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Research article

An assembly of two slender anisotropic beams through a very thin adhesive layer

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Abstract. We derive various models of assemblies of slender anisotropic linearly elastic beams through an asymptotic analysis taking into account a triplet of small parameters associated with the slenderness of the beams but also the thinness and the stiffness of a very thin third body which connects them. Our models allow the description of the mechanical constraint of the linkage between two beams which strongly depends on the relative orders of magnitude of the previous parameters.

Keywords. assemblies of slender anisotropic beams, asymptotic analysis, variational convergence, mechanical constraints.

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1. Setting the problem

Using the same asymptotic analysis by variational convergence, we continue the studies [1,2] by considering the case of several kinds of assemblies of two slender anisotropic beams through a very thin layer. As usual we do not distinguish between $\mathbb{R}^3$ and the Euclidean physical space whose basis is $\{e_1, e_2, e_3\}$. Throughout the paper, lower Greek (resp. Latin) indices run from 1 to 2 (resp. 1 to 3) and for every $\xi = (\xi_1, \xi_2, \xi_3)$ of $\mathbb{R}^3$, $\widehat{\xi}$ denotes (ξ_1, ξ_2). The space of symmetric 3×3 matrices is denoted by $\mathbb{S}^3$ and we write $\otimes_s$ for the symmetrized tensor product, i.e. given ξ and ζ in $\mathbb{R}^3$, the element $\xi \otimes_s \zeta$ of $\mathbb{S}^3$ is defined componentwise by $(\xi \otimes_s \zeta)_{ij} := \frac{1}{2}(\xi_i \zeta_j + \xi_j \zeta_i)$. Finally $\mathrm{Lin}(\mathbb{S}^3)$ stands for the space of linear mappings from $\mathbb{S}^3$ into itself. Seeking to address several geometric and mechanical situations in a unified manner, we opted for the introduction of upper left superscripts. This makes it possible to consider various cases indexed by $\mathfrak{c} = (\mathfrak{p}, \mathfrak{q})$ in $\{1, 2\}^2$.

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Given two positive real numbers $L^{\pm}$ and a domain ω of $\mathbb{R}^2$ with a Lipschitz-continuous boundary $\partial\omega$ such that

$$\int_{\omega} x_\alpha \, d\widehat{x} = \int_{\omega} x_1 x_2 \, d\widehat{x} = 0,$$

for all pairs of numbers (ε, h) in $(0,+\infty)\times(0,\mathrm{Min}(L^+,L^-))$, let

$$\begin{cases} \Omega^\varepsilon := \varepsilon\omega \times (-L^-, L^+), & \Omega^{\varepsilon\pm} := \Omega^\varepsilon \cap \{x^\varepsilon \in \mathbb{R}^3;\ \pm x_3^\varepsilon > 0\} \\ B_h^\varepsilon := \varepsilon\omega \times (-h,h) & \\ {}^{\mathfrak{p}}\Omega_h^\varepsilon := {}^{\mathfrak{p}}\Omega_h^{\varepsilon+} \cup {}^{\mathfrak{p}}\Omega_h^{\varepsilon-} & \\ {}^{1}\Omega_h^{\varepsilon\pm} := \Omega^{\varepsilon\pm} \cap \{\pm x_3^\varepsilon > h\}, & {}^{2}\Omega_h^{\varepsilon\pm} := \Omega^{\varepsilon\pm} \pm h e_3 \\ S^\varepsilon := \varepsilon\omega \times \{0\}, & S_\eta^\varepsilon := \varepsilon\omega\times\{\eta\} \quad \forall\, \eta \in \mathbb{R}\setminus\{0\} \\ {}^{\mathfrak{p}}G_h^\varepsilon := \varepsilon\omega \times \{\pm x_3 \in [0, L^{\pm} + (\mathfrak{p}-1)h]\}. & \end{cases} \tag{1}$$

The domains ${}^{\mathfrak{p}}\Omega_h^{\varepsilon\pm}$ are occupied by two linearly elastic slender beams which are connected by a very thin linearly elastic layer occupying B_h^ε. The case $\mathfrak{p}=1$ corresponds to the analysis of the influence of the stiffness and thickness of an inclusion in a slender structure, whereas the case $\mathfrak{p}=2$ corresponds to the analysis of the influence of the stiffness and the thickness of an adhesive joint on the bonding of two given slender beams.

We assume that the "upper part" $S^\varepsilon_{L^+ + (\mathfrak{p}-1)h} =: {}^{\mathfrak{p}}S^\varepsilon_{\mathrm{up}}$ is clamped on a rigid body while the "lower part" $S^\varepsilon_{-(L^- + (\mathfrak{p}-1)h)} =: {}^{\mathfrak{p}}S^\varepsilon_{\mathrm{low}}$ is either clamped on a rigid body ($\mathfrak{q}=1$) or subjected to surface forces ($\mathfrak{q}=2$)! Moreover the beams are subjected to body forces in ${}^{\mathfrak{p}}\Omega_h^\varepsilon$ and surface forces on $\varepsilon\partial\omega\times\{h < \pm x_3^\varepsilon < L^{\pm} + (\mathfrak{p}-1)h\}$. More precisely, if

$$^{(\mathfrak{p},1)}\Gamma^\varepsilon_{h\mathrm{D}} := {}^{\mathfrak{p}}S^\varepsilon_{\mathrm{up}} \cup {}^{\mathfrak{p}}S^\varepsilon_{\mathrm{low}}, \quad ^{(\mathfrak{p},2)}\Gamma^\varepsilon_{h\mathrm{D}} := {}^{\mathfrak{p}}S^\varepsilon_{\mathrm{up}}, \tag{2}$$

we assume that the assembly is clamped along ${}^{\mathfrak{c}}\Gamma^\varepsilon_{h\mathrm{D}}$, subjected to body forces of density ${}^{\mathfrak{p}}f_h^\varepsilon$ in $L^2({}^{\mathfrak{p}}\Omega_h^\varepsilon,\mathbb{R}^3)$ and surface forces of density ${}^{\mathfrak{c}}g_h^\varepsilon$ in $L^2({}^{\mathfrak{c}}\Gamma^\varepsilon_{h\mathrm{N}},\mathbb{R}^3)$ with ${}^{\mathfrak{c}}\Gamma^\varepsilon_{h\mathrm{N}} := \partial\, {}^{\mathfrak{p}}\Omega_h^\varepsilon \setminus ({}^{\mathfrak{c}}\Gamma^\varepsilon_{h\mathrm{D}} \cup S^\varepsilon_{\pm h})$ whereas the thin layer is free of loading and perfectly stuck to the beams along $S^\varepsilon_{\pm h}$. The following Figure 1 summarizes the above information:

Lastly we denote the elasticity tensors of the beams and of the layer by ${}^{\mathfrak{p}}a_h^\varepsilon$, μb respectively. They satisfy

$$\begin{cases} \bullet\ \mu > 0 \\ \bullet\ ({}^{\mathfrak{p}}a_h^\varepsilon, b) \in L^\infty({}^{\mathfrak{p}}\Omega_h^\varepsilon, \mathrm{Lin}(\mathbb{S}^3)) \times \mathrm{Lin}(\mathbb{S}^3) \\ \quad \text{with } {}^{\mathfrak{p}}a_h^\varepsilon, b \text{ symmetric } \exists\, \alpha > 0 \text{ s.t.} \\ \quad \left.\begin{array}{l} 2\mathcal{W}_{{}^{\mathfrak{p}}a_h^\varepsilon}(\xi) := {}^{\mathfrak{p}}a_h^\varepsilon(x^\varepsilon)\xi\cdot\xi \geq \alpha|\xi|^2_{\mathbb{S}^3} \quad \text{a.e. } x^\varepsilon \in {}^{\mathfrak{p}}\Omega_h^\varepsilon \\ 2\mathcal{W}_b(\xi) := b\xi\cdot\xi \geq \alpha|\xi|^2_{\mathbb{S}^3} \end{array}\right\} \forall\, \xi \in \mathbb{S}^3. \end{cases} \tag{3}$$

In the following for any domain $\mathcal{G}$ of $\mathbb{R}^d, 1 \leq d \leq 3$, $H^1_\gamma(\mathcal{G},\mathbb{R}^{d'}), 1 \leq d' \leq 3$, stands for the subspace of $H^1(\mathcal{G},\mathbb{R}^{d'})$ of the elements with vanishing trace on a subset γ of the boundary $\partial\mathcal{G}$. Hence the problem of the equilibrium of the assembly involves a triplet $s := (\varepsilon, h, \mu)$ of data and can be formulated as:

$$\mathrm{Min}\Big\{ \int_{{}^{\mathfrak{p}}\Omega_h^\varepsilon} \mathcal{W}_{{}^{\mathfrak{p}}a_h^\varepsilon}(e^\varepsilon(v))\,dx^\varepsilon + \mu \int_{B_h^\varepsilon} \mathcal{W}_b(e^\varepsilon(v))\,dx^\varepsilon - \int_{{}^{\mathfrak{p}}\Omega_h^\varepsilon} {}^{\mathfrak{p}}f_h^\varepsilon \cdot v\,dx^\varepsilon - \int_{{}^{\mathfrak{c}}\Gamma^\varepsilon_{h\mathrm{N}}} {}^{\mathfrak{c}}g_h^\varepsilon\cdot v\, d\mathcal{H}_2^\varepsilon;\quad v \in {}^{\mathfrak{c}}\mathcal{V}_h^\varepsilon := H^1_{{}^{\mathfrak{c}}\Gamma^\varepsilon_{h\mathrm{D}}}({}^{\mathfrak{p}}G_h^\varepsilon,\mathbb{R}^3)\Big\} \tag{${}^{\mathfrak{c}}\mathrm{P}^s$}$$

where $e_{ij}^\varepsilon(v) = (\partial v_i/\partial x_j^\varepsilon + \partial v_j/\partial x_i^\varepsilon)/2$ is the strain tensor associated with the displacement v and $\mathcal{H}_2^\varepsilon$ is the 2-dimensional Hausdorff-measure. Obviously $({}^{\mathfrak{c}}\mathrm{P}^s)$ has a unique solution ${}^{\mathfrak{c}}u^s$.

