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Short paper / *Note*

Asymptotic approach to a rotational Taylor swimming sheet

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Abstract. The interaction of a viscous fluid and a circular, pre-stressed active shell is studied in the limit of low Reynolds numbers. A seminal paper of Taylor represents a benchmark for this class of problems. Here, inspired by the same approach, we determine with asymptotic techniques the possible swimming motions of the shell for the particular changes of curvature that it can achieve when actuated. We confirm numerical results obtained previously, and highlight the structure of a problem that turns out to be similar to that of Taylor, and as such represents a simple example of Stokesian swimming.

Keywords. Stokes flow, Micromotility, Morphing shells, Perturbation series, Low Reynolds swimming, Circular disk.

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1. Introduction

The problem of swimming at low Reynolds numbers has stimulated a great amount of research, resulting in a huge body of literature (see [1–3] and references cited therein), and was originally proposed by Taylor [4] in a celebrated paper. Taylor considered the problem of the self-propulsion of a two-dimensional infinite sheet immersed in a viscous fluid. The fluid flow is governed by the Stokes equations, and the sheet undergoes periodic deformation due to the propagation of waves of transversal displacement. Assuming these waves have small amplitude, Taylor was able (with a perturbative expansion of the boundary conditions) to obtain the corresponding solution of the Stokes problem, and therefore determine the motion of the body. The model is one of the simplest examples of self-propulsion (no external forces are acting on the body) in viscous flow that can be treated analytically. In order to give more details about Taylor's solution, we consider an unbounded 3D fluid domain where the undeformed sheet coincides with the plane $y = 0$. A traveling wave propagating in the x direction will then cause a periodic vertical displacement:

$$y_0 = b\sin(k(x - vt)), \tag{1}$$

where b is the amplitude of the wave, and k, v, are the wave number and wave phase speed, respectively. Choosing, as is customary in this kind of problems, a reference frame moving with the sheet, the boundary condition on the fluid velocity at the body surface will be [1]:

$$\boldsymbol{u}(x, y_0(x,t)) = -bkv\cos(k(x-vt))\boldsymbol{e}_y, \tag{2}$$

while the other boundary condition is that infinitely far from the sheet the flow be uniform and steady:

$$\lim_{y\to\infty}\boldsymbol{u} = -U\boldsymbol{e}_x. \tag{3}$$

It should be noted that U, which is the swimming speed, is not known a priori; therefore, the value at the boundary will be obtained as a result of the solution of the Stokes flow. Indeed, Taylor showed that by expanding the boundary condition (2) in powers of the dimensionless parameter bk, the problem can be solved for each power, approximating U with increasing accuracy. Considering terms of the solution up to second order in bk, the swimming velocity in (2) is

$$U = -\tfrac{1}{2}v(bk)^2. \tag{4}$$

The first important result following from (4) is that, when deformations are small, traveling waves of bending cause translation in the direction opposite to their propagation. This solution defines a paradigm for swimming at low Reynolds numbers that inspired many applications, as for example in [5], where the mechanism described by Taylor is applied to micron-sized artificial systems. More recently, a strong interest was oriented toward active structures and, notably among these, active shells. In [6], a thin circular shell with spontaneous self-curvature is studied. In particular, the shell is neutrally stable, that is, the curvature axis can be oriented arbitrarily by actuation with very weak (ideally vanishing) forces. Therefore, with proper actuation, one can obtain a particular pattern of periodic deformation, namely the precession of the axis of curvature on the plane of the disc in the flat configuration (see [6] for details). A representation of this periodic deformation, that can be seen as a periodic wave of curvature, is given in Figure 1, together with a comparison with what happens in the case of Taylor. The pattern of this particular deformation suggests that a circular disc, deformed in this way, might be seen as a circular analog of the Taylor sheet, leaving a question open concerning what kind of motions could be achieved. In [7], the idealized problem of a disc undergoing such deformations in Stokes flow was addressed numerically. The numerical results pointed at a similarity with the case of Taylor in that a rotational motion of the shell was generated, in the opposite direction with respect to that of precession of the axis of curvature (a sample of these results is reported in Figure 2). On the other hand, no net displacement could be achieved. The results for a wide range of values of curvature were considered (up to the disc sides almost touching) and a relation between precession and induced rotation rate was obtained, by numerical fitting. Indeed, the general behavior of the solution can be estimated by dimensional arguments, both in the case of [7] and that of Taylor, and leads to the same prediction: a quadratic dependence of the swimming velocity on the dimensionless parameter of wave amplitude, which in the case of the disc is given by the (nondimensionalized) curvature. One important result from Taylor's relation (4) is that the coefficient of this dependence is obtained analytically. This completely determines the relation between the actuation, identified with the wave phase speed v and the dimensionless amplitude bk, and the resulting swimming speed U. In [7], the equivalent swimming action was identified with the rotation rate of the shell ($\dot{\alpha}$), and the actuation with the rate of rotation of the axis of curvature (the precession) $\dot{\varphi}$ and the curvature c (equivalent to the amplitude of the wave), where the curvature c is nondimensionalized with the radius of the flat disc R. A relation similar to that of Taylor was then obtained numerically. Here, we proceed with an asymptotic approach, aiming to reproduce Taylor's procedure, in order to confirm analytically some results

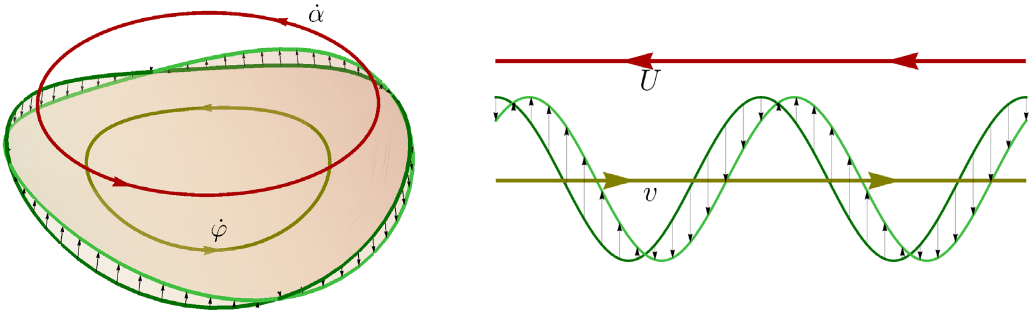

Figure 1. Left: representation of the periodic deformation of the shell under actuation, due to change in orientation of the axis of curvature (described by the rate $\dot{\varphi}$). Right: comparison with the periodic deformation of the Taylor swimming sheet, due to the propagation of a wave of bending.

