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Short paper / *Note*

A hierarchy of reduced models to approximate Vlasov–Maxwell equations for slow time variations

Une hiérarchie de modèles asymptotiques paraxiaux pour approcher les équations de Vlasov–Maxwell non relativistes

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Abstract. We introduce a new family of paraxial asymptotic models that approximate the Vlasov–Maxwell equations in non-relativistic cases. This formulation is nth order accurate in a parameter η, which denotes the ratio between the characteristic velocity of the beam and the speed of light. This family of models is interesting, first because it is simpler than the complete Vlasov–Maxwell equation and then because it allows us to choose the model complexity according to the expected accuracy.

Résumé. On introduit une nouvelle famille de modèles asymptotiques paraxiaux pour approcher le système d'équations de Vlasov–Maxwell dans le cas non relativiste. Cette formulation est précise à un ordre n (que l'on peut choisir) par rapport à un paramètre η, qui désigne le quotient de la vitesse caractéristique du faisceau par celle de la lumière. L'intéret de cette famille de modèle est, d'une part, qu'elle est plus simple que le système complet des équations de Vlasov–Maxwell, tout en permettant, d'autre part, de choisir le degré de complexité du modèle, en fonction de la précison désirée.

Keywords. Vlasov–Maxwell equations, Asymptotic analysis, Paraxial model, Reduced models, Non-relativistic.

Mots-clés. Équations de Vlasov–Maxwell, Analyse asymptotique, Modèle paraxial, Modèles réduits, Non relativiste.

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On considère un faisceau de particules chargées non colisionnelles, qui se déplacent dans un champ électromagnétique. Ce système peut être modélisé par le système d'équations couplées de Vlasov–Maxwell. Cependant, sa résolution numérique s'avère souvent assez lourde et coûteuse. Il est donc important de pouvoir construire, pour des hypothèses physiques données, des modèles d'équations plus facile à résoudre, qui approchent cependant le modèle initial avec une précision choisie.

Dans cette Note, nous nous intéressons au cas d'un faisceau de particules non relativistes. L'exemple typique envisagée ici est le cas d'un faisceau court de particules chargées, se déplaçant (voir (1)) dans un tube cylindrique parfaitement conducteur. Nous proposons alors une nouvelle famille de modèles asymptotiques paraxiaux pour approcher le système d'équations de Vlasov–Maxwell.

Pour dériver cette famille de modèles, on commence par effectuer un changement de variables dans le système d'équations de Vlasov–Maxwell (1)–(6), en transformant la variable longitudinale z en la variable $\zeta = \beta ct - z$, où c désigne la vitesse de la lumière, $0 < \beta < 1$ un paramètre donné, z l'axe du faisceau et t le temps. En décomposant les composantes de toutes les quantités (position, vitesse, champs, ...) en parties transverses et longitudinales, nous obtenons alors le système (9)–(16).

La seconde étape consiste à introduire une mise à l'échelle des équations, adaptée à la physique considérée. Pour ce faire, on suppose que le faisceau est court, que sa vitesse v_z est de l'ordre de βc et que sa vitesse transverse $\mathbf{v}_\perp$ est petite devant βc. On introduit alors un petit paramètre η défini par (22) et on déduit le système d'équations adimensionnées (24)–(33).

Ensuite, on considère les développements asymptotiques des quantités f, n, $\mathbf{j}$ et $\mathbf{E}$, $\mathbf{B}$, $\mathscr{E}_\perp$, $\mathbf{F}$ en fonction de ce petit paramètre η. On montre ainsi que pour obtenir une approximation de la solution $f(\mathbf{x},\mathbf{v},t)$ de l'équation de Vlasov en $\mathcal{O}(\eta^n)$, $n \in N$, il suffit de connaître le développement asymptotique de la force électromagnétique $\mathbf{F}_\perp, F_z$ à l'ordre n. En utilisant l'expression de ces forces (35)–(36), on détermine quels sont les termes dans le développement asymptotique des champs électromagnetiques nécessaires pour fermer le système.

La dernière étape consiste alors à déterminer des équations caractéristiques pour ces champs électromagnétiques, c'est-à-dire des termes intervenant dans le développement asymptotique. C'est l'objet des lemmes 4.0.1 à 4.0.5, où sont caracterisés les termes d'ordre n de ces champs.

On dérive ensuite la nouvelle famille de modèles paraxiaux en revenant aux variables physiques initiales, (cf. équations (53)–(57)). Cette nouvelle famille de modèles, après discrétisation, devrait conduire à une méthode numérique, rapide et facile à implémenter, pour laquelle on peut choisir le degré de complexité du modèle, en fonction de la précison désirée.

1. Introduction

Charged particle beams are very useful in a variety of scientific and technological applications. After the discovery that both magnetic and electric fields can act as lenses for electron rays, this field experienced rapid development with industrial applications such as welding [1], micromachining and lithography [2], thermonuclear fusion [3], and so on. More recent developments use intense electron beams as electromagnetic radiation sources such as the gyrotron or the free-electron laser (see for instance [4, 5]). More details can be found in [6] and [7]. Hence, there is great interest in mathematical and numerical modeling of these phenomena.

Considering non-collisional beams, a well-accepted method for describing the transport of bunches of particles is the Vlasov equation [8, 9]. Since the particles are electrically charged, the force field that governs their movement is the Lorentz force, which in turn depends on both electric and magnetic fields, which are solutions to the well-known Maxwell equations [10]. This set of equations coupled together is known as the time-dependent Vlasov–Maxwell system of equations.

However, the numerical solution of this model, which is unavoidable in many situations [11, 12], requires a large computational effort, usually based on a combination of finite elements or finite volume discretization with particle-in-cell methods. Therefore, whenever possible, it is worth taking into account the geometric and physical characteristics of the problem to derive approximate models with different degrees of accuracy and complexity, leading to cheaper simulations (see [13–17]).

Hence, in [13], the authors presented a study of a paraxial model as an approximation of the stationary Vlasov–Maxwell equations. The particles of the beam remain close to its optical axis, so that the transverse width of the beam is very small compared to a characteristic length, and have approximately the same kinetic energy. By different assumptions, the authors in [14] used an asymptotic expansion technique to treat the case of high-energy short beams, considering a bunch of highly relativistic charged particles. In the same spirit, in [15], the authors considered a steady-state beam, that is, all partial derivatives with respect to time are a priori set equal to zero. In addition, the beam is assumed to be sufficiently long so that longitudinal self-consistent forces can be neglected, and it propagates at a constant velocity along the propagation axis. In particular, assuming $\partial/\partial t = 0$ allows writing equations in a transverse plane for which the component z plays the role of time. This gives a system in which only four dimensions are involved.

