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Correspondence between de Saint-Venant and Boussinesq 5: Viscosity and hydraulic resistance

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Abstract. Fluid viscosity is a main feature of fluids; an inviscid fluid does not exist even though a large number of theories has been advanced for flows of such fluids. The velocity of fluid flow may considerably be reduced due to the presence of fluid viscosity both for laminar and turbulent flows. The letters exchanged between de Saint-Venant (dSV) and Boussinesq mainly refer to laminar flow. The most peculiar statement of dSV is that, if the flow in a typical lowland river would be laminar, its velocity would be larger than the speed of sound. It is evident that this statement is wrong because laminar flow has stringent limitations typically expressed by the Reynolds number.

The hydraulic resistance is another peculiar feature in hydrodynamics, given that several theories have been advanced which do not at all reflect the everyday experience. One of these is the d'Alembert paradox, stating that the resistance is equal to zero for a steady flow of an inviscid and incompressible fluid. Typically, a body suspended in a large pipe is considered. Applying the momentum equation in the axial direction sufficiently up- and downstream of the body, the resistance would indeed become zero, as occurs for potential flows. Again, this is far away from everybody's experience, such as for a swimmer or a walker under heavy wind.

This paper mainly explores the letters exchanged between the two late 19th century scientists dSV and Boussinesq. Their considerations indirectly advanced the turbulence theory later developed by Boussinesq, based on the Boussinesq turbulent exchange coefficient. It also rectifies the role of dSV in improving the fundamental equations currently referred to as the Navier–Stokes equations. Both aspects of turbulence and hydraulic resistance are by now still under considerable research activity.

Keywords. Académie des Sciences, History of hydraulics, Hydrodynamics, Institut de France, Resistance, Viscosity.

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1. Introduction

The d'Alembert paradox corresponds to a contradiction reached by the French mathematician d'Alembert [1]. For the irrotational motion of an inviscid and incompressible fluid, that is, potential flows, he proved that the drag force is zero on a body moving with constant velocity relative to the fluid. This finding is in direct contradiction to the observation of the substantial drag on bodies moving relative to fluids, such as air or water, especially at high velocities. It is a particular example of the reversibility paradox.

Accordingly, fluid mechanics was thus discredited by the engineers from the very start. It resulted in an unfortunate split between the field of hydraulics, observing phenomena, which could not be explained on the one hand, and theoretical fluid mechanics explaining phenomena, which could not be observed on the other hand.

The occurrence of the paradox is due to the neglect of viscous effects. Based on scientific experiments, there were huge advances in the theory of viscous fluid friction during the 19th century. This development culminated in the discovery and description of thin boundary layers by Ludwig Prandtl in 1904. Even at very high Reynolds numbers, these layers remain due to the viscous forces. They cause friction drag on streamlined bodies, and flow separation along with a low-pressure wake for bluff bodies, leading to form drag.

From a practical point of view, the paradox is considered to be solved along the lines of Prandtl. A formal mathematical proof is lacking, and difficult to provide, as in so many other fluid-flow problems involving the Navier–Stokes equations by which viscous flows are generally described.

de Saint-Venant (dSV) [2] provided the first steps toward solving the paradox, by including viscous fluid friction. He states: "Another result emerges if, instead of considering an ideal fluid . . . , a real fluid is taken. It is composed of a finite number of molecules exerting in its state of motion unequal pressure forces or forces having components tangential to the surface elements through which they act. This is referred to as fluid friction, a denotation given since Descartes and Newton until Venturi."

Stokes [3] determined the drag on a sphere in Stokes flow, known as Stokes' law. This flow corresponds to the low Reynolds-number limit of the Navier–Stokes equations describing the motion of a viscous liquid. The details will be considered below in discussing the letters exchanged in the Correspondence.

However, when the problem is put into non-dimensional form, the viscous Navier–Stokes equations converge for increasing Reynolds numbers toward the inviscid Euler equations. This suggests that the flow should converge toward the inviscid solutions of the potential flow theory, involving zero drag of the d'Alembert paradox. No evidence is found in experimental measurements of drag and flow visualizations. This again raised questions concerning the applicability of fluid mechanics in the second half of the 19th century.

Prandtl [4] suggested that the effects of a thin viscous boundary layer could be the source of substantial drag. He put forward the idea that, at high Reynolds numbers, a no-slip boundary condition causes a strong variation of the flow speeds over a thin layer near the surface of the body. This generates vorticity and viscous dissipation of kinetic energy in the boundary layer. The energy dissipation, which is lacking in the inviscid theories, results for bluff bodies in flow separation. The low pressure in the wake region causes form drag, and this can be larger than the friction drag due to the viscous shear stress at the wall.

Prandtl hypothesized that the viscous effects are important in boundary layers adjacent to solid boundaries, yet that viscosity has no relevance outside of these. The full problem of viscous flow, described by the non-linear Navier–Stokes equations, is mathematically not solvable. However, using his hypothesis and experimental data, Prandtl derived an approximate model for the flow inside the boundary layer, called boundary-layer theory. In addition, he suggested that the flow outside the boundary layer could be treated by using the inviscid flow theory. The boundary-layer theory is amenable to the method of matched asymptotic expansions for deriving approximate solutions. The simplest case involves a flat plate parallel to the approach flow, for which the boundary-layer theory results in the (friction) drag, whereas all inviscid flow theories predict zero drag. Prandtl's theory applies also to streamlined bodies like airfoils, where, in addition to the surface-friction drag, there is a form drag. Form drag is due to the effect of the boundary layer and a thin wake on the pressure distribution around the airfoil. Given this short account on fluid resistance, the reader realizes the important role of dSV in this still vital question of fluid mechanics. As in other fields of research, he felt unable to present the complete theory, so that Joseph Boussinesq (JB) published dSV's results posthumously [5].

Part 5 of the Correspondence between dSV and JB has the same restrictions as the previous papers. The first author (WHH) had to sign a document in which publication of any original letter is excluded. The letters available at the Academy of Sciences, Paris, were thus photographed first, then transcribed, and finally translated into English. The authors were allowed to use the original sketches shown in this paper, yet the lettering had to be improved for better reading. The following refers exclusively to letters exchanged between dSV and JB dealing with the effects of fluid viscosity and the hydraulic resistance, two eminently important issues in hydromechanics still today.

2. Letters from 1872 to 1885

The first letter dealing with friction and viscosity was written by dSV to JB on July 12, 1868. Letter 10 reads: "I find that Navier [6] was wrong when claiming his formulas suitable only for viscous fluids. Perhaps he also spoke only about an order of the approval by the Academy of which he was not yet a member. Its reviewers were no doubt all supporting the idea that the laws of the so-called perfect viscous fluids were completely represented by the equations of d'Alembert found at the end of the treaty of Poisson's mechanics. Viscosity may increase the inner friction of fluids or the tangential components of pressures through plane elements, but it is not necessary for this friction to be a component with location. The air is not viscous [Sic.: Obviously a mistake of dSV], and yet uniform movement in channels and pipes, the phenomena of divergent orifices, lateral addition of flow, erosive cements etc. prove that friction is considerable. The air behaves like water. [...]. In any fluid that is considered to be constituted by a finite number of molecules, friction, in the state of motion, is at least a property as essential as pressure. There might be, in a continuous mass, no friction; but a fluid so constituted would make a sort of physical nonsense, for it would be indeterminate. There were, in its mass, undeformed processes to these maneuvers to a host of others in the least circumstances. In this regard, I send you a Memoir on the Issue of Continuous Masses, and a Notice on Du Buat where a note on the same subject is at the bottom of pages 48 to 49. I assume that you do not share the opinions that I express (which I think you would adopt). I think it will show you that fluid friction does not stop at this viscosity contained in syrup, in heavy oil, in the stupidity to sag on itself a kind of notion between fluidity and solidity.

If you maintained the description of viscosity in your title and elsewhere, I think the reviewer would be an observer with a reserve for his approval. Therefore, could you not change the title for example to: The influence of inner and outer frictions (or tangential actions) in the movement of fluids?"

The answer to this letter is given by JB in Letter 11, dated July 14, 1868. It reads: "Sir and dear Master, I have not yet received the two memoirs that you will send me as referred to in your letter of July 12. I think I must tell you right away that there is no opposition to you in my manuscript dealing with the friction of fluids. If you have before your eyes, not the copy, but the original text of my work, you will see that I use the term friction everywhere instead of the term viscosity. I had believed, in the copy, that I should use the latter, because, based on Mr. Bresse [Sic.: Charles Bresse, 1822–1883, French civil engineer], who keeps using it, I thought he was privileged for its use. I would therefore gladly replace it with the term friction, which reflects much better my thoughts, because water, air, etc., endowed with friction coefficients of sensitive size, are nevertheless devoid of viscosity, this quasi-solidity which, if manifested in an appreciable way, would not allow them to obey the laws of hydrostatic pressure."