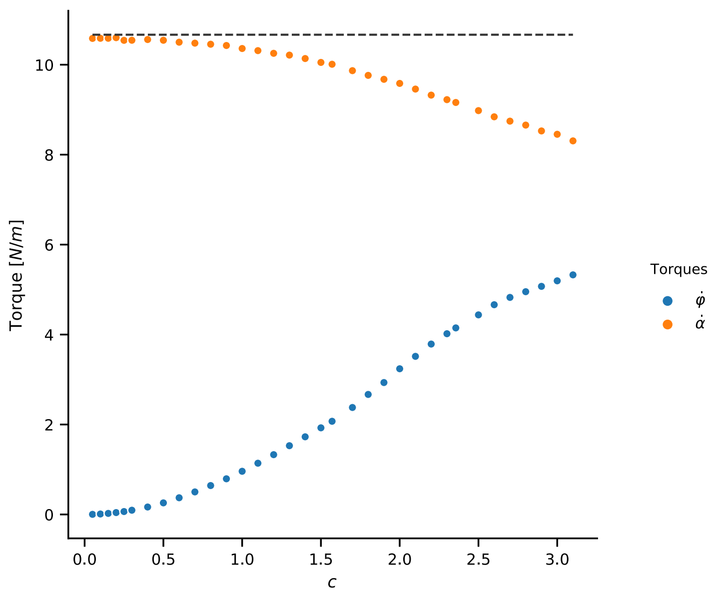

Figure 2. Torques induced on a circular shell for imposed unitary precession of the curvature axis, $\dot{\varphi} = 1$, or unitary rotation rate, $\dot{\alpha} = 1$, respectively, from [7]. Dashed line: torque acting on a flat disc for unitary rotation rate, known to be 32/3.

from [7], namely the behavior of rotations and translations resulting from the precession of the axis of curvature, the quadratic dependence of the induced rotation rate on curvature, and the value of the coefficient for this dependence. This coefficient is the final result of our analytical argument, and determines a relation that is equivalent to that obtained in the case of Taylor, as will be proved:

$$\dot{\alpha} = \frac{1}{10} c^2 \dot{\varphi}. \tag{5}$$

These results allow us to back up the numerical results and give an analytical confirmation, as well as to further support the analogy of this particular shell as a real-life example of rotational Taylor's sheet.

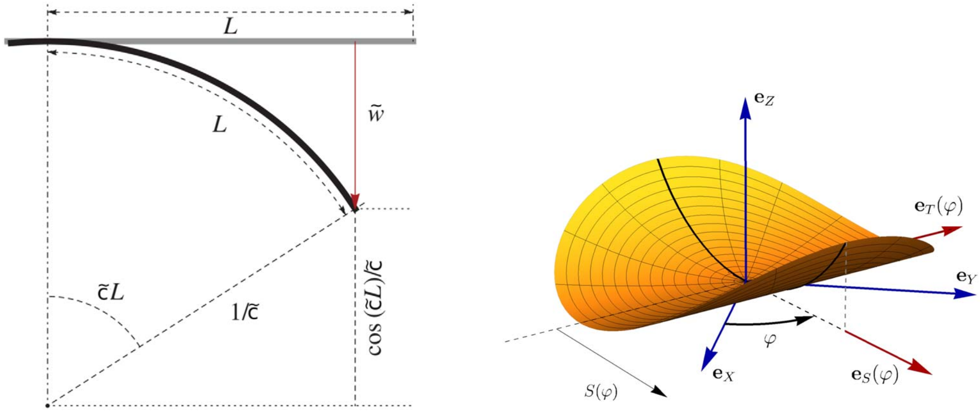

Figure 3. Left: shape of the shell intersecting a plane perpendicular to the axis of curvature, adapted from [6]. Right: description of a generic shell configuration, from [7].

2. Statement of the problem

We consider, as in [7], a circular thin shell in an unbounded domain, filled by a viscous fluid. The reference configuration for the shell is the flat one, which coincides with a circular disk with center at the origin in the plane $Z = 0$. The shell is deformed out of this plane, due to the spontaneous curvature of the object, in a time-dependent manner due to the precession of the axis of curvature [6]. In a plane perpendicular to this axis, intersecting the shell, the shape of the body is that of a circular arc. In this plane, as shown in Figure 3 (left), it is straightforward to define the curvature $\tilde{c}$ as the inverse of the radius of the circular arc. In the figure, it is also shown that the deformation of the shell is inextensible (we refer again to [6] for details about the production of such shells). We also introduce two sets of coordinate axes, one fixed ($\boldsymbol{e}_X$, $\boldsymbol{e}_Y$, and $\boldsymbol{e}_Z$) and one following the rotation of the curvature axis ($\boldsymbol{e}_S$, $\boldsymbol{e}_T$, and $\boldsymbol{e}_Z$). The coordinate values along these vectors will be denoted as X, Y, Z, S, and T, respectively. These systems are shown in Figure 3 (right), where the angle φ, being the (time-dependent) angle between the axis of curvature and the fixed X axis, is also introduced. The axis will rotate staying within the XY plane. The displacement along the Z axis, which is the same for the two coordinate systems, will be denoted as $\tilde{w}$ (also shown in Figure 3, left), and is the predominant effect of the deformation due to curvature. Indeed, assuming small values of curvature from now on, the other components of the displacement will be of smaller order of magnitude, and from simple geometrical considerations, the value of $\tilde{w}$ can be straightforwardly calculated as:

