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Research article

A comparison between extended energy flux and extended entropy flux formulations of nonlocal constitutive models

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Abstract. Nonlocal constitutive models have been widely developed to account for size/gradient effects or to circumvent the difficulties associated with excessive spatial localization. Two main thermodynamic frameworks have been proposed to incorporate some information regarding the spatial distribution of internal variables in a consistent manner: formulations based on an extended energy flux and those relying on an extended entropy flux. Although these approaches are often regarded as equivalent, their respective implications on the thermodynamic structure of the governing equations remain insufficiently understood. In this work, a comparison between nonlocal models formulated with an extended energy flux and with an extended entropy flux is conducted within the general framework of continuum thermodynamics. Both gradient-based and integral-based nonlocal approaches are considered in a unified manner. The corresponding energy and entropy balance equations are derived, along with the associated dissipation inequalities, evolution equations for internal degrees of freedom, and heat diffusion equations. It is shown that the two formulations lead to identical evolution equations under uniform temperature conditions. However, some differences arise in the presence of temperature gradients, notably through the coupling between nonlocal interactions and the temperature field in the extended entropy flux formulation. The analysis also clarifies under which assumptions some formulations presented as extended entropy flux approaches should rather be interpreted as extended energy flux frameworks. The results provide a theoretical basis to assess the relevance and limitations of extended energy and entropy flux formulations of nonlocal constitutive models in thermomechanically coupled problems.

Keywords. Thermodynamics, internal variable, nonlocal, constitutive model.
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Introduction

The behavior of solid materials is largely influenced by the history effects resulting from the microstructural transformations (e.g., hardening, damage) that occur during a thermomechanical process. In the context of constitutive modeling, history effects are typically described with some specific state variables commonly referred to as internal variables [1,2]. When some internal variables are introduced, a constitutive model typically consists of a set of state equations, which provide some fundamental quantities such as the stress tensor or the entropy density, and a set of evolution equations, which govern the evolution of internal variables as functions of the current state. The procedure for obtaining a consistent set of constitutive equations from the viewpoint of continuum thermodynamics is largely documented [3–5].

In recent years, nonlocal formulations of constitutive models with internal variables have attracted increasing attention. These models are based on the idea that the state of a given material point depends not only on the local values of some internal variables but also on their spatial distribution. In the terminology of [6], such variables are called internal degrees of freedom. By contrast, within the framework of local theories, internal variables reduced to standard state variables. As discussed by [7], nonlocal models can be classified as either weakly (aka gradient-based) or strongly (aka integral-based) nonlocal. The former introduce the spatial gradients of internal variables into the set of constitutive relations while the latter treat the spatial averages of internal variables as additional state variables. These nonlocal models are particularly relevant for modeling of small scale plasticity [8–10], damage [11–13] or phase transitions [14].

The conventional thermodynamic setting used for local constitutive models does not allow nonlocality (i.e., the spatial gradients and/or spatial averages of some internal variables) to be introduced within constitutive relations. To overcome this limitation, two main thermodynamic formulations of nonlocal constitutive models have emerged. The most widely used formulation uses an extended energy flux, in which the classical energy balance is augmented by additional contributions associated with internal degrees of freedom [15–17]. This approach has been widely used in the context of strain-gradient plasticity [18,19] and phase-field models of damage [20–22]. The second formulation relies on an extended entropy flux [23–25], where additional entropy flux terms are introduced to ensure compatibility with the second law in the presence of nonlocal effects. While these two viewpoints can be equivalent under specific assumptions, they may lead to different evolution equations for internal degrees of freedom in the general case.

The objective of this work is to provide a systematic comparison between nonlocal models formulated with an extended energy flux and those based on an extended entropy flux. Particular emphasis is placed on the thermodynamic structure of the governing equations and the conditions under which the two formulations can be considered equivalent. The analysis is developed within the general framework of continuum thermodynamics, allowing for applications to a wide range of dissipative materials exhibiting nonlocal behavior.

The present paper is organized as follows. The definitions of the spatial gradient and the spatial average of an internal variable are first recalled. The mass and momentum conservation equations, which display their classical form, are then briefly presented. The construction of constitutive relations relies on some thermodynamic potentials whose general form is discussed in the third section. The features of nonlocal constitutive models based on an extended energy flux and an extended entropy flux are detailed in the fourth and fifth sections. The application of the principle of virtual power, which provides an alternative path toward the balance equations governing a system with internal degrees of freedom, is then briefly presented. The extended energy flux and an extended entropy flux formulations are finally compared to each other in the final section.

1. Nonlocal variables

Many constitutive models rely on some internal variables to consider the microstructural transformations that affect the thermomechanical behavior of solid materials (e.g., plasticity, damage). For simplicity, a single scalar internal variable, denoted by α, is considered hereafter.[1] The description of the constitutive behavior can be further enriched by considering some additional information regarding the spatial distribution of the internal degree of freedom. For this purpose,

[1] The introduction of additional and/or tensorial internal variables is straightforward and does not add significant insight to the present study.

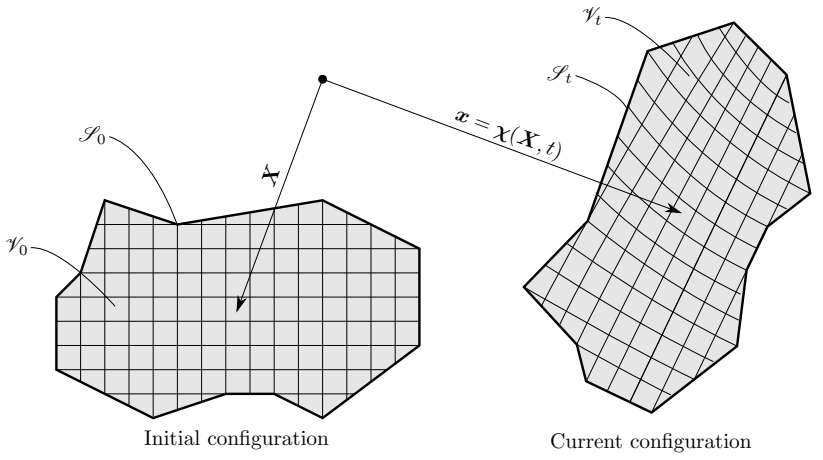

Figure 1. Schematic diagram showing the initial and current configurations of a system. The initial position of a material point is denoted by $\boldsymbol{X}$ while the current position is denoted by $\boldsymbol{x}$.

two types of approaches are commonly used. The first type of approach, largely used in the context of plasticity [26–28] and damage [29–31], consists of introducing the gradient $\nabla_0 \alpha$ of the internal degree of freedom α as an additional state variable. In a material description, this gradient is obtained from

$$\nabla_0 \alpha = \frac{\partial \alpha}{\partial \boldsymbol{X}}, \tag{1}$$

where $\boldsymbol{X}$ is the initial position vector (see Figure 1).