Here, the two notions "fluid viscosity" and "friction" are considered. Friction is a more general term given that it includes also the effect of fluid viscosity. Note that dSV was one of the first having distinguished between internal and external friction. The first action is due to fluid viscosity, manifested by turbulence. The second action is due to boundary roughness varying from extremely smooth as for glass pipes to extremely rough in mountain brooks. For uniform free surface flow, the two effects are represented by the Colebrook–White formula set up in 1937.

In Letter 24, dated November 25, 1868, dSV writes the following to JB: "Dear Sir, in writing a 'Note on hydrodynamics and its future', where I intend to say that for all slow and steady movements without swirl, it is appropriate to stick to Navier's equations [Sic.: Navier (1822)] [6]. In addition, the assumption of zero speed against wet walls applies. However, I find myself stopped by a remark you made at the conclusion of your memoir 'On the role of viscosity (or inner friction) of fluids', of which I do not have the first draft."

"You say [Sic.: Boussinesq (1867)] [7]: 'Take the second as time unit, the millimeter for the length unit and the milligram for the force unit. Then, the equations of Mr. Poiseuille [Sic.: Jean Poiseuille 1797–1869, French scientist] result for water in $H = 1/481 = 0.0001937$ [this value is deduced from Equation (19) (of slope $= (\pi/8H)(-\mathrm{d}p/\mathrm{d}y)R^4$) compared to that given by Mr. Jamin on page 329, paragraph I of his course in physics]." If the length unit becomes the same as the force unit per kilogram, this value remains the same in an open, semi-circular channel with a radius of 1 m. Tilted only by 0.0029 m, Equation (12) $[v = (L/2)[a^2b^2/(a^2+b^2)](1-(x^2/a^2)-(z^2/b^2)]$ provides for the steady, axial flow velocity with $L = \rho g \tan\alpha/H = 7'481'000$ on 1 and $a = b = 1$, $v = (7481,000{:}1)/4 = 544$ m/s. Considerable velocities are therefore required for friction to balance gravity in a liquid of these dimensions when u, v, w vary continuously from point to point. However, long before such speeds have been established, the smallest swirls produced by the inequalities of the bottom must produce a loss of force capable to neutralize the acceleration due to the action of gravity or variations in pressure."

"I do not have the details of Mr. Poiseuille's experiences. Could you, dear Sir, tell me the numbers that you have extracted from the two so that I can recalculate and extend it to other examples? Can you state limit diameters not to be exceeded for *regular* movement and based on the assumption $v = 0$ against the walls? I believe, moreover, that even against *very polished walls*, the friction of the other layers on the continuous layer, provided the speeds are large, implore to the elements of this layer of rotations that detach them in small vortices floating in the remainder of the fluid. Accordingly, a certain speed of finite translation is established against the walls. And from where does $L = \rho g \tan\alpha/H$ originate (with ρg doubtless the specific weight of water)?" There was no immediate response by JB on these questions, however.

Both JB and dSV clearly realized that the above example is beyond the possibilities of realistic flows. It was only in 1883 when Osborne Reynolds (1842–1912, British physicist) introduced the so-called Reynolds number. For pipe flow, he proposed as the upper limit of laminar flow a Reynolds number of $\mathsf{R} = VD/\nu = 2\times 10^3$. Here, V is the pipe flow velocity, D the pipe diameter, and ν the kinematic viscosity of water, of the order of 10^{-6} m^2/s. With $V = \mathsf{R}\nu/D$, the upper limit of the flow velocity would be $2\times 10^3 \times 10^{-6}/2 = 10^{-3}$ m/s, indeed an extremely small flow velocity. This topic was no longer considered during the following years. Then, problems in the theory of elasticity, of orifice flow, of the molecular action, of permanent waves and wind

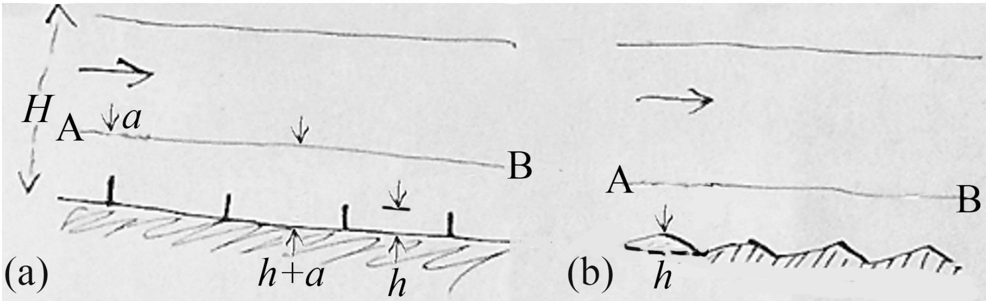

Figure 1. Flow over vertical thin walls (a), and triangular bottom elements (b).

waves, liquid waves in general and both, the unsteady flow equations (referred also to the dSV equations) and the Boussinesq equations accounting for streamline curvature and slope, were of main interest.

In Letter 37, dated October 9, 1872, dSV writes the following to JB: "Dear Sir, I note that in one of my last letters, the question of calculating resistance to movement produced in a water channel by a series of slopes and counter-slopes of the bottom was addressed. Fabre [Sic.: Pierre Jean Fabre 1588–1658, French physician and alchemist, referring to the lines on human hands] calls these 'chasms'. To avoid the trouble of doing so much research, I want to tell you the result of these, which I have done during these days. If they are only very small unevennesses, or bottom ripples, it must be assumed that they do not alter the flow velocity over the entire depth H. Instead, this refers only to the depth or thickness h of a thin layer, thus the velocity becomes $u_o = u_o[h/(h+a)]$ above these protrusions whose heights I call a (Figure 1)."

"The plane AB separating the bottom layer from the remainder of the liquid can be viewed as a pipe wall. You know that this is similar to what Poncelet [8] does in chapter 'Resistances' at the end of his Introduction to *Industrial Mechanics*. In a current, he assumes, based on the tests of Bossut [Sic.: Charles Bossut 1730–1814, French experimental hydraulician] on the resistance of boats in narrow channels, that it is in a limited extent AC that the presence of a solid body BF augments the flow velocity along its sides. Its value would be u_o[AC/(AC − EF)] along the lateral sides BE, FD if it were u_o on the approach flow section AC (Figure 2)."

"Well, above the intervals and protruding from one corner of height a, the velocity (being V above the protrusions) will be $V[h/(h+a)]$. By applying the theorem of Borda, the head loss is $h(u_o^2/2g)[1-h/(h+a)]^2$ for each protruding interval, and, if these intervals are e, one has $(h/e)(u_o^2/2g)[a/(h+a)]^2$, or $(a/e)(u_o^2/2g)(h/a)/[(h/a+1)^2]$ as head loss per square meter of bottom (Figure 2). It is, as one sees, proportional to the velocity square, and varies directly with the protrusion heights and reverses with their intervals."

"Yet, if the bottom roughness elements are large enough for the entire section to participate in the regulations and if this decreases all alternatives to the average flow speed, one cannot reason this way (Figure 3). While the open surface nearly maintains an even slope, the pressures do not vary as in pipes. They vary hydrostatically if centrifugal forces are neglected. The demonstration given by Borda's theorem does no longer apply and my above formula is false. As you say very well, the Borda principle is no longer applicable; it only applies to hydraulic jumps, or it has to be assumed that these large bottom elements produce them on the surface. I think, this is what you are actually doing. It is even likely that you have already thought of all what I just said. I will gladly see the richness of your memoir, Yours dSV."

Note that dSV attempted to solve a question that the general hydraulic scientists started to work only from about the 1960s. The effect of large bottom roughness elements have attracted

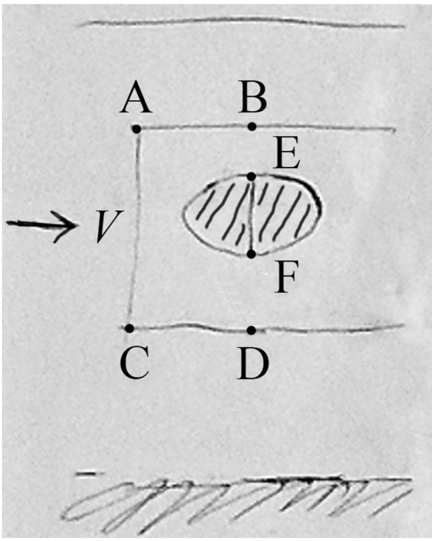

Figure 2. Head loss due to the presence of a solid body.