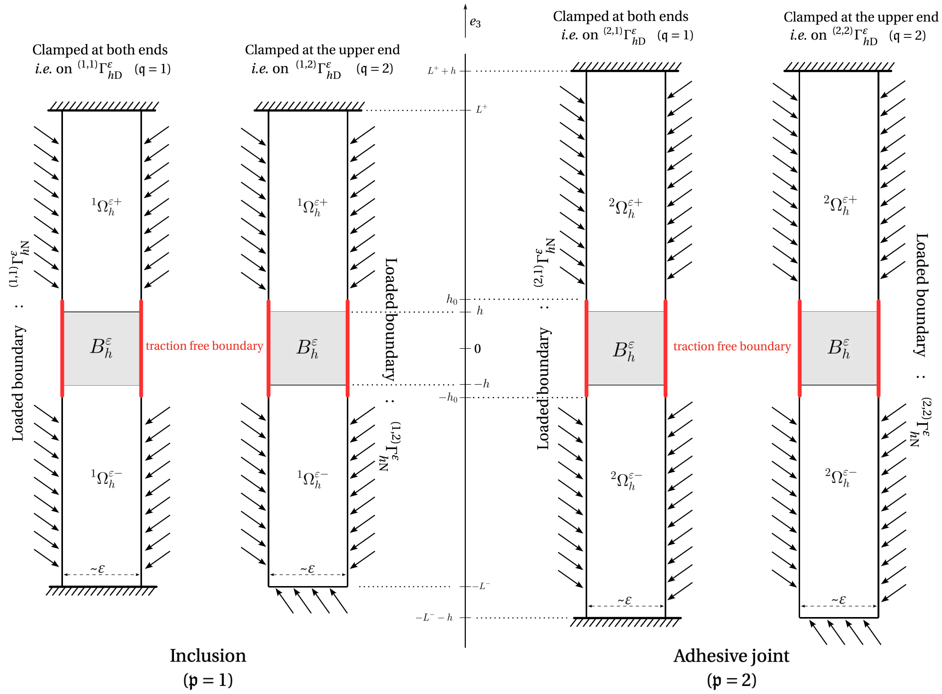

Figure 1. Four configurations of abutting slender beams with diameter of order ε. The left panels correspond to slender beams containing a very small inclusion, while the right panels represent a junction through a very small adhesive layer. The traction-free boundary is shown in red; its size is discussed in assumption (H_1) below.

2. Asymptotic behavior

As is customary in the analysis of slender beams (see, e.g., [3,4]), we introduce:

- a change of coordinates π^{ε}:

$$x \in \mathbb{R}^3 \longmapsto x^{\varepsilon} = \pi^{\varepsilon} x := (\varepsilon\hat{x}, x_3) \in \mathbb{R}^3 \tag{4}$$

  and drop the index ε for the image by $(\pi^{\varepsilon})^{-1}$ of the previous domains while, in the sequel, the space variables x^{ε} in ${}^{\mathfrak{p}}G^{\varepsilon}_h$ and x in ${}^{\mathfrak{p}}G_h$ are systematically connected by $x^{\varepsilon} = \pi^{\varepsilon} x$.
- An operator $\mathscr{S}_{\varepsilon}$ which transforms any element w of ${}^{\mathfrak{c}}\mathcal{V}^{\varepsilon}_h$ into an element of ${}^{\mathfrak{c}}\mathcal{V}_h := H^1_{{}^{\mathfrak{c}}\Gamma_{h\mathrm{D}}}({}^{\mathfrak{p}}G_h, \mathbb{R}^3)$ such that:

$$\widehat{\mathscr{S}_{\varepsilon} w}(x) = \hat{w}(x^{\varepsilon}), \quad (\mathscr{S}_{\varepsilon} w)_3(x) = \frac{1}{\varepsilon} w_3(x^{\varepsilon}) \tag{5}$$

  and

$$\left.\begin{aligned} e^{\varepsilon}(w)(x^{\varepsilon}) &= \varepsilon\, e^{\mathrm{beam}}(\varepsilon, \mathscr{S}_{\varepsilon} w)(x), \\ e^{\mathrm{beam}}(\varepsilon, z) &:= \begin{bmatrix} e_{\alpha\beta}(z)/\varepsilon^2 & e_{\alpha 3}(z)/\varepsilon \\ \mathrm{sym} & e_{33}(z) \end{bmatrix} \\ e_{ij}(z) &:= \tfrac{1}{2}\left(\partial_i z_j + \partial_j z_i\right), \quad 1 \le i, j \le 3 \end{aligned}\right\} \quad \forall\, z \in H^1\left({}^{\mathfrak{p}}G_h, \mathbb{R}^3\right). \tag{6}$$

- Some assumptions on the loading and the stiffness of the beams. To this end we introduce:

$${}^1\Gamma_{\mathrm{D}} := S_{L^+} \cup S_{L^-}, \quad {}^2\Gamma_{\mathrm{D}} := S_{L^+}, \quad {}^{\mathfrak{q}}\Gamma_{\mathrm{N}} := \partial\Omega \setminus {}^{\mathfrak{q}}\Gamma_{\mathrm{D}}, \quad {}^{\mathfrak{c}}\Gamma_{h\mathrm{N}} := \partial\, {}^{\mathfrak{p}}G_h \setminus {}^{\mathfrak{c}}\Gamma_{h\mathrm{D}}$$

and, for any function φ defined on a subset of $\overline{\Omega}$, the mapping ${}^{\mathfrak{p}}\tau_h\varphi$ such that:

$$ {}^{\mathfrak{p}}\tau_h\varphi(\widehat{x},x_3)=\varphi\big(\widehat{x},x_3-\operatorname{sgn}(x_3)(\mathfrak{p}-1)h\big)\quad\forall\, x\in{}^{\mathfrak{p}}\Omega_h \tag{7}$$

where $\operatorname{sgn}(x_3)$ denotes the sign of x_3. We assume that:

$$\begin{cases}\text{There exist } h_0>0,\ f \text{ in } L^2(\Omega,\mathbb{R}^3),\ g \text{ in } L^2({}^{\mathfrak{q}}\Gamma_{\mathrm{N}},\mathbb{R}^3) \text{ with } (\operatorname{supp}(f),\operatorname{supp}(g))\cap B_{h_0}=\emptyset\\ \text{and } a \text{ symmetric in } L^\infty(\Omega,\operatorname{Lin}(\mathbb{S}^3)) \text{ such that}\\ \exists\ \alpha>0;\quad 2\mathscr{W}_a(\xi):=a(x)\xi\cdot\xi\geq\alpha|\xi|^2\quad\text{a.e. } x \text{ in } \Omega,\quad\forall\,\xi\in\mathbb{S}^3\\ \bullet\ \widehat{{}^{\mathfrak{p}}f^{\varepsilon}_h}(x^\varepsilon)=\varepsilon^2\,{}^{\mathfrak{p}}\tau_h\widehat{f}(x),\quad {}^{\mathfrak{p}}f^{\varepsilon}_{h3}(x^\varepsilon)=\varepsilon\,{}^{\mathfrak{p}}\tau_h f_3(x)\quad\forall\, x\in{}^{\mathfrak{p}}\Omega_h\\ \bullet\ \widehat{{}^{\mathfrak{c}}g^{\varepsilon}_h}(x^\varepsilon)=\varepsilon^3\,{}^{\mathfrak{p}}\tau_h\widehat{g}(x),\quad {}^{\mathfrak{c}}g^{\varepsilon}_{h3}(x^\varepsilon)=\varepsilon^2\,{}^{\mathfrak{p}}\tau_h g_3(x)\quad\forall\, x\in{}^{\mathfrak{c}}\Gamma_{h\mathrm{N}}\cap\big\{\pm x_3\in\big[0,L^\pm+(\mathfrak{p}-1)h\big)\big\}\\ \bullet\ \widehat{{}^{\mathfrak{c}}g^{\varepsilon}_h}(x^\varepsilon)=\varepsilon^2\,{}^{\mathfrak{p}}\tau_h\widehat{g}(x),\quad {}^{\mathfrak{c}}g^{\varepsilon}_{h3}(x^\varepsilon)=\varepsilon\,{}^{\mathfrak{p}}\tau_h g_3(x)\quad\forall\, x\in{}^{\mathfrak{c}}\Gamma_{h\mathrm{N}}\cap\big\{x_3=-(L^-+(\mathfrak{p}-1)h)\big\}\\ \bullet\ {}^{\mathfrak{p}}a^{\varepsilon}_h(x^\varepsilon)={}^{\mathfrak{p}}\tau_h a(x)\quad\forall\, x\in{}^{\mathfrak{p}}\Omega_h\end{cases}\tag{H$_1$}$$