The second type of approach, often referred to as an integral-based approach [7,32], consists in including the average $\bar{\alpha}$ of the internal degree of freedom α in the list of state variables. The average $\bar{\alpha}$ for a material point with initial position $\boldsymbol{X}$ is given by

$$\bar{\alpha}(\boldsymbol{X}) = \begin{cases} \alpha(\boldsymbol{X}) + \frac{1}{W} \int_{\mathscr{V}_0} w(\boldsymbol{X} - \boldsymbol{X}') \big(\alpha(\boldsymbol{X}') - \alpha(\boldsymbol{X})\big) \, \mathrm{d}V' & \text{constant,} \\ \alpha_{\text{ext}} + \frac{1}{W} \int_{\mathscr{V}_0} w(\boldsymbol{X} - \boldsymbol{X}') \big(\alpha(\boldsymbol{X}') - \alpha_{\text{ext}}\big) \, \mathrm{d}V' & \text{fixed-value.} \end{cases} \tag{2}$$

In the above equation, w is an even function, commonly referred to as the weight function and W is a normalization factor. The weight function measures the strength of interactions between two material points from their relative initial positions. As discussed by [33] in the context of plasticity, two different options can be used to evaluate the average of the internal variable (see Figure 2). These options, which allow the treatment of boundary effects, differ from each other by the treatment of exterior points, i.e., the points that lie outside the region over which the variable α is defined. Indeed, when computing the average of an internal variable near a boundary, a specific treatment must be adopted to consider the interactions of a material point with exterior points. The first option, which was proposed by [12], assumes that the value of the internal variable for exterior points is the same as that of the material point of interest. This option is referred to as local constant extension in the following. The second option, referred to as fixed-value extension, considers that the body of interest is surrounded by a fictitious medium for which the internal variable is assigned a specific value α_{ext}.

Though gradient-based and integral-based approaches are rarely used together, they are not mutually exclusive of each other. For the purpose of generality, a unified treatment, which consists in treating both the gradient $\nabla_0 \alpha$ and the average $\bar{\alpha}$ as additional state variables, is therefore proposed in the following.

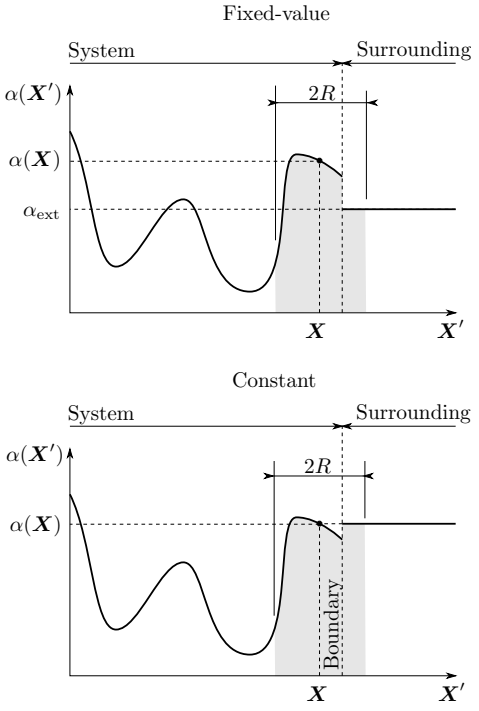

Figure 2. Schematic diagram showing the procedure used to calculate the spatial average of an internal variable α for a near-boundary material point with initial position $\boldsymbol{X}$ using the fixed-value (top) and constant (bottom) assumptions.

2. Mass and momentum conservation equations

In the following, we restrict attention to closed systems for which the mass and momentum conservation equations take their classical form. Specifically, in the absence of mass transfer, the referential and current mass densities, denoted by ρ_0 and ρ, satisfy

$$\dot{\rho}_0 = \dot{\rho} + \rho \operatorname{tr}(\mathbf{D}) = 0, \tag{3}$$

where $\mathbf{D}$ is the eulerian strain rate tensor. Also, for linear and angular momenta to be conserved, the first Piola–Kirchoff stress tensor $\mathbf{P}$ must verify

$$\mathbf{P}\cdot\nabla_0 + \boldsymbol{B} = \rho_0\,\ddot{\boldsymbol{x}} \qquad \text{and} \qquad \mathbf{P}\cdot\mathbf{F}^{\mathrm{t}} = \mathbf{F}\cdot\mathbf{P}^{\mathrm{t}}, \tag{4}$$

where $\boldsymbol{B}$ is the body force density and $\mathbf{F} = \boldsymbol{x}\otimes\nabla_0$ is the deformation gradient tensor.

3. Thermodynamic potentials

In a thermomechanical context, the state of a material point is represented by a set of state variables that typically includes the absolute temperature T and the deformation gradient tensor $\mathbf{F}$. Also, the variable α, its average $\bar{\alpha}$ and its gradient $\nabla_0\alpha$ are additional state variables. With such a set of state variables, the rate of the specific Helmholtz free energy f is

$$\begin{aligned}\dot{f} &= \frac{\partial f}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial f}{\partial T}\dot{T} + \frac{\partial f}{\partial \alpha}\dot{\alpha} + \frac{\partial f}{\partial \bar{\alpha}}\dot{\bar{\alpha}} + \frac{\partial f}{\partial \nabla_0\alpha}\cdot\nabla_0\dot{\alpha} \\ &= \frac{\mathbf{P}_{\text{rev}}}{\rho_0} : \dot{\mathbf{F}} - s\dot{T} + \frac{A_{\text{rev}}}{\rho_0}\dot{\alpha} + \frac{M_{\text{rev}}}{\rho_0}\dot{\bar{\alpha}} + \frac{\boldsymbol{G}_{\text{rev}}}{\rho_0}\cdot\nabla_0\dot{\alpha},\end{aligned} \tag{5}$$

where $s = -\partial f / \partial T$ is the specific entropy. For conciseness, the above equation uses the following state equations:

$$\mathbf{P}_{\text{rev}} = \varrho_0 \frac{\partial f}{\partial \mathbf{F}}, \tag{6}$$

$$A_{\text{rev}} = \varrho_0 \frac{\partial f}{\partial \alpha}, \tag{7}$$

$$M_{\text{rev}} = \varrho_0 \frac{\partial f}{\partial \bar{\alpha}}, \tag{8}$$

$$\boldsymbol{G}_{\text{rev}} = \varrho_0 \frac{\partial f}{\partial \nabla_0 \alpha}. \tag{9}$$

The subscript "rev" is used to emphasize the fact that the thermodynamical forces $\mathbf{P}_{\text{rev}}$, A_{rev}, M_{rev} and $\boldsymbol{G}_{\text{rev}}$ control the non-dissipative part of the process of interest. For subsequent developments, it is convenient to introduce the dissipative counterparts $\mathbf{P}_{\text{irr}}$, A_{irr}, M_{irr} and $\boldsymbol{G}_{\text{irr}}$ of the above thermodynamical forces. These dissipative forces are responsible for the irreversible character of a process in that the specific dissipation source d takes the following form:

$$d = \underbrace{\frac{\mathbf{P}_{\text{irr}}}{\varrho_0} : \dot{\mathbf{F}} + \frac{A_{\text{irr}}}{\varrho_0}\dot{\alpha} + \frac{M_{\text{irr}}}{\varrho_0}\dot{\bar{\alpha}} + \frac{\boldsymbol{G}_{\text{irr}}}{\varrho_0} \cdot \nabla_0 \dot{\alpha}}_{d_{\text{in}}} \underbrace{- \frac{\nabla_0 T}{\varrho_0 T} \cdot \boldsymbol{J}_{\text{q}}}_{d_{\text{th}}}, \tag{10}$$

where $\boldsymbol{J}_{\text{q}}$ is the heat flux density vector. The above equation indicates that the specific dissipation source can be decomposed into two contributions, denoted by d_{in} and d_{th}, usually referred to as the intrinsic and thermal dissipation sources.[2] The former results from microstructural transformations while the latter is caused by heat conduction.