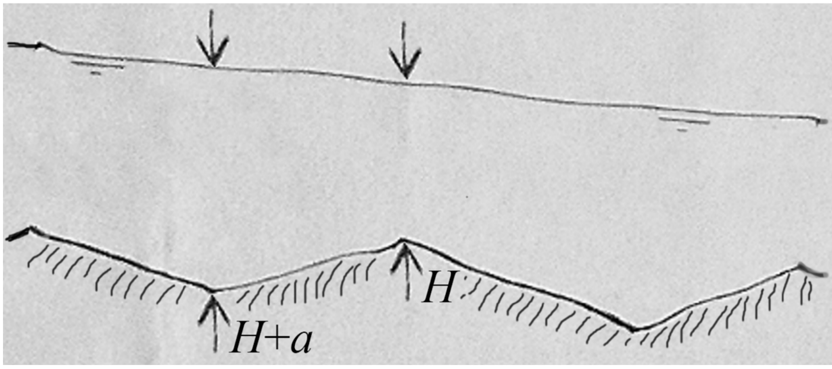

Figure 3. Effect of bottom roughness on the free surface.

scientists mainly in the USA using adequate experimental setups by which the gross effects could be observed. Even today, such a complex system, often present in steep mountain rivers, has not been fully explored despite the support of computational fluid mechanics.

In Letter 38, dated October 12, 1872, dSV writes to JB: "Dear Sir, I did not want to mean by head loss $\Delta\xi$ or $\xi_1-\xi$ in open channel flow, that the additional slope, required between two neighboring points, to imply an extraordinary resistance. This may result from the conversion into a fast-moving or unsteady force of a part of the acquired force. This slope is lost to overcome the ordinary resistance of the bottom and the walls to maintain the flow. I do not like this expression, which in fact may be inappropriate when it applies to streams of which the surface is in contact with the atmosphere."

"The comparison in my letter of the 3rd [Sic.: October], by a calculation that I completed in the note of the 6th, the slope, assessed by the formula of permanent movement and by Borda's theorem, I wanted to call your attention to consider that I thought would help you in clarifying the matter. However, it seems to me from your letter of the 8th, that you have already fixed it on

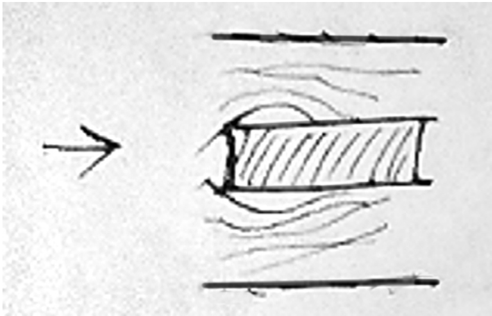

Figure 4. Flow around a bluff body.

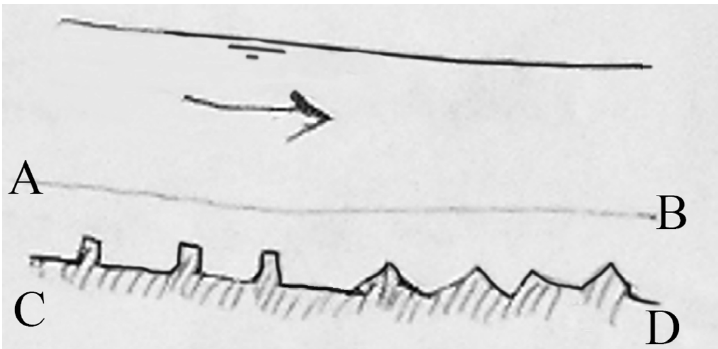

Figure 5. Flow over two types of bottom roughness elements.

this letter, and on the separation limit beyond which one of the two modes of assessing the loss of load, or the slope overcoming the resistances, gives a different result than the other."

"In my letter of the 9th (written quickly and without checking my calculations), you can see that I have come to agree with you regarding the inapplicability of Borda's principle in streams and the need to replace it, in the development by something like the principle of the hydraulic jump (Figure 4).

An exception was proposed by Poncelet to assess the impact of water against a small, submerged body. He surrounded it by a fictitious pipe ABCD having only about 2.5 times the diameter of this supposed circular cylindrical body, or having only about 2.5^2 the size of this section (Figure 5). This is in accordance with the experiments of Bossut (and those of Colonel Duchemin [Sic.: Albert Duchemin 1837–1907, French general and hydraulician] who would have found only twice times instead of 2.5 times the diameter, by examining dust mixed with water). In such a pipe one can still, as Poncelet did, apply the principle of Borda as in a nozzle when the front of the body is not rounded. As I told you, this should be done, or to assume a fictitious pipe ABCD at the bottom, when roughness elements are not very prominent."

"Whereas for large roughness elements such as in Figure 6, given the large slopes AB, the flow will separate at ACB. This would be by the assumption of fixed disturbances on the free surface, whose influence on the additional losses $\Delta\xi$ could be assessed. I will leave it to you. However, the main thing is to rest for a few days and to recover, which I mainly say to Mrs. Boussinesq."

dSV continued to engage JB on this problem. In Letter 39, dated October 15, 1872, he states: "I do not wish, dear Sir, to tire you with some news to solve, and to divert you from the final setting

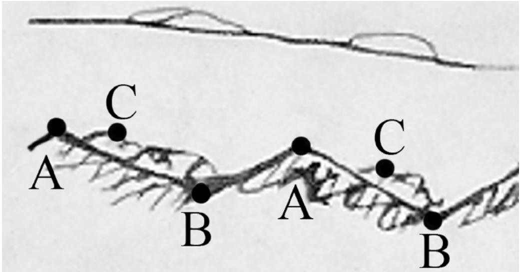

Figure 6. Flow separation at crests A.

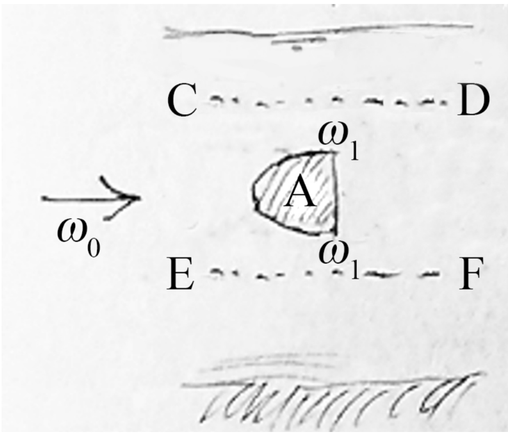

Figure 7. Fixed body in a stream.

of your great memoir [Sic.: Boussinesq's Memoir on water flow, published only in 1877]. However, I answer immediately, for fear of forgetting, at the end of your last letter (of the 11th). This is a topic that you probably find interesting to deal with than others at this time more of urgency."

"I wish to ask you to consider what you think of Poncelet's method in his 'Introduction to industrial mechanics' [Sic.: Poncelet (1841)] [8], which is entirely based on the assumption of a fictitious pipe CDEF enveloping a motionless body A in a current, to assess the impact of the flow on that body (Figure 7)."

"Suppose, for simplicity, a horizontal cross-border stick A having the width of the current. CD and EF will be two horizontal planes (or better, parallel to the free surface and the bottom). The pressure will be hydrostatic: On CD from what you say very well in your letter of the 11th. It will also be hydrostatic on the bottom portion EF if one assumes with Poncelet that the presence of body A does not seem to be recognized according to what you observe. This means that the Borda principle is applicable, because Poncelet's method would have a radical defect. And this imperfection does not disappear if, instead of assuming with Poncelet, that the impulse is $R = A(p_0 - p_1) = A[(\rho V_1^2/2) - (\rho V_0^2/2)] = (\rho A V_0^2/2)\,[(\omega_0^2 - \omega_1^2)/\omega_1^2]$. [Sic.: With R as the resistance, A as cross-sectional area of the body, ω_0 and ω_1 as the approach flow and contracted sections, p_0 and p_1 as the corresponding pressures, V_0 and V_1 as the corresponding velocities, and ρ as the fluid density.] Instead, one takes, as did Bélanger for pipes, the more exact expression

$$R = \int_0^A \rho\,\mathrm{d}(\omega_0 - \omega) = (\rho V_0^2/2)(\omega_0 A)/\omega_1^2.$$

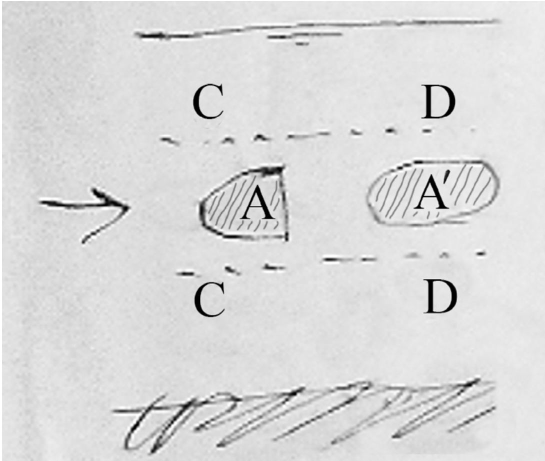

Figure 8. Fictitious pipe CDCD around bodies A and A′.

