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Historical article / *Article historique*

# About Tresca's Memoirs on the fluidity of solids (1864–1870)

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**Abstract.** One hundred and fifty years ago, Henri-Édouard Tresca submitted a series of Memoirs to the French Academy of Sciences, which were devoted to recording the extensive series of experiments he had carried out. In these experiments, he had investigated punching, rolling, forging, stamping and extruding processes of various materials, and he had definitely identified a phenomenon that he called the “fluidity” of solids subjected to very high pressures. In addition to the precise description of his experiments, he proposed a mechanical theory for the observed phenomenon. In this theory, the concepts now called yield criterion and plastic flow rule were introduced for the first time.

**Keywords.** History of mechanics, Plasticity, Tresca's experiments, Tresca's theoretical approach, Tresca's legacy.

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## 1. Henri-Édouard Tresca (1814–1885)

As a brief biography, we may mention that Tresca graduated from the *École Polytechnique* in 1833 and *École des ponts et chaussées* in 1835. After practicing civil and construction engineering, he became a talented professor of applied mechanics in prestigious French engineering schools such as the *Conservatoire des arts et métiers* in Paris, where he was also in charge of setting up a laboratory for industrial mechanics. It is worth mentioning that in this laboratory, among many achievements, he invented and actually realised the X-shape standard metre that was to be used as a global reference. It was also there that he carried out the extensive series of experiments recorded in the Memoirs he sent to the academy about 150 years ago. He was such a renowned scientist that Gustave Eiffel wanted his name to be written on the great frieze of the north face of the 300-metre tower he had built for the 1889 World Exhibition:

“Pour exprimer d'une manière frappante que le monument que j'élève sera placé sous l'invocation de la Science, j'ai décidé d'inscrire en lettres d'or sur la grande frise du premier étage et à la place d'honneur, les noms des plus grands savants qui ont honoré la France depuis 1789 jusqu'à nos jours”.<sup>1</sup>

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<sup>1</sup>“To express in a striking way that the monument I am raising will be placed under the invocation of Science, I have decided to write in golden letters on the great frieze of the first floor and in the place of honour, the names of the greatest scholars who have graced France from 1789 till the present day”.

## 2. Tresca's Memoirs and results

“Tresca's paper on the flow of solids may approximately be regarded as the birth of the mathematical theory of plasticity” (Koiter [1]).

From a simplified historical viewpoint, Tresca's scientific contribution can be considered as filling a gap between Coulomb's stability analyses [2] and Cauchy's elasticity theory [3]. In the first study, based on experimental results, stability analyses of civil engineering structures were performed from the only knowledge of the resistance of the constituent material. In the second study, reversible behaviour of materials was described through the mathematical theory of elasticity. Both approaches had proven highly efficient regarding civil and construction engineering issues but could not be applied to issues such as metal forming and processing.

As a genuine physicist, Tresca performed, described and recorded his experiments (Figure 1), with the utmost precision, in not less than six Memoirs [4–12] addressed to the Academy of Sciences from 1864 to 1870. In the Memoirs, he definitely identified three stages in the behaviour of the various materials he had been testing (lead, copper, iron, tin, zinc, ceramics, modelling wax etc.):