In the context of standard materials [2], the dissipative forces, which are the dual variables of flux-like variables in the sense of dissipation, are obtained from a dissipation potential ϕ such that:[3]

$$\mathbf{P}_{\text{irr}} = \varrho_0 \frac{\partial \phi}{\partial \dot{\mathbf{F}}}, \tag{11}$$

$$A_{\text{irr}} = \varrho_0 \frac{\partial \phi}{\partial \dot{\alpha}}, \tag{12}$$

$$M_{\text{irr}} = \varrho_0 \frac{\partial \phi}{\partial \dot{\bar{\alpha}}}, \tag{13}$$

$$\boldsymbol{G}_{\text{irr}} = \varrho_0 \frac{\partial \phi}{\partial \nabla_0 \dot{\alpha}}, \tag{14}$$

$$\frac{\nabla_0 T}{T} = -\varrho_0 \frac{\partial \phi}{\partial \boldsymbol{J}_{\text{q}}}. \tag{15}$$

It should be mentioned that, for some applications, the framework of standard materials is too restrictive in the sense that the resulting set of constitutive equations does not allow the reproduction of experimental observations. In such a situation, constitutive equations relating the dissipative forces to the flux-like variables may be introduced without resorting to a dissipation potential, provided that consistency with the second law of thermodynamics is ensured.

[2] The expression of the thermal dissipation source is provided in equation (10) by anticipation. Strictly speaking, one does not need to make any assumption regarding the thermal dissipation source at this stage. The upcoming developments indicate that the expression of this contribution to the total dissipation source can be deduced from the entropy production inequality.

[3] In the non-differentiable case, the constitutive relation between the fluxes and the dissipative forces is expressed in terms of the subdifferential of the dissipation potential.

Due to the purely dissipative character of heat conduction, the constitutive equation for the heat flux density vector solely derives from the dissipation potential, as indicated by equation (15). At the opposite, the constitutive equation for the first Piola–Kirchoff, which includes both non-dissipative and dissipative contributions, uses both the state and dissipation potentials:

$$\begin{aligned}\mathbf{P} &= \mathbf{P}_{\text{rev}} + \mathbf{P}_{\text{irr}} \\ &= \varrho_0 \frac{\partial f}{\partial \mathbf{F}} + \varrho_0 \frac{\partial \phi}{\partial \dot{\mathbf{F}}} \qquad \text{(standard materials)}.\end{aligned} \tag{16}$$

In a similar fashion to the first Piola–Kirchoff stress tensor $\mathbf{P}$, it is convenient to introduce the quantities A, M and $\boldsymbol{G}$ that incorporate both dissipative and non-dissipative contributions such that

$$A = A_{\text{rev}} + A_{\text{irr}}, \tag{17}$$

$$M = M_{\text{rev}} + M_{\text{irr}}, \tag{18}$$

$$\boldsymbol{G} = \boldsymbol{G}_{\text{rev}} + \boldsymbol{G}_{\text{irr}}. \tag{19}$$

In the context of a nonlocal approach, the evolution of the internal degree of freedom α is governed by a constitutive equation that involves A, M and $\boldsymbol{G}$. Such a constitutive equation however depends on how nonlocality is introduced in conservation equations.

4. Extended energy flux formulation

4.1. *Energy conservation*

A possible approach to incorporate the gradient and/or the average of an internal degree of freedom in constitutive relations in a thermodynamically consistent manner consists of adopting a non-conventional form of the first law of thermodynamics. In contrast to local constitutive models, this specific form includes an extended energy flux that results from nonlocality. Specifically, in the context of an extended energy flux approach, the global form of the first law of thermodynamics is given by

$$\dot{\mathscr{U}} = \dot{\mathscr{W}} + \dot{\mathscr{Q}} + \dot{\mathscr{N}}, \tag{20}$$

where $\mathscr{U}$ is the total internal energy of the system of interest, $\dot{\mathscr{W}}$ is the mechanical work rate, $\dot{\mathscr{Q}}$ is the heat flux and $\dot{\mathscr{N}}$ is the nonlocality contribution to global energy transfer.

To obtain the corresponding local statement of the first law of thermodynamics, it is useful to introduce the specific internal energy $u = f + sT$ such that

$$\mathscr{U} = \int_{\mathscr{V}_0} \varrho_0 \, u \, \mathrm{d}V. \tag{21}$$

Also, for a deformable body, the mechanical work rate is obtained from

$$\dot{\mathscr{W}} = \int_{\mathscr{V}_0} \mathbf{P} : \dot{\mathbf{F}} \, \mathrm{d}V. \tag{22}$$

The heat flux includes the contributions of the heat flux density vector $\boldsymbol{J}_{\text{q}}$ and the specific heat source r_{q}. The latter accounts for external heat sources while the former represents heat transfer through the external boundary. The heat flux is thus given by

$$\dot{\mathscr{Q}} = \int_{\mathscr{V}_0} \varrho_0 \, r_{\text{q}} \, \mathrm{d}V - \int_{\mathscr{S}_0} \boldsymbol{J}_{\text{q}} \cdot \mathrm{d}\boldsymbol{S}. \tag{23}$$

The nonlocality contribution to global energy transfer takes a similar form:

$$\dot{\mathscr{N}} = \int_{\mathscr{V}_0} \varrho_0 \, r_{\text{nl}} \, \mathrm{d}V - \int_{\mathscr{S}_0} \boldsymbol{J}_{\text{nl}} \cdot \mathrm{d}\boldsymbol{S}, \tag{24}$$

where $\boldsymbol{J}_{\mathrm{nl}}$ is the extended energy flux density vector and r_{nl} is the specific extended energy source.[4]

The local form of the first law of thermodynamics equation is obtained by requiring that the global form holds not only for the entire system but for any arbitrary control volume within it. The corresponding local conservation equation for energy is therefore given by

$$\rho_0 \dot{u} = \mathbf{P} : \dot{\mathbf{F}} - \boldsymbol{J}_{\mathrm{q}} \cdot \nabla_0 + \rho_0 r_{\mathrm{q}} - \boldsymbol{J}_{\mathrm{nl}} \cdot \nabla_0 + \rho_0 r_{\mathrm{nl}}. \tag{25}$$

The definition of the extended energy flux requires some constitutive equations for the extended energy flux density vector and the specific extended energy source. A convenient choice is to adopt the following form for these quantities:

$$\boldsymbol{J}_{\mathrm{nl}} = -\boldsymbol{G}\dot{\alpha}, \tag{26}$$

$$r_{\mathrm{nl}} = \frac{1}{\rho_0}\big(M\dot{\tilde{\alpha}} - \widetilde{M}\dot{\alpha} + B\dot{\alpha}\big), \tag{27}$$

where B is a prescribed body force that affects the evolution of the internal degree of freedom. Also, the forces $\widetilde{M}$ and M are such that

$$\int_{\mathscr{V}_0} M\,\dot{\tilde{\alpha}}\,\mathrm{d}V = \int_{\mathscr{V}_0} \widetilde{M}\,\dot{\alpha}\,\mathrm{d}V. \tag{28}$$

Depending on the retained assumption for the treatment of exterior points, one thus obtains that $\widetilde{M}$ is related to M according to

$$\widetilde{M}(\boldsymbol{X}) = \begin{cases} M(\boldsymbol{X}) + \frac{1}{W}\int_{\mathscr{V}_0} w(\boldsymbol{X}-\boldsymbol{X}')\big(M(\boldsymbol{X}') - M(\boldsymbol{X})\big)\,\mathrm{d}V' & \text{constant,} \\ \frac{1}{W}\int_{\mathscr{V}_0} w(\boldsymbol{X}-\boldsymbol{X}')M(\boldsymbol{X}')\,\mathrm{d}V' & \text{fixed-value.} \end{cases} \tag{29}$$