In the model we propose here, these last three assumptions—steady state, longitudinal self-consistent forces, and propagation at a constant velocity along the propagation axis—are no longer required. Following the principle revealed in [14], our approach relies on the introduction of a moving frame, which travels along the optical axis at a given velocity. Many noticeable research works have been conducted in this field. In the case of high-energy, ultra-relativistic short beams, Laval *et al.* [14] derived a paraxial approximation of the Vlasov–Maxwell equations by introducing a moving frame, which travels along the optical axis at the speed of light c.

This idea of changing variables to follow the moving frame is not new; it can be found elsewhere, for instance in [18, 19]. Similar work was carried out for the case of a laminar beam in [20]. A different paraxial model was also derived for the case of high-energy short beams [21], and it was typically related to free-electron lasers or particle accelerators. This work takes into account the specific geometric features of devices, thus leading to a somewhat different dimensional analysis. Numerical applications were also proposed in [22], whereas comparison methods of these models, based on data mining techniques, were proposed in [23].

The aim of this paper is to derive a new family of paraxial asymptotic models that approximate the Vlasov–Maxwell equations in non-relativistic cases. Section 2 gives a short overview of the equations and the change of variables for the beam frame. The scaling of the equations is presented in Section 3, whereas the asymptotic expansion of the relevant parameters to derive a new family of paraxial models is proposed in Section 4. Finally, the resulting paraxial models, which allow us to choose the model complexity according to the expected accuracy, are given in Section 5.

2. Vlasov–Maxwell model

Consider a beam of charged particles with mass m and charge q moving in a perfectly conducting cylindrical tube, whose axis is constituted by the z-axis. We denote by Ω the transverse section of boundary Γ, $\nu = (\nu_x, \nu_y, 0)$ denoting the unit exterior normal to the tube. We assume that an external magnetic field $\mathbf{B}^e$ confines the beam in a neighborhood of the z-axis, which may be therefore chosen as the optical axis of the beam. Let $\mathbf{x} = (x, y, z)$ be the position of the particle and $\mathbf{v} = (v_x, v_y, v_z)$ be its velocity. We assume that the beam is non-relativistic and non-collisional so that its distribution function $f = f(\mathbf{x}, \mathbf{v}, t)$ in the phase space $(\mathbf{x}, \mathbf{v})$ is a solution to the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \mathbf{grad}_{\mathbf{x}} f + \frac{1}{m} \mathbf{F} \cdot \mathbf{grad}_{\mathbf{v}} f = 0. \tag{1}$$

Here, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ denotes the electromagnetic force acting on the particles. The electric field $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ and the magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ are solutions to the Maxwell equations

$$\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} - \mathbf{curl}\, \mathbf{B} = -\mu_0 \mathbf{J}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\, \mathbf{E} = \mathbf{0}, \tag{3}$$

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{4}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{5}$$

where the charge $\rho(\mathbf{x}, t)$ and the current density $\mathbf{J}(\mathbf{x}, t)$ are obtained from the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ with

$$\rho(\mathbf{x}, t) = q \int_{\mathbb{R}^3_{\mathbf{v}}} f \,\mathrm{d}\mathbf{v}, \quad \mathbf{J}(\mathbf{x}, t) = q \int_{\mathbb{R}^3_{\mathbf{v}}} \mathbf{v} f \,\mathrm{d}\mathbf{v}. \tag{6}$$

Now, we introduce a fixed parameter $0 < \beta < 1$, and we consider that the particle longitudinal velocity v_z satisfies $v_z \simeq \beta c$ for any particle in the beam. Hence, we rewrite the Vlasov–Maxwell equations in a frame that moves along the z-axis with the velocity βc, that is, a fraction of the light velocity. For this purpose, we set $\zeta = \beta c t - z$, $v_\zeta = \beta c - v_z$ and we perform the change of variables (and not a change of reference frame) $(x, y, z, v_x, v_y, v_z, t) \to (x, y, \zeta, v_x, v_y, v_\zeta, t)$ so that

$$\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial v_z}, \frac{\partial}{\partial t} \right) \to \left(-\frac{\partial}{\partial \zeta}, -\frac{\partial}{\partial v_\zeta}, \frac{\partial}{\partial t} + \beta c \frac{\partial}{\partial \zeta} \right). \tag{7}$$

Remark 1. Note that the parameter β can be understood as a “degree of freedom” that allows us to choose the most appropriate change of variables according to the average particle longitudinal velocity. In practical applications, it could be taken equal (for instance) to 0.25, 0.5, but in any case, neither close to 0 nor close to 1. Indeed, following the arbitrary classification, β up to a value of 0.4 is non-relativistic, whereas it is semi-relativistic from 0.4 to 0.8. Under these conditions, we cannot have $\beta \ll 1$. In addition, as β is a fixed parameter, we do not consider here the transition from the relativistic to the non-relativistic case.

It is also convenient to introduce the transverse quantities

$$\mathbf{x}_\perp = (x, y), \quad \mathbf{v}_\perp = (v_x, v_y)$$

and to define the transverse operators

$$\mathbf{grad}_\perp \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \right), \quad \mathbf{curl}_\perp \varphi = \left(\frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x} \right), \quad \Delta_\perp \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2},$$

where $\varphi = \varphi(x, y)$ is a scalar function. Similarly, for $\mathbf{A}_\perp = (A_x, A_y)$ denoting a transverse vector field, we set

$$\operatorname{div}_\perp \mathbf{A}_\perp = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}, \quad \operatorname{curl}_\perp \mathbf{A}_\perp = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}.$$

We define $\mathbf{A}_\perp \times \mathbf{e}_z = (A_y, -A_x)$, and we readily obtain the following identities:

$$\operatorname{div}_\perp (\mathbf{A}_\perp \times \mathbf{e}_z) = \operatorname{curl}_\perp A, \quad \operatorname{curl}_\perp (\mathbf{A}_\perp \times \mathbf{e}_z) = -\operatorname{div}_\perp A, \quad \operatorname{curl}_\perp \mathbf{curl}_\perp \varphi = -\Delta_\perp \varphi. \tag{8}$$

Moreover, denoting by $\boldsymbol{\tau} = (-\nu_y, \nu_x)$ the unit tangent along Γ, we have the relation $\mathbf{curl}_\perp \varphi \cdot \boldsymbol{\tau} = -(\partial \varphi / \partial \boldsymbol{\nu})$.