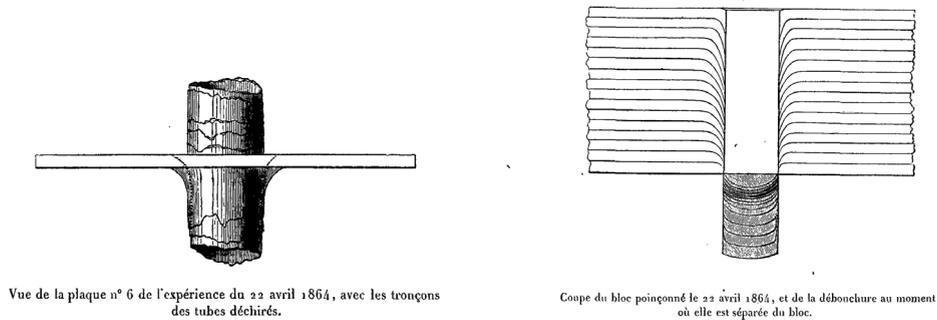
- the first elastic stage, which he called “perfect elasticity”, which corresponds to linear elasticity;
- then, “imperfect elasticity”, characterised by a non-linear relationship and where he observed partly permanent deformation; and
- finally, the phenomenon, which he called the “fluidity” of materials subjected to very high pressures, that was to be his main topic of interest. “This state of fluidity obviously follows, in compression phenomena, the period of perfect elasticity and even the period of imperfect elasticity, which is characterized by a change, partly permanent, in the length of the compressed fibres” (Ref. [10], p. 776).<sup>2</sup> “The period of fluidity would thus take place after the period of perfect elasticity, during which the resistance is proportional to the elastic spread, and following the period of imperfect elasticity, which is characterized by the non-proportionality between spread and resistance”. (Ref. [10], p. 824).<sup>3</sup> (In this last sentence, the “spread” Tresca is referring to relates to the displacement between “neighbouring molecules”).

In addition to his phenomenological approach, Tresca proposed a mechanical theory for the fluidity phenomenon, where he observed the constancy of the stress state when this phenomenon took place and defined what he called the fluidity resistance: “Beyond this limit, we propose to consider, for the state of fluidity, that the forces that are developed are absolutely constant and entirely independent of the relative displacements, which amounts to admitting that they can always be evaluated by means of a resistance coefficient  $K$  per square meter or per square centimetre, this coefficient  $K$  remaining the same for the assessment of any molecular deformation

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<sup>2</sup>“Cet état de fluidité fait évidemment suite, dans les phénomènes de compression, à la période d'élasticité parfaite et même à la période d'élasticité imparfaite, qui est caractérisée par une modification, en partie permanente, dans la longueur des fibres comprimées”.

<sup>3</sup>“La période de fluidité viendrait ainsi prendre place à la suite de la période d'élasticité parfaite, pendant laquelle la résistance est proportionnelle à l'écartement élastique, et à la suite de la période d'élasticité imparfaite, qui est caractérisée par la non-proportionnalité entre l'écartement et la résistance”.



**Figure 1.** Excerpts from [4].

developed during the fluidity period” (Ref. [10], p. 777).<sup>4</sup> “Shear resistance is equal to fluidity resistance” (Ref. [10], p. 827).<sup>5</sup>

Moreover, regarding the “flow” of matter, Tresca established that it obeyed geometrical conditions, among which volume preservation: “We have shown that matter can flow under the action of this pressure and that the direction of flow is subjected to geometrical conditions that could already be expressed by formulas in several circumstances” (Ref. [10], p. 618).<sup>6</sup> “The already indicated verification of the preservation of primitive density in the openings resulting from punching could leave no doubt about the constancy of the total volume of the metal engaged in each of our experiments” (Ref. [4] p. 771).<sup>7</sup>

### 3. Saint-Venant’s contribution

As Tresca was not yet a member of the Académie des sciences, his Memoirs had to be approved before summaries, written by the author, were published in the *Comptes rendus* [5, 7, 9, 11, 12]. Saint-Venant was one of the commissioners in charge of reviewing them and, in 1870, he published a Note, whose rather long title may be considered as a perfect definition of what is now called the plastic behaviour of materials [13, 14].<sup>8</sup> “On the equations of internal movements operated in ductile solids beyond the limits where elasticity could bring them back to their initial state”. In this Note, considering the particular case of a two-dimensional problem, Saint-Venant derived five equations that governed what he called “*hydrostéréodynamique*” or “*plasticodynamique*”. They consisted of two differential equations of dynamics, one equation expressing the fluidity condition, another equation expressing incompressibility of matter and a last one expressing the coincidence of the principal directions of the two-dimensional strain-rate and stress tensors.