Hence ${}^{\mathfrak{c}}u_s:=\mathscr{S}_\varepsilon\,{}^{\mathfrak{c}}u^s$ is the unique solution of

$$\operatorname{Min}\left\{\int_{{}^{\mathfrak{p}}\Omega_h}\mathscr{W}_{{}^{\mathfrak{p}}\tau_h a}\Big(e^{\text{beam}}(\varepsilon,v)\Big)\,dx+\mu\int_{B_h}\mathscr{W}_b\Big(e^{\text{beam}}(\varepsilon,v)\Big)\,dx-\int_{{}^{\mathfrak{p}}\Omega_h}{}^{\mathfrak{p}}\tau_h f\cdot v\,dx\right.$$
$$\left.-\int_{{}^{\mathfrak{c}}\Gamma_{h\mathrm{N}}}{}^{\mathfrak{p}}\tau_h g\cdot v\,d\mathcal{H}_2;\quad v\in{}^{\mathfrak{c}}\mathcal{V}_h:=H^1_{{}^{\mathfrak{c}}\Gamma_{h\mathrm{D}}}\big({}^{\mathfrak{p}}G_h,\mathbb{R}^3\big)\right\}.\tag{${}^{\mathfrak{c}}$P$_s$}$$

We will consider $s=(\varepsilon,h,\mu)$ as a triplet of *parameters* taking values in a countable set $\mathfrak{S}$ of $(0,+\infty)\times(0,h_0)\times(0,+\infty)$ with a unique cluster point $\bar{s}$ in $\{0\}\times\{0\}\times[0,+\infty]$ and distinguish various cases of relative behaviors of the elements of s characterized by $\mathfrak{r}$:

$$\begin{cases}\text{For all } k \text{ in } \{1,2\} \text{ there exists } {}^k\overline{\mu} \text{ in } [0,+\infty] \text{ such that } \lim\limits_{s\to\bar{s}}\mu/2h\varepsilon^{2(2-k)}={}^k\overline{\mu},\\ \bullet\ \mathfrak{r}=1\quad\text{if } \big({}^1\overline{\mu},{}^2\overline{\mu}\big)\in[0,+\infty)\times\{0\},\\ \bullet\ \mathfrak{r}=2\quad\text{if } \big({}^1\overline{\mu},{}^2\overline{\mu}\big)\in\{+\infty\}\times[0,+\infty),\\ \bullet\ \mathfrak{r}=3\quad\text{if } \big({}^1\overline{\mu},{}^2\overline{\mu}\big)\in\{+\infty\}^2,\end{cases}\tag{H$_2$}$$

with the additional assumption:

$$\text{If } \mathfrak{q}=2,\quad\int_{\Omega^-}f\cdot\rho\,dx+\int_{{}^{\mathfrak{q}}\Gamma_{\mathrm{N}}\cap\{x_3<0\}}g\cdot\rho\,d\mathcal{H}_2=0,\quad\forall\,\rho\in\mathscr{R}\tag{H$_3$}$$

where $\mathscr{R}$ denotes the set of rigid displacements ($\rho\in\mathscr{R}\iff e(\rho)=0$). Of course when μ remains bounded from below, this assumption may be dropped.

In the sequel, C will denote various constants (independent of s!) which may vary from lines to lines. So, classically, ${}^{\mathfrak{c}}u_s$ does satisfy:

$$\int_{{}^{\mathfrak{p}}\Omega_h}\Big|e^{\text{beam}}\big(\varepsilon,{}^{\mathfrak{c}}u_s\big)\Big|^2\,dx\leq C\tag{8}$$

$$\mu\int_{B_h}\Big|e^{\text{beam}}\big(\varepsilon,{}^{\mathfrak{c}}u_s\big)\Big|^2\,dx\leq C\tag{9}$$

In the following, for any measurable function φ defined in a subset ϕ of $\mathbb{R}^3$, we will denote the restriction of φ to $\{\pm x_3 > 0\} \cap \phi$ by $\varphi^\pm$. As estimation (8) involves a domain which varies with h, it is convenient to associate with all v in ${}^{\mathfrak{c}}\mathcal{V}_h$, an element ${}^{\mathfrak{c}}\mathcal{T}_h v$ in $H^1_{{}^{\mathfrak{q}}\Gamma_{\mathrm{D}}}(\Omega \setminus S, \mathbb{R}^3)$ given by:

$$\begin{cases} \bullet \text{ when } \mathfrak{c} = (1,1), \\ \left({}^{\mathfrak{c}}\mathcal{T}_h v\right)^\pm(x) := \begin{cases} v\left(\hat{x}, x_3 + \operatorname{sgn}(x_3)h\right) & \text{if } \pm x_3 \in \left(0, L^\pm - h\right), \\ 0 & \text{if } \pm x_3 \in \left(L^\pm - h, L^\pm\right), \end{cases} \\ \bullet \text{ when } \mathfrak{c} = (1,2), \\ \left({}^{\mathfrak{c}}\mathcal{T}_h v\right)^+(x) \text{ is as } \mathfrak{c} = (1,1), \\ \left({}^{\mathfrak{c}}\mathcal{T}_h v\right)^-(x) := \begin{cases} v(\hat{x}, x_3 - h) & \text{if } x_3 \in (-L^- + h, 0), \\ v(\hat{x}, -2L^- - x_3 + h) & \text{if } x_3 \in (-L^-, -L^- + h), \end{cases} \\ \bullet \text{ when } \mathfrak{c} = (2,1) \text{ or } (2,2), \\ \left({}^{\mathfrak{c}}\mathcal{T}_h v\right)^\pm(x) := v\left(\hat{x}, x_3 + \operatorname{sgn}(x_3)h\right). \end{cases} \tag{10}$$

Let

$$\begin{cases} x^R := (-x_2, x_1), \quad \forall\, x \in \mathbb{R}^3, \\ \hat{e}_{\alpha\beta} := e_{\alpha\beta} \quad \forall\, e \in \mathbb{S}^3, \\ H^1_m(w) := \left\{ w \in H^1(\omega);\ \int_\omega w(\hat{x})\, d\hat{x} = 0 \right\}, \\ \mathcal{V}^\pm_{\mathrm{BN}} := \left\{ w \in H^1\left(\Omega^\pm, \mathbb{R}^3\right);\ e_{\alpha\beta}(w) = e_{\alpha 3}(w) = 0 \right\}, \\ {}^{\mathfrak{q}}\mathcal{U}^+_{\mathrm{D}} := \left\{ w \in H^1\left(\Omega^+, \mathbb{R}^3\right);\ w = 0 \text{ on } S_{L^+} \right\}, \\ {}^{\mathfrak{q}}\mathcal{U}^-_{\mathrm{D}} := \left\{ w \in H^1\left(\Omega^-, \mathbb{R}^3\right); w = 0 \text{ on } S_{L^-} \text{ if } \mathfrak{q} = 1 \right\}, \\ {}^{\mathfrak{q}}\mathcal{U}^{0\pm} := V^\pm_{\mathrm{BN}} \cap {}^{\mathfrak{q}}\mathcal{U}^\pm_{\mathrm{D}}, \\ {}^{\mathfrak{q}}\mathcal{U}^{1+} := \left\{ w;\ \exists\, c^+ \in H^1_{L^+}(0, L^+) \text{ s.t. } \hat{w}(x) = c^+(x_3) x^R,\ w_3 \in L^2\left(0, L^+; H^1_m(\omega)\right) \right\}, \\ {}^{\mathfrak{q}}\mathcal{U}^{1-} := \left\{ w;\ \exists\, c^- \in H^1(-L^-, 0),\ c^-(-L^-) = 0 \right. \\ \qquad\qquad \left. \text{if } \mathfrak{q} = 1 \text{ s.t. } \hat{w}(x) = c^-(x_3) x^R, w_3 \in L^2\left(-L^-, 0; H^1_m(\omega)\right) \right\}, \\ {}^{\mathfrak{q}}\mathcal{U}^{2+} := \left\{ w;\ \hat{w} \in L^2\left(0, L^+;\ H^1_m\left(\omega, \mathbb{R}^2\right)\right) \text{ s.t.} \right. \\ \qquad\qquad \left. \int_\omega x^R \cdot \hat{w}(\hat{x})\, d\hat{x} = 0 \quad \text{a.e. } x_3 \in (0, L^+),\ w_3 = 0 \right\}, \\ {}^{\mathfrak{q}}\mathcal{U}^{2-} := \left\{ w;\ \hat{w} \in L^2\left(-L^-, 0; H^1_m\left(\omega, \mathbb{R}^2\right)\right) \text{ s.t.} \right. \\ \qquad\qquad \left. \int_\omega x^R \cdot \hat{w}(\hat{x})\, d\hat{x} = 0 \quad \text{a.e. } x_3 \in (-L^-, 0),\ w_3 = 0 \right\}, \\ {}^{\mathfrak{q}}\mathcal{U}^i := \left\{ u; u^\pm \in {}^{\mathfrak{q}}\mathcal{U}^{i\pm} \right\}, \quad {}^{\mathfrak{q}}\mathcal{U} := {}^{\mathfrak{q}}\mathcal{U}^0 \times {}^{\mathfrak{q}}\mathcal{U}^1 \times {}^{\mathfrak{q}}\mathcal{U}^2, \\ E^\pm_u := \begin{bmatrix} \hat{e}\left(u^{2\pm}\right) & e_{\alpha 3}\left(u^{1\pm}\right) \\ \text{sym} & e_{33}\left(u^{0\pm}\right) \end{bmatrix}, \quad \forall\, u^\pm = \left(u^{0\pm}, u^{1\pm}, u^{2\pm}\right) \in {}^{\mathfrak{q}}\mathcal{U}^{0\pm} \times {}^{\mathfrak{q}}\mathcal{U}^{1\pm} \times {}^{\mathfrak{q}}\mathcal{U}^{2\pm}. \end{cases} \tag{11}$$