$$\tilde{w} = \frac{1-\cos(\tilde{c}S)}{\tilde{c}} \simeq \frac{\tilde{c}S^2}{2}, \tag{6}$$

where the arc length is $S = X\cos\varphi + Y\sin\varphi$, and coincides with the coordinate value along the $\boldsymbol{e}_S$ vector, since the deformation is inextensible. Introducing cylindrical coordinates, $\tilde{r}$, θ, and Z:

$$\tilde{w} \simeq \frac{\tilde{c}(X\cos\varphi + Y\sin\varphi)^2}{2} = \tilde{c}\frac{(\tilde{r}\cos\theta\cos\varphi + \tilde{r}\sin\theta\sin\varphi)^2}{2} = \frac{\tilde{c}\tilde{r}^2}{2}\cos^2(\theta-\varphi). \tag{7}$$

Differentiating this displacement in time now yields the velocity perpendicular to the plane of the flat configuration:

$$\tilde{v} = \frac{\tilde{c}\tilde{r}^2}{2}\cdot 2\cos(\theta-\varphi)\sin(\theta-\varphi)\dot{\tilde{\varphi}} = \frac{\tilde{c}\tilde{r}^2}{2}\sin(2\theta-2\varphi)\dot{\tilde{\varphi}}. \tag{8}$$

In the case of a circular shell, the value of φ can be taken arbitrarily due to rotational invariance; thus we fix $\varphi = 0$ and the fields simplify to

$$\tilde{w} = \frac{\tilde{c}\tilde{r}^2}{2}\cos^2\theta, \tag{9}$$

$$\tilde{v} = \frac{\tilde{c}\tilde{r}^2}{2}\sin 2\theta \cdot \dot{\tilde{\varphi}}. \tag{10}$$

Considering the flow problem now, it will be governed by the Stokes equations:

$$\nabla\tilde{p} + \mu\Delta\tilde{\boldsymbol{u}} = 0, \quad \nabla\cdot\tilde{\boldsymbol{u}} = 0, \tag{11}$$

with μ the dynamic viscosity. We now proceed to make the problem nondimensional. We take the radius of the disc R as reference length, so that the mathematical problem will be set up on a circular domain with unitary radius. Then, we take a reference time value, T_R, and $V = R/T_R$ as reference velocity. We can then introduce the nondimensional variables $r = \tilde{r}/R$, $\boldsymbol{u} = \tilde{\boldsymbol{u}}/V$, $p = \tilde{p}R/(\mu V)$, $x = X/R$, $y = Y/R$, $z = Z/R$, $\dot{\varphi} = \dot{\tilde{\varphi}}T_R$ and the nondimensional curvature $c = \tilde{c}R$. Since the velocity expression (10) is linear in $\dot{\varphi}$ (in nondimensional form), we can fix $\dot{\varphi} = 1$ and focus only on the coefficient. After the proper substitutions in the previous equations, and simplification, the following problem can now be stated: solve the Stokes equations

$$\nabla p + \Delta\boldsymbol{u} = 0, \quad \nabla\cdot\boldsymbol{u} = 0, \tag{12}$$

with the following boundary conditions

$$\boldsymbol{u}\left(r,\theta,z = \frac{cr^2}{2}\cos^2\theta\right) = \frac{cr^2}{2}\sin 2\theta\cdot\boldsymbol{e}_Z \quad \text{for } r \le 1, \tag{13}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{for } r \to \infty. \tag{14}$$

Here, as in Taylor's case, the no-slip condition on the velocity at the surface of the shell results in a boundary condition on a moving, and deformed, boundary. Following Taylor's approach, we expand the solution in powers of the nondimensional parameter $c = \tilde{c}R$:

$$u_z = cu_z^{(1)} + c^2u_z^{(2)} + c^3u_z^{(3)} + \cdots, \tag{15}$$

$$u_\theta = cu_\theta^{(1)} + c^2u_\theta^{(2)} + c^3u_\theta^{(3)} + \cdots, \tag{16}$$

$$u_r = cu_r^{(1)} + c^2u_r^{(2)} + c^3u_r^{(3)} + \cdots. \tag{17}$$

Moreover, we expand the boundary condition (13) in z (the expansion is done for small values of the curvature, and therefore, equivalently around $z = 0$):

$$u_z = u_z(r,\theta,0) + \left.\frac{\partial u_z(r,\theta,z)}{\partial z}\right|_{z=0}\cdot\frac{cr^2}{2}\cos^2\theta + \cdots = \frac{cr^2}{2}\sin 2\theta \quad (r \le 1), \tag{18}$$

$$u_\theta = u_\theta(r,\theta,0) + \left.\frac{\partial u_\theta(r,\theta,z)}{\partial z}\right|_{z=0}\cdot\frac{cr^2}{2}\cos^2\theta + \cdots = 0 \quad (r \le 1), \tag{19}$$

$$u_r = u_r(r,\theta,0) + \left.\frac{\partial u_r(r,\theta,z)}{\partial z}\right|_{z=0}\cdot\frac{cr^2}{2}\cos^2\theta + \cdots = 0 \quad (r \le 1). \tag{20}$$

From this system, matching the terms of equal power of c, we obtain two problems to solve (choosing to stop at second order), where the interior boundary condition is now defined on a flat disc about the origin of unitary radius:

First order:

$$u_z^{(1)} = \frac{cr^2}{2}\sin 2\theta, \quad u_\theta^{(1)} = 0, \quad u_r^{(1)} = 0 \quad (r \le 1). \tag{21}$$

Second order:

$$u_z^{(2)} + \frac{\partial u_z^{(1)}(r,\theta,z)}{\partial z}\Big|_{z=0} \cdot \frac{cr^2}{2}\cos^2\theta = 0 \quad (r \le 1), \tag{22}$$

$$u_\theta^{(2)} + \frac{\partial u_\theta^{(1)}(r,\theta,z)}{\partial z}\Big|_{z=0} \cdot \frac{cr^2}{2}\cos^2\theta = 0 \quad (r \le 1), \tag{23}$$

$$u_r^{(2)} + \frac{\partial u_r^{(1)}(r,\theta,z)}{\partial z}\Big|_{z=0} \cdot \frac{cr^2}{2}\cos^2\theta = 0 \quad (r \le 1). \tag{24}$$