Using the above notations, the Vlasov equation in the new variables can be written as

$$\frac{\partial f}{\partial t}+\mathbf{v}_\perp\cdot\mathbf{grad}_\perp f+v_\zeta\frac{\partial f}{\partial \zeta}+\frac{1}{m}\mathbf{F}_\perp\cdot\mathbf{grad}_{\mathbf{v}_\perp} f-\frac{F_z}{m}\frac{\partial f}{\partial v_\zeta}=0. \tag{9}$$

Additionally, setting $\mathscr{E}_\perp=(E_x-\beta cB_y, E_y+\beta cB_x)$ and $J_\zeta=\rho\beta c-J_z=q\int_{\mathbb{R}^3_{\mathbf{v}}}v_\zeta f\,\mathrm{d}\mathbf{v}$, we obtain the following expressions for the Maxwell equations. First, we consider the Gauss law, which takes the form

$$\mathrm{div}_\perp\mathbf{E}_\perp-\frac{\partial E_z}{\partial\zeta}=\frac{\rho}{\varepsilon_0}, \tag{10}$$

and the Gauss law for magnetism is expressed as

$$\mathrm{div}_\perp\mathbf{B}_\perp-\frac{\partial B_z}{\partial\zeta}=0. \tag{11}$$

In the same way, the Ampere law can be written as

$$\perp\colon \frac{1}{c^2}\frac{\partial\mathbf{E}_\perp}{\partial t}+\frac{1}{\beta c}\frac{\partial}{\partial\zeta}(\mathscr{E}_\perp-(1-\beta^2)\mathbf{E}_\perp)-\mathbf{curl}_\perp B_z=-\mu_0\mathbf{J}_\perp, \tag{12}$$

$$\zeta: \frac{1}{c^2}\frac{\partial E_z}{\partial t}+\frac{1}{\beta c}\mathrm{div}_\perp(\mathscr{E}_\perp-(1-\beta^2)\mathbf{E}_\perp)=\mu_0 J_\zeta, \tag{13}$$

and the Faraday law becomes

$$\perp\colon \frac{\partial\mathbf{B}_\perp}{\partial t}+\frac{\partial}{\partial\zeta}(\mathscr{E}_\perp\times\hat{\mathbf{e}}_z)+\mathbf{curl}_\perp E_z=\mathbf{0}, \tag{14}$$

$$\zeta: \frac{\partial B_z}{\partial t}+\mathrm{curl}_\perp\mathscr{E}_\perp=0. \tag{15}$$

Finally, the electromagnetic force becomes

$$\mathbf{F}_\perp=q(\mathscr{E}_\perp+(B_z\mathbf{v}_\perp+v_\zeta\mathbf{B}_\perp)\times\hat{\mathbf{e}}_z), \tag{16}$$

$$F_z=q(E_z+\mathbf{v}_\perp\cdot(\mathbf{B}_\perp\times\hat{\mathbf{e}}_z)). \tag{17}$$

Let us formulate now the boundary conditions. Assuming that the particles, which drift in the direction $\zeta>0$, remain inside a fixed domain $\Omega\times(0,Z)$ during the time interval $(0,T)$ when we study the behavior of the beam, we assume that no particle is emitted at the boundary of the domain. Hence, for f, we can write

$$f=0\quad\text{for}\begin{cases}(\mathbf{x}_\perp,\zeta)\in\Gamma\times]0,Z[, & \mathbf{v}\cdot\nu<0,\\ \mathbf{x}_\perp\in\Omega,\zeta=0, & v_\zeta>0,\\ \mathbf{x}_\perp\in\Omega,\zeta=Z, & v_\zeta<0.\end{cases}$$

For the initial conditions, we simply assume that the initial distribution of particles is a known function that satisfies the boundary conditions $f_{|t=0}=f_0$.

Regarding the electromagnetic fields, as the surface of the tube is a perfect conductor, the tangential components of the electric field vanish for $\mathbf{x}_\perp\in\Gamma$, $\zeta\in(0,Z)$. Then, we have

$$\mathbf{E}_\perp\cdot\boldsymbol{\tau}=0,\quad E_z=0.$$

For the artificial boundary $\zeta=0$, assuming that there is no external electric field and that the static electromagnetic fields that exist ahead of the beam cannot be modified by the electromagnetic waves generated by the beam, we have for $\mathbf{x}_\perp\in\Omega$, $\zeta=0$

$$\mathbf{E}=0,\quad \mathbf{B}=\mathbf{B}^e,\quad \text{where } \mathbf{B}^e \text{ denotes a given external field.}$$

We also assume given initial conditions $\mathbf{E}|_{t=0}=\mathbf{E}_0$, $\mathbf{B}|_{t=0}=\mathbf{B}_0$, where $\mathbf{E}_0$ and $\mathbf{B}_0$ satisfy both the Maxwell equations and the boundary conditions specified above.

For what follows, let us note some important consequences of these boundary conditions. Taking the inner product of $\mathscr{E}_\perp$ and $\boldsymbol{\tau}$ for $\mathbf{x}_\perp\in\Gamma$, $\zeta\in(0,Z)$, we obtain

$$\mathscr{E}_\perp\cdot\boldsymbol{\tau}=\beta c\mathbf{B}_\perp\cdot\boldsymbol{\nu}. \tag{18}$$

Next, taking the dot product of (14) by $\boldsymbol{\nu}$ and using the definition of $\mathbf{curl}_\perp$, for $\mathbf{x}_\perp \in \Gamma, \zeta \in (0, Z)$, we obtain

$$\left(\frac{\partial}{\partial t} + \beta c \frac{\partial}{\partial \zeta}\right)(\mathbf{B}_\perp \cdot \boldsymbol{\nu}) = 0. \tag{19}$$

Similarly, integrating (15) over Ω and applying the Green theorem for $\zeta \in (0, Z)$, we obtain

$$\int_\Omega \frac{\partial B_z}{\partial t} \mathrm{d}\mathbf{x}_\perp + \beta c \oint_\Gamma \mathbf{B}_\perp \cdot \boldsymbol{\nu} \, \mathrm{d}l = 0. \tag{20}$$

In the same spirit as above, we obtain using (11), for $\zeta \in (0, Z)$,

$$\int_\Omega \left(\frac{\partial}{\partial t} + \beta c \frac{\partial}{\partial \zeta}\right) B_z \, \mathrm{d}\mathbf{x}_\perp = 0. \tag{21}$$

3. Scaling of equations

The second step to derive the paraxial model is to introduce an ad hoc scaling of the equations. Assuming that we deal with a short beam, we introduce a scaling of the equations by considering the following properties of the beam:

(i) The beam dimension is small compared to the longitudinal length L of the device.
(ii) The transverse particle velocities $\mathbf{v}_\perp$ are comparable to v_ζ; so we have $v_\zeta \simeq \mathbf{v}_\perp \ll v_z \simeq \beta c$.