The term $B\,\dot{\alpha}$, which appears in equation (27) can be interpreted as a density of power resulting from the action of the body force B on the system of interest. Indeed, the condition (28) implies that

$$\int_{\mathscr{V}_0} \rho_0\, r_{\mathrm{nl}}\,\mathrm{d}V = \int_{\mathscr{V}_0} B\,\dot{\alpha}\,\mathrm{d}V. \tag{30}$$

In a similar fashion, one may consider that nonlocality contributes to energy transfer through the external boundary of the system of interest. The energy flux surface density $J = \boldsymbol{J}_{\mathrm{nl}} \cdot \boldsymbol{N}$ is typically controlled with appropriate natural or essential boundary conditions:

$$\boldsymbol{G}\cdot\boldsymbol{N} = G_{\mathrm{bc}},\ \forall\, \boldsymbol{X} \in \mathscr{S}_{\mathrm{n}} \qquad \text{and} \qquad \dot{\alpha} = \dot{\alpha}_{\mathrm{bc}},\ \forall\, \boldsymbol{X} \in \mathscr{S}_{\mathrm{e}}, \tag{31}$$

where $\mathscr{S}_{\mathrm{n}}$ and $\mathscr{S}_{\mathrm{e}}$ denote the portions of the external boundary where either natural or essential boundary conditions are imposed (with $\mathscr{S}_{\mathrm{n}} \cup \mathscr{S}_{\mathrm{e}} = \mathscr{S}_0$ and $\mathscr{S}_{\mathrm{n}} \cap \mathscr{S}_{\mathrm{e}} = \emptyset$).

Remark 1. A common assumption consists of imposing that the first law of thermodynamics (20) takes its conventional global form, i.e., $\dot{\mathscr{U}} = \dot{\mathscr{W}} + \dot{\mathscr{Q}}$, in which case the nonlocality contribution $\dot{\mathscr{N}}$ is forced to vanish. With such an assumption, sometimes referred to as a constitutive insulation assumption [37], the body force density satisfies the following condition:

$$B = 0 \quad \Longrightarrow \quad \int_{\mathscr{V}_0} \rho_0\, r_{\mathrm{nl}}\,\mathrm{d}V = 0. \tag{32}$$

Also, the following boundary conditions are commonly imposed:

$$\boldsymbol{G}\cdot\boldsymbol{N} = 0,\ \forall\, \boldsymbol{X} \in \mathscr{S}_{\mathrm{n}} \quad \text{and} \quad \dot{\alpha} = 0,\ \forall\, \boldsymbol{X} \in \mathscr{S}_{\mathrm{e}} \qquad \Longrightarrow \qquad \int_{\mathscr{S}_0} \boldsymbol{J}_{\mathrm{nl}} \cdot \mathrm{d}\boldsymbol{S} = 0. \tag{33}$$

The constitutive insulation assumption is not a fundamental principle in the sense that it does not allow representing some situations. For instance, in the context of crystal plasticity, some

[4] In the context of integral-based nonlocal models, the specific extended energy source is sometimes referred to as the nonlocality residual [34,35]. For an infinitesimal surface element with external unit normal $\boldsymbol{N}$, the quantity $J = \boldsymbol{J}_{\mathrm{nl}} \cdot \boldsymbol{N}$ is referred to as the interstitial working by [36].

defects (e.g., dislocations, vacancies) may be transferred to the system through the external surface. The corresponding transfer of internal energy cannot be accommodated with the constitutive insulation assumption.

4.2. *Entropy production and dissipation*

In contrast with the first law of thermodynamics, the second law of thermodynamics imposes the entropy production rate to be non-negative. The entropy production rate is defined as the difference between the entropy rate $\dot{\mathscr{S}}$ and the entropy flux $\dot{\mathscr{J}}$, that is

$$\dot{\mathscr{S}} - \dot{\mathscr{J}} \geq 0. \tag{34}$$

The total entropy $\mathscr{S}$ of the system is obtained from

$$\mathscr{S} = \int_{\mathscr{V}_0} \rho_0 \, s \, \mathrm{d}V. \tag{35}$$

In the context of an extended energy flux formulation, the flux of entropy for a closed system takes its classical form, i.e., it is equal to the heat flux divided by the temperature:

$$\dot{\mathscr{J}} = \int_{\mathscr{V}_0} \rho_0 \, \frac{r_\mathrm{q}}{T} \, \mathrm{d}V - \int_{\mathscr{S}_0} \frac{\boldsymbol{J}_\mathrm{q}}{T} \cdot \mathrm{d}\boldsymbol{S}. \tag{36}$$

Using the above expressions, one finds that the corresponding local statement of the second law of thermodynamics is

$$\rho_0 \, \sigma = \rho_0 \, \dot{s} + \left(\frac{\boldsymbol{J}_\mathrm{q}}{T} \right) \cdot \nabla_0 - \rho_0 \, \frac{r_\mathrm{q}}{T} \geq 0, \tag{37}$$

where σ is the specific entropy production rate. The expression of the specific dissipation source $d = \sigma \, T$ is obtained by combining the above expression of the specific entropy production rate, the state equations (6) to (9) and the energy conservation equation (25), which leads to

$$\rho_0 \, d = (\mathbf{P} - \mathbf{P}_\mathrm{rev}) : \dot{\mathbf{F}} - \boldsymbol{J}_\mathrm{nl} \cdot \nabla_0 + \rho_0 \, r_\mathrm{nl} - \frac{\nabla_0 T}{T} \cdot \boldsymbol{J}_\mathrm{q} - A_\mathrm{rev} \, \dot{\alpha} - \boldsymbol{G}_\mathrm{rev} \cdot (\nabla_0 \, \dot{\alpha}) - M_\mathrm{rev} \, \dot{\tilde{\alpha}}. \tag{38}$$

Introducing relations (26) and (27) in the expression (38) of the specific dissipation source, one obtains

$$\rho_0 \, d = \mathbf{P}_\mathrm{irr} : \dot{\mathbf{F}} + (\boldsymbol{G} \cdot \nabla_0) \, \dot{\alpha} - \frac{\nabla_0 T}{T} \cdot \boldsymbol{J}_\mathrm{q} - A_\mathrm{rev} \, \dot{\alpha} + \boldsymbol{G}_\mathrm{irr} \cdot (\nabla_0 \, \dot{\alpha}) - \widetilde{M} \, \dot{\alpha} + B \, \dot{\alpha} + M_\mathrm{irr} \, \dot{\tilde{\alpha}}. \tag{39}$$

4.3. *Evolution equation for an internal degree of freedom*

The evolution equation for the internal degree of freedom α is obtained by remarking that the expressions (10) and (39) of the dissipation source should be equivalent for any evolution of the system of interest. The dissipative force A_irr thus has the following expression:

$$A_\mathrm{irr} = \boldsymbol{G} \cdot \nabla_0 - A_\mathrm{rev} - \widetilde{M} + B. \tag{40}$$

Using the above relation, the evolution equation for the internal degree of freedom α is conveniently written as

$$A + \widetilde{M} = \boldsymbol{G} \cdot \nabla_0 + B. \tag{41}$$

In contrast with conventional theories, the above evolution equation of the internal degree of freedom α depends on the spatial distributions of forces associated with its spatial gradient and its spatial average.

4.4. *Heat diffusion equation*

For thermomechanical problems, it is necessary to establish the heat diffusion equation, which controls the evolution of the temperature field within the studied system. The procedure used to obtain this equation is briefly described here.