where there is no doubt we will write indifferently $E_{u^\pm}$ or $E^\pm_u$.

Recalling that $\mathscr{R}$ stands for the space of the rigid displacements, we let:

$$\begin{cases} {}^{\mathfrak{q}}\mathscr{R}^+ = \{0\}, \quad \forall\, \mathfrak{q} \in \{1,2\}, \\ {}^{\mathfrak{q}}\mathscr{R}^- = \begin{cases} \{0\} & \text{if } \mathfrak{q}=1, \\ \{\rho^- \text{ s.t. } \rho \in \mathscr{R}\} & \text{if } \mathfrak{q}=2. \end{cases} \end{cases} \tag{12}$$

To obtain our main results (see Theorems 4–5 below) we will use the method of variational convergence by first establishing a lower bound for the strain energy of the assembly and next an upper bound for the strain energy of the assembly for sequences of scaled displacements satisfying (8)–(9). This will be done under the additional assumption

$$\sup_{s\in\mathfrak{S}} h/\varepsilon^2 < +\infty \tag{H$_4$}$$

which is precisely the premise that underpins the title of our article.

Lemma 1. *For all sequences $\{{}^{\mathfrak{c}}v_s\}_{s\in\mathfrak{S}}$ in ${}^{\mathfrak{c}}\mathcal{V}_h$ such that $\int_{{}^{\mathfrak{p}}\Omega_h} |e^{\text{beam}}(\varepsilon, {}^{\mathfrak{c}}v_s)|^2\,dx \le C$, there exists $v = (v^0, v^1, v^2)$ in ${}^{\mathfrak{q}}\mathscr{U}$ and a not relabeled subsequence such that $({}^{\mathfrak{c}}\mathscr{T}_h\, {}^{\mathfrak{c}}v_s)^{\pm}$ converges weakly in $H^1(\Omega^{\pm},\mathbb{R}^3)/{}^{\mathfrak{q}}\mathscr{R}^{\pm}$ and*

(i) $\liminf_{s\to\bar{s}} \int_{{}^{\mathfrak{p}}\Omega_h} \mathscr{W}_{{}^{\mathfrak{p}}\tau_h a}(e^{\text{beam}}(\varepsilon, {}^{\mathfrak{c}}v_s))\,dx \ge \int_{\Omega\setminus S} \mathscr{W}_a(E_v)\,dx$,

(ii) *the traces $\gamma_{\pm h}({}^{\mathfrak{c}}v_s)$ on $S_{\pm h}$ converge strongly in $L^2(S,\mathbb{R}^3)/{}^{\mathfrak{q}}\mathscr{R}^{\pm}$ toward the traces $\gamma_{0\pm}(v^{0\pm})$ of $v^{0\pm}$ on S.*

Proof. Clearly there exists some v^{0+} in $H^1_{S_{L^+}}(\Omega^+,\mathbb{R}^3)$ and a not relabeled subsequence such that $({}^{\mathfrak{c}}\mathscr{T}_h\, {}^{\mathfrak{c}}v_s)^+$ weakly converges in $H^1(\Omega^+,\mathbb{R}^3)$ toward v^{0+}. Note that when $\mathfrak{p}=1$, there exists w in $H^1(\Omega^+_\eta,\mathbb{R}^3)$ such that, up to a new subsequence, $({}^{\mathfrak{c}}v_s)^+$ converges weakly in $H^1(\Omega^+_\eta,\mathbb{R}^3)$ toward w for all η in $(0,L^+)$. As $w = v^{0+}$ in Ω^+_η, one has $w = v^{0+}$ which then belongs to $H^1_{S_{L^+}}(\Omega^+,\mathbb{R}^3)$. Moreover the trace on S of $({}^{\mathfrak{c}}\mathscr{T}_h\, {}^{\mathfrak{c}}v_s)^+$ converges strongly in $L^2(S,\mathbb{R}^3)$ toward the trace $\gamma_{0+}(v^{0+})$ of v^{0+} on S. Lastly as $\int_{\Omega^+} |e^{\text{beam}}(\varepsilon, ({}^{\mathfrak{c}}\mathscr{T}_h\, {}^{\mathfrak{c}}v_s)^+)|^2\,dx = \int_{{}^{\mathfrak{p}}\Omega^+_h} |e^{\text{beam}}(\varepsilon, {}^{\mathfrak{c}}v_s)|^2\,dx$, by proceeding as in [3,5,6], one deduces that there exists $v^+ = (v^{0+}, v^{1+}, v^{2+})$ in ${}^{\mathfrak{q}}\mathscr{U}^{0+} \times {}^{\mathfrak{q}}\mathscr{U}^{1+} \times {}^{\mathfrak{q}}\mathscr{U}^{2+}$ such that $e^{\text{beam}}(\varepsilon, ({}^{\mathfrak{c}}\mathscr{T}_h\, {}^{\mathfrak{c}}v_s)^+)$ converges weakly in $L^2(\Omega^+,\mathbb{S}^3)$ toward some E_{v^+} so that

$$\liminf_{s\to\bar{s}} \int_{{}^{\mathfrak{p}}\Omega^+_h} \mathscr{W}_{{}^{\mathfrak{p}}\tau_h a}\Big(e^{\text{beam}}\big(\varepsilon, {}^{\mathfrak{c}}v_s\big)\Big)\,dx \ge \int_{\Omega^+} \mathscr{W}_a(E_{v^+})\,dx.$$

We may proceed in the same way in ${}^{\mathfrak{p}}\Omega^-_h$ when $\mathfrak{q}=1$, while when $\mathfrak{q}=2$, one has to consider a rigid displacement. □

For any v of $H^1(\Omega\setminus S,\mathbb{R}^d)$, $1\le d\le 3$, let $[v] = \gamma_{0+}(v^+) - \gamma_{0-}(v^-)$ be the jump of v across S where $\gamma_{0\pm}$ is the trace operator from $H^1(\Omega^{\pm},\mathbb{R}^d)$ into $L^2(S,\mathbb{R}^d)$ and

$${}^{\mathfrak{q},\mathfrak{r}}\mathcal{V} := \Big\{ w \in H^1_{{}^{\mathfrak{q}}\Gamma_D}\big(\Omega\setminus S;\mathbb{R}^3\big); \quad [\widehat{w}] = 0 \text{ if } \mathfrak{r}=2, \quad [w]=0 \text{ if } \mathfrak{r}=3 \Big\}. \tag{13}$$

Let ${}^{\mathfrak{q},\mathfrak{r}}\mathcal{J}$ and $\mathscr{W}_{\mathfrak{q},\mathfrak{r}\,\tilde{b}}$ respectively defined on ${}^{\mathfrak{q},\mathfrak{r}}\mathcal{V}$ and $\mathbb{R}^3$ by:

$${}^{\mathfrak{q},\mathfrak{r}}\mathcal{J}(w) := \begin{cases} {}^{\mathfrak{r}}\overline{\mu} \displaystyle\int_S \mathscr{W}_{\mathfrak{q},\mathfrak{r}\,\tilde{b}}([w])\,d\widehat{x} & \text{if } (\mathfrak{q},\mathfrak{r}) \in \{1,2\}^2, \\ 0 & \text{if } (\mathfrak{q},\mathfrak{r}) \in \{1,2\}\times\{3\}, \end{cases} \tag{14}$$

$$\mathscr{W}_{\mathfrak{q},\mathfrak{r}\,\tilde{b}}(z) := \operatorname{Min}\left\{ \mathscr{W}_b(\zeta \otimes_s e_3); \begin{cases} \zeta = (\widehat{z}, 0) & \text{if } (\mathfrak{q},\mathfrak{r}) = (1,1) \\ \widehat{\zeta} = \widehat{z} & \text{if } (\mathfrak{q},\mathfrak{r}) = (2,1) \\ \zeta_3 = z_3 & \text{if } \mathfrak{r} = 2 \end{cases} \right\}. \tag{15}$$