Thus, by following a classical approach in dimensional analysis (see for instance [24, 25]), we introduce two characteristic quantities that “reflect” the geometry and the physics of our problem:

(i) l, the characteristic dimension of the beam;
(ii) $\overline{v}$, the characteristic velocity of the particles.

Note that in contrast to the case described in [14], [22], or [26], we do not require here the longitudinal particle velocities v_z to be close to the light velocity c since we consider a non-relativistic case. For this reason, we set $v_z \simeq \beta c, 0 < \beta < 1$, which allows us to play with the value of the parameter β.

Now, defining a small parameter η and a characteristic time T with

$$\eta \equiv \frac{\overline{v}}{c} \ll 1, \quad T = \frac{l}{\bar{v}}, \tag{22}$$

we can write

$$x = lx', \quad y = ly', \quad \zeta = l\zeta', \quad t = Tt', \quad v_x = \bar{v} v'_x, \quad v_y = \bar{v} v'_y, \quad v_\zeta = \bar{v} v'_\zeta, \tag{23}$$

where the primes represent dimensionless quantities. Using the physical units of the physical quantities and based on the Vlasov–Maxwell equations, we can introduce the following scaling factors. For the electric field, we can define $\bar{E} = m\bar{v}^2/ql$ so that from the Gauss law we can set $\bar{\rho} = \varepsilon_0 m\bar{v}^2/ql^2$. From the definition of ρ, we obtain $\bar{f} = \varepsilon_0 m/q^2 l^2 \bar{v}$. Similarly, using the physical units of the other quantities, we obtain that $\bar{J} = \varepsilon_0 mc\bar{v}^2/ql^2$, $\bar{F} = m\bar{v}^2/l$, and $\bar{B} = m\bar{v}^2/qcl$. This allows us to write $f(\mathbf{x}_\perp, \zeta, \mathbf{v}_\perp, v_\zeta, t) = \bar{f} f'(\mathbf{x}'_\perp, \zeta', \mathbf{v}'_\perp, v'_\zeta, t')$, $\mathbf{E}(\mathbf{x}_\perp, \zeta, t) = \bar{E}\mathbf{E}'(\mathbf{x}'_\perp, \zeta', t')$, $\mathbf{B}(\mathbf{x}_\perp, \zeta, t) = \bar{B}\mathbf{B}'(\mathbf{x}'_\perp, \zeta', t')$, and $\mathbf{F}(\mathbf{x}_\perp, \zeta, \mathbf{v}_\perp, v_\zeta, t) = \bar{F}\mathbf{F}'(\mathbf{x}'_\perp, \zeta', \mathbf{v}'_\perp, v'_\zeta, t')$.

Now, defining $\rho' = \int_{\mathbb{R}^3_{\mathbf{v}}} f' \, \mathrm{d}\mathbf{v}'$ and $\mathbf{J}' = \int_{\mathbb{R}^3_{\mathbf{v}}} \mathbf{v}' f' \, \mathrm{d}\mathbf{v}'$, it is convenient to introduce $\rho = \bar{\rho}\rho'$ for the charge density and $\mathbf{J}_\perp = \bar{J}\eta \mathbf{J}'_\perp, J_\zeta = \bar{J}\eta J'_\zeta$ for the current density.

Hence, we are able to write the Vlasov–Maxwell equations using these dimensionless variables. Dropping the primes for simplicity, the Vlasov equation in dimensionless variables is

$$\frac{\partial f}{\partial t} + \mathbf{v}_\perp \cdot \mathbf{grad}_\perp f + v_\zeta \frac{\partial f}{\partial \zeta} + \mathbf{F}_\perp \cdot \mathbf{grad}_{\mathbf{v}_\perp} f - F_z \frac{\partial f}{\partial v_\zeta} = 0. \tag{24}$$

Next, defining the quantity $\mathscr{E}'_\perp = (E'_x - \beta B'_y, E'_y + \beta B'_x)$, we easily verify that $\mathscr{E}_\perp = \bar{E}\mathscr{E}'_\perp$. Accordingly, applying these dimensionless variables and still dropping the primes, the Ampere law (Equations (12)–(13)) and the Poisson equation (10) give

$$\eta \frac{\partial \mathbf{E}_\perp}{\partial t} + \frac{1}{\beta}\frac{\partial}{\partial \zeta}(\mathscr{E}_\perp - (1-\beta^2)\mathbf{E}_\perp) - \mathbf{curl}_\perp B_z = -\eta \mathbf{J}_\perp, \tag{25}$$

$$\eta \frac{\partial E_z}{\partial t} + \frac{1}{\beta}\mathrm{div}_\perp(\mathscr{E}_\perp - (1-\beta^2)\mathbf{E}_\perp) = \eta J_\zeta, \tag{26}$$

$$\mathrm{div}_\perp \mathbf{E}_\perp - \frac{\partial E_z}{\partial \zeta} = \rho, \tag{27}$$

whereas the Faraday law (Equations (14)–(15)) and the absence of monopole equation (11) are written as

$$\eta \frac{\partial \mathbf{B}_\perp}{\partial t} + \frac{\partial}{\partial \zeta}(\mathscr{E}_\perp \times \hat{\mathbf{e}}_z) + \mathbf{curl}_\perp E_z = \mathbf{0}, \tag{28}$$

$$\eta \frac{\partial B_z}{\partial t} + \mathrm{curl}_\perp \mathscr{E}_\perp = 0, \tag{29}$$

$$\mathrm{div}_\perp \mathbf{B}_\perp - \frac{\partial B_z}{\partial \zeta} = 0. \tag{30}$$

In the above equations, the right-hand sides ρ and $(\mathbf{J}_\perp, J_\zeta)$ fulfill the charge conservation equation

$$\eta\left(\frac{\partial \rho}{\partial t} + \mathrm{div}_\perp \mathbf{J}_\perp + \frac{\partial J_\zeta}{\partial \zeta}\right) = 0. \tag{31}$$