Well, even Colonel Duchemin observed that after a certain distance from the submerged body, this disturbance is no longer relevant. Do you think that we should also abandon the assumption of Poncelet's fictitious pipe? I am inclined to believe so. And this appears to be also Bélanger's opinion, followed as well by Bresse and Collignon, who merely cite the results of experiments on the impact in an indefinite fluid."

Within less than a week, JB was again contacted by dSV. A letter, written in the meantime by JB to dSV, is not available, however. Letter 40, written on October 20, 1872, reads: "Given my age of 75, and the multiplicity of things I have started, I will certainly have no time to complete them all. However, I gladly make you adorned of my ideas that perhaps you will see some topics of research that would honor you and advance science."

Indeed, the more practical scientist dSV was able to motivate the more theoretical JB for having a closer look at a multitude of then-relevant research questions. In most cases, JB was able to provide an adequate solution, except for instance for the problem of flood flows over large distances and long durations. Despite various attempts, JB finally was unable to find a solution by which dSV would have been able to solve his engineering problems [9].

"What I said in my last letter was provoked by one of your own observations. I say, Poncelet's theory of fluid resistance or the impact of an open and indefinite current on the submerged bodies A or A′, must render the given proof as an inacceptable hypothesis of a fictitious pipe CDCD. Then, it would be necessary to look for a different theory than that of Poncelet (Figure 8)."

"I have, in a large memoir of February 15, 1847 [Sic.: dSV (1847)] [2], entitled 'On the resistance of fluids', containing a complete history of the subject, and which I would send to you if I had it here, you will be able to read it (for nothing has been printed yet than the Abstract you have). It demonstrates, by the energy and momentum equations, that the resulting impact of a river on a solid body would be rigorously zero for steady movement. If the fluid would not transmit friction onto the body, in other words, if it were what is called a perfect fluid, it would have only normal pressures on the faces. d'Alembert, Euler, Borda, Bossut had reached the same result without knowing how to find anything else. Well, I was showing that, if the fluid were restored to its real and physical state (which I do not consider an imperfection), the friction and the resulting pressure of the fluid on the body were obtained. This is not only because of the tangential friction or adhesion exerted on the surface. It is also, and above all, because the friction of the fluid layers on each other, more or less impeding fluid flow from upstream A to downstream B, causes that normal pressures on the body are stronger on the A-side than on the B-side."

"In my opinion, this would therefore be for you a worthy subject of research. The result, even if incomplete, would be honoring you. It would at least have its beginning of great usefulness, that the calculation of what should be this resulting actions of fluid impact on a body. You would first consider the case of a vessel, or rather a fish, that is, a body of a form elongated enough, allowing to overlook that there are no other swirls than those generated by the slippage of fluid on the solid. This would even prevent you from considering first a fluid with friction without swirl or a constant friction coefficient of 1/7488 [Sic.: As previously found by JB] with zero speed against the solid body as in the hair tubes of Poiseuille [Sic.: (1840)] [10–12]."

"You would start from the fact, provided mainly by the experiments of Duchemin, that at distances AB (Figure 9) equal to both the plane half the body width AA, the influence of pressure is substantially zero. Accordingly, as in Poncelet's theory, you calculate what happens between the lines LM. However, instead of assuming with Poncelet nearly constant velocity and pressure in the flow section AB, assume that the CDEF curve gives BC, GD, HE, AF. This will determine the velocities. This means, that it is at C perpendicular to LM, that it has an inflection at D, a maximum in E, because the speed at level H must necessarily be greater than that at BC in the fluid environment between the two levels LM. Without knowing this curve, it can be composed hypothetically by two or three portions of parabolas."

"It seems to me that this would be a problem similar to that you solved by the varied movement, especially if you assume the body to be very elongated, so as to neglect the square of the inclinations of its surface on the axis of the current. The body could be a cylinder with OAPA as the base and transverse horizontal edges in the current, or a basic OAPA′ cycloid with vertical edges, rising from the bottom to the surface of a rectangular-section of indefinite width (Figure 9)."

"Sometimes we rest from a hard work, thinking at times of another work that is planned, and therefore we reflect on the basics and the means of solutions in moments of walking, or insomnia. It is only in this regard, and not by asking you by tiring answers, that I propose this. Of course, it could give me only approximations based on assumptions."

As previously, dSV is quite clear in his statements to JB, the 45 years younger person than his master. Whether JB exactly followed the way proposed by dSV is certainly not always the case. Much more important for JB and the hydraulics community were the excellent ideas of the elder exposed to the younger, which he would hardly have had without this precious input. As stated already in our other writings, the collaboration of the two top scientists of the late 19th century was fortunate for hydraulics, and water science and its culture in general.

In Letter 41, dated October 30, 1872, JB writes to dSV: "I have just studied Poncelet's theory, at the end of his Industrial Mechanics [Sic.: Poncelet (1841)] [8], and I find it inacceptable (while admitting that it can adapt to certain questions, because of the unspecified coefficients it contains). This is not only because of the two reasons given in one of my previous letters. It is also because Poncelet believed that he could replace the pressure exerted by a unit section on the upstream side of the body, by that which is applied at all sections at a certain distance upstream, where the streamlines are parallel. However, it seems impossible to accept that these pressures are equal. Moreover, this equality would not occur anyway if the fictitious pipe would exist, or if one would study what happens not in an indefinite fluid but in a pipe."

"The first of these reasons is that, for the pipe considered by Poncelet's method having an arbitrary diameter, the results to which it leads by assuming this diameter a little smaller or a little larger, and even by assuming it to be infinite, which would be the most natural hypothesis. However, this most natural hypothesis would give $R = 0$. The second of these reasons is which, by assuming hydrostatic pressure along the walls of the fictitious pipe, the quantities that Poncelet calls P_0 and P_1, must be equal, unless these amounts are extremely variable pressures from one

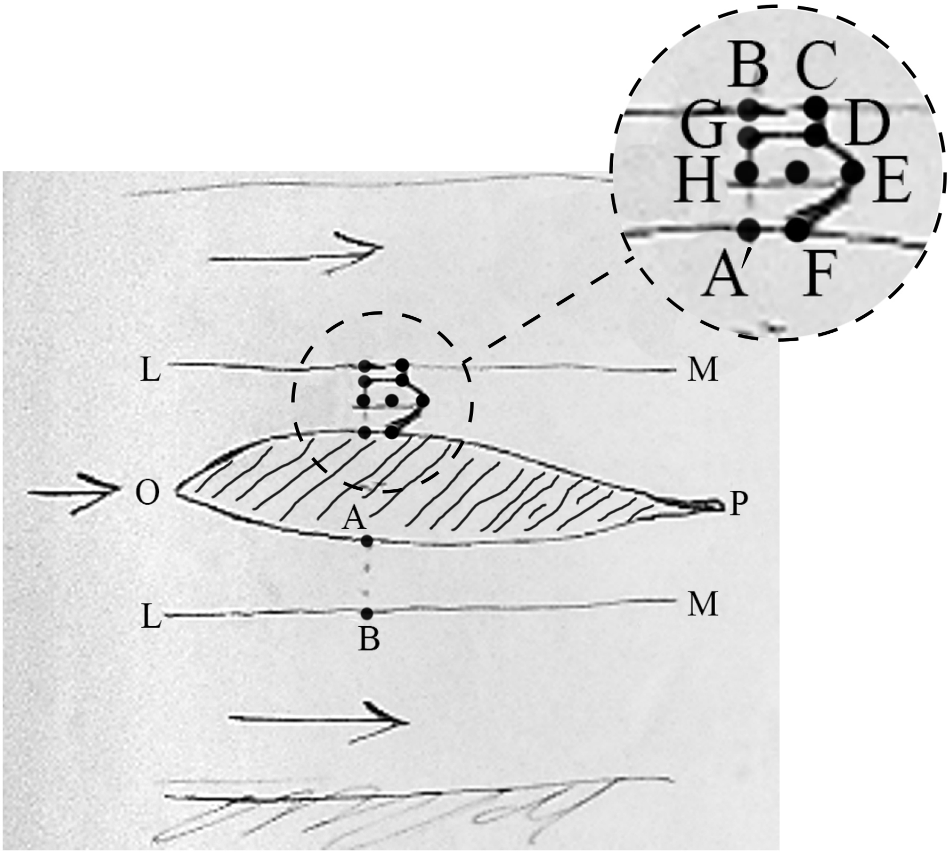

Figure 9. Resistance of a fish-shaped body.

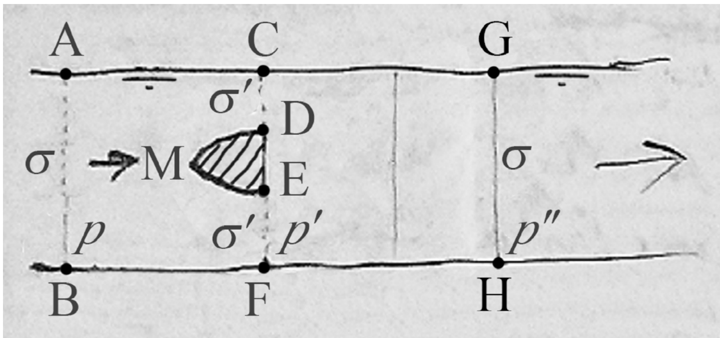

Figure 10. Resistance due to the presence of a streamlined body.

point to another in the same section, which seems inappropriate. I do not see what relation these quantities might have with the resistance to be calculated."