For both the systems the boundary equation for the far field remains the standard one from (14). The problem has thus been reformulated as the solution of a sequence of subproblems, for increasing powers of the dimensionless curvature, where each problem involves boundary conditions defined on a flat disc instead of the deformed shell surface. In Taylor's case, the setting was two dimensional: expressing the velocity as the derivative of a potential, and knowing the general solution for the potential of a Stokes flow, the problem then simplified in determining the constants of the terms in the expression of the potential, by imposing the boundary conditions. Here, instead, the problem cannot be as easily reduced to a 2D one, and a method is needed to find a solution for the systems at hand given the boundary conditions.

3. Solution strategy

To solve the Stokes equations, we start from the approach of Tanzosh and Stone [8], who solved the problem of a disc in an unbounded viscous fluid, undergoing a generic rigid motion. The idea is to use the same technique, but with the slightly more complex boundary conditions that arise from the problem at hand. We briefly recall the method in Appendix A for the reader's convenience. After introducing cylindrical coordinates, the fluid velocity $\boldsymbol{u} = (u_r, u_\theta, u_z)$ and pressure p are given as an expansion in Fourier modes:

$$u_r + \mathrm{i}u_\theta = \sum_{n=0}^{\infty}\left[U_{-n}(r,z)\mathrm{e}^{-\mathrm{i}n\theta} + U_n(r,z)\mathrm{e}^{\mathrm{i}n\theta}\right], \tag{25}$$

$$u_z = \sum_{n=0}^{\infty}\left[\overline{W}_n(r,z)\mathrm{e}^{-\mathrm{i}n\theta} + W_n(r,z)\mathrm{e}^{\mathrm{i}n\theta}\right], \tag{26}$$

$$p = \sum_{n=0}^{\infty}\left[\overline{P}_n(r,z)\mathrm{e}^{-\mathrm{i}n\theta} + P_n(r,z)\mathrm{e}^{\mathrm{i}n\theta}\right], \tag{27}$$

where the coefficients are in general complex and satisfy, for $z > 0$, the following integrals involving Bessel functions:

$$W_n(r,z) = \int_0^\infty k\,[A_1(k) + zA_2(k)]\,\mathrm{e}^{-kz} J_n(kr)\,\mathrm{d}k, \tag{28}$$

$$P_n(r,z) = \int_0^\infty 2kA_2(k)\mathrm{e}^{-kz} J_n(kr)\,\mathrm{d}k, \tag{29}$$

$$U_n(r,z) = \int_0^\infty k\left[A_3(k) + A_1(k) - A_2(k)\frac{1-kz}{k}\right]\mathrm{e}^{-kz} J_{n+1}(kr)\,\mathrm{d}k, \tag{30}$$

$$U_{-n}(r,z) = \int_0^\infty k\left[\bar{A}_3(k) - \bar{A}_1(k) + \bar{A}_2(k)\frac{1-kz}{k}\right]\mathrm{e}^{-kz} J_{n-1}(kr)\,\mathrm{d}k. \tag{31}$$

These expressions are obtained by applying the Hankel transform to the Stokes equations in cylindrical coordinates, and after various simplifications (see Appendix A). The functions W_n, P_n, U_n, U_{-n} also have to satisfy the appropriate boundary condition related to the corresponding Fourier mode (on the circular disc). The problem has been reduced to finding the functions $A_i(k)$ $(i = 1,2,3)$ that appear in the integral forms. In order to do this, one needs to calculate (or approximate) the integrals involving Bessel functions in (28)–(31). These equations are divided in two contributions: on the domain $r < 1$, $z = 0$, the functions will need to satisfy the given boundary conditions due to the no-slip condition on the surface of the shell. Outside this region, for $r > 1$, $z = 0$, the solution will need to satisfy the continuity of normal stresses [9]. Therefore, the coefficients $A_1(k)$, $A_2(k)$, and $A_3(k)$ (where we recall that a dependence on the Fourier mode n is implicit), will follow from the solution of a series of dual integral equations. Provided that a solution to these problems can be found, the Stokes solution will then be known, and examination of the stresses acting on the surface of the structure will allow to determine the swimming motions.

3.1. *First-order iteration*

The boundary condition (21) has to be expressed in terms of the Fourier coefficients

$$\bar{W}_n e^{-in\theta} + W_n e^{in\theta} = 2W_{n,R}\cos n\theta - 2W_{n,C}\sin n\theta, \tag{32}$$

to solve for $n = 2$, we enforce the boundary condition (21):

$$-2W_{2C} = \frac{cr^2}{2} \Rightarrow W_{2C} = -\frac{cr^2}{4}, \quad W_{2R} = 0. \tag{33}$$

This boundary condition will be imposed in the disc $r \leq 1$, $z = 0$. For the domain $r > 1$, the conditions, if the velocity is perpendicular to the plane of the disc, are $\boldsymbol{e}_z \cdot \boldsymbol{T} \cdot \boldsymbol{e}_z = 0$, which translates in terms of Fourier modes to $-P_2 + 2W_2' = 0$. From (21), instead we obtain $U_{-2} = U_2 = 0$ at $z = 0$, and thus we can take advantage of the simplifications $A_3 = 0$, $A_2 = kA_1$. The problem then is to determine the function $A_1(k)$ for the mode $n = 2$, and is of the form:

$$W_2(r,0): \quad -i\frac{cr^2}{4} = \int_0^\infty kA_1(k)J_2(kr)\,dk \quad \text{for } r < 1, \tag{34}$$

$$-P_2 + 2W_2': \quad 0 = \int_0^\infty k^2 A_1 J_2(kr)\,dk \quad \text{for } r > 1. \tag{35}$$