Finally, the electromagnetic force $\mathbf{F} = (\mathbf{F}_\perp, F_z)$ takes the form

$$\mathbf{F}_\perp = \mathscr{E}_\perp + \eta(B_z \mathbf{v}_\perp + v_\zeta \mathbf{B}_\perp) \times \hat{\mathbf{e}}_z, \tag{32}$$

$$F_z = E_z + \eta(v_x B_y - v_y B_x). \tag{33}$$

We turn to the boundary conditions. The scaled electric field $\mathbf{E}$ obeys the same boundary conditions on the perfectly conducting boundary of the tube together with the analogous to (18), that is, $\mathscr{E}_\perp \cdot \boldsymbol{\tau} = \beta \mathbf{B}_\perp \cdot \boldsymbol{\nu}$. Regarding the scaled magnetic field $(\mathbf{B}_\perp, B_z)$, we obtain from (19)–(21)

$$\left(\eta\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial \zeta}\right)\mathbf{B}_\perp \cdot \boldsymbol{\nu} = 0, \quad \eta\int_\Omega \frac{\partial B_z}{\partial t}\,\mathrm{d}\mathbf{x}_\perp + \beta\oint_\Gamma \mathbf{B}_\perp \cdot \boldsymbol{\nu}\,\mathrm{d}l = 0, \quad \int_\Omega \left(\eta\frac{\partial}{\partial t} + \frac{1}{\eta}\frac{\partial}{\partial \zeta}\right) B_z\,\mathrm{d}\mathbf{x}_\perp = 0,$$

whereas, for $\mathbf{x}_\perp \in \Omega, \zeta = 0$, we obtain $\mathbf{E} = \mathbf{0}, \mathbf{B} = \mathbf{B}^e$ and for $\mathbf{x}_\perp \in \Omega, \zeta = Z$, we obtain $\mathscr{E}_\perp = 0$.

4. Asymptotic expansion

To derive a paraxial model, let us now rewrite the scaled Vlasov–Maxwell equations using expansions of the quantities f, ρ, $\mathbf{J}$, $\mathbf{E}$, $\mathbf{B}$, $\mathscr{E}_\perp$, and $\mathbf{F}$ in powers of the small parameter η, namely,

$$f = f^0 + \eta f^1 + \eta^2 f^2 + \cdots, \quad \rho = \rho^0 + \eta\rho^1 + \eta^2\rho^2 + \cdots, \quad \mathbf{J} = \mathbf{J}^0 + \eta\mathbf{J}^1 + \eta^2\mathbf{J}^2 + \cdots,$$
$$\mathbf{E} = \mathbf{E}^0 + \eta\mathbf{E}^1 + \eta^2\mathbf{E}^2 + \cdots, \quad \mathbf{B} = \mathbf{B}^0 + \eta\mathbf{B}^1 + \eta^2\mathbf{B}^2 + \cdots, \quad \mathscr{E}_\perp = \mathscr{E}^0_\perp + \eta\mathscr{E}^1_\perp + \eta^2\mathscr{E}^2_\perp + \cdots,$$
$$\mathbf{F} = \mathbf{F}^0 + \eta\mathbf{F}^1 + \eta^2\mathbf{F}^2 + \cdots.$$

Then, in the scaled Vlasov–Maxwell equations, we replace formally the functions by their asymptotic expansions; we identify the coefficients of η^0, η^1, and so on. We begin by applying these expansions to the Vlasov equation (24). We obtain

– at the zeroth order

$$\frac{\partial f^0}{\partial t} + \mathbf{v}_\perp \cdot \mathbf{grad}_\perp f^0 + v_\zeta \frac{\partial f^0}{\partial \zeta} + \mathbf{F}^0_\perp \cdot \mathbf{grad}_{\mathbf{v}_\perp} f^0 + F^0_z \frac{\partial f^0}{\partial v_\zeta} = 0$$

or

– at the first order

$$\frac{\partial f^1}{\partial t}+\mathbf{v}_\perp\cdot\mathbf{grad}_\perp f^1+v_\zeta\frac{\partial f^1}{\partial\zeta}+\mathbf{F}_\perp^0\cdot\mathbf{grad}_{\mathbf{v}_\perp}f^1+\mathbf{F}_\perp^1\cdot\mathbf{grad}_{\mathbf{v}_\perp}f^0+F_z^0\frac{\partial f^1}{\partial v_\zeta}+F_z^1\frac{\partial f^0}{\partial v_\zeta}=0.$$

More generally, we can expand this equation for powers of η, that is, for the nth order:

$$\frac{\partial f^n}{\partial t}+\mathbf{v}_\perp\cdot\mathbf{grad}_\perp f^n+v_\zeta\frac{\partial f^n}{\partial\zeta}+\sum_{i=0}^{n}\mathbf{F}_\perp^i\cdot\mathbf{grad}_{\mathbf{v}_\perp}f^{n-i}+\sum_{i=0}^{n}F_z^i\frac{\partial f^{n-i}}{\partial v_\zeta}=0. \tag{34}$$

In (34), we use the convention that the negative superscripts vanish.

Hence, for determining the asymptotic expansion of the distribution function f up to a given order n in η, it is sufficient to know the expansion of the transverse and longitudinal electromagnetic forces $\mathbf{F}_\perp$ and F_z, respectively, up to their nth order. Then, using expressions (32) and (33) for the forces, we obtain, with the same convention on the negative superscript,

$$\mathbf{F}_\perp^n=\mathscr{E}_\perp^n+(B_z^{n-1}\mathbf{v}_\perp+v_\zeta\mathbf{B}_\perp^{n-1})\times\hat{\mathbf{e}}_z, \tag{35}$$

$$F_z^n=E_z^n+\mathbf{v}_\perp\cdot(\mathbf{B}_\perp^{n-1}\times\hat{\mathbf{e}}_z). \tag{36}$$

Under these conditions, the asymptotic expressions of these forces are entirely determined if we know the expansions of $\mathscr{E}_\perp$ and E_z up to the nth order and $\mathbf{E}_\perp$,[1] $\mathbf{B}_\perp$, and B_z up to the $(n-1)$th order. Our aim now is to determine equations that characterize these "required" electromagnetic asymptotic fields.

For this purpose, we apply these expansions to the Maxwell equations. We remark that all the terms where a time derivative is involved are multiplied by η; so they do not appear at the zeroth order. Hence, we obtain

– for the Ampere law and the Poisson equation (Equations (25)–(27))

$$\frac{\partial}{\partial\zeta}(\mathscr{E}_\perp^0-(1-\beta^2)\mathbf{E}_\perp^0)-\beta\,\mathbf{curl}_\perp B_z^0=0,$$
$$\mathrm{div}_\perp(\mathscr{E}_\perp^0-(1-\beta^2)\mathbf{E}_\perp^0)=0,$$
$$\mathrm{div}_\perp\mathbf{E}_\perp^0-\frac{\partial E_z^0}{\partial\zeta}=\rho^0,$$

whereas the Faraday law and the absence of monopole equation (Equations (28)–(30)) yield

$$\frac{\partial}{\partial\zeta}(\mathscr{E}_\perp^0\times\hat{\mathbf{e}}_z)+\mathbf{curl}_\perp E_z^0=\mathbf{0},$$
$$\mathrm{curl}_\perp\mathscr{E}_\perp^0=0,$$
$$\mathrm{div}_\perp\mathbf{B}_\perp^0-\frac{\partial B_z^0}{\partial\zeta}=0.$$