"Indeed, let AGBH be such a pipe, MDE a body with a frontal shape by which no contraction effect results downstream of DE (Figure 10). It should be such that the contracted section σ' is the annular space CD and EF, σ the pipe section AB = GH, and AB, GH are the two sections up- and downstream of CF. Their spacing should be sufficiently large so that the streamlines are parallel. (In addition, we assume also uniform velocity distributions.) Let p, p', p'' be the pressures on AB, on CDEF and on GH, V is the velocity at AB, V' that at CD or at EF. The total head remains constant from section AB to sections CD, EF. It reduces by $(1/2g)(V'-V)^2$, however, based on Borda's principle, when passing from this last section to section GH. Excluding the gravity effect, the total head on section AB is $p/(\rho g)+V^2/(2g)$, $p'/(\rho g)+V'^2/(2g)$ on sections CD, EF, and $p''/(\rho g)+V^2/(2g)$ on section GH. One then has

$$p/(\rho g)+V^2/(2g)=p'/(\rho g)+V'^2/(2g)=p''/(\rho g)+V^2/\,(2g(1/2g)(V'-V)^2, \qquad (1_1)$$

or

$$p-p''=(\rho/2)(V'-V)^2=\rho V^2/2[(\sigma/\sigma_1)-1]^2, \qquad (2_1)$$

$$p''-p'=(\rho/2)\,(V'^2-V^2-(V'-V)^2=\rho V^2[(\sigma/\sigma_1)-1]. \qquad (3_1)$$

Applying the momentum equation on the liquid comprised between the two extreme sections AB and GH, one finds that the sum of the axial forces on this liquid has to be zero, or that the resultant of the pressure $(p-p'')\sigma$ is just equal to the reaction R of the body. Therefore, one has

$$R=(p-p'')\sigma=(\rho V^2/2)\sigma[(\sigma/\sigma_1)-1]^2=(\sigma-\sigma')(\rho V^2/2)\sigma/(\sigma-\sigma')[(\sigma/\sigma_1)-1]^2. \qquad (1_2)$$

The resistance of the body, based on the unit section $(\sigma-\sigma')$, thus is

$$R/(\sigma-\sigma')=(\rho V^2/2)\sigma/(\sigma-\sigma')/\sigma'^2. \qquad (2_2)$$

This expression is due to Bélanger. With p' as the pressure applied on section CF, including the space DE, the difference $p''-p'$ is then the value of the non-pressure of du Buat, by unit section. The dynamic pressure P, by unit section in the axial direction applied on the body front, thus is the excess of the resistance $R/(\sigma-\sigma')$ on this non-pressure. One, therefore, has

$$P=(\rho V'^2/2)[(\sigma-\sigma')(\sigma-2\sigma')/\sigma'^2]. \qquad (3_2)$$

Poncelet believed that he had identified the pressure applied on the body front, namely that relative to section AB, which would demand $(p-p'')=P$, or

$$[(\sigma/\sigma')-1]^2=[(\sigma-\sigma')(\sigma-2\sigma')]/\sigma'^2, \quad \text{or finally } (\sigma-\sigma')/\sigma'=(\sigma-2\sigma')/\sigma'.$$

This is obviously impossible. This analysis evidences that, if the annular space σ' is smaller than half of σ, the dynamic pressure becomes negative or is changed into a non-pressure [Sic.: Better a 'tension'], even ahead of the body. Instead of being pushed by the flow, the body would be attracted by the upstream flow, a curious result for concave streamlines, which removes large pressures from the body and throws them either on section AB or on the pipe walls."

In Letter 42, dated November 4, 1872, dSV writes the following to JB: "Dear Sir, the expression $R/(\sigma-\sigma')=(\rho V^2/2)[\sigma/(\sigma-\sigma')/\sigma'^2]$, which you found in your letter of October 30th, is exactly that of which Poncelet told me he had regretted to have published instead of his own. As I had already found in 1847, and as I told you in my letter of October 15th, I wrote $R=(1/2)A\rho V_0^2(\omega_0 A/\omega_1^2)$ (because $A=\sigma_0-\sigma'$, $\omega_0=\sigma_0$, $\omega_1=\sigma'$). This result obtains if instead of assuming constant pressure p_0 on the entire body EMD (dashed in Figure 10), variable pressure is admitted based on Bernoulli's theorem, by taking $R=\int_0^A p\,\mathrm{d}(\sigma_0-\sigma')-p'A$ with $p=p_0+(\rho/2)(V_0^2-V^2)$, $p'=p_0+(\rho/2)(V_0^2-V'^2)$, and $V=V_0\sigma_0/\sigma$, $V'=V_0\sigma_0/\sigma'$, $A=\sigma_0-\sigma'$."

"As Poncelet then lent me the lithographed course notes of Mr. Bélanger (he is still alive), I realized that Mr. Bélanger had found precisely the same expression in another way, by usage of the momentum equation combined with Bernoulli's theorem. He was only doing it for the pressure of water against a solid body in a pipe, real-walled and non-fictitious. What he stated was reproduced by Bresse [Sic.: (1860)] [13], page 392 (he speaks only of the plate itself. It is necessary to have the contraction coefficient $m=1$ for a rounded body), and by Collignon, p. 332."

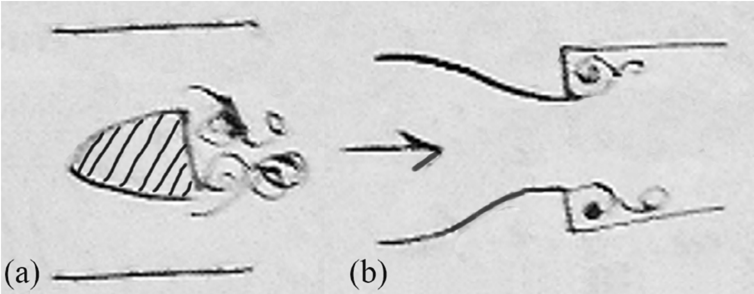

Figure 11. Flow features downstream of a local contraction (a) and local expansion (b).

"I said in my memoir [Sic.: de Saint-Venant (1847)] [2] that the same expression results at once by using the energy head equation for the relative movement. With this correction of Poncelet's theory, only the fictitious pipe objection remains against it, and it is this which you point out very well. It is sufficiently serious to give up the concept, because it was only admitted that there are different pressures along the walls of this pipe p_0, p', p''. Therefore, the theory of resistance, or the impact of an indefinite fluid onto a solid, has to be reconsidered. It would be zero for a frictionless fluid when permanence is established. It depends on the friction of the fluid, not only on the body itself."

"You can calculate the pressure distribution of a basic vertical cylinder or for a channel contraction as you determined the features of the gradually varied flow. This is equivalent to determine the extra free surface slope required for the fluid to flow with the same discharge in this narrowed section (Figure 11). It will be the natural complement to your research this year, and I believe you will succeed. There is also the case of abrupt enlargement."

For a long period, the question of hydraulic resistance was not advanced. Only in 1876, dSV presents his ideas on the problem of d'Alembert [Sic.: Jean-Baptiste le Rond d'Alembert 1717–1783, French mathematician, physicist, philosopher]. In Letter 21, dated July 22, 1876, dSV writes to JB: "Dear Sir, I have read with interest what you have done in your additions [Sic.: to JB's Essay on the theory of flowing water (1877)], to assess head losses in elbows and turns. We must be grateful for your efforts, especially since you arrive at formulas which experience does not reject, and for which, no doubt, you have sometimes bent your reasoning. However, this is not known in fluid theory! Even by assuming that the differential equations are settled, analytical solutions are not available, not even by successive approximation. These digital and graphic solutions, are a business of time, patience, and trial & error, which would apply to a series of particular cases. It may be shortened by the inexpensive consumption of a number of sheets, and by paying for calculators and designers, if you don't have an employee in your office."

"It seems to me that a general thesis, as soon as the equations of a problem are posed (even in form of differentials, or partial differences), or, as is the same, as soon as its solution is verifiable, it is possible to find this solution. This can of course be numerical (or graphical, what amounts to the same), because there can be no question of a solution given by general formulas. Methods should be created given for instruction to calculators, as Prony had logarithms calculated by aides who had previously been involved by the new style of hairdressing."