These dual integral equation problems are well known (see [10]) and, in some particular cases, solution strategies are available to attempt finding a solution in closed form. In the particular case at hand, the problem can be reformulated in such a way that the method exposed by Tranter [11] applies (see also Appendix B). The solution will be:

$$A_1(k) = -i\frac{2c}{3}k^{-\frac{3}{2}}J_{\frac{5}{2}}(k)\sqrt{\frac{2}{\pi}} \tag{36}$$

and therefore, the coefficients for the velocities [12] are:

$$U_2(r,z) = \int_0^\infty k^2 A_1(k) z e^{-kz} J_3(kr)\,dk, \tag{37}$$

$$U_{-2}(r,z) = -\int_0^\infty k^2 \bar{A}_1 z e^{-kz} J_1(kr)\,dk, \tag{38}$$

$$W_2(r,z) = \int_0^\infty kA_1(1+kz)e^{-kz}J_2(kr)\,dk, \tag{39}$$

$$P_2(r,z) = \int_0^\infty 2k^2 A_1 e^{-kz} J_2(kr)\,dk. \tag{40}$$

Knowing these coefficients for the Fourier modes, the velocity fields can be obtained, applying (A.8)–(A.10) and retaining only the nonzero terms:

$$u_r^{(1)} = -\sin 2\theta (U_{2C} - U_{-2C}) = -\sin 2\theta \int_0^\infty k^2 A_1(k) z \mathrm{e}^{-kz} [J_3(kr) - J_1(kr)] \,\mathrm{d}k, \tag{41}$$

$$u_\theta^{(1)} = \cos 2\theta (U_{2C} + U_{-2C}) = \cos 2\theta \int_0^\infty k^2 A_1(k) z \mathrm{e}^{-kz} [J_3(kr) + J_1(kr)] \,\mathrm{d}k, \tag{42}$$

$$u_z^{(1)} = -\sin 2\theta (2W_{2C}) = -2\sin 2\theta \int_0^\infty k A_1 (1+kz) \mathrm{e}^{-kz} J_2(kr) \,\mathrm{d}k, \tag{43}$$

$$p^{(1)} = -\sin 2\theta (2P_{2C}) = -2\sin 2\theta \int_0^\infty 2k^2 A_1 \mathrm{e}^{-kz} J_2(kr) \,\mathrm{d}k. \tag{44}$$

We note that $A_1(k)$ is complex in this case.

3.2. *Second-order iteration*

From the first section we see now that, to pass to a second-order approximation for the problem, we have

$$u_z^{(2)} = -\frac{\partial u_z^{(1)}(r,\theta,z)}{\partial z}\Big|_{z=0} \cdot \frac{cr^2}{2} \cos^2\theta \quad (r \le 1), \tag{45}$$

$$u_\theta^{(2)} = -\frac{\partial u_\theta^{(1)}(r,\theta,z)}{\partial z}\Big|_{z=0} \cdot \frac{cr^2}{2} \cos^2\theta \quad (r \le 1), \tag{46}$$

$$u_r^{(2)} = -\frac{\partial u_r^{(1)}(r,\theta,z)}{\partial z}\Big|_{z=0} \cdot \frac{cr^2}{2} \cos^2\theta \quad (r \le 1). \tag{47}$$

We see immediately that the fields will have a different dependence on the azimuthal variable θ: the velocities u_z and u_r will be of the form $\cos^2\theta \sin 2\theta$, while u_θ will be of the form $\cos^2\theta \cos 2\theta$. Therefore, the average in θ will not be zero for this last component, since $\cos^2\theta \cos 2\theta = 1/4 + \cos 2\theta/2 + \cos 4\theta/4$. This is in analogy to what happens in Taylor's case, where at second order, the solution has a nonzero average, and that component is precisely the net swimming effect [13]. From (46), an integral has to be calculated after the partial derivative in z (evaluated at 0). By using the well-known recurrence identity

$$J_{n-1}(kr) + J_{n+1}(kr) = \frac{2n}{kr} J_n(kr), \tag{48}$$

and using the already known solution (36), we can easily calculate the integral

$$\frac{\partial u_\theta^{(1)}}{\partial z} = \int_0^\infty k^2 A_1(k) [J_3(kr) + J_1(kr)] \,\mathrm{d}k = \frac{4}{r} \int_0^\infty k A_1(k) J_2(kr) \,\mathrm{d}k = r. \tag{49}$$

Taking only the term with nonzero average over 2π for the θ variable:

$$\langle u_\theta^{(2)} \rangle = \left\langle -\frac{\partial u_\theta^{(1)}(r,\theta,z)}{\partial z}\Big|_{z=0} \frac{cr^2}{2} \cos^2\theta \cos 2\theta \right\rangle = \left\langle \frac{c^2 r^3}{2} \cos^2\theta \cos 2\theta \right\rangle = \frac{c^2 r^3}{8}, \tag{50}$$

$$\langle u_z^{(2)} \rangle = \langle u_r^{(2)} \rangle = 0. \tag{51}$$

The average problem, together with appropriate boundary conditions, yields another dual integral problem, which is similar to that of an in-plane rotation of the disk [8]. The boundary conditions enforced can be obtained as in the previous iteration and, in terms of Fourier modes, will be $U_0 = \mathrm{i}(c^2 r^3/8)$, $W_0 = 0$, for $r < 1$, and $U_0' = 0$, for $r > 1$. Based on these constraints it follows that now $A_1 = A_2 = 0$, and the problem reduces to the dual integral equations with the unknown $A_3(k)$:

$$U_0(r,0): \quad \mathrm{i}\frac{c^2 r^3}{8} = \int_0^\infty k A_3(k) J_1(kr)\,\mathrm{d}k \quad \text{for } r < 1, \tag{52}$$

$$U_0'(r,0): \quad 0 = \int_0^\infty k^2 A_3(k) J_1(kr)\,\mathrm{d}k \quad \text{for } r > 1. \tag{53}$$