Finally, the charge conservation equation (31) leads to

$$\frac{\partial\rho^0}{\partial t}+\mathrm{div}_\perp\mathbf{J}_\perp^0+\frac{\partial J_\zeta^0}{\partial\zeta}=0.$$

On the contrary, at the first order, the terms with a time derivative do appear with an index ^{0}. More precisely, we have, for the Ampere law,

$$\frac{\partial\mathbf{E}_\perp^0}{\partial t}+\frac{1}{\beta}\frac{\partial}{\partial\zeta}(\mathscr{E}_\perp^1-(1-\beta^2)\mathbf{E}_\perp^1)-\mathbf{curl}_\perp B_z^1=-\mathbf{J}_\perp^0,$$
$$\frac{\partial E_z^0}{\partial t}+\frac{1}{\beta}\mathrm{div}_\perp(\mathscr{E}_\perp^1-(1-\beta^2)\mathbf{E}_\perp^1)=J_\zeta^0,$$

[1] $\mathbf{E}_\perp$ does not appear explicitly in forces (Equations (35)–(36)), but it is required to compute B_z.

and for the Faraday law,

$$\frac{\partial \mathbf{B}_\perp^0}{\partial t} + \frac{\partial}{\partial \zeta}(\mathscr{E}_\perp^1 \times \hat{\mathbf{e}}_z) + \mathbf{curl}_\perp E_z^1 = \mathbf{0},$$

$$\frac{\partial B_z^0}{\partial t} + \mathrm{curl}_\perp \mathscr{E}_\perp^1 = 0.$$

The other equations have the same expression simply by replacing index 0 with index 1. More generally, these expansions can be written by general expressions for the nth order. For the electric field, we obtain, still using the convention on the negative superscript,

$$\frac{\partial \mathbf{E}_\perp^{n-1}}{\partial t} + \frac{1}{\beta}\frac{\partial}{\partial \zeta}(\mathscr{E}_\perp^n - (1-\beta^2)\mathbf{E}_\perp^n) - \mathbf{curl}_\perp B_z^n = -\mathbf{J}_\perp^{n-1}, \tag{37}$$

$$\frac{\partial E_z^{n-1}}{\partial t} + \frac{1}{\beta}\mathrm{div}_\perp(\mathscr{E}_\perp^n - (1-\beta^2)\mathbf{E}_\perp^n) = J_\zeta^{n-1}, \tag{38}$$

$$\mathrm{div}_\perp \mathbf{E}_\perp^n - \frac{\partial E_z^n}{\partial \zeta} = \rho^n, \tag{39}$$

whereas, for the magnetic field, we obtain

$$\frac{\partial \mathbf{B}_\perp^{n-1}}{\partial t} + \frac{\partial}{\partial \zeta}(\mathscr{E}_\perp^n \times \hat{\mathbf{e}}_z) + \mathbf{curl}_\perp E_z^n = 0, \tag{40}$$

$$\frac{\partial B_z^{n-1}}{\partial t} + \mathrm{curl}_\perp \mathscr{E}_\perp^n = 0, \tag{41}$$

$$\mathrm{div}_\perp \mathbf{B}_\perp^n - \frac{\partial B_z^n}{\partial \zeta} = 0, \tag{42}$$

and the charge conservation equation is expressed as

$$\frac{\partial \rho^n}{\partial t} + \mathrm{div}_\perp \mathbf{J}_\perp^n + \frac{\partial J_\zeta^n}{\partial \zeta} = 0. \tag{43}$$

For the sake of completeness, we finally present the boundary conditions for $\mathbf{x}_\perp \in \Gamma, \zeta \in (0, Z)$, which are written as

$$\mathbf{E}_\perp^n \cdot \boldsymbol{\tau} = 0, \quad E_z^n = 0, \quad \mathscr{E}_\perp^n \cdot \boldsymbol{\tau} = \beta \mathbf{B}_\perp^n \cdot \boldsymbol{\nu}, \tag{44}$$

$$\left(\frac{\partial \mathbf{B}_\perp^{n-1}}{\partial t} + \beta \frac{\partial \mathbf{B}_\perp^n}{\partial \zeta}\right)\cdot \boldsymbol{\nu} = 0, \int_\Omega \frac{\partial B_z^{n-1}}{\partial t}\,\mathrm{d}\mathbf{x}_\perp + \beta \oint_\Gamma \mathbf{B}_\perp^n \cdot \boldsymbol{\nu}\,\mathrm{d}l = 0, \int_\Omega \left(\frac{\partial B_z^{n-1}}{\partial t} + \beta \frac{\partial B_z^n}{\partial \zeta}\right)\mathrm{d}\mathbf{x}_\perp = 0. \tag{45}$$

As a consequence, we can obtain the following lemmas that characterize different field components at a given order n. First, for the longitudinal electric component E_z^n, we have the lemma that follows.

Lemma 4.0.1. *The nth order component E_z^n is the unique solution to*

$$\begin{cases} \Delta_\perp E_z^n + (1-\beta^2)\dfrac{\partial^2 E_z^n}{\partial \zeta^2} = \\ \qquad \dfrac{\partial}{\partial t}\left(\beta \dfrac{\partial E_z^{n-1}}{\partial \zeta} + \mathrm{curl}_\perp \mathbf{B}_\perp^{n-1}\right) - \dfrac{\partial}{\partial \zeta}(\beta J_\zeta^{n-1} + (1-\beta^2)\rho^n) \quad \mathit{in}\ \Omega, \\ E_z^n = 0 \quad \mathit{on}\ \Gamma. \end{cases} \tag{46}$$

Proof. Inserting (39) into (38) gives

$$\mathrm{div}_\perp \mathscr{E}_\perp^n - (1-\beta^2)\frac{\partial E_z^n}{\partial \zeta} = (1-\beta^2)\rho^n + \beta J_\zeta^{n-1} - \beta \frac{\partial E_z^{n-1}}{\partial t}. \tag{47}$$

Then, differentiating this relation with respect to ζ and adding the $\mathrm{curl}_\perp$ of (40) give the desired result. □

Then, E_z^n and quantities of the previous order $n-1$ are used to compute the pseudo-field $\mathscr{E}_\perp^n$.