"Fortunately, for the work you did, Borda's theory is available. You have made a wide and bold use of it, especially when you replaced a curve by a polygon. However, it seems to me that the invocation of this theorem, for which reason and scope are not always remembered, would advantageously be replaced by that of its proof, following my general method of simplifying the demonstrations of 1855, which I reproduced in my lecture of the 10th of this month (last *Compte rendu*, page 102). In the same issue, at p. 139, you find Mr. Bazin's analysis of Mr. Allan Cunningham's hydraulic expressions [Sic.: On experiments of uniform flow]. I have the brochure no. 8 of this English–Indian person. As I may also send it to the Academy, to which it is intended, in six weeks than right now, do you want me to send it to you to read it during your holidays to see a pocket book, and to prepare a list of symbols for the English expressions in French? It seems to me that you have already done it, on page 608 [Sic.: of Boussinesq (1877)] [14, 15], to explain the result of Péclet [Sic.: Eugène Péclet 1793–1857, French physicist]. This is the principle of the momentum equation, which is precisely what one uses to give a demonstration as it is of Borda's theorem of loss of energy, especially of the loss that takes place when a non-elastic body comes to collide obliquely with a plane."

"It seems to me, that all of the above would gain clarity and admissibility, if everywhere you use the principle of momentum (which amounts to the same as balancing dynamic forces and inertia) by looking at the water (as Borda did) as a soft or non-reflecting body, without invoking Borda. This is just a simple indication; do with it what you want. Using the momentum principle, it is given by Bélanger, taken from Borda's theorem, the most plausible demonstration. It was reproduced by Bresse and Collignon. You probably have the lithographed course, 1846–47, of Bélanger in your library. Doesn't your solution of the spillway problem, in case it has a certain length, have an analogy? If you don't have Bélanger available, do you want me to send it to you?"

"I had thought or hoped, for a long time, that when one would possess equations describing the liquid movement, more applicable to streams than those of Navier, one could, without invoking head losses, determine the flow in river bends. I thought, in the parts where the streamlines are curved, to explain how velocities are greater on the concave bank than on the convex bank. Yet, it appears that this is not the way things really are. As to the solution you gave in 1869, based on Navier's equations, assuming zero velocity toward the wall, you revealed pseudo-helicoidal movements in the opposite direction. This is what also occurs with friction depending on the state of vorticity that do not prevent velocities from being reduced close to walls? Another enigma! Moreover, would these less extraordinary frictions be less turbulent than those that occur at an abrupt sectional enlargement? Wouldn't these frictions advantageously be replaced in curves with wide radii, the non-bounces of water considered as a soft body impacting a plane."

"Since you use the Borda principle to explain the head loss due to bends, you must even with stronger reasoning use it to explain the impulse of the fluids against a submerged body or what has been taken for a long time as fluid resistance. This is what occurs from a resting fluid to a solid moving there. In other words, you probably adopt the last part of Poncelet's 'Introduction to the industrial mechanics' [Sic.: (1841)] [8], and improve it as I did with regard to the evaluation of the supposedly rounded front (as Bélanger had done pretty much I think before me). It is very good when the stern has convergent faces. Yet, when the body plunged at its downstream part as elongated as a tail, it seems to me that it must be different, and that the resistance or impulse must be much less. Using the Borda theorem, I hope, you will recognize its value. My 1847-memoir, which you had in your hands, is nearly reduced to proving this result. It strongly surprised d'Alembert, Euler and Bossut; they regarded it as a paradox or enigma bequeathed to posterity. It states that a perfect fluid exerts on a submerged body only actions whose total sum is zero. This result, I say, is all natural, and so to speak obvious."

"As a consequence, the resistance of fluids depends entirely on their friction on themselves as well as on the submerged body. Well, a fluid with friction as it opposes all, but a friction with imperceptible swirls, and even without whirlwind [Sic.: Today's turbulence], like these that lead into the tubes of Poiseuille, can therefore exert, by its only quiet frictions, an impulse on a submerged body. Couldn't you determine it by assuming the body to be very elongated up- and downstream so that the fluid, between that body and the walls, has a gradually varied movement, and is therefore likely to be represented by your equations? This would be a problem at least curious, and one that would solve more completely than I did the Paradox proposed by d'Alembert and on which he, Euler, Robins, Bossut and also Navier saw nothing but fire, as one says."

"An easier solution, but less curious and less flowing, is obtained by assuming that the fluid has only regular [Sic.: laminar] movements represented by the equations of Navier with constant ε [Sic.: kinematic viscosity], and zero velocities against the body, as well as against the walls, that could even be removed to infinity. Would that, then, not be something analogous to Mr. Stokes' pendulum? All this would excellently be received by the *Journal de Liouville*, or even by the *Journal de l'Ecole polytechnique*, because you need no more, since for a long time, neither a Report on your studies, nor the *Mémoires des Savants étrangers* had published your works [Sic.: Given that JB had successfully submitted a large number of research studies]."

An answer to all these proposals of dSV to JB is not given in the Correspondence. The present authors suppose that JB was heavily involved in the preparation of mathematical textbooks, so that he postponed the project on the research of hydraulic resistance. He may certainly also have realized that this problem counts to the essence in hydrodynamics, whose solution required a large input before the first papers were at the stage of publication. Finally, in Letter 10, dated March 23, 1885, JB writes the following to dSV: "Sir and dear Master, despite the writing of my course notes, which presses, I could not resist the obsession with this problem of the pendulum, treated by Stokes [Sic.: (1851)] [3], finding his analysis or that of Mr. Kirchhoff extremely laborious. I left it all there, and by simply generalizing my solution to the case of uniform movement, obtained almost without calculations the general solution, for the case of a rather slow varied translation of the sphere. I come to a formula of resistance that shows with obviousness and simplicity the part that is proportional to the speed, that which is the acceleration, and that which depends on the effects not yet erased from the increase of speed of the sphere prior to the time considered."

"I think you will like what I've done there for your memoir (which you probably will print soon) on the mutual pressures of fluids and solids, an excellent memoir, whose publication, however, should have been done thirty years ago. [Sic.: Note here that dSV was unable to finalize this project, but it was brought to publication by JB, as stated in [16]. The English translation of the preface of JB is available in the Appendix]."

In Letter 11, dated March 24, 1885, JB continues as follows: "I am astonished that the case $\varepsilon = 0$ (perfect fluid), treated implicitly in my note [Sic.: Submitted to the *Comptes rendus*], is in all physics courses not as simple as it is. This provides a first approximation and its theory requires only classical equations [Sic.: of fluid mechanics, namely] $\mathrm{d}p/[\mathrm{d}(x,y,z)] = \rho\,\mathrm{d}(u,v,w)/\mathrm{d}t$, $\mathrm{d}u/\mathrm{d}x + \mathrm{d}v/\mathrm{d}y + \mathrm{d}w/\mathrm{d}z = 0$. As I have no doubt that you do not treat it separately in your memoir, before giving the general case explained by my note: here it is."

"The hypothesis $\varepsilon = 0$ reduces the solution to

$$u = \Delta_2\varphi - \mathrm{d}^2\varphi/\mathrm{d}x^2, \quad v = -\mathrm{d}^2\varphi/\mathrm{d}x\,\mathrm{d}y, \quad w = -\mathrm{d}^2\varphi/\mathrm{d}x\,\mathrm{d}z, \quad p = \rho\mathrm{d}^2\varphi/\mathrm{d}x\,\mathrm{d}t + \text{const.} \qquad (\alpha)$$

The equation in φ, $\Delta_2(\mathrm{d}\varphi/\mathrm{d}t - \varepsilon/\rho\Delta_2\varphi) = 0$, reduces for $\varepsilon = 0$ to $\Delta_2\,\mathrm{d}\varphi/\mathrm{d}t = 0$. This is solved by the expression

$$\Delta_2\varphi = 0, \quad \text{i.e., } \mathrm{d}^2\varphi/\mathrm{d}r^2 + (n-1)/r(\mathrm{d}\varphi/\mathrm{d}r) = 0. \qquad (\beta)$$

Here, n counts the number of coordinates x, y, z and of the velocity components u, v, w, which is 3 for a sphere, and 2 for a cylinder. At the surface $y = R$ of the sphere or the cylinder, the plane tangent at M is displaced by unit time, in its normal direction by MN, if MM′ is the actual velocity V of the center O of the sphere or the cylinder, so that MN = $V\cos A$ (Figure 12). However, the fluid molecules adjacent to the surface do not leave it and have, according to MN (whose cosines

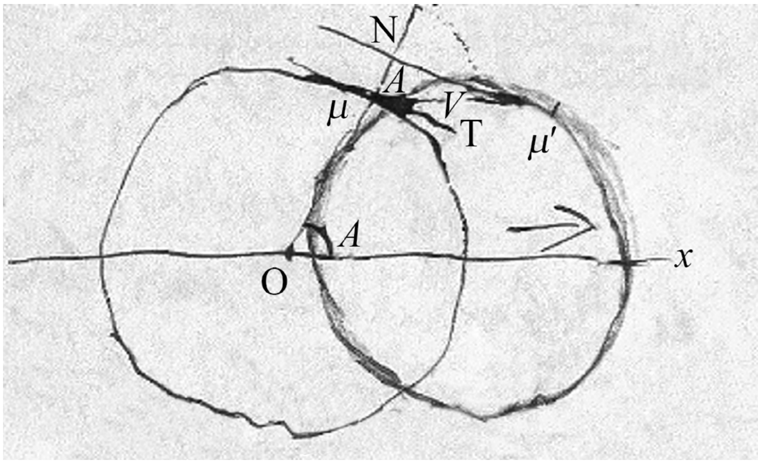

Figure 12. Stokes problem of the pendulum.