The method presented by Copson [14] (see also Appendix B) can be employed here to obtain:

$$A_3(k) = \frac{2}{\pi}\frac{-k(-6+k^2)\cos k + 3(-2+k^2)\sin k}{3k^5} = \frac{1}{2}\sqrt{\frac{2}{\pi}}k^{\frac{1}{2}}\left(-\frac{2}{3}J_{\frac{7}{2}}(k) + \frac{2}{k}J_{\frac{5}{2}}(k)\right). \tag{54}$$

Now, following Sherwood [12], if we use the notation $A_3(k) = \mathrm{i}B_3(k)$, then the velocity will be

$$u_\theta = \int_0^\infty k B_3(k) \mathrm{e}^{-kz} J_1(kr)\,\mathrm{d}k; \tag{55}$$

then the stress component of interest will be given by

$$\sigma_{\theta z} = \frac{\partial u_\theta}{\partial z} = \int_0^\infty k^2 B_3(k) \mathrm{e}^{-kz} J_1(kr)\,\mathrm{d}k. \tag{56}$$

If, introducing another approximation, we integrate this stress field over the flat disc $r < 1$, $z = 0$, then, exchanging integrals and using the relation

$$\int_0^1 J_1(kr) r^2\,\mathrm{d}r = \frac{J_2(k)}{k}, \tag{57}$$

we obtain the dependence of the torque on the value of the curvature:

$$L_z = 4\pi \int_{r=0}^1 \sigma_{\theta z} r^2\,\mathrm{d}r = 4\pi \int_0^1 \int_0^\infty k^2 B_3(k) J_1(kr)\,\mathrm{d}k r^2\,\mathrm{d}r = \frac{16}{15}c^2. \tag{58}$$

3.3. *Results and discussion*

The problem was recently studied numerically in [7]: the Stokes problem was solved with a finite element discretization using the standard *P2/P1* elements for velocity and pressure. The system was solved monolithically with the *FEniCS* [15] finite element library, and forces and torques acting on the shell were determined for a wide range of curvature values. It was expected by means of dimensional analysis that the torque acting on the solid would scale quadratically with the curvature, similar to what happened in Taylor's case. Since the motion generated is of interest, an important result is to calculate the rotational rate induced, denoted in the following as $\dot{\alpha}$, for a given precession motion $\dot{\varphi}$. To obtain this value, one needs to know the resistive torque acting on the shell for a unitary value of rigid rotation rate $\dot{\alpha}$. For a flat disc, this value is known [16] to be 32/3. Therefore, for small curvature values, we can use this value without introducing new approximations: given an actuation generating a precession $\dot{\varphi}$, the resulting net torque acting on the surface of the shell will be given by (58): $L_z = (16/15)c^2$, where we recall that the setting is nondimensional, as described in Section 2, and that we had fixed $\dot{\varphi} = 1$ for simplicity. Then, dividing this torque by the rotational resistance coefficient 32/3 yields the induced rotational rate, that is:

$$\dot{\alpha} = \tfrac{1}{10}c^2\dot{\varphi}. \tag{59}$$

This value can then be compared with the numerical results, and to this end, in Figure 4, we report the comparison between the analytic prediction and the numerical data from [7], showing a good agreement when the curvature is reasonably small. Several interesting points can be taken from the analysis above: the problem is similar to that of the Taylor sheet, in that the solution corresponding to the expansion of the boundary condition, at first order, results in no motion.

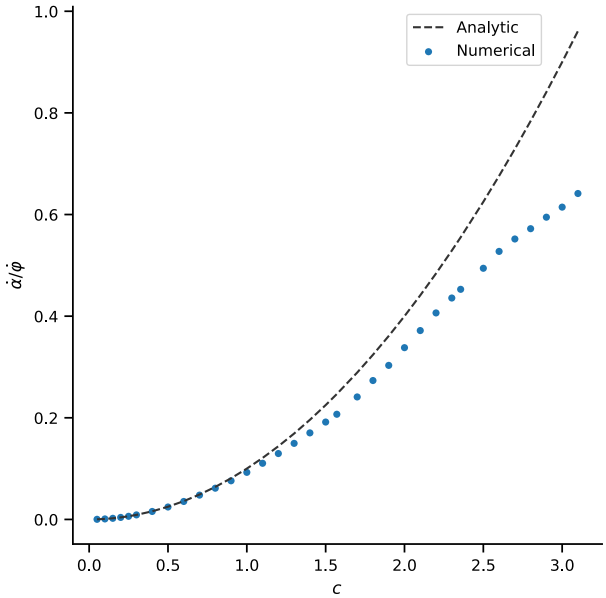

Figure 4. Rotational rate induced by the precession motion for a given value of the curvature (nondimensional). Dashed line: analytical estimate. Scatter plot: data from numerical simulations from [7].

At second order instead, the result is a term with nonzero average in one component of the velocity, specifically the direction in which the wave of displacement travels (u_θ here). This results in a rotation or equivalently, in our setting, in a net torque with z axis. It is also confirmed that the other components of the velocity are zero, at least to this order of approximation: the main effect of the precession of the curvature axis is thus to generate a rotation. The analysis also confirms that, as expected by dimensional arguments, the dependence of the induced torque on the curvature is quadratic, at least for small values. From the comparison in Figure 4, it is evident that the agreement of analytical and numerical data decreases for higher values of curvature. This is expected, since in the numerical study, values of curvature up to the geometrical limit, when the shell closes on itself and becomes a cylinder, were considered. For such cases, the small curvature approximation clearly does not hold anymore. Nonetheless, an interesting conclusion from the analysis above is that a neutrally stable shell, under the proper actuation, can be a good candidate for a real-life realization of a circular version of Taylor's sheet.