Under the convention $\infty \times 0 = 0$, one has:

$$ {}^{\mathfrak{q},\mathfrak{r}}\mathcal{J}(w) = \sum_{\mathrm{i}=1}^{2} {}^{\mathrm{i}}\overline{\mu} \int_S \mathcal{W}_{\mathfrak{q},\mathrm{i}\,\tilde{b}}([w])\, d\widehat{x}. \tag{16} $$

Lemma 2. *For all sequences* $\{{}^{\mathfrak{c},\mathfrak{r}}v_s\}_{s\in\mathfrak{S}}$ *satisfying* (8)–(9)*, the field* v^0 *supplied by Lemma 1 belongs to* ${}^{\mathfrak{q},\mathfrak{r}}\mathcal{V}$ *and* v^{0-} *is defined up to an element of*

$$ {}^{\mathfrak{q},\mathfrak{r}}\mathcal{R}_0^- := \left\{\rho^- \in {}^{\mathfrak{q}}\mathcal{R}^-;\quad \gamma_{0-}(\widehat{\rho^-}) = 0 \ \textit{if}\ \mathfrak{r}=1 \quad \textit{and} \quad {}^1\overline{\mu} > 0,\ \rho^- = 0\, d \ \textit{if}\ \mathfrak{r}=2,3\right\}. \tag{17} $$

We have

$$ \liminf_{s\to\bar{s}} \mu \int_{B_h} \mathcal{W}_b\Big(e^{\mathrm{beam}}\big(\varepsilon, {}^{\mathfrak{c},\mathfrak{r}}v_s\big)\Big)\, dx \geq {}^{\mathfrak{q},\mathfrak{r}}\mathcal{J}\big(v^0\big). \tag{18} $$

Note that as $v^{0\pm}$ belongs to ${}^{\mathfrak{q}}\mathcal{U}^{0\pm}$, $\mathcal{W}_{\mathfrak{q},\mathfrak{r}\,\tilde{b}}([v^0])$ reduces either to a constant or to a quadratic function if $\mathfrak{r} = 1$ or 2, respectively.

Proof. Following [7] we proceed to the following change of coordinates and fields:

$$ \begin{cases} x = (\widehat{x}, x_3) \in B_h \mapsto y = (\widehat{x}, x_3/h) \in B_1 := \omega \times (-1,1), \\ v \in H^1(B_h, \mathbb{R}^3) \mapsto V \in H^1\big(B_1, \mathbb{R}^3\big) \ \text{ s.t. }\ v(x) = \big(h\widehat{V}(y), V_3(y)\big), \end{cases} \tag{19} $$

so that

$$ \begin{cases} e^{\mathrm{beam}}(\varepsilon, v)(x) = h/\varepsilon^2\, e^{\mathrm{plate}}(h/\varepsilon, V)(y), \\ e^{\mathrm{plate}}(t, V) := \begin{bmatrix} \widehat{e}(V) & e_{\alpha 3}(V)/t \\ \mathrm{sym} & e_{33}(V)/t^2 \end{bmatrix}, \quad \forall\, V \in H^1\big(B_1, \mathbb{R}^3\big), \\ e_{ij}(V) := \dfrac{1}{2}\left(\dfrac{\partial V_i}{\partial y_j} + \dfrac{\partial V_j}{\partial y_i}\right), \quad 1 \leq i,j \leq 3, \\ \displaystyle\int_{B_h} \mathcal{W}_b\Big(e^{\mathrm{beam}}(\varepsilon, v)\Big)\, dx = \frac{h}{\varepsilon^4} \int_{B_1} \mathcal{W}_b\Big(e^{\mathrm{plate}}(h/\varepsilon, W)\Big)\, dy, \\ W := hV. \end{cases} \tag{20} $$

Let ${}^{\mathfrak{c},\mathfrak{r}}W_s := h\ {}^{\mathfrak{c},\mathfrak{r}}V_s$, its traces $\gamma_{\pm 1}({}^{\mathfrak{c},\mathfrak{r}}W_s)$ on $S_{\pm 1}$ satisfy:

$$ \begin{cases} \gamma_{\pm 1}\big(\widehat{{}^{\mathfrak{c},\mathfrak{r}}W_s}\big) = h\gamma_{\pm 1}\big(\widehat{{}^{\mathfrak{c},\mathfrak{r}}V_s}\big) = h\gamma_{\pm h}\big(\widehat{{}^{\mathfrak{c},\mathfrak{r}}v_s}/h\big) \longrightarrow \gamma_{0\pm}\big(\widehat{v^{0\pm}}\big), & \text{with } \partial_\alpha \widehat{v^{0\pm}} = 0 \\ & \text{in } L^2\big(S, \mathbb{R}^3\big)/{}^{\mathfrak{q}}\mathcal{R}^{\pm}. \\ \gamma_{\pm 1}\big({}^{\mathfrak{c},\mathfrak{r}}W_{s_3}\big) = h\gamma_{\pm 1}\big({}^{\mathfrak{c},\mathfrak{r}}V_{s_3}\big) = h\gamma_{\pm h}\big({}^{\mathfrak{c},\mathfrak{r}}v_{s_3}\big) \longrightarrow 0 & \end{cases} \tag{21} $$

• **Case** $\mathfrak{r} = 1$ $\big(({}^1\overline{\mu}, {}^2\overline{\mu}) \in [0,+\infty) \times \{0\}\big)$**.** The result being obvious if ${}^1\overline{\mu} = 0$, we assume ${}^1\overline{\mu} > 0$. Denoting ${}^{\mathfrak{c},1}\Psi_s := h/\varepsilon\ {}^{\mathfrak{c},1}W_s$, (9)–(20) imply:

$$ C\frac{2h\varepsilon^2}{\mu} \geq \int_{B_1} \left|e^{\mathrm{plate}}\big(h/\varepsilon, {}^{\mathfrak{c},1}\Psi_s\big)\right|^2 dy. \tag{22} $$

Hence, up to a non-relabeled subsequence, $e^{\mathrm{plate}}(h/\varepsilon, {}^{\mathfrak{c},1}\Psi_s)$ weakly converges in $L^2(B_1, \mathbb{S}^3)$ so that by Jensen inequality one has:

$$ \liminf_{s\to\bar{s}} \mu \int_{B_h} \mathcal{W}_b\Big(e^{\mathrm{beam}}\big(\varepsilon, {}^{\mathfrak{c},1}v_s\big)\Big)\, dx \geq {}^1\overline{\mu} \int_S \mathcal{W}_b\big({}^{\mathfrak{c},1}\ell\big)\, d\widehat{x}, \tag{23} $$

where ${}^{\mathfrak{c},1}\ell$ is the weak limit in $L^2(S, \mathbb{S}^3)$ of $\int_{-1}^{+1} e^{\mathrm{plate}}(h/\varepsilon, {}^{\mathfrak{c},1}\Psi_s)\, dy_3$. Moreover, because of (20)–(21), ${}^{\mathfrak{c},1}\Psi_s$ does converge weakly in $H^1(B_1, \mathbb{R}^3)$ toward some Kirchhoff–Love displacement ${}^{\mathfrak{c},1}\Psi$ with vanishing traces on S_1, so that ${}^{\mathfrak{c},1}\Psi = 0$. Hence ${}^{\mathfrak{c},1}\ell_{\alpha\beta}$ vanishes. Next, let τ in $C_0^\infty(S, \mathbb{S}^3)$ such that $\tau_{\alpha\beta} = \tau_{33} = 0$. The following estimation (see [8, eq. (4.20) p. 695]):

$$ \int_{B_h} \left|{}^{\mathfrak{c},1}v_{s3}\right|^2 dx \leq C\left[h\int_{S_h} \left|{}^{\mathfrak{c},1}v_{s3}\right|^2 d\widehat{x} + h^2 \int_{B_h} \left|e\big({}^{\mathfrak{c},1}v_s\big)\right|^2 dx + h^2 \int_{{}^{\mathfrak{p}}\Omega_h^+} \left|e\big({}^{\mathfrak{c},1}v_s\big)\right|^2 dx\right] \tag{24} $$

and an integration by parts in $\int_S \tau(\hat{x}) \int_{-1}^{+1} e^{\text{plate}}(h/\varepsilon, {}^{\mathfrak{c},1}\Psi_s)\, dy_3\, d\hat{x}$ implies that $\gamma_{+1}(\widehat{{}^{\mathfrak{c},1}W_s}) - \gamma_{-1}(\widehat{{}^{\mathfrak{c},1}W_s})$ has a weak limit in $L^2(S,\mathbb{R}^2)$. Therefore, modulo a modification of the field v^{0-} implied in the proof of Lemma 1 (and defined up to a rigid displacement), we get ${}^{\mathfrak{c},1}\ell_{\alpha 3} = \frac{1}{2}[v^0_\alpha]$ by due account of (21). Now by taking τ in $C_0^\infty(S,\mathbb{S}^3)$ such that $\tau_{\alpha\beta} = \tau_{\alpha 3} = 0$, one deduces that ${}^{\mathfrak{c},1}\ell_{33} = 0$ if $q = 1$ and the results stated in Lemma 2 stem from the very definition of $\mathcal{W}_{\mathfrak{q},1\tilde{b}}$.