are x/r, y/r, z/r), the same velocity of the tangent plane, $V\cos A$. Their effective velocities have the components u, v, w; it is, based on MN, $u(x/r)+v(y/r)+w(z/r)$. Accordingly, one has $V\cos A = u(x/r)+v(y/r)+w(z/r)$ or, with Equation (α), (and by assuming $\Delta_2\varphi = 0$)

$$V\cos A = \frac{x}{r}\Delta_2\varphi - \left[\frac{x}{r}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)+\frac{y}{r}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)+\frac{z}{r}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)\right] = (\cos A)\Delta_2\varphi - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right).$$

However, $\mathrm{d}\varphi/\mathrm{d}x = \mathrm{d}\varphi/\mathrm{d}r(x/r)\cos A$ in which $\cos A$ does not vary along the length $\mathrm{d}r$. Therefore, one has

$$V\cos A = (\Delta_2\varphi)\cos A - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\cos A\right) = \left(\Delta_2\varphi - \frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2}\right)\cos A,$$

or

$$V = \left(\Delta_2\varphi - \frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2}\right) = \left(\frac{n-1}{r}\right)\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \left(\frac{n-1}{R}\right)\frac{\mathrm{d}\varphi}{\mathrm{d}R}.$$

Thus, the special condition at $r = R$ is

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{R}{n-1}V, \quad \text{for } r = R. \tag{γ}$$

Given this result, one has $\mathrm{d}\varphi/\mathrm{d}x = (\mathrm{d}\varphi/\mathrm{d}r)\cos A$. In Equation ($\alpha$) for the pressure p,

$$p = \text{const.} + \rho(\cos A)\frac{\mathrm{d}\frac{\mathrm{d}\varphi}{\mathrm{d}r}}{\mathrm{d}t}.$$

At the surface $r = R$, based on Equation (γ), one has

$$p = \rho\frac{R}{n-1}\frac{\mathrm{d}V}{\mathrm{d}t}\cos A + \text{const.}$$

The pressure component in the negative x-direction thus is

$$p\cos A = \rho\frac{R}{n-1}\frac{\mathrm{d}V}{\mathrm{d}t}\cos^2 A + \text{const.}$$

On the average over the entire surface, with $\mathrm{M}_{oy}(\cos A) = 0$ and $\mathrm{M}_{oy}(\cos^2 A) = 1/n$, the average resistance per unit surface is

$$\frac{\rho R}{n(n-1)}\frac{\mathrm{d}V}{\mathrm{d}t}.$$

The total resistance over the area $\sigma = 2\pi R$ or $4\pi R^2$ for the sphere and the cylinder, respectively, therefore is

$$\frac{1}{n-1}\left(\rho\sigma\frac{R}{n}\right)\frac{\mathrm{d}V}{\mathrm{d}t}.$$

The expression $\sigma(R/n)$ for the sphere or the cylinder given by half or a third of the radius, equals the volume of these bodies, times the product by φ representing the mass m of the displaced fluid. Therefore, the result equals the admiringly simple expression

$$\begin{aligned}\text{Resistance} &= \frac{m}{n-1}\frac{\mathrm{d}V}{\mathrm{d}t}\\ &= \frac{m}{2}\frac{\mathrm{d}V}{\mathrm{d}t} \quad \text{for the sphere,}\\ &= m\frac{\mathrm{d}V}{\mathrm{d}t} \quad \text{for the cylinder.}\end{aligned} \tag{δ}$$

The first result is due to Poisson for the pendular movement, but its proof is not complete, and would give z/r for the constant, referred to n by du Buat, a number that he found a little bit larger (due to the term in ε or rather in $\varepsilon^{1/2}$). The second result for the pendulum was found by [3]. You see that it remains the same regardless of the nature of the movement (supposed only to be slow)."

"It remains to determine φ. One has the equation $\Delta_2\varphi = 0$, i.e., $\mathrm{d}^2(r\varphi)/\mathrm{d}r^2 = 0$ for the sphere, and $\mathrm{d}/\mathrm{d}r(r\mathrm{d}\varphi/\mathrm{d}r) = 0$ for the cylinder. Given that φ is independent of the term in r, which gives nothing in the equations (α) for (u, v, w, p), one finds

$$\varphi = \begin{matrix}\mathrm{f}(t)/r \text{ for the sphere,}\\ \mathrm{f}(t)\log r \text{ for the cylinder,}\end{matrix} \qquad \text{resulting indeed in } \Delta_2\varphi = 0. \tag{ε}$$

The arbitrary function $\mathrm{f}(t)$ to be determined by condition (γ), namely by only posing $\mathrm{d}\varphi/\mathrm{d}r$, i.e. $-\mathrm{f}(t)/r^2$ or $\mathrm{f}(t)/r$, equal to $RV/2$ or RV for $r = R$. The result is

$$\mathrm{f}(t) = \begin{cases}(1/2)R^3V & \text{for the sphere,}\\ R^2V & \text{for the cylinder.}\end{cases}$$

Accordingly,

$$\varphi = \begin{cases}-\dfrac{V}{2}\dfrac{R^3}{r} & \text{for the sphere}\\ VR^2\log r & \text{for the cylinder.}\end{cases} \tag{ζ}$$

Inserting these expressions into (α) gives the values of u, v, w of order $1/r^3$ for the sphere and of order $1/r^2$ for the cylinder. As to the pressure

$$p = \text{const.} + \rho\frac{\mathrm{d}^2\varphi}{\mathrm{d}x\mathrm{d}t}.$$

Except for the constant, it is of the inverse order of r^2 for the sphere, and of r for the cylinder. Accordingly, the conditions at infinity are well satisfied. You realize the extreme simplicity of the theory in this particular case. It is too simple for being novel and must already have been deduced. At the limit the problem is that considered by Clebsch for the movement of an ellipsoid in a fluid, because if the sphere is an ellipsoid, the cylinder may be considered making a part of another ellipsoid infinitely long in the direction of z."

"In this regard, let me tell you that it is not Mr. Kirchhoff who should be attributed with the problem of the slowly moving sphere from a uniform movement in a liquid with friction. I found the same problem, and the same solution, in [3]. Stokes treats it exactly as does Mr. Kirchhoff for the sphere, the two cases of pendulum movement and uniform movement (which, moreover, is included in that of the pendulum movement, because $\cos kt$ tends to $V = 1$ for $k = 0$). As to the cylinder, Stokes only considers the case of pendulum movement, with series developments. I have the solution for the general case, but only implicitly in a defined integral of my Lille volume [Sic.: JB's courses on higher mathematics], and there would have to be a kind of inextricable elimination of arbitrary functions. I do not think that it can be treated differently than the pendulum movement. It would necessarily be possible and simpler than the tangles of series considered by Stokes, and he even had the patience to do the calculation. Yet I think it is useless, because in the movement of a pendulum, the resistance experienced by the same cylindrical rod that supports it is always small compared to that of the sphere. It will be taken into account to a certain point by admitting the simple formula of resistance $m(\mathrm{d}V/\mathrm{d}t)$ based on the hypothesis $\varepsilon = 0$."

"I forgot to tell you that I had the idea last night (because I only slept after 1 AM, so much was I obsessed with the study of the cylinder moving in a fluid), to look what would become the resistance if its movement started from rest. This is always assumed after a sufficiently long time by which the movement becomes uniform. I recognized that, in this case, the liquid so disturbed over an infinite length by the movement of the supposedly indefinite cylinder in length, could not come to a permanent regime, because the agitated part of the fluid would extend indefinitely, and the amount of movement of the fluid would grow without limit. The permanent state could be established only by communication with the ambient fluid (up to an ever-increasing limit) of the liquid's own uniform speed, and the resistance formula would be for resistance $\sim V + \mathrm{d}V/\mathrm{d}t$. I suspect this is a bad trick based on the assumption of incompressibility or the exact preservation of fluid volume. It leads to an infinite speed of propagation for the sound, and it gave me the idea to look whether Stokes would not have thought of this difficulty. I realized that he had asked the question and recognized as me the impossibility of establishing a permanent fluid flow around the body when there are only two dimensions. He devoted to this difficulty an entire issue of his Memoir (N° 46, p. 47). This reminds me of a fact observed by Mr. de Caligny [Sic.: Anatole de Caligny 1811–1892, French private scientist]. Consider a wide and very large body moving along the canal, immersed only to a third or a quarter of the depth. It had all the difficulties to produce a solitary wave, while a cylinder of small diameter in relation to the width, plunged to the bottom in the middle of the canal, vertically produces immediately a solitary wave. It therefore communicated its speed much better to the fluid, over a large extent, than a much larger mass but leaving room below for the passage of fluid."