<sup>4</sup>“Au delà de cette limite, nous proposons de considérer, pour l’état de fluidité, les forces développées comme absolument constantes et entièrement indépendantes des déplacements relatifs, ce qui revient à admettre qu’elles pourront toujours être évaluées au moyen d’un coefficient K de résistance par mètre carré ou par centimètre carré, ce coefficient K restant le même pour l’appréciation de toute déformation moléculaire développée dans la période de fluidité”.

<sup>5</sup>“La résistance au cisaillement est égale à la résistance de fluidité”.

<sup>6</sup>“Nous avons montré que la matière peut s’écouler sous l’action de cette pression, et que le sens de l’écoulement est assujéti à des conditions géométriques qui, dans plusieurs circonstances déjà, ont pu être exprimées par des formules”.

<sup>7</sup>“La vérification que nous avons indiquée déjà, de la conservation de la densité primitive dans les débouchures provenant du poinçonnage ne pouvait nous laisser aucun doute sur la constance du volume total du métal engagé dans chacune de nos expériences”.

<sup>8</sup>“Sur les équations des mouvements intérieurs opérés dans les solides ductiles au delà des limites où l’élasticité pourrait les ramener à leur premier état”.

Thus, as stated by Koiter, the cornerstones of plastic modelling, which consist of the yield criterion and the plastic flow rule, were laid for the first time.

#### 4. An energetic viewpoint

According to Timoshenko [15], the origin of the criterion usually called the von Mises criterion seems to start with a letter written by Maxwell to Lord Kelvin in 1856 saying: “I have strong reasons for believing that when the strain energy of distortion reaches a certain limit, then the element will begin to give way”. Beltrami’s contribution [16] proposed that the total linear elastic energy be retained as a criterion, with the critical energy being estimated from a traction test. Huber [17] observed that this criterion could not account for the insensitivity of ductile materials to hydrostatic stress. This seems to have triggered von Mises to write the criterion in terms of the deviatoric elastic energy.

#### 5. The flow rule issue

We observed that the flow rule issue was already addressed in Tresca’s Memoirs and, subsequently, in Saint-Venant’s contribution although the set of equations in Saint-Venant’s Note remains incomplete and requires the addition of the *consistency equation*. As stated by Hill [18], a landmark contribution to that topic is due to von Mises [19] with the concept of a plastic potential: “In 1928, von Mises, in a brilliant paper on the plastic distortion of crystals, introduced the fruitful concept of a plastic potential”.<sup>9</sup>

With the yield criterion being identified as the plastic potential, it follows that the plastic strain rate is collinear with the outward normal to the yield surface with the *consistency equation*, written in its specific form corresponding to a perfectly plastic material, accounting for the irreversibility of the material behaviour.

Obviously, a problem arises when the yield criterion is not continuously differentiable: The gradient of the yield criterion is not uniquely defined, which is the case, for instance, along the edges of the Tresca yield surface. In 1948, Hill [22] proposed an alternative formulation to the concept of a plastic potential in the form of the *principle of maximum plastic work* (viz., [23]). This statement implies that the yield criterion is convex and that the plastic strain rate belongs to the convex cone of outward normals to the yield surface at the concerned stress point, be it regular or singular. Materials that obey the principle of maximum plastic work have sometimes been called *standard* plastic materials.

The problem was then addressed by Koiter in 1953 [24], who completed the answer through the *multiple plastic potential* theory, where the consistency equation means that the plastic strain rate is a convex combination of *activated* elementary plastic potentials. It results in an orthogonality relationship between the stress rate and the plastic strain rate. This relationship had, independently, already been introduced as a postulate, valid for any non-continuously differentiable convex plastic potential, by Drucker in 1951 [25].

#### 6. A convex analysis viewpoint

Convexity associated with normality explains why the plastic model can be considered from a convex analysis viewpoint. This was already implicitly present in Prager’s work (e.g., [26]).

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<sup>9</sup>Cf. Geiringer’s contribution [20]. Incidentally, this gives us an opportunity to recall the role played by German scientists in exile in Turkey in building up the theory of plasticity: Richard von Mises, William Prager and Hilda Geiringer (cf. [21]).