• **Case** $\mathfrak{r} = 2$ $\big(({}^1\overline{\mu}, {}^2\overline{\mu}) \in \{+\infty\} \times [0,+\infty)\big)$. We first establish that ${}^1\overline{\mu} = +\infty$ implies $[\widehat{v^0}] = 0$. Inequality (22) implies that $e^{\text{plate}}(h/\varepsilon, {}^{\mathfrak{c},1}\Psi_s)$ converges strongly to zero in $L^2(B_1,\mathbb{S}^3)$ and ${}^{\mathfrak{c},1}\ell = 0$, thus $[v^0_\alpha]$ which has been seen equal to ${}^{\mathfrak{c},1}\ell_{\alpha 3}$ vanishes. This proves the result when ${}^2\overline{\mu} = 0$.

Next when ${}^2\overline{\mu} > 0$, (9), (20), (21) and assumption (H$_4$) imply that $e^{\text{plate}}(h/\varepsilon, {}^{\mathfrak{c},2}\Psi_s)$ is bounded in $L^2(B_1,\mathbb{S}^3)$ if ${}^{\mathfrak{c},2}\Psi_s := h/\varepsilon^2\, {}^{\mathfrak{c},2}W_s$ and, up to a non relabeled subsequence, ${}^{\mathfrak{c},2}\Psi_s$ converges in $H^1(B_1,\mathbb{R}^3)$ toward some Kirchhoff–Love displacement ${}^{\mathfrak{c},2}\Psi$ whose third component of the traces on S_{+1} vanishes, so that the flexural part ${}^{\mathfrak{c},2}\Psi^{\text{F}}$ of ${}^{\mathfrak{c},2}\Psi$ is equal to zero. Moreover the trace of $\widehat{{}^{\mathfrak{c},2}\Psi_s}$ on S_{+1} converges in $L^2(S,\mathbb{R}^2)$ toward a constant by (21) hence $e({}^{\mathfrak{c},2}\Psi)$ vanishes.

Finally (20)–(21) imply that $\int_{-1}^{+1} (\frac{\varepsilon}{h})^2 \frac{\partial}{\partial y_3}\big(\frac{h}{\varepsilon^2}\, {}^{\mathfrak{c},2}W_{s_3}\big)\, dy_3$ which is equal to $\gamma_{+h}({}^{\mathfrak{c},2}v_{s_3}) - \gamma_{-h}({}^{\mathfrak{c},2}v_{s_3})$ converges strongly in $L^2(S)$ toward $[v^0_3]$ and the result is obtained through the definition of $\mathcal{W}_{\mathfrak{q},2\tilde{b}}$.

• **Case** $\mathfrak{r} = 3$ $\big({}^1\overline{\mu} = {}^2\overline{\mu} = +\infty\big)$. As $e^{\text{plate}}(h/\varepsilon, h/\varepsilon^2\, {}^{\mathfrak{c},2}W_s)$ converges strongly to zero in $L^2(B_1,\mathbb{S}^3)$, one has $[v^0_3] = 0$ and the proof of Lemma 2 is complete. □

Lemma 3. *For all $v = (v^0, v^1, v^2)$ in ${}^{\mathfrak{q}}\mathcal{U}$ such that v^0 belongs to ${}^{\mathfrak{q},\mathfrak{r}}\mathcal{V}$ there exists a sequence $\{{}^{\mathfrak{c},\mathfrak{r}}v_s\}_{s\in\mathfrak{S}}$ in ${}^{\mathfrak{c}}\mathcal{V}_h$ such that*

$$\limsup_{s\to\bar{s}} \int_{{}^{\mathfrak{p}}\Omega_h} \mathcal{W}_{{}^{\mathfrak{p}}\tau_h a}\Big(e^{\text{beam}}\big(\varepsilon, {}^{\mathfrak{c},\mathfrak{r}}v_s\big)\Big)\, dx + \mu \int_{B_h} \mathcal{W}_b\Big(e^{\text{beam}}\big(\varepsilon, {}^{\mathfrak{c},\mathfrak{r}}v_s\big)\Big)\, dx \leq \int_{\Omega\setminus S} \mathcal{W}_a(E_v)\, dx + {}^{\mathfrak{q},\mathfrak{r}}\mathcal{J}(v^0). \quad (25)$$

Proof. As ${}^{\mathfrak{q},\mathfrak{r}}\mathcal{J}$ is continuous on ${}^{\mathfrak{q},\mathfrak{r}}\mathcal{V}$, it is enough to assume that $(v^{0\pm}, \widehat{v^{1\pm}})$ belongs to $C^\infty(\overline{\Omega}^\pm;\mathbb{R}^5)$ and that $(v^1_3, \widehat{v^2})$ belongs to $H^1(-L^-, L^+; H^1_m(\omega,\mathbb{R}^3))$ with a compact support in $(-L^-,0) \cup (0,L^+)$. For any w in $H^1_m(\Omega;\mathbb{R}^3)$ let $R_h w$ be defined by

$$R_h w = \begin{cases} w^{\text{S}}(x) + \dfrac{|x_3|}{h} w^{\text{A}}(x), & \forall\, x \in B_h, \\ w(x), & \forall\, x \in {}^{\mathfrak{p}}\Omega_h, \end{cases} \quad (26)$$

where

$$w^{\text{S}}(x) = \frac{1}{2}\big(w(\hat{x}, x_3) + w(\hat{x}, -x_3)\big), \quad w^{\text{A}}(x) = \frac{1}{2}\big(w(\hat{x}, x_3) - w(\hat{x}, -x_3)\big), \quad \forall\, x \in B_h. \quad (27)$$

Let η in $C_0^\infty(-L^-, L^+)$ with $\eta = 1$ if $|x_3| < h_0$, $\eta = 0$ if $|x_3| > (1/2)(h_0 + \text{Min}(L^+, L^-))$ and ρ in $C_0^\infty(-1,1)$ such that $2\int_{-1}^{+1} (\rho'(y))^2\, dy = 1$. Considering first the cases $\mathfrak{r} = 1, 2$, as there exists some (χ, ξ) in $L^2(S,\mathbb{R}^2) \times L^2(S)$ such that $\mathcal{W}_{\mathfrak{q},1\tilde{b}}([v^0]) = \mathcal{W}_b(([\widehat{v^0}], \xi) \otimes_s e_3)$, $\mathcal{W}_{\mathfrak{q},2\tilde{b}}([v^0]) = \mathcal{W}_b((\chi, [v^0_3]) \otimes_s e_3)$, let ${}^{\mathfrak{c},\mathfrak{r}}\tilde{v}_s$ be such that

$${}^{\mathfrak{c},\mathfrak{r}}\tilde{v}_s := \begin{cases} R_h\big(v^0 + \varepsilon v^1\big) + {}^{\mathfrak{r}}\theta_s & \text{in } B_h, \\ {}^1\theta_s := \varepsilon^{-1}\, \rho(\cdot/h)(0,\xi), \quad {}^2\theta_s := \varepsilon\, \rho(\cdot/h)\Big(\chi - [\widehat{v^1}], 0\Big), & \\ {}^2\tau_h\Big(\eta\big(v^0 + \varepsilon v^1 + \varepsilon^2 v^2\big)\Big) + {}^{\mathfrak{p}}\tau_h\Big((1-\eta)\big(v^0 + \varepsilon v^1 + \varepsilon^2 v^2\big)\Big) & \text{in } {}^{\mathfrak{p}}\Omega_h. \end{cases} \quad (28)$$

Clearly $\int_{{}^{\mathfrak{p}}\Omega_h} \mathcal{W}_{{}^{\mathfrak{p}}\tau_h a}(e^{\text{beam}}(\varepsilon, {}^{\mathfrak{c},\mathfrak{r}}\widetilde{v}_s))\,dx$ converges to $\int_{\Omega\setminus S} \mathcal{W}_a(E_v)\,dx$ while a rather tedious computation using *all the previous assumptions* on v, η, ρ and the proof of [8, Lemma 4.1] shows that:

$$\lim_{s\to\bar{s}} \quad \mu \int_{B_h} \mathcal{W}_b\Big(e^{\text{beam}}\big(\varepsilon, {}^{\mathfrak{c},\mathfrak{r}}\widetilde{v}_s\big)\Big)\,dx = {}^{\mathfrak{c},\mathfrak{r}}\mathcal{J}\big(v^0\big). \tag{29}$$