4. Conclusions

We have shown that an asymptotic approach in the fashion of Taylor can be applied to the case of a disc. The solution, to second power of the (nondimensional) curvature, results in a rotational motion, with amplitude depending quadratically on the curvature. The analytic result shows good agreement with the numerical data for small values of the curvature, supporting the predictions in [7]. As expected, the analytic prediction loses accuracy at higher values of curvature. The range of validity of the analysis could be widened by considering further terms in the power expansion, as is done in [17] for the case of Taylor. As directions of further investigation, we point out that different shapes of the shell could be considered, in particular, the elliptic shape would be the natural extension to a case in which the shell is not rotationally symmetric, while still simple enough that an analytical approach might still be feasible. Indeed, simulations for an elliptical shape in [7] show a richer behavior with respect to the circular case. In addition, the effects of oscillatory motions and unsteadiness could be considered, as is done in [18], for flat discs, with methods involving dual integral equations, to solve the unsteady Stokes problem.

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Appendix A. Solution method for Stokes equations

We now recall the approach in [8]. The Stokes equations in cylindrical coordinates are:

$$0 = -\frac{\partial p}{\partial r} + \mathscr{L}_{-1} u_r + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2}, \tag{A.1}$$

$$0 = -\frac{\partial p}{\partial \theta} + \mathscr{L}_{-1} u_\theta + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}, \tag{A.2}$$

$$0 = -\frac{\partial p}{\partial z} + \mathscr{L}_{0} u_z + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}, \tag{A.3}$$

$$0 = -\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}, \tag{A.4}$$

the last one being the continuity equation, and having introduced the operator

$$\mathscr{L}_{-n} = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}. \tag{A.5}$$

For convenience, u_r and u_θ are taken together by summing (A.1) ±i (A.2):

$$0 = -\left(\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial \theta}\right) p + \left(\mathscr{L}_{-1} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} + \frac{2\mathrm{i}}{r^2}\frac{\partial}{\partial \theta}\right)(u_r + \mathrm{i}u_\theta), \tag{A.6}$$

$$0 = -\left(\frac{\partial}{\partial r} - \frac{\mathrm{i}}{r}\frac{\partial}{\partial \theta}\right) p + \left(\mathscr{L}_{-1} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} - \frac{2\mathrm{i}}{r^2}\frac{\partial}{\partial \theta}\right)(u_r - \mathrm{i}u_\theta), \tag{A.7}$$

where one resulting equation is the complex conjugate of the other. Since the boundary conditions involve trigonometric dependencies on the azimuthal coordinate θ, and since the governing equations are linear, the velocity and pressure fields can be represented as expansions in complex Fourier modes $\mathrm{e}^{\mathrm{i}n\theta}$. The general expression for the solutions will therefore be:

$$(u_r + \mathrm{i}u_\theta) = \sum_{n=0}^{\infty}\left[U_{-n}(r,z)\mathrm{e}^{-\mathrm{i}n\theta} + U_n(r,z)\mathrm{e}^{\mathrm{i}n\theta}\right], \tag{A.8}$$

$$p = \sum_{n=0}^{\infty}\left[\bar{P}_n(r,z)\mathrm{e}^{-\mathrm{i}n\theta} + P_n(r,z)\mathrm{e}^{\mathrm{i}n\theta}\right], \tag{A.9}$$

$$u_z = \sum_{n=0}^{\infty}\left[\bar{W}_n(r,z)\mathrm{e}^{-\mathrm{i}n\theta} + W_n(r,z)\mathrm{e}^{\mathrm{i}n\theta}\right], \tag{A.10}$$

where it is already taken into account that p and u_z have to be real functions. It is important to note that if the different expansion orders are not coupled in the governing equations, nor in the boundary conditions, then a single mode characterizes the solution, with all other modes being identically zero [8]. Substituting the expansions (A.8)–(A.10) into (A.1)–(A.4), we obtain:

$$0 = -r^{-n}\frac{\partial}{\partial r}(r^n \bar{P}_n) + \left(\mathscr{L}_{-(n-1)} + \frac{\partial^2}{\partial z^2}\right)U_{-n}, \tag{A.11}$$

$$0 = -r^{n}\frac{\partial}{\partial r}(r^{-n} P_n) + \left(\mathscr{L}_{-(n+1)} + \frac{\partial^2}{\partial z^2}\right)U_{n}, \tag{A.12}$$

$$0 = -\frac{\partial P_n}{\partial z} + \left(\mathscr{L}_{-n} + \frac{\partial^2}{\partial z^2}\right)W_{n}, \tag{A.13}$$

$$0 = r^{-(n+1)}\frac{\partial}{\partial r}(r^{n+1} U_n) + r^{(n-1)}\frac{\partial}{\partial r}(r^{-(n-1)} \bar{U}_{-n}) + 2\frac{\partial W_n}{\partial z}. \tag{A.14}$$

Next, the system of PDEs is to be reduced to a system of ODEs in the variable z, that can be solved in closed form. To this end the integral Hankel transform is employed. The definition of Hankel transform of order n and wave number k, and related inverse transform are, respectively:

$$\Phi(k) = \mathscr{H}_n[\phi(r);k] \equiv \int_0^\infty r\phi(r)J_n(kr)\,\mathrm{d}r, \tag{A.15}$$

$$\phi(r) = \mathscr{H}_n^{-1}[\Phi(k);r] \equiv \int_0^\infty k\Phi(k)J_n(kr)\,\mathrm{d}k, \tag{A.16}$$

where J_n indicates a Bessel function of first kind of order n. This transform is useful for the case at hand thanks to its property of converting the radial operator $\mathscr{L}_{-n}$ into an algebraic expression:

$$\mathscr{H}_n\left[\mathscr{L}_{-n}\phi\right] = -k^2\mathscr{H}_n(\phi). \tag{A.17}$$

Other properties of the Hankel transform, that will be needed in order to simplify the system of equations are:

$$\mathscr{H}_{n+1}\left[r^n\frac{\partial}{\partial r}(r^{-n}\phi)\right] = -k\mathscr{H}_n[\phi], \tag{A.18}$$

$$\mathscr{H}_{n-1}\left[r^{-n}\frac{\partial}{\partial r}(r^{n}\phi)\right] = k\mathscr{H}_n[\phi], \tag{A.19}$$

$$\mathscr{H}_{n}\left[\frac{n\phi}{r}\right] = \frac{k}{2}\left(\mathscr{H}_{n-1}[\phi] + \mathscr{H}_{n+1}[\phi]\right). \tag{A.20}$$