Moreau's seminal contributions [27–29] provided the fundamental concepts to be implemented such as the subgradient and subdifferential of a convex function (or functional). Within this mathematical formalism, the principle of maximum plastic work results in a plastic flow rule where the plastic strain rate belongs to the convex cone of outward normals generated by the subdifferential of the plastic potential, which also provides the expression of the consistency equation, leaving aside Drucker's postulate.

## 7. More about Drucker's postulate

Within the framework of convex analysis, it is possible to revisit Drucker's postulate in the context of the solution of quasistatic elastoplastic processes with the results by Brezis [30] stated as follows: At any instant of time, the *solution* stress rate and plastic strain rate fields comply with Drucker's orthogonality relationship without it being imposed on the plastic flow rule. Hence, the orthogonality relationship is no longer a postulate within this context but a result from the solution process.

## 8. (Work or strain) hardening

In Tresca's description of the three stages in the behaviour of the materials he was testing, "imperfect elasticity", characterised by a non-linear relationship, was not a focal issue. In fact, this stage corresponds to what is presently known as a (work or strain)-hardening elastoplastic behaviour. An important contribution is due to Bauschinger [31], with the phenomenon currently being associated with his name: the Bauschinger effect observed in one-parameter loading processes (e.g., tension–compression tests).

Generally speaking, the work- or strain-hardening phenomenon can be defined as the change in size and shape of the boundary of the current elastic domain that is concomitant with the plastic deformation of the material element. This can be symbolically described through the introduction of hardening parameters that characterise the hardening state of the material. The issue then is to define these parameters and the corresponding hardening rule as a counterpart to the plastic flow rule. Two landmark contributions usually referred to are as follows.

*Isotropic hardening:* Here the hardening parameter is a scalar and generates an isotropic expansion of the current elastic domain following the stress point of the material (Taylor and Quinney [32]).

*Kinematic hardening:* Here the hardening parameter is the "back stress" tensor, which governs the translation of the current elastic domain without any deformation, as it is driven by the loading point along an increasing loading arc (Melan [33], Prager [34, 35]), with the hardening rule proposed by Melan.

Although these two historical models do not account for all aspects of experimental results, they have been used, even in computational software, often in the form of a mixed model "isotropic–kinematic", thus providing relevant results for practical applications. A large body of literature has been devoted to the development of (work or strain)-hardening models, based on numerous reliable experimental data and theoretical analyses, as required by industrial applications.

Assuming the existence of a countable number of hardening parameters, Halphen and Nguyen [36] introduced the concept of *generalised standard materials*, whose basis lies in describing the current elastic domain of the material by a convex function  $f(\underline{\sigma}, \underline{\alpha})$  in the space  $\{\underline{\sigma} \times \underline{\alpha}\}$  defined by the stress tensor  $\underline{\sigma}$  and hardening parameters  $\underline{\alpha}$ . In addition, it is assumed that the hardening parameters are derived from a set of internal parameters  $\underline{\beta}$  in the form of the

components of the gradient of a *scalar convex function*  $\varphi(\underline{\beta})$ . The plastic flow and hardening rules are then derived from  $f(\underline{\sigma}, \underline{\alpha})$  through a generalised plastic potential principle. This model turned out to be a convenient approach for well-specified practical applications and processes (e.g., in computer codes). An important outcome relates to a particular class of generalised standard materials, for which it is possible to prove existence and uniqueness theorems, which are similar to those obtained in the case of standard elastic and perfectly plastic materials, for the solution to quasistatic elastoplastic loading processes within the small perturbation hypothesis framework.

## 9. Conclusion

As a brief sketch of landmark contributions to the classical theory of plasticity in the form it has been commonly applied to engineering problems, this contribution is meant to show how mathematical models were progressively built up following the seminal experimental results presented by Tresca. Illustrating Tresca's legacy, it is restricted to the *small perturbation framework* and does not mention prestigious contributions to the theory of *large deformation* plasticity.

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