When $\mathfrak{r} = 3$ (and also $\omega \times (-h, h) \cap \operatorname{supp}(v^2) = \emptyset$), taking into account that $v^{0\pm}$ belongs to $\mathcal{V}^{\pm}_{\text{BN}}$ with $[v^0] = 0$ and that consequently there exists (v^{0T}, v^{0L}) in $H^2(-L^-, L^+; \mathbb{R}^2) \times H^1(-L^-, L^+)$ such that

$$v^0(x) = \left(v^{0T}(x_3),\ v^{0L}(x_3) - x_\alpha \frac{dv^{0T}_\alpha}{dx_3}(x_3)\right),$$

we introduce ${}^{\mathfrak{c},3}\widetilde{v}_s$ as:

$$ {}^{\mathfrak{c},3}\widetilde{v}_s := \begin{cases} \left(v^{0T}(0) + x_3\left(\dfrac{dv^{0T}}{dx_3}\right)(0), v^{0L}(0) - x_\alpha \dfrac{dv^{0T}_\alpha}{dx_3}(0)\right), & \text{if } x \in B_h, \\ {}^{2}\tau_h\Big(\eta\big(v^0 + \varepsilon v^1 + \varepsilon^2 v^2\big)\Big) + {}^{\mathfrak{p}}\tau_h\Big((1-\eta)\big(v^0 + \varepsilon v^1 + \varepsilon^2 v^2\big)\Big) & \\ \qquad\qquad + h\operatorname{sgn}(x_3)\left(0, \dfrac{dv^{0T}}{dx_3}(0)\eta(x_3)\right) & \text{if } x \in {}^{\mathfrak{p}}\Omega_h. \end{cases} \tag{30}$$

As $e^{\text{beam}}(\varepsilon, {}^{\mathfrak{c},3}\widetilde{v}_s)$ vanishes in B_h, one has the expected result. □

We therefore classically deduce:

Theorem 4. *Problem* $({}^{\mathfrak{q},\mathfrak{r}}\mathrm{P}_{\bar{s}})$ *defined by*

$$\operatorname{Min}\left\{ \int_{\Omega\setminus S} \mathcal{W}_a(E_v)\,dx + {}^{\mathfrak{q},\mathfrak{r}}\mathcal{J}(v^0) - \int_\Omega f\cdot v^0\,dx - \int_{{}^{\mathfrak{q}}\Gamma_{\mathrm{N}}} g\cdot v^0\,d\mathcal{H}_2;\ v \in {}^{\mathfrak{q}}\mathcal{U}\ \ s.t.\ \ v^0 \in {}^{\mathfrak{q},\mathfrak{r}}\mathcal{V} \right\} \qquad ({}^{\mathfrak{q},\mathfrak{r}}\mathrm{P}_{\bar{s}})$$

has a unique solution ${}^{\mathfrak{q},\mathfrak{r}}u_{\bar{s}}$ *with* ${}^{\mathfrak{q},\mathfrak{r}}u^{0-}_{\bar{s}}$ *defined up to an element of* ${}^{\mathfrak{q},\mathfrak{r}}\mathcal{R}^-_0$ *(see* (17)*) and when* s *goes to* $\bar{s}$ *the solution* ${}^{\mathfrak{c}}u_s$ *of* $({}^{\mathfrak{c}}\mathrm{P}_s)$ *converges toward* ${}^{\mathfrak{q},\mathfrak{r}}u_{\bar{s}}$ *in the sense*

$$\lim_{s\to\bar{s}} \int_{{}^{\mathfrak{p}}\Omega_h} \mathcal{W}_{{}^{\mathfrak{p}}\tau_h a}\Big(e^{\text{beam}}\big(\varepsilon, {}^{\mathfrak{c}}u_s\big) - {}^{\mathfrak{p}}\tau_h\, E_{{}^{\mathfrak{q},\mathfrak{r}}u_{\bar{s}}}\Big)\,dx$$
$$+ \mu \int_{B_h} \mathcal{W}_b\left(e^{\text{beam}}\big(\varepsilon, {}^{\mathfrak{c}}u_s\big) - \left(\frac{\Big[\Big(\widehat{{}^{\mathfrak{q},\mathfrak{r}}u^0_{\bar{s}}}/\varepsilon,\ {}^{\mathfrak{q},\mathfrak{r}}u^0_{\bar{s}_3}c\Big)\Big] + \big({}^{\mathfrak{r}}\bar{\chi}, {}^{\mathfrak{q},\mathfrak{r}}\bar{\xi}/\varepsilon\big)}{2h} \right) \otimes_{\mathrm{s}} e_3 \right) dx = 0, \tag{31}$$

where

$${}^{\mathfrak{r}}\bar{\chi} = 0 \quad \textit{if} \quad \mathfrak{r} \in \{1,3\}, \quad {}^{2}\bar{\chi} = \operatorname{Argmin}\Big\{\mathcal{W}_b\big((\chi, [{}^{\mathfrak{q},2}u^0_{\bar{s}_3}]) \otimes_{\mathrm{s}} e_3\big),\ \chi \in \mathbb{R}^2\Big\},$$
$${}^{1,1}\bar{\xi} = 0, \quad {}^{2,1}\bar{\xi} = \operatorname{Argmin}\Big\{\mathcal{W}_b\big(([\widehat{{}^{2,1}u^0_{\bar{s}}}], \xi) \otimes_{\mathrm{s}} e_3\big),\ \xi \in \mathbb{R}\Big\}$$

and

$${}^{\mathfrak{q},\mathfrak{r}}\bar{\xi} = 0 \quad \textit{if} \quad (\mathfrak{q}, \mathfrak{r}) \in \{1,2\} \times \{2,3\}.$$

Even if $({}^{\mathfrak{q},\mathfrak{r}}\mathrm{P}_{\bar{s}})$ involves *abstract* fields defined in "abstract beams" occupying $\Omega \setminus S$, we will use the language of Mechanics to comment it. Problem $({}^{\mathfrak{q},\mathfrak{r}}\mathrm{P}_{\bar{s}})$ accounts for the equilibrium of two solids filling the cylindrical domains Ω^+ and Ω^- made of a "generalized linearly elastic" material whose state is described by its field of displacement v^0 and two additional variables v^1 and v^2. Each element of $v := (v^0, v^1, v^2)$ has a specific kinematics, especially v^0 which is of Bernoulli–Navier type, while the stress tensor Σ does satisfy:

$$\Sigma = a E_v, \quad E_v = \begin{bmatrix} \widehat{e}(v^2) & e_{\alpha 3}(v^1) \\ \text{sym} & e_{33}(v^0) \end{bmatrix}. \tag{32}$$

These solids are subjected to forces f, g and clamped on ${}^{\mathfrak{q}}\Gamma_{\mathrm{D}}$ with a *constraint* on S which reads as:

- ${}^1\overline{\mu} = 0$: free to separate.
- ${}^1\overline{\mu} > 0$: elastic pull-back with $(2\Sigma_{\mathrm{T}}, \Sigma_{\mathrm{N}}) = {}^1\overline{\mu}\, D\mathscr{W}_{\mathfrak{q},1\tilde{b}}([v^0])$, with:
  - $\Sigma_{\mathrm{T}} = \widehat{\Sigma e_3}$: the tangential stress,
  - $\Sigma_{\mathrm{N}} = (\Sigma e_3)_3$: the normal stress.
- $({}^1\overline{\mu}, {}^2\overline{\mu}) \in \{+\infty\} \times [0, +\infty)$:
  (1) tangential adhesion i.e. $[\widehat{v^0}] = 0$,
  (2) elastic normal pull-back i.e. $\Sigma_{\mathrm{N}} = {}^2\overline{\mu}\, D\mathscr{W}_{\mathfrak{q},2\tilde{b}}([v^0])$.
- $({}^1\overline{\mu}, {}^2\overline{\mu}) \in \{+\infty\}^2$: perfect adhesion i.e. $[v^0] = 0$.

Roughly speaking, the thin layer B_h at the limit shrinks to the surface S and is replaced by a mechanical constraint which does not involve $\mathfrak{p}$ which only accounts for the length of the cylindrical domains occupied by the scaled beams.

We moreover obviously have:

Theorem 5. *When s goes to $\bar{s}$, the solution ${}^{\mathfrak{c}}u^s$ of $({}^{\mathfrak{c}}\mathrm{P}^s)$ satisfies:*

$$\lim_{s\to\bar{s}} \frac{1}{\varepsilon^4}\left[\int_{{}^{\mathfrak{p}}\Omega_h^\varepsilon} \mathscr{W}_{{}^{\mathfrak{p}}a_h^\varepsilon}\left(e^\varepsilon({}^{\mathfrak{c}}u^s) - {}^{\mathfrak{p}}E^\varepsilon_{\mathfrak{q},\mathfrak{r}_{u_{\bar{s}}}}\right)dx^\varepsilon \right. \\ \left. + \mu \int_{B_h^\varepsilon} \mathscr{W}_b\left(e^\varepsilon({}^{\mathfrak{c}}u^s) - \frac{\left[{}^{\mathfrak{q},\mathfrak{r}}\overline{u}^{0\varepsilon}\right] + \left({}^{\mathfrak{r}}\overline{\chi}^\varepsilon, {}^{\mathfrak{q},\mathfrak{r}}\overline{\xi}^\varepsilon\right)}{2h} \otimes_s e_3\right)dx^\varepsilon\right] = 0,$$

where ${}^{\mathfrak{p}}E^\varepsilon_{\mathfrak{q},\mathfrak{r}_{u_{\bar{s}}}}(x^\varepsilon) = \varepsilon\ {}^{\mathfrak{p}}\tau_h E_{\mathfrak{q},\mathfrak{r}_{u_{\bar{s}}}}(x)$ for all x in ${}^{\mathfrak{p}}\Omega_h$, ${}^{\mathfrak{q},\mathfrak{r}}\overline{u}^{0\varepsilon} := \mathscr{S}_\varepsilon^{-1}\ {}^{\mathfrak{q},\mathfrak{r}}u^0_{\bar{s}}$ and $({}^{\mathfrak{r}}\overline{\chi}^\varepsilon, {}^{\mathfrak{q},\mathfrak{r}}\overline{\xi}^\varepsilon)(\widehat{x}^\varepsilon) = \varepsilon\,({}^{\mathfrak{r}}\overline{\chi}, {}^{\mathfrak{q},\mathfrak{r}}\overline{\xi})(\widehat{x})$ for all $\widehat{x}$ in ω.

