or δ , other than the symbol which denotes the vector field (here the symbol which denotes the vector field is f). When the author considers several vector fields, he uses symbols such as d, d_1, δ, δ_1 , etc to denote the corresponding derivations. Of course, these conventions are perfectly logical and coherent. However, their use may make the reading of some formulae rather difficult. Moreover, since the symbol d is used for derivations associated to vector fields, the author cannot use it for the exterior derivation of differential forms, as done by most mathematicians. For the exterior derivation of differential forms, the author uses the symbol ∇ , which may disconcert some readers.

Let us finally indicate a few peculiarities of the notations used by the author in exterior differential calculus. He does not use the symbol $\wedge$ for the exterior product of differential forms, which makes some formulae heavy. He denotes by $\eta(X)$ the interior product of a differential form η by a vector field X , while many other mathematicians denote it by $i(X)\eta$, $i_X\eta$ or $i_X(\eta)$. It is probably for this reason that he denotes by $\eta(X_1)(X_2) \cdots (X_p)$ the evaluation of the p -form η on the vector fields $X_1, X_2, \dots, X_p$, instead of writing more simply $\eta(X_1, \dots, X_p)$. At first glance, this convention may seem simple and convenient. However, it makes some formulae heavy, such as the Cartan formula $\mathcal{L}(X) = i(X)d + di(X)$, which expresses the Lie derivative with respect to a vector field X as the anticommutator of the interior product by X with the exterior derivative.

3. Novel research ideas in Jean-Marie Souriau's footsteps

Every research book is a survey at a given time of the state of knowledge obviously limited on the issue. Just as the author considered that the book by Lagrange *Mécanique analytique* was unfinished, we believe that *Structure des systèmes dynamique* is a very in-depth work opening new research paths.

4. Conclusion

We cannot fail to be impressed when reading this book by the extent and the thoroughness of the author's knowledge, as well in mathematics as in mechanics or in physics, and by the originality and the depth of his thoughts. Jean-Marie Souriau is the author of two other very remarkable books, *Géométrie et Relativité* [16] and *Calcul linéaire* [7], which are very rich and original too and deserve to be read, and read again. We believe that among the paths for research he indicates in these books, many still are not yet fully explored.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

- [1] J.-M. Souriau, *Structure des systèmes dynamique*, Dunod, 1970.
- [2] B. Kostant, "Quantization and unitary representations. I. Prequantization", in *Lectures in Modern Analysis and Applications, III*, Lecture Notes in Mathematics, vol. 170, Springer, 1970, pp. 87–208.
- [3] J.-L. Lagrange, "Mécanique analytique. Première édition chez la veuve Desaint, Paris, 1808. Réimprimé par Jacques Gabay, Paris, 1989. Deuxième édition par Mme veuve Courcier, Paris, 1811. Réimprimé par Albert Blanchard, Paris. Quatrième édition (la plus complète) en deux volumes, avec des notes par M. Poincot, M. Lejeune-Dirichlet, J. Bertrand, G. Darboux, M. Puiseux, J. A. Serret, O. Bonnet, A. Bravais", in *Œuvres de Lagrange, volumes XI et XII*, Gauthier-Villars, 1888.
- [4] F. Gallissot, "Les formes extérieures en mécanique", *Ann. Inst. Fourier* **4** (1952), pp. 145–297.
- [5] V. Bargmann, "On unitary ray representations of continuous groups", *Ann. Math. (2)* **59** (1954), pp. 1–46.
- [6] J. Hermann, "Extrait d'une lettre de M. Herman à M. Bernoulli datée de Padoüe le 12 Juillet 1710", in *Histoire de l'académie royale des sciences*, Imprimerie royale, 1710, pp. 519–521.
- [7] J.-M. Souriau, *Calcul linéaire, tome I et II*, Euclide, Éditions Jacques Gabay, 1992.
- [8] J.-M. Souriau, "Mécanique statistique, groupes de Lie et cosmologie", in *Géométrie symplectique et physique mathématique*, Colloques internationaux du CNRS, vol. 237, Éditions du CNRS, 1974, pp. 59–113.
- [9] E. Noether, "Invariante Variationsprobleme", in *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, *Mathematisch-physikalische Klasse*, Weidmannsche Buchhandlung, 1918, pp. 235–257.
- [10] S. Smale, "Topology and mechanics. I", *Invent. Math.* **10** (1970), pp. 305–331.
- [11] S. Smale, "Topology and mechanics. II", *Invent. Math.* **11** (1970), pp. 45–64.
- [12] W. R. Hamilton, "Theory of systems of rays, Part First and Part Second, Supplement to an essay on the theory of systems of rays, Second supplement to an essay on the theory of systems of rays and Third supplement to an essay on the theory of systems of rays", in *Sir William Rowan Hamilton mathematical Works, vol. I, chapters 1 to IV*, Cambridge University Press, 1830.
- [13] V. Guillemin and S. Sternberg, *Symplectic techniques in Physics*, Cambridge University Press, 1984.
- [14] G. W. Mackey, *The mathematical foundations of quantum mechanics: A lecture-note volume*, W. A. Benjamin, Inc., 1963, pp. x+137.
- [15] J.-M. Souriau, *Structure of Dynamical Systems. A Symplectic View of Physics. Translated by C. H. Cushman-de Vries. Translation Editors R. H. Cushman and G. M. Tuynman*, Progress in Mathematics, vol. 149, Birkhäuser, 1997.
- [16] J.-M. Souriau, *Géométrie et relativité*, Collection Enseignement des Sciences, Éditions Jacques Gabay, 2008.