This is the last letter of the Correspondence dealing with the hydraulic resistance. Until the passing of dSV, most letters dealt with the various election processes for the *Académie des sciences*, given the various members who had passed away in 1885. Details on this topic are highlighted by [16].

3. Conclusions

Hydraulic resistance was and still is a major concern in hydromechanics. The common name friction was shown by dSV and others to be composed of two main sources, namely due to boundary roughness (also referred to as the skin friction), and internal friction due to fluid viscosity. The latter is detailed at the beginning of this work, culminating in the absurd of Boussinesq's example when assuming laminar river flow instead of effectively fully turbulent flow with an enormously large Reynolds number.

Saint-Venant's interest into fluid resistance started in the 1830s while being an assistant to Navier. He published on the topic and particularly on the d'Alembert paradox, noting that it has no physical foundation for viscous fluid flow. The letters of the Correspondence shed new light on this problem, given that dSV was unable to publish after having found in Boussinesq an excellent person to exchange their knowledge. Accordingly, as Boussinesq wrote to his master, he accepted to publish the available notes of dSV thereby adding his own findings, which Boussinesq previously had shared with him. The wish of dSV and his son Raoul thus came true, as also stated by [16, 17]. Of particular relevance is the derivation of Stokes' problem relating to the pendulum moving in a fluid. Here, Boussinesq is able to find the correct result in a much simpler and physically correct approach. Based on the letters presented in this work, the resistance problem in hydromechanics is thus brought to a much-improved level, allowing future scientists to further advance our knowledge.

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Appendix A. Boussinesq's preface to dSV's unpublished works

Fluid resistance

Historical, physical, and practical considerations relating to the problem of the mutual dynamic action of a fluid and a solid, especially in the state of permanence, assumed by their movements by

Mr. de Saint-Venant, Member of the Academy
Recueil des Mémoires de l'Académie, vol. 46 (7), 1-271.

This Memoir on fluid resistance, composed as early as in 1847 and largely based on material already developed in 1836, had not been able, until 1885, to receive from its eminent author, who was always occupied by many other subjects of study, the last retouch that was to precede its publication. Mr. dSV had finally begun, around March 1885, to put his last hand on it, and he had more than once communicated to me his intention to ask for it to be included in the Academy's Memoirs Collection. Unfortunately, when, on January 6, 1886, his friends and the learned world had the grief of losing him, he had not yet been able to fulfill this desire. Yet, I found, on the margins of his manuscript, indications quite explicit to complete this work of revision, in turn, what essentials it contained, without really putting anything due to me, so that his wish can be fulfilled.

This extensive work, relating to a subject that is most useful but most despairing in applied mechanics, consists of three parts.

The first (Chapters I and II), resulting from a patient and profound study of the history of the issue, is an interesting presentation of the research of our predecessors of the last two centuries, touching the impulse of moving fluids on the solids they encounter. Mr. dSV made a special effort to bring out the insoluble paradox of a zero total impulse or resistance (once the permanence of the regime is established). For a submerged solid, surrounded completely by a continuous fluid including its rear, it leads to the hypothesis of the so-called perfect fluidity, which only the surveyors knew alone to implement.

In the second part (Chapters III to VI), he shows that the impulse in question is, in fact, only the so-called imperfection of fluidity. Accordingly, the production of friction (especially inside) of the fluid to be overcome, requires a greater pressure on the upstream than on the downstream surface of the submerged body. Thus, he finds for the value of this impulse, by the speed animating the current, the ratio of the total work absorbed in the time unit and the friction of the fluid both on itself and on the body. Mr. dSV explains ingeniously the existence of friction by the inequalities experienced by mutual actions, as function of distance, exercised between neighboring molecules, during the multiplied passages of these in front of each other. These inequalities introduce in the trajectories of molecules of countless sinousnesses, rendered asymmetrical by inertia, and which are, between contiguous layers sliding over each other, the equivalent of a kind of molecular gearing to divert, in favor of the vortex movements constantly dissipated by communication to the surrounding environment, a significant fraction of the translating force and, therefore, the energy expended to maintain it.

The part of which it is concerned dates from 1836. Although purely theoretical, it borrows a great interest in this original idea, however natural, to make reason for the friction of fluids by the discontinuity of their matter. Its current division into molecules requires a large number of elementary leaps in any passage of the fluid from one state to another, that is, in any discernible deformation of its mass.

Finally, the third part of the memoir (Chapters VII to XI) aims at the practical calculation of the impulse experienced by a body in the center of an indefinite fluid current. For this purpose, Mr. dSV assumes the solid and fluid surrounding nets, up to those that are no longer significantly deviated by the presence of the body, contained in a polished cylindrical pipe, of a section about four to five times of the maximum cross section of the body. According to quite numerous experiments, which long ago led us recognize the absence of any sensitive deviation of the streamlines beyond those distances of the body, in its extended maximum section, which hardly exceed half its diameter.

Poncelet already had, it is true, the idea of bringing back the case of a laterally indefinite fluid to that of a fluid enclosed in a pipe of limited transverse dimensions, a much simpler case, because the velocities of all streamlines, on the contour of the body, are only little different. This allows for evaluating by the principle of Borda the location of maximum of frictional work, that is, downstream of the body. Yet Mr. dSV corrects Poncelet on an essential point, no longer neglecting the variations of the pressure at various points of the upstream face of the body. He even calculates simply these variations for a body equipped with a rounded bow. This allows the fluid, on its outline, to move in roughly normal slices at the axis. He also perfected the method on other no less important points. He, for example, tried to account for the inequalities of velocities in the narrowest sections, an inequality not very influential for a fluid filling a pipe of a moderately greater width than that of the submerged body. However, these are so much in the considered case of a purely fictitious pipe, enveloping many streamlines whose velocity shows no noticeable increase as the parade formed around the body, where other more interior nets accelerate significantly.

With these perfections, and admitting, according to the experiments mentioned, a section of the fictitious pipe four to five times as large as the maximum cross section of the body, this theoretical attempt satisfactorily explains the results of the observation, in case of rounded-bowed bodies, where there is no contraction coefficient to be introduced. In the more complicated case of a body without a bow, having the normal upstream face against the current, it still delivers a good result for the resistance, by attributing to the contraction coefficient a reasonable value. In addition, however imperfect it may be, it is and will probably remain the best work for a long time on this issue of the resistance of a laterally undefined fluid current.

Since the year 1847 in which this theory was composed, the problem of uniform relative movement of a solid, in a liquid surrounding it from all sides up to great distances, has been the subject of many beautiful theoretical researches, especially in England and Germany. Yet most of them, based on the double assumption of perfect fluidity and streamlines joining at the back of the body, could only lead, once again, to the paradox, already reported by d'Alembert, of zero total resistance. Only one of these works, published in 1851, by the academy's illustrious correspondent, Mr. Stokes, took into account friction for a spherical body, but only for the case, in which the integrations seem unaffordable, of a movement slow enough to neglect the non-linear terms of the equations. However, this suppression precisely removes the impulse of the liquid, which almost exclusively the hydraulicians have in sight, namely, this resistance, proportional to the density of the fluid, the cross section of the body, and the square of its velocity, which, on a submerged body of a certain size, is predominant. It allows only for a resistance usually very incidental, which the fluid opposes, because of its inertia. However, its molecular constitution, to be divided, and which, the same (with equal friction coefficient) for a dense liquid as for a light liquid, is proportional to the contour of the section of the body and its speed.

These works, useful from other points of view of interest to physicists, have therefore done little to advance the question of fluid momentum. They do not diminish the interest of the present Memoir, in which the only possible method is developed. However, despite the imperfection of its principle, it sheds some light on such an important problem. That is why I hope, by publishing it under the auspices of the academy, to do a useful work. At the same time, I am fulfilling a duty of gratitude to a master whose remembrance is very thoughtful to me.

I thought I had to insert two other hydrodynamic works, much less extensive but also unpublished, by Mr. dSV. The first, "On the loss of living force of a fluid in places where its section rises sharply or rapidly", is of some interest, by the historical exposure of the subject, that is, in comparing the various demonstrations given to the principle of Borda, and also by a delicate discussion of the conditions under which this principle is applicable. The second, "On the consideration of centrifugal forces in the calculation of the movement of flowing water and on the distinction of torrents and rivers", deserved to be born, although it is unfinished, and despite the defects of an equation of permanent movement (of Coriolis) that served it as the starting point. It dates from 1851 and is probably the first test in which the differences of pressure, at various points of a vertical, were compared with the hydrostatic law, whenever the bottom or surface of a current affects sensitively longitudinal curvatures.

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