It should be noted that similarly to the modeling of a single slender beam [9], it is the limit ${}^{\mathfrak{q},\mathfrak{r}}u^0_{\bar{s}}$ of the scaled displacement which is a Bernoulli–Navier displacement and that the true strain tensor is not close to a Bernoulli–Navier strain tensor but should involve ${}^{\mathfrak{q},\mathfrak{r}}u^1_{\bar{s}}$ and ${}^{\mathfrak{q},\mathfrak{r}}u^2_{\bar{s}}$. Here however it is the *sole* jump of the displacement ${}^{\mathfrak{q},\mathfrak{r}}u^0_{\bar{s}}$ which is involved in the energetic relative approximation of the real strain tensor in the very thin layer, with two critical sizes corresponding to the jump of the $\widehat{\cdot}$ component and the third component respectively, a "natural extension" of the case of the assembly of two three-dimensional linearly elastic bodies linked by a soft layer [1]. But contrary to the case of an assembly of two 3-dimensional bodies linked by a stiff layer [10], $1/\varepsilon^4 \int_{B_h^\varepsilon} \mathscr{W}_b(e^\varepsilon({}^{\mathfrak{c}}u^s))\,dx^\varepsilon$ vanishes when $\mu/2h \to +\infty$!

Finally, we might also consider a more complex situation where the elasticity tensor of the joint presents some partial dependencies with respect to a quadruple of parameters $\tilde{s} := (\varepsilon, h, \mu_1, \mu_2)$. We may assume as an example that $\mathscr{W}_b := \mu_1 \mathscr{W}_{b_1} + \mu_2 \mathscr{W}_{b_2}$, with at least one of the two quadratic forms $\mathscr{W}_{b_1}$ and $\mathscr{W}_{b_2}$ being positive definite. Introducing ${}^k\overline{\mu}_\alpha$ as in (H_2), we therefore obtain up to 9 different limit energy densities, i.e. such a case leads to additional asymptotic regimes in which the constraint on S would be of a mixed nature: adhesion in certain directions but elastic pull-back or free separation in others.

3. A proposal of model

Similarly to [9] we do not use ${}^{\mathfrak{q},\mathfrak{r}}\overline{u}^{0\varepsilon}$ as a simple and accurate enough model for the physical situation described by $({}^{\mathfrak{c}}\mathrm{P}^s)$ but we have to take into account two new facts. First the field in question has to be an element of ${}^{\mathfrak{c}}V_h^\varepsilon$, the space of admissible displacements for $({}^{\mathfrak{c}}\mathrm{P}^s)$. Second, in practice we do not know what are the limits $({}^1\overline{\mu}, {}^2\overline{\mu})$, we only know what is $s = (\varepsilon, h, \mu)$! We therefore follow a two-step process:

(1) Taking into account (14), (16) and (25), the problem $({}^{\mathfrak{q}}\mathrm{Q}_s)$

$$\mathrm{Min}\left\{\int_{\Omega\setminus S}\mathscr{W}_a(E_v)\,dx+\sum_{\mathfrak{r}=1}^{2}\frac{\mu}{2h\varepsilon^{2(2-\mathfrak{r})}}\int_S \mathscr{W}_{\mathfrak{q},\mathfrak{r}\,\tilde{b}}\big([v^0]\big)\,d\hat{x}-\int_\Omega f\cdot v^0\,dx-\int_{{}^{\mathfrak{q}}\Gamma_{\mathrm{N}}} g\cdot v^0\,d\mathcal{H}_2 \right.$$
$$\left. \text{with } v=\big(v^0,v^1,v^2\big)\in {}^{\mathfrak{q}}\mathscr{U} \text{ s.t. } v^0\in H^1_{{}^{\mathfrak{q}}\Gamma_{\mathrm{D}}}\big(\Omega\setminus S,\mathbb{R}^3\big)\right\} \qquad ({}^{\mathfrak{q}}\mathrm{Q}_s)$$

has a unique solution (from which it is easy to get a numerical approximation by standard finite element methods) ${}^{\mathfrak{q}}\breve{u}_s$ which is obviously close to ${}^{\mathfrak{q},\mathfrak{r}}u_{\bar{s}}$.

(2) One approximates ${}^{\mathfrak{q}}\breve{u}_s$ by a smooth enough field in order to proceed to the construction used in the proof of Lemma 3 to obtain through $\mathscr{S}_\varepsilon^{-1}$ a field ${}^{\mathfrak{c}}\tilde{u}_s^\varepsilon$ in ${}^{\mathfrak{c}}\mathcal{V}_h^\varepsilon$ which is close to ${}^{\mathfrak{c}}u^s$ in terms of relative energy gap (and similarly for the associated stresses due to (3) and (H_1)). A "numerical version" of this step is again easy to perform.

4. A slight variant: two hollow beams connected by a solid joint

In the previous analysis we did not require the cross-section ω to be simply-connected. To handle more realistic cases, we now assume that the joint still occupies the domain $B_h^\varepsilon := \varepsilon\omega\times(-h,h)$, while the two beams occupy the domains ${}^{\mathfrak{p}}\Omega_h^\varepsilon$ defined via (1) from Ω^ε which now reads as $\varepsilon(\omega\setminus\kappa)\times(-L^-,L^+)$. Here, κ is a compact subset of ω whose interior $\mathring{\kappa}$ is a simply-connected domain with a Lipschitz boundary and satisfying $\int_\kappa x_\alpha\,d\hat{x} = \int_\kappa x_1x_2\,d\hat{x} = 0$, three conditions henceforth also met by ω.

Concerning the limit problem $({}^{\mathfrak{q},\mathfrak{r}}\mathrm{P}_{\bar{s}})$, which is key to understanding the asymptotic behavior of the mechanical system, the sole change implied by this new configuration lies in the substitutions of Ω by $\Omega^{\mathrm{eff}} := (\omega\setminus\kappa)\times(-L^-,L^+)$ and of S by $S^{\mathrm{eff}} := (\omega\setminus\kappa)\times\{0\}$ in the integrals involved in the formulation of $({}^{\mathfrak{q},\mathfrak{r}}\mathrm{P}_{\bar{s}})$ and the definition (14) of ${}^{\mathfrak{q},\mathfrak{r}}\mathcal{J}$! The proof of Lemma 1 is indeed left unchanged while in the proof of Lemma 2 it suffices to bound the energy from below by zero in $B_h\setminus B_h^{\mathrm{eff}} := \kappa\times(-h,h)$. As to the proof of Lemma 3, it has to be adapted somewhat by introducing, for δ positive and small enough, the cut-off function φ_δ defined on ω by $\varphi_\delta(\hat{x}) = \mathrm{Max}(1-\mathrm{dist}(\hat{x},\omega\setminus\kappa)/\delta, 0)$. The fields ${}^{\mathfrak{c},\mathfrak{r}}\overline{v}_s$ defined in B_h^{eff} are then extended to $B_h\setminus B_h^{\mathrm{eff}}$ through formulas such as (28) and (30) so that introducing

$$ {}^{\mathfrak{c},1}\overline{\overline{v}}_s := \varphi_\delta\,{}^{\mathfrak{c},1}\overline{v}_s,\quad {}^{\mathfrak{c},2}\overline{\overline{v}}_s := \varphi_\delta\left({}^{\mathfrak{c},2}\overline{v}_s-\left(\widehat{v^0(\cdot,0)},0\right)\right)+\left(\widehat{v^0(\cdot,0)},0\right),\quad {}^{\mathfrak{c},3}\overline{\overline{v}}_s := {}^{\mathfrak{c},3}\overline{v}_s \tag{33}$$

leads to

$$\lim_{s\to\bar{s}}\mu\int_{B_h\setminus B_h^{\mathrm{eff}}}\mathscr{W}_b\Big(e^{\mathrm{beam}}\Big(\varepsilon,{}^{\mathfrak{c},\mathfrak{r}}\overline{\overline{v}}_s\Big)\Big)\,dx = 0 \tag{34}$$

by choosing δ of order ε.

Thus, although the "scaled joint" geometrically shrinks to the entire surface S, from a mechanical standpoint the limit surface energy remains concentrated on the effective part S^{eff}. In particular, the limit surface energy density is the same as the one identified in the previous section and corresponds to the *effective limit constraint* between the beams discussed after Theorem 4.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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