With the present choice of state variables, the rate of change of specific entropy is given by

$$\dot{s} = \frac{\partial s}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial s}{\partial T}\dot{T} + \frac{\partial s}{\partial \alpha}\dot{\alpha} + \frac{\partial s}{\partial \bar{\alpha}}\dot{\bar{\alpha}} + \frac{\partial s}{\partial \nabla_0 \alpha} \cdot \nabla_0 \dot{\alpha}. \tag{42}$$

Also, the specific intrinsic dissipation source d_{in} can be expressed as

$$d_{\text{in}} = \frac{\nabla_0 \cdot \boldsymbol{J}_{\text{q}}}{\varrho_0} - r_{\text{q}} + \dot{s}T. \tag{43}$$

Combining the above expression with equation (42) provides the usual form of the heat diffusion equation:

$$c\dot{T} = r_{\text{c}} + d_{\text{in}} - \frac{\nabla_0 \cdot \boldsymbol{J}_{\text{q}}}{\varrho_0} + r_{\text{q}}. \tag{44}$$

For the purpose of conciseness, the above equation uses the specific heat capacity at constant strain c and the specific coupling heat source r_{c}. The specific heat capacity is defined according to

$$c = T\frac{\partial s}{\partial T} = -T\frac{\partial^2 f}{\partial T^2}. \tag{45}$$

The specific coupling heat source r_{c} is such that

$$\begin{aligned} r_{\text{c}} &= -T\frac{\partial s}{\partial \mathbf{F}} : \dot{\mathbf{F}} - T\frac{\partial s}{\partial \alpha}\dot{\alpha} - T\frac{\partial s}{\partial \bar{\alpha}}\dot{\bar{\alpha}} - T\frac{\partial s}{\partial \nabla_0 \alpha} \cdot \nabla_0 \dot{\alpha} \\ &= T\frac{\partial^2 f}{\partial \mathbf{F}\partial T} : \dot{\mathbf{F}} + T\frac{\partial^2 f}{\partial \alpha \partial T}\dot{\alpha} + T\frac{\partial^2 f}{\partial \bar{\alpha}\partial T}\dot{\bar{\alpha}} + T\frac{\partial^2 f}{\partial \nabla_0 \alpha\, \partial T} \cdot \nabla_0 \dot{\alpha}. \end{aligned} \tag{46}$$

The above equation indicates that the coupling heat source results from energetic couplings between temperature and other state variables.

5. Extended entropy flux formulation

5.1. *Energy conservation*

An alternative strategy consists of constructing a nonlocal approach based on the assumption that nonlocality is introduced in the form of an extended entropy, rather than extended energy, flux. This assumption does not affect the first law of thermodynamics, which displays its conventional form:

$$\dot{\mathscr{U}} = \dot{\mathscr{W}} + \dot{\mathscr{Q}}. \tag{47}$$

The corresponding local statement is

$$\varrho_0 \dot{u} = \mathbf{P} : \dot{\mathbf{F}} - \boldsymbol{J}_{\text{q}} \cdot \nabla_0 + \varrho_0 r_{\text{q}}. \tag{48}$$

The above equation indicates that internal energy changes because of the mechanical work produced by internal forces and/or heat transfer.

5.2. *Entropy production and dissipation*

Because of the assumption of an additional contribution to the entropy flux, the second law of thermodynamics takes the following modified form:

$$\dot{\mathscr{S}} - \dot{\mathscr{J}} - \dot{\mathscr{Z}} \geq 0, \tag{49}$$

where $\dot{\mathscr{Z}}$ represents the flux of entropy that results from nonlocal interactions. In a similar fashion as for the extended energy flux formulation, this flux contains both a volumetric contribution and a surface contribution such that

$$\dot{\mathscr{Z}} = \int_{\mathscr{V}_0} \varrho_0 \, z_{\mathrm{nl}} \, \mathrm{d}V - \int_{\mathscr{S}_0} \boldsymbol{K}_{\mathrm{nl}} \cdot \mathrm{d}\boldsymbol{S}, \tag{50}$$

where $\boldsymbol{K}_{\mathrm{nl}}$ is the extended entropy flux density vector and z_{nl} is the specific extended entropy source.

The corresponding local statement of the second law of thermodynamics becomes

$$\varrho_0 \, \sigma = \varrho_0 \, \dot{s} + \left(\frac{\boldsymbol{J}_{\mathrm{q}}}{T} \right) \cdot \nabla_0 - \varrho_0 \, \frac{r_{\mathrm{q}}}{T} + \boldsymbol{K}_{\mathrm{nl}} \cdot \nabla_0 - \varrho_0 \, z_{\mathrm{nl}} \geq 0. \tag{51}$$

In the context of an extended entropy flux approach, the constitutive equations for the extended entropy flux density vector and the specific extended entropy source are selected as follows:

$$\boldsymbol{K}_{\mathrm{nl}} = \frac{1}{T} \boldsymbol{G} \, \dot{\alpha}, \tag{52}$$

$$z_{\mathrm{nl}} = \frac{1}{\varrho_0 T} \left(\widehat{M} \, \dot{\alpha} - M \, \dot{\tilde{\alpha}} - B \, \dot{\alpha} \right). \tag{53}$$

The operator that provides the rate of the average of a variable is self-adjoint. It is thus convenient to introduce the quantity $\widehat{M}$ that satisfies the following condition:

$$\int_{\mathscr{V}_0} \frac{M}{T} \, \dot{\tilde{\alpha}} \, \mathrm{d}V = \int_{\mathscr{V}_0} \frac{\widehat{M}}{T} \, \dot{\alpha} \, \mathrm{d}V. \tag{54}$$

Depending on the retained assumption for the treatment of exterior points, the condition (54) is met if the force $\widehat{M}$ is connected to M according to

$$\widehat{M}(\boldsymbol{X}) = \begin{cases} M(\boldsymbol{X}) + \frac{T(\boldsymbol{X})}{W} \int_{\mathscr{V}_0} w(\boldsymbol{X} - \boldsymbol{X}') \left(\frac{M(\boldsymbol{X}')}{T(\boldsymbol{X}')} - \frac{M(\boldsymbol{X})}{T(\boldsymbol{X})} \right) \mathrm{d}V' & \text{constant,} \\ \frac{T(\boldsymbol{X})}{W} \int_{\mathscr{V}_0} w(\boldsymbol{X} - \boldsymbol{X}') \frac{M(\boldsymbol{X}')}{T(\boldsymbol{X}')} \, \mathrm{d}V' & \text{fixed-value.} \end{cases} \tag{55}$$

It is worth mentioning that the above expressions are very similar to that obtained (see (29)) with the extended energy flux assumption. However, the value of $\widehat{M}$ for a given material point not only depends on the spatial distribution of M, but also on that of temperature. The condition (54) implies that the action of the body force density B contributes to the flow of entropy between the system and its environment according to

$$\int_{\mathscr{V}_0} \varrho_0 \, z_{\mathrm{nl}} \, \mathrm{d}V = - \int_{\mathscr{V}_0} \frac{B}{T} \, \dot{\alpha} \, \mathrm{d}V. \tag{56}$$

The transfer of entropy between the system of interest and its surrounding can also be achieved through the external boundary. The entropy flux surface density $K = \boldsymbol{K}_{\mathrm{nl}} \cdot \boldsymbol{N}$ on any material point lying on the boundary of the system is classically controlled with either natural or essential boundary conditions:

$$\left(\frac{\boldsymbol{G}}{T} \right) \cdot \boldsymbol{N} = Z_{\mathrm{bc}}, \ \forall \, \boldsymbol{X} \in \mathscr{S}_{\mathrm{n}} \qquad \text{and} \qquad \dot{\alpha} = \dot{\alpha}_{\mathrm{bc}}, \ \forall \, \boldsymbol{X} \in \mathscr{S}_{\mathrm{e}}. \tag{57}$$

In contrast with the extended energy flux formulation, the application of essential boundary conditions on the external boundary uses the ratio $\boldsymbol{G}/T$, instead of $\boldsymbol{G}$.