Lemma 4.0.2. *The nth-order component $\mathscr{E}_\perp^n$ is the unique solution to*

$$\begin{cases} \operatorname{curl}_\perp \mathscr{E}_\perp^n = -\dfrac{\partial B_z^{n-1}}{\partial t}, \\ \operatorname{div}_\perp \mathscr{E}_\perp^n = (1-\beta^2)\left(\dfrac{\partial E_z^n}{\partial \zeta} + \rho^n\right) + \beta\left(J_\zeta^{n-1} - \dfrac{\partial E_z^{n-1}}{\partial t}\right) \quad in\, \Omega, \\ \displaystyle\oint_\Gamma \mathscr{E}_\perp^n \cdot \boldsymbol{\tau}\, \mathrm{d}l = -\int_\Omega \dfrac{\partial B_z^{n-1}}{\partial t}\, \mathrm{d}\mathbf{x}_\perp. \end{cases} \tag{48}$$

Proof. Since E_z^n is known from (46), obtaining the equations from (41) and (47) is straightforward. The boundary conditions are easily obtained from their expressions above. □

Similarly, we obtain the system that solves the transverse electric field $\mathbf{E}_\perp^n$, which is required to obtain the transverse magnetic field $\mathbf{B}_\perp^n$ (see Lemma 4.0.4).

Lemma 4.0.3. *The nth-order component $\mathbf{E}_\perp^n$ is the solution to*

$$\begin{cases} \mathbf{curl}_\perp(\operatorname{curl}_\perp \mathbf{E}_\perp^n) - (1-\beta^2)\dfrac{\partial^2 \mathbf{E}_\perp^n}{\partial \zeta^2} \\ \qquad = -\dfrac{\partial^2 \mathscr{E}_\perp^n}{\partial \zeta^2} - \mathbf{curl}_\perp\left(\dfrac{\partial B_z^{n-1}}{\partial t}\right) - \beta\dfrac{\partial}{\partial \zeta}\left(\dfrac{\partial \mathbf{E}_\perp^{n-1}}{\partial t} + \mathbf{J}_\perp^{n-1}\right) \quad in\, \Omega, \\ \operatorname{div}_\perp \mathbf{E}_\perp^n = \dfrac{\partial E_z^n}{\partial \zeta} + \rho^n \quad in\, \Omega, \\ \mathbf{E}_\perp^n \cdot \boldsymbol{\tau} = 0 \quad on\, \Gamma. \end{cases} \tag{49}$$

Proof. Computing $\operatorname{curl}_\perp \mathscr{E}_\perp^n := \operatorname{curl}_\perp(\mathbf{E}_\perp^n - \beta \mathbf{B}_\perp^n \times \hat{\mathbf{e}}_z)$ and using (41) and (42) give

$$\frac{\partial B_z^n}{\partial \zeta} = -\frac{1}{\beta}\frac{\partial B_z^{n-1}}{\partial t} - \frac{1}{\beta}\operatorname{curl}_\perp \mathbf{E}_\perp^n. \tag{50}$$

Combining (50) with the derivative of (37) with respect to ζ gives the result, where $\mathscr{E}_\perp^n$ is known from (48). □

The last two results deal with the magnetic field. First, we have for the transverse component the following lemma.

Lemma 4.0.4. *The nth-order component $\mathbf{B}_\perp^n$ is the unique solution to*

$$\begin{cases} \operatorname{curl}_\perp \mathbf{B}_\perp^n = \dfrac{\partial E_z^{n-1}}{\partial t} + \beta \operatorname{div}_\perp \mathbf{E}_\perp^n - J_\zeta^{n-1}, \\ \operatorname{div}_\perp \mathbf{B}_\perp^n = -\dfrac{1}{\beta}\left(\operatorname{curl}_\perp \mathbf{E}_\perp^n + \dfrac{\partial B_z^{n-1}}{\partial t}\right) \quad in\, \Omega, \\ \displaystyle\oint_\Gamma \mathbf{B}_\perp^n \cdot \boldsymbol{\nu}\, \mathrm{d}l = -\dfrac{1}{\beta}\int_\Omega \dfrac{\partial B_z^{n-1}}{\partial t}\, \mathrm{d}\mathbf{x}_\perp \quad on\, \Gamma. \end{cases} \tag{51}$$

Proof. Computing $\operatorname{div}_\perp \mathscr{E}_\perp^n := \operatorname{div}_\perp(\mathbf{E}_\perp^n - \beta \mathbf{B}_\perp^n \times \hat{\mathbf{e}}_z)$ in combination with (38) gives one of the equations. The second is obtained by combining $\operatorname{curl}_\perp \mathscr{E}_\perp^n$ with (41), where the boundary condition is (45). □

Finally, the longitudinal component B_z^n is entirely determined by the magnetic field and is characterized by the following lemma.

Lemma 4.0.5. *The nth-order component B_z^n is the unique solution to*

$$\begin{cases} \dfrac{\partial B_z^n}{\partial \zeta} = \mathrm{div}_\perp \mathbf{B}_\perp^n \quad \textit{in}\ \Omega, \\ \displaystyle\int_\Omega \frac{\partial B_z^n}{\partial \zeta} \mathrm{d}\mathbf{x}_\perp = -\frac{1}{\beta}\int_\Omega \frac{\partial B_z^{n-1}}{\partial t} \mathrm{d}\mathbf{x}_\perp. \end{cases} \tag{52}$$

Proof. As $\mathbf{B}_\perp^n$ is known from (51), the equation is given by (42). The boundary condition is straightforward to obtain. □

5. Paraxial model

We are now ready to introduce the paraxial model, which provides an approximation of the distribution function f, which is formally nth order accurate in η. This means that the asymptotic expansions of f in the Vlasov–Maxwell model and in the paraxial model coincide up to the order n in η. We derive this model by going back to the physical variables by using the scaling factors as introduced in Section 3.[2]

To begin with, let us derive the equations satisfied by E_z^n. Assuming the knowledge of the data $(\rho, \mathbf{J})$ and of the fields up to the order $n-1$, from Lemma 4.0.1, we obtain

$$\begin{cases} \Delta_\perp{}^2 E_z^n + (1-\beta^2)\dfrac{\partial^2 E_z^n}{\partial \zeta^2} = \dfrac{1}{c}\left[\dfrac{\partial}{\partial t}\left(\beta \dfrac{\partial E_z^{n-1}}{\partial \zeta} + \mathrm{curl}_\perp c\mathbf{B}_\perp^{n-1}\right)\right. \\ \qquad\qquad \left. -\dfrac{1}{\varepsilon_0}\dfrac{\partial}{\partial \zeta}(\beta J_\zeta^{n-1} + (1-\beta^2)c\rho^n)\right] \quad \text{in}\ \Omega, \\ E_z^n = 0 \quad \text{on}\ \Gamma. \end{cases} \tag{53}$$