To take advantage of these properties, the transformed variables will be defined accordingly:

$$\mathscr{U}_n(k,z) \equiv \mathscr{H}_{n+1}[U_n(r,z)], \tag{A.21}$$

$$\mathscr{P}_n(k,z), \mathscr{W}_n(k,z) \equiv \mathscr{H}_{n}[P_n(r,z), W_n(r,z)], \tag{A.22}$$

$$\bar{\mathscr{U}}_{-n}(k,z) \equiv \mathscr{H}_{n-1}[\bar{U}_{-n}(r,z)]. \tag{A.23}$$

Now, to obtain some simplifications, the $\mathscr{H}_{n-1}$ transform is applied to (A.11), and the complex conjugate of the result is taken. Furthermore, the $\mathscr{H}_n$ transform is applied to (A.13) and (A.14), and the $\mathscr{H}_{n+1}$ transform to (A.12). Thanks to the properties of the Hankel transform (reported above), it follows:

$$0 = -k\mathscr{P}_n(k,z) + \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\bar{\mathscr{U}}_{-n}(k,z), \tag{A.24}$$

$$0 = k\mathscr{P}_n(k,z) + \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\mathscr{U}_{n}(k,z), \tag{A.25}$$

$$0 = -\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{P}_n(k,z) + \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\mathscr{W}_{n}(k,z), \tag{A.26}$$

$$0 = 2\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{W}_n(k,z) + k\mathscr{U}_n(k,z) - k\bar{\mathscr{U}}_{-n}(k,z). \tag{A.27}$$

Combining these gives

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2\right)(k\mathscr{U}_n - k\bar{\mathscr{U}}_{-n}) = -2k^2\mathscr{P}_n \Rightarrow 2\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2\right)\mathscr{W}_n'' = \mathscr{P}_n', \tag{A.28}$$

which allows to obtain a fourth-order ODE in terms of $\mathscr{W}_n$ alone:

$$0 = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2\right)^2 \mathscr{W}_n, \tag{A.29}$$

while the relations that give the other (transformed) expansion coefficients are:

$$\mathscr{P}_n = \frac{1}{2k^2}(\mathscr{W}_n''' - k^2\mathscr{W}_n'), \tag{A.30}$$

$$\mathscr{U}_n - \bar{\mathscr{U}}_{-n} = -\frac{2}{k}\mathscr{W}_n', \tag{A.31}$$

$$(\mathscr{U}_n + \bar{\mathscr{U}}_{-n})'' - k^2(\mathscr{U}_n + \bar{\mathscr{U}}_{-n}) = 0. \tag{A.32}$$

All derivatives ($'$) here are in the variable z. These ODEs can then be solved easily, and only the terms that do not diverge for $z \to \infty$ are retained (that is, terms of the kind e^{-kz}). The solutions will be defined up to three unknown functions $A_1(k)$, $A_2(k)$, and $A_3(k)$, that depend implicitly on the expansion order n. These functions will be determined by the boundary conditions, once the appropriate inverse Hankel transform, as in (A.16), is taken. Applying the inverse Hankel transform then yields (28)–(31).

Appendix B. Solution methods for dual integral equations

The solution methods of Tranter and Copson [11, 14] are an example of strategies that allow to solve dual integral problems, when the conditions imposed on the integrals are simple enough that closed form solutions can be calculated. The idea is to express the solution in an alternate form involving integration with Bessel functions, such that the outer condition part of the dual integral equation is automatically satisfied. To better fix ideas, if we consider the generic problem of the form

$$\int_0^\infty \xi^{2\alpha}\psi(\xi)J_\nu(\xi r)\,\mathrm{d}\xi \;=\; f(r) \quad 0<r<1, \tag{B.1}$$

$$\int_0^\infty \psi(\xi)J_\nu(\xi r)\,\mathrm{d}\xi \;=\; 0 \quad r>1, \tag{B.2}$$

then a good solution candidate would be given by introducing a further unknown function ϕ to have a solution ψ of the form:

$$\psi(\xi) = \xi^{1-\alpha}\int_0^1 \phi(t)J_{\nu+\alpha}(\xi t)\,\mathrm{d}t. \tag{B.3}$$

This solution automatically satisfies condition (B.2). To see this, we need a simple lemma regarding properties of Bessel functions from [19]: If $\lambda > \mu > -1$,

$$\int_0^\infty J_\lambda(at)J_\mu(bt)t^{1+\mu-\lambda}\,\mathrm{d}t = \begin{cases} 0, & (0<a<b) \\ \dfrac{b^\mu(a^2-b^2)^{\lambda-\mu-1}}{2^{\lambda-\mu-1}a^\lambda\Gamma(\lambda-\mu)}, & (0<b<a). \end{cases} \tag{B.4}$$

The next step of the method is substituting the solution candidate in the inner problem, that is the first of the two integral equations, and obtain after manipulations an integral problem of which the closed form solution is known. In particular, Tranter and Copson reduce the problem, after algebraic manipulations, to an integral Abel problem which is solvable for certain combinations of exponents of the special functions and the outer data. In more generality [9], integral equations of the kind

$$\int_0^r g(t)(r^2 - t^2)^{-\alpha}\,\mathrm{d}t = h(r) \quad 0 < r < R, \quad 0 < \alpha < 1, \tag{B.5}$$

arise. The solution to this problem is of the form

$$g(r) = \frac{2}{\pi}\sin(\alpha\pi)\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r h(t)(r^2 - t^2)^{\alpha-1}\,t\,\mathrm{d}t. \tag{B.6}$$

Once this solution has been calculated, going back through all of the substitutions yields the original function searched for. Clearly, it is assumed that all of the functions in the computations have sufficient regularity and decay properties, for the manipulations to be justified.

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