Remark 2. By analogy with the extended energy flux formulation, one may postulate that nonlocality should not contribute to the global production of entropy, in which case the second law of thermodynamics (49) displays its standard global form (i.e., $\dot{\mathcal{Z}} = 0$). For the nonlocality contribution to the extended flux of entropy to vanish, the body force density B should be such that

$$B = 0 \quad \Longrightarrow \quad \int_{\mathcal{V}_0} \varrho_0 \, z_{\mathrm{nl}} \, \mathrm{d}V = 0. \tag{58}$$

The condition $\dot{\mathcal{Z}} = 0$ also necessitates the following boundary conditions to be met:

$$\frac{\boldsymbol{G}}{T} \cdot \boldsymbol{N} = 0, \ \forall \, \boldsymbol{X} \in \mathscr{S}_{\mathrm{n}} \quad \text{and} \quad \dot{\alpha} = 0, \ \forall \, \boldsymbol{X} \in \mathscr{S}_{\mathrm{e}} \qquad \Longrightarrow \qquad \int_{\mathscr{S}_0} \boldsymbol{K}_{\mathrm{nl}} \cdot \mathrm{d}\boldsymbol{S} = 0. \tag{59}$$

The expression of the specific dissipation source $d = \sigma \, T$ is deduced from the local statements of the first and second laws of thermodynamics (48) and (51) and the state equations (6) to (9), which leads to

$$\varrho_0 \, d = (\mathbf{P} - \mathbf{P}_{\mathrm{rev}}) : \dot{\mathbf{F}} + (\boldsymbol{K}_{\mathrm{nl}} \cdot \nabla_0) \, T - \varrho_0 \, z_{\mathrm{nl}} \, T - \frac{\nabla_0 T}{T} \cdot \boldsymbol{J}_{\mathrm{q}} - A_{\mathrm{rev}} \, \dot{\alpha} - \boldsymbol{G}_{\mathrm{rev}} \cdot (\nabla_0 \, \dot{\alpha}) - M_{\mathrm{rev}} \, \dot{\tilde{\alpha}}. \tag{60}$$

The introduction of equations (52) and (53) in the expression (60) of the specific dissipation source leads to

$$\varrho_0 \, d = \mathbf{P}_{\mathrm{irr}} : \dot{\mathbf{F}} + \left(\left(\frac{\boldsymbol{G}}{T} \right) \cdot \nabla_0 \right) \dot{\alpha} \, T - \frac{\nabla_0 T}{T} \cdot \boldsymbol{J}_{\mathrm{q}} - A_{\mathrm{rev}} \, \dot{\alpha} + \boldsymbol{G}_{\mathrm{irr}} \cdot (\nabla_0 \, \dot{\alpha}) - \widehat{M} \, \dot{\alpha} + B \, \dot{\alpha} + M_{\mathrm{irr}} \, \dot{\tilde{\alpha}}. \tag{61}$$

5.3. *Evolution equation for an internal degree of freedom*

The evolution equation for the internal degree of freedom α is deduced from the comparison between the general expression of the specific dissipation source (10) with the expression (61). Since these two expressions should be equivalent for any thermodynamically admissible process, one finds that the dissipative force A_{irr} must satisfy

$$A_{\mathrm{irr}} = \left(\left(\frac{\boldsymbol{G}}{T} \right) \cdot \nabla_0 \right) T - A_{\mathrm{rev}} - \widehat{M} + B. \tag{62}$$

The evolution of the internal degree of freedom α is thus governed by the following equation:

$$A + \widehat{M} = \left(\left(\frac{\boldsymbol{G}}{T} \right) \cdot \nabla_0 \right) T + B. \tag{63}$$

The above expression is very similar to that obtained in the context of an extended energy flux formulation, except that it explicitly includes the role of temperature.

5.4. *Heat diffusion equation*

The definitions (45) and (46) of the specific heat capacity and the specific coupling heat source remain unchanged with respect to the extended energy flux approach. Using these definitions, the product of the specific entropy rate and the temperature can be written as

$$\dot{s} \, T = c \, \dot{T} - r_{\mathrm{c}}. \tag{64}$$

When an additional contribution to the entropy flux is included, the specific intrinsic dissipation source can be expressed as follows:

$$d_{\mathrm{in}} = \frac{\nabla_0 \cdot \boldsymbol{J}_{\mathrm{q}}}{\varrho_0} - r_{\mathrm{q}} + \frac{\nabla_0 \cdot \boldsymbol{K}_{\mathrm{nl}}}{\varrho_0} \, T - z_{\mathrm{nl}} \, T + \dot{s} T. \tag{65}$$

The combination of equation (64) with the above relation leads to the following heat diffusion equation:

$$c \, \dot{T} = r_{\mathrm{c}} + d_{\mathrm{in}} - \frac{\nabla_0 \cdot \boldsymbol{J}_{\mathrm{q}}}{\varrho_0} + r_{\mathrm{q}} - \frac{\nabla_0 \cdot \boldsymbol{K}_{\mathrm{nl}}}{\varrho_0} \, T + z_{\mathrm{nl}} \, T. \tag{66}$$

The heat diffusion equation displays a non-standard form when the extended entropy flux approach is adopted. Indeed, the heat diffusion equation includes an additional contribution that arises from the entropy transfer associated with nonlocality when the extended entropy flux approach is adopted.

6. Generalized principle of virtual power

6.1. *Balance equations*

A common approach to construct a model with some internal degrees of freedom consists of deriving the corresponding balance equations from a generalized principle of virtual power [38]. This type of approach is largely used in the context of strain-gradient plasticity [39–41] and damage [16]. The idea is to consider that the variables $\boldsymbol{x}$ and α satisfy an extended virtual power principle in the sense that

$$\mathscr{P}^{\star}_{\text{ext}} = \dot{\mathscr{K}}^{\star} + \mathscr{P}^{\star}_{\text{int}}, \quad \forall\, \dot{\boldsymbol{x}}^{\star},\, \dot{\alpha}^{\star}. \tag{67}$$

In the above equation, $\mathscr{P}^{\star}_{\text{int}}$ is the power developed by internal forces. In contrast with conventional theories, the power developed by internal forces includes a contribution associated with the evolution of internal degrees of freedom. When both the gradient and the average of the internal degree of freedom α are considered, the power developed by internal forces is thus expressed as follows:

$$\mathscr{P}^{\star}_{\text{int}} = \int_{\mathscr{V}_0} \left(\mathbf{P} : \dot{\mathbf{F}}^{\star} + A\,\dot{\alpha}^{\star} + \boldsymbol{G} \cdot (\nabla_0 \dot{\alpha}^{\star}) + M\,\dot{\tilde{\alpha}}^{\star} \right) \mathrm{d}V, \tag{68}$$

with $\dot{\mathbf{F}}^{\star} = \dot{\boldsymbol{x}}^{\star} \otimes \nabla_0$. The power developed by external forces $\mathscr{P}^{\star}_{\text{ext}}$ incorporates the contribution of body and surface forces to the transfer of energy between the system of interest and its surroundings. When some internal degrees of freedom are introduced, the power developed by external forces $\mathscr{P}^{\star}_{\text{ext}}$ is written as

$$\mathscr{P}^{\star}_{\text{ext}} = \int_{\mathscr{V}_0} (\boldsymbol{B} \cdot \dot{\boldsymbol{x}}^{\star} + B\,\dot{\alpha}^{\star})\, \mathrm{d}V + \int_{\mathscr{S}_0} (\boldsymbol{T} \cdot \dot{\boldsymbol{x}}^{\star} + G\,\dot{\alpha}^{\star})\, \mathrm{d}S, \tag{69}$$

where $\boldsymbol{B}$ and B are body force densities while $\boldsymbol{T}$ and G are surface force densities. Finally, the power of inertial forces is

$$\dot{\mathscr{K}}^{\star} = \int_{\mathscr{V}_0} \rho_0\, \ddot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}^{\star}\, \mathrm{d}V. \tag{70}$$