Let us now deal with the transverse electric field. From E_z^n, we can compute $\mathscr{E}_\perp^n$ by solving a quasi-static model, following Lemma 4.0.2, which is written as

$$\begin{cases} \mathrm{curl}_\perp \mathscr{E}_\perp^n = -\dfrac{\partial B_z^{n-1}}{\partial t}, \\ \mathrm{div}_\perp \mathscr{E}_\perp^n = (1-\beta^2)\left(\dfrac{\partial E_z^n}{\partial \zeta} + \dfrac{\rho^n}{\varepsilon_0}\right) + \dfrac{\beta}{\varepsilon_0 c} J_\zeta^{n-1} - \dfrac{\beta}{c}\dfrac{\partial E_z^{n-1}}{\partial t} \quad \text{in}\ \Omega, \\ \displaystyle\oint_\Gamma \mathscr{E}_\perp^n \cdot \boldsymbol{\tau}\, \mathrm{d}l = -\int_\Omega \frac{\partial B_z^{n-1}}{\partial t} \mathrm{d}\mathbf{x}_\perp. \end{cases} \tag{54}$$

In our paraxial model, even if $\mathbf{E}_\perp^n$ does not appear explicitly in the expression of the forces, there is yet a need to compute it as it is required to obtain B_z. Following Lemma 4.0.3, we have

$$\begin{cases} \mathbf{curl}_\perp(\mathrm{curl}_\perp \mathbf{E}_\perp^n) - (1-\beta^2)\dfrac{\partial^2 \mathbf{E}_\perp^n}{\partial \zeta^2} \\ \qquad = -\dfrac{\partial^2 \mathscr{E}_\perp^n}{\partial \zeta^2} - \mathbf{curl}_\perp\left(\dfrac{\partial B_z^{n-1}}{\partial t}\right) - \dfrac{\beta}{c}\dfrac{\partial}{\partial \zeta}\left(\dfrac{\partial \mathbf{E}_\perp^{n-1}}{\partial t} + \dfrac{\mathbf{J}_\perp^{n-1}}{\varepsilon_0}\right) \quad \text{in}\ \Omega, \\ \mathrm{div}_\perp \mathbf{E}_\perp^n = \dfrac{\partial E_z^n}{\partial \zeta} + \dfrac{\rho^n}{\varepsilon_0} \quad \text{in}\ \Omega, \\ \mathbf{E}_\perp^n \cdot \boldsymbol{\tau} = 0 \quad \text{on}\ \Gamma. \end{cases} \tag{55}$$

[2]Recall that we dropped $'$ in Section 3.

This allows us to compute now the transverse magnetic field $\mathbf{B}_\perp^n$, following Lemma 4.0.4, by solving the quasi-static system of equations

$$\begin{cases} \mathrm{curl}_\perp \mathbf{B}_\perp^n = \dfrac{1}{c^2}\dfrac{\partial E_z^{n-1}}{\partial t} + \dfrac{\beta}{c}\mathrm{div}_\perp \mathbf{E}_\perp^n - \mu_0 J_\zeta^{n-1} \quad \text{in } \Omega, \\ \mathrm{div}_\perp \mathbf{B}_\perp^n = -\dfrac{1}{\beta c}\left(\mathrm{curl}_\perp \mathbf{E}_\perp^n + \dfrac{\partial B_z^{n-1}}{\partial t}\right) \quad \text{in } \Omega, \\ \displaystyle\oint_\Gamma \mathbf{B}_\perp^n \cdot \boldsymbol{\nu}\, \mathrm{d}l = -\frac{1}{\beta c}\int_\Omega \frac{\partial B_z^{n-1}}{\partial t}\mathrm{d}\mathbf{x}_\perp \quad \text{on } \Gamma. \end{cases} \tag{56}$$

Finally, we can obtain the longitudinal magnetic field of order n by solving the simple equation deduced from Lemma 4.0.5:

$$\begin{cases} \dfrac{\partial B_z^n}{\partial \zeta} = \mathrm{div}_\perp \mathbf{B}_\perp^n \quad \text{in } \Omega, \\ \displaystyle\int_\Omega \frac{\partial B_z^n}{\partial \zeta}\mathrm{d}\mathbf{x}_\perp = -\frac{1}{\beta}\int_\Omega \frac{\partial B_z^{n-1}}{\partial t}\mathrm{d}\mathbf{x}_\perp. \end{cases} \tag{57}$$

We can summarize our main result in the following theorem.

Theorem 5.1. *Equations (53)–(57) determine the triple* $(\mathbf{E}^n, \mathbf{B}^n, \mathscr{E}_\perp^n)$ *from the data* $(\rho, \mathbf{J})$ *and* $(\mathbf{E}^l, \mathbf{B}^l, \mathscr{E}_\perp^l)$ *for* $0 \le l \le n-1$ *in a unique way. Moreover, the paraxial model provides an approximation of the distribution function* f*, which is formally* n*th order accurate in* η*. Namely, the asymptotic expansions of* f *in the Vlasov–Maxwell model and in the paraxial model coincide up to the* n*th order in* η*.*

The paraxial model proposed here is hierarchical and closed for each order: The zeroth-order fields allow us to solve the first order and so on. In addition, the nth-order fields are required only for $\mathscr{E}_\perp$ and E_z, whereas it is sufficient to know the other fields up to the $(n-1)$th order. Last but not least, note that as the time derivatives are only on the right-hand side, the model is quasi-static and not *explicitly* time-dependent. From a computational point of view, this point is very important. Indeed, the underlying idea is to use a particle-in-cell method, which means solving the Vlasov equation by a particle method and the electromagnetic fields by a grid method (finite differences, finite element, etc.). In this context, it is well known that the difficult point from a computational point of view is the time-dependent aspect of the equations, which is sometimes unavoidable. In other words, solving the full system of time-dependent Vlasov–Maxwell equations can be extremely expensive in terms of computation time. One advantage of the proposed model is that it is not time-dependent even if it approximates the time-dependent system of equations at a given order n.

6. Conclusion

In this note, we proposed a new family of paraxial asymptotic models that approximate the non-relativistic Vlasov–Maxwell equations. It has been derived by introducing a small parameter $\eta = \overline{v}/c$, and it is nth order accurate for $n \in \mathbb{N}$. Under these conditions, we can easily choose the complexity of the model we need to use according to the expected accuracy. In addition, this family of models is simpler than the Vlasov–Maxwell equations—for instance, they are not time-dependent but only static or quasi-static—which allows us to implement simple and efficient numerical schemes such as particle-in-cell techniques. Hence, this approach would be very powerful in its ability to obtain fast and easy-to-implement algorithms.

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