For the condition (67) to be valid for any admissible fields $\dot{\boldsymbol{x}}^{\star}$ and $\dot{\alpha}^{\star}$, the first Piola–Kirchoff stress tensor must satisfy:

$$\mathbf{P} \cdot \nabla_0 + \boldsymbol{B} = \rho_0\, \ddot{\boldsymbol{x}}, \quad \forall\, \boldsymbol{X} \in \mathscr{V}_0, \tag{71}$$

$$\mathbf{P} \cdot \boldsymbol{N} = \boldsymbol{T}, \quad \forall\, \boldsymbol{X} \in \mathscr{S}_0. \tag{72}$$

Also, since a rigid body rotation should not contribute to power developed by internal forces, the first Piola–Kirchoff stress tensor is such that

$$\mathbf{P} \cdot \mathbf{F}^{\mathrm{t}} = \mathbf{F} \cdot \mathbf{P}^{\mathrm{t}}, \quad \forall\, \boldsymbol{X} \in \mathscr{V}_0. \tag{73}$$

Finally, the evolution of the integral degree of freedom α is controlled by the following conditions:

$$A + \widetilde{M} = \boldsymbol{G} \cdot \nabla_0 + B, \quad \forall\, \boldsymbol{X} \in \mathscr{V}_0, \tag{74}$$

$$\boldsymbol{G} \cdot \boldsymbol{N} = G, \quad \forall\, \boldsymbol{X} \in \mathscr{S}_0. \tag{75}$$

6.2. *Energy conservation*

The equilibrium condition (74) is identical to that (41) obtained in the context of an extended energy flux approach, which suggests that the application of the principle of virtual power is an alternative path toward an extended energy flux approach. This can be further confirmed by inspecting the local energy conservation equation resulting from the principle of virtual power. Specifically, the first law of thermodynamics stipulates that the change in total (i.e., kinetic and internal) energy of the system results from the work of external forces and the heat flux:

$$\dot{\mathscr{Q}} + \mathscr{P}_{\text{ext}} = \dot{\mathscr{U}} + \dot{\mathscr{K}}. \tag{76}$$

Using the condition (67), the first law of thermodynamics can equivalently be stated as

$$\dot{\mathscr{U}} = \dot{\mathscr{W}} + \dot{\mathscr{Q}}, \tag{77}$$

where the power of internal forces is identified as the mechanical work rate, i.e., $\dot{\mathscr{W}} = \mathscr{P}_{\text{int}}$. The corresponding local conservation equation for the specific internal energy is obtained by noting that the above global conservation equation should be valid not only for the system of interest but also for any subpart of it. The local statement of the first law of thermodynamics is thus given by

$$\varrho_0 \, \dot{u} = \mathbf{P} : \dot{\mathbf{F}} + A \, \dot{\alpha} + \boldsymbol{G} \cdot (\nabla_0 \dot{\alpha}) + M \, \dot{\tilde{\alpha}} - \boldsymbol{J}_{\text{q}} \cdot \nabla_0 + \varrho_0 \, r_{\text{q}}. \tag{78}$$

Using the equilibrium condition (74), the above equation can be recast as

$$\varrho_0 \, \dot{u} = \mathbf{P} : \dot{\mathbf{F}} + \underbrace{(\boldsymbol{G} \, \dot{\alpha})}_{-\boldsymbol{J}_{\text{nl}}} \cdot \nabla_0 + \underbrace{M \, \dot{\tilde{\alpha}} - \widetilde{M} \, \dot{\alpha} + B \, \dot{\alpha}}_{\varrho_0 r_{\text{nl}}} - \boldsymbol{J}_{\text{q}} \cdot \nabla_0 + \varrho_0 \, r_{\text{q}}. \tag{79}$$

The above form of the first law of thermodynamics shows that the application of the generalized principle of virtual power leads to an extended energy flux approach. Indeed, the above equation allows retrieving the divergence of what can be interpreted as an energy flux surface density $\boldsymbol{J}_{\text{nl}}$ and a body source of energy r_{nl}. The expressions of these two quantities are strictly equivalent to those proposed in the context of an extended energy flux formulation. The application of the principle of virtual power is therefore an alternative but equivalent path toward an extended energy flux formulation. Such a path can be qualified as an a priori approach. Indeed, following a strategy widely adopted in the mechanics literature, the principle of virtual power is taken as a foundational axiom and the balance equations are subsequently inferred from it. The use of this principle requires defining the contribution of internal degrees of freedom to the power expended by external actions. The presentation of the extra energy flux approach in Section 4 is based on an a posteriori approach. The conservation laws of mass, energy and momentum are first established. Constitutive assumptions are then introduced, notably to characterize the additional energy flux needed to accommodate nonlocality.

7. Discussion

7.1. *The role of temperature*

In the specific, but rather common, situation of a process for which the temperature field is uniform, equations (29) and (55) show that the forces $\widetilde{M}$ and $\widehat{M}$ are equivalent. Also, in such a situation, the right-hand side of equation (63) is the same as that of equation (41) since

$$\left(\left(\frac{\boldsymbol{G}}{T} \right) \cdot \nabla_0 \right) T = \boldsymbol{G} \cdot \nabla_0 \quad \text{if } \| \nabla_0 T \| = 0. \tag{80}$$

Consequently, the evolution equation for the internal degree of freedom α (equation (41) or equation (63)) does not depend on whether nonlocality is accommodated with an extended energy or extended entropy flux when the temperature is uniform.

Differences between extended energy flux and extended entropy flux approaches arise only when important temperature differences exist within the system of interest. The role of such temperature differences is expected to be significant in the cryogenic regime while only limited differences should be observed in the high temperature regime. To highlight this aspect, it is convenient to decompose the absolute temperature T at a given position $\boldsymbol{X}$ as follows:

$$T(\boldsymbol{X}) = T_{\text{avg}} + \Delta T(\boldsymbol{X}), \tag{81}$$

where T_{avg} is the average temperature of the system while ΔT represents the temperature fluctuation with respect to the average temperature. Using the above decomposition, one finds that

$$\lim_{T_{\text{avg}}\to\infty}\left(\left(\frac{\boldsymbol{G}}{T}\right)\cdot\nabla_0\right)T = \boldsymbol{G}\cdot\nabla_0 \qquad \text{and} \qquad \lim_{T_{\text{avg}}\to\infty}\widehat{M} = \widetilde{M}. \tag{82}$$

The above analysis indicates that the balance equation (63) obtained in the context of an extended entropy flux approach is equivalent to that derived from an extended energy flux formulation (41) when the average temperature approaches infinity. In the high temperature regime, the formulation based on an extended entropy flux is thus expected to provide similar results to those derived from an extended energy flux approach.

The evolution of the temperature field is controlled by the heat diffusion equation. Though the intrinsic dissipation source includes a nonlocality contribution, the heat diffusion equation (44) obtained with the extended energy flux formulation displays a form similar to that derived with a local constitutive model [42]. The extended entropy flux formulation leads to a different form of the heat diffusion equation. Indeed, as indicated by equation (66), it involves an additional contribution that corresponds to the product between the absolute temperature and the nonlocality contribution to the entropy flux. As a consequence, similar constitutive models, i.e., with identical state and dissipation potentials, will evolve toward different configurations depending on whether nonlocality is accommodated with an extended energy flux or extended entropy flux formulation. The differences between both formulations result not only from different evolution equations for internal degrees of freedom but also from the effect of nonlocality on the evolution of the temperature field.

7.2. *An extended energy flux approach disguised as an extended entropy flux approach*

To treat the gradient of an internal variable as an additional state variable, [24] proposed an approach presented as an extended entropy flux approach.[5] However, this approach leads to an evolution equation for the internal degree of freedom α that is similar to the one (41) obtained in the context of an extended energy flux approach, which appears to be contradictory. To explain how confusion regarding the entropic or energetic nature of the extended flux can be avoided, one may consider an extended energy flux formulation. The corresponding local energy conservation equation (25) can be reformulated to include the nonlocality contribution in an alternative definition of the heat flux:

$$\varrho_0\,\dot{u} = \mathbf{P} : \dot{\mathbf{F}} - \boldsymbol{J}'_{\text{q}}\cdot\nabla_0 + \varrho_0\, r'_{\text{q}}, \tag{83}$$

where $\boldsymbol{J}'_{\text{q}}$ and r'_{q} are alternative definitions of the heat flux surface density vector and the specific heat source:

$$\boldsymbol{J}'_{\text{q}} = \boldsymbol{J}_{\text{q}} + \boldsymbol{J}_{\text{nl}}, \tag{84}$$

$$r'_{\text{q}} = r_{\text{q}} + r_{\text{nl}}. \tag{85}$$

[5] The original paper of [24] is restricted to the case of a gradient-based model. For the sake of generality, the scope of the present discussion is extended to the situation of a constitutive model that also treats the average of the variable α as an additional state variable.

Using such definitions, the entropy production inequality can be presented as

$$\varrho_0 \sigma = \varrho_0 \dot{s} + \left(\frac{\boldsymbol{J}'_{\mathrm{q}}}{T}\right) \cdot \nabla_0 - \varrho_0 \frac{r'_{\mathrm{q}}}{T} + \boldsymbol{K}_{\mathrm{nl}} \cdot \nabla_0 - \varrho_0 z_{\mathrm{nl}} \geq 0, \tag{86}$$

with

$$\boldsymbol{K}_{\mathrm{nl}} = -\frac{\boldsymbol{J}_{\mathrm{nl}}}{T} = \frac{\boldsymbol{J}_{\mathrm{q}} - \boldsymbol{J}'_{\mathrm{q}}}{T}, \tag{87}$$

$$z_{\mathrm{nl}} = -\frac{r_{\mathrm{nl}}}{T} = \frac{r_{\mathrm{q}} - r'_{\mathrm{q}}}{T}. \tag{88}$$

Equations (83) and (86) suggest that an extended energy flux formulation can be presented as an extended entropy flux formulation, provided that the alternative definition of heat flux (based on $\boldsymbol{J}'_{\mathrm{q}}$ and r'_{q}) is used instead of the original one (based on $\boldsymbol{J}_{\mathrm{q}}$ and r_{q}). As a consequence, there is some ambiguity between extended energy flux and extended entropy flux formulations. Indeed, it is possible to switch from one to another by adopting either the original or the alternative definition of the heat flux. However, such an ambiguity can be removed by deciding which quantities actually define the heat flux. For this purpose, it is instructive to introduce the evolution equation associated with the internal degree of freedom α within the entropy production inequality.

On the one hand, if the equation derived from an extended energy flux formulation (41) is used, one finds that the entropy production rate is given by

$$\varrho_0 \sigma = \frac{\mathbf{P}_{\mathrm{irr}}}{T} : \dot{\mathbf{F}} + \frac{A_{\mathrm{irr}}}{T} \dot{\alpha} + \frac{\boldsymbol{G}_{\mathrm{irr}}}{T} \cdot (\nabla_0 \dot{\alpha}) + \frac{M_{\mathrm{irr}}}{T} \dot{\tilde{\alpha}} - \frac{\nabla_0 T}{T^2} \cdot \boldsymbol{J}_{\mathrm{q}}. \tag{89}$$

The above equation thus indicates that the flux variable associated with heat conduction is the original heat flux density vector $\boldsymbol{J}_{\mathrm{q}}$, in which case nonlocality is accommodated through an extended energy flux. Indeed, the entropy production inequality takes its standard form while the energy conservation equation is enriched with an additional contribution associated with internal degrees of freedom.

On the other hand, the introduction of the evolution equation obtained in the context of an extended entropy flux formulation (63) in (86) leads to

$$\varrho_0 \sigma = \frac{\mathbf{P}_{\mathrm{irr}}}{T} : \dot{\mathbf{F}} + \frac{A_{\mathrm{irr}}}{T} \dot{\alpha} + \frac{\boldsymbol{G}_{\mathrm{irr}}}{T} \cdot (\nabla_0 \dot{\alpha}) + \frac{M_{\mathrm{irr}}}{T} \dot{\tilde{\alpha}} - \frac{\nabla_0 T}{T^2} \cdot \boldsymbol{J}'_{\mathrm{q}}. \tag{90}$$

The flux variable associated with heat conduction is thus the alternative heat flux density vector $\boldsymbol{J}'_{\mathrm{q}}$. The resulting formulation is based on an extended entropy flux since, with such an alternative definition of the heat flux, nonlocality is absorbed through the entropy production inequality.

The above analysis indicates that the balance equation associated with the internal degree of freedom needs to be inspected to determine how nonlocality is accommodated. Maugin [24] proposed an approach presented as an extended entropy flux formulation. However, since it relies on an evolution equation that takes form (41), the nature of the extended flux is energetic rather than entropic.

8. Conclusion

In this work, nonlocal constitutive models formulated within two distinct thermodynamic frameworks — namely extended energy flux and extended entropy flux formulations — are compared. Both approaches aim to incorporate the spatial gradients and/or spatial averages of some internal variables into constitutive relations while preserving compatibility with the fundamental laws of thermodynamics. Starting from a unified set of state variables, the governing balance

laws, dissipation inequalities, evolution equations for internal degrees of freedom, and heat diffusion equations have been systematically derived for each formulation. The comparison reveals that, under isothermal or spatially uniform temperature conditions, the two approaches lead to identical evolution equations for internal degrees of freedom, thereby justifying their equivalence in a wide range of practical situations.

However, the analysis also highlights that this equivalence no longer holds in the presence of temperature gradients. In particular, the extended entropy flux formulation introduces an explicit coupling between nonlocal interactions and the temperature field, which modifies both the evolution equation of the internal degrees of freedom and the heat diffusion equation. These effects become especially significant in regimes where large temperature variations are expected, such as cryogenic conditions, while they tend to vanish at higher temperatures.

Furthermore, it has been shown that certain formulations initially presented as extended entropy flux approaches can be reinterpreted as extended energy flux models through a redefinition of the heat flux, without altering the underlying thermodynamic structure. This observation emphasizes the importance of carefully identifying the true fluxes and conjugate forces in the sense of dissipation when formulating nonlocal constitutive models. Overall, the present work provides a clear thermodynamic interpretation of extended energy and entropy flux formulations and establishes precise conditions under which they are equivalent or fundamentally different. These results offer useful guidelines for the formulation and selection of nonlocal constitutive models in thermomechanically coupled problems involving dissipative materials.

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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