Modal energetic analysis and dynamic response of worm gear drives with a new developed dynamic model

© Académie des sciences, Paris and the authors, 2021.
Some rights reserved.

This article is licensed under the
Creative Commons Attribution 4.0 International License.
http://creativecommons.org/licenses/by/4.0/

Les Comptes Rendus. Mécanique sont membres du
Centre Mersenne pour l’édition scientifique ouverte
www.centre-mersenne.org
Modal energetic analysis and dynamic response of worm gear drives with a new developed dynamic model

Ala Eddin Chakroun\textsuperscript{a, b}, Ahmed Hammami\textsuperscript{a, c}, Ana De-Juan\textsuperscript{b, c}, Fakher Chaari\textsuperscript{a, c, d}, Alfonso Fernandez\textsuperscript{b, c, d}, Fernando Viadero\textsuperscript{b, c, d} and Mohamed Haddar\textsuperscript{a, e}

\textsuperscript{a} Mechanics, Modelling and Production Laboratory, National School of Engineers Sfax, Tunisia

\textsuperscript{b} Department of Structural and Mechanical Engineering, University of Cantabria, Spain

E-mails: Chakroun.al.eeddin@gmail.com (A. E. Chakroun), ahmed.hammami2109@gmail.com (A. Hammami), ana.dejuan@unican.es (A. De-Juan), fakher.chaari@gmail.com (F. Chaari), alfonso.fernandez@unican.es (A. Fernandez), fernando.viadero@unican.es (F. Viadero), mohamed.haddar@enis.rnu.tn (M. Haddar)

Abstract. In order to investigate the behaviour of worm drives, a new dynamic model, composed of two blocks, is established and used to extract numerical results. The tooth deflection of the worm drive, bearings, and wheels inertias are taken into consideration. Newmark solving method is applied to solve motion equations. The state of contact of teeth is what enables these signals to manifest themselves. Modal analysis is developed to investigate the different natural modes of the model. Furthermore, modal energetic analysis is used to understand the distribution of strain and kinetic energies. It is also applied to classify natural models into “teeth modes” and “bearing modes”. These two modes constitute two different frequency bands. The dynamic coefficient is measured simultaneously with the gradual increase of the turning speed of the motor. This allows for the evaluation of the overload of the system.

Keywords. Dynamic model, Natural frequencies, Natural modes, Worm, Worm gear, Stiffness.

Funding. This work has been supported by project DPI2017-85390-P funded by the Spanish Ministry of Economy, Industry, and Competitiveness.

Manuscript received 30th January 2021, revised 12th March 2021, accepted 22nd March 2021.
1. Introduction

Over time, many types of gears, from simple spur gears to complex hypoid gears, have been established. Worm drives, for example, are characterized by transmission axes that do not cross but can put into different angles. Magyar and Sauer [1] take into account most of the factors of power loss in order to investigate the average efficiency of transmission. They bring to the fore a tribological calculation method that is complex but reliable. Other studies claim that it is important to investigate other factors. The studies of Sharif et al. [2] and Sharif et al. [3] focus on the pattern and rate of wear. They use lubricant and predict the evolution of wear over time. Some studies were theoretical. Falah and Elkholy [4] used their original slicing method to determine the stiffness, load, and stress distribution. Chen and Tsay developed a mathematical model to generate the geometry of a unique type of worm gear [5]. Some studies tried to find a better geometrical characteristics of worm gears by creating new forms. In the study of Simon [6], a new type of cylindrical worm gear drive has been developed. The idea is having a worm gear with a double arc profile while the worm has a concave profile. This type of gear showed a favourable position of the instantaneous contact line, a higher load carrying capacity, and a lower power loss in the oil film compared with common worm drives.

Other researchers found it interesting to study the behaviour of worm drives made with plastic. In the study of Marshek and Chan [7], it was found that pitting and ridging are major wear types. Both were followed with microcracks. This conclusion is the result of investigating the sliding contact surface of the gear pair using a scanning electron microscope. In the studies of Hiltcher et al. [8], the generalized Kelvin model was used to compute the viscoelastic displacement of the worm gear made of polymer. The quasi-static load sharing was investigated. The result of this study is that load sharing, meshing stiffness, and loaded transmission error depends on the speed of the rotation of the gears. de Vaujany et al. [9] used the same model and found correct correlations between simulation and measurements. They showed that the revolution speed modifies the mean value of the transmission error while the amplitude stayed the same. They also found that temperature have a big influence on the shape and the amplitude of the transmission error signal.

Regarding the existing dynamic models show reliable results but with many limitations. Benabid and Mansouri [10] introduce an original eight-degrees-of-freedom model. They try not to overload the model and to keep it as simple as possible. This model does not take into consideration the behaviour of bearings and other inertias in the system. Chung and Shaw [11] used a model with a flywheel. They solved a dynamic equation of transmission to predict the dynamic behaviour of such model. No bearings have been added and the flywheel remains very specific to this model. Liu et al. [12] attempted to use the spur gear dynamic model proposed by Tamminana et al. [13] in order to predict the dynamic performance of a worm drive. Jiang et al. [14] used the computer aided design model and import it into the ADAMS dynamic model in order to investigate the dynamic response to friction of a spur gear couple. All these models do not consider the bearing behaviour, the geometry of the worm drive, or the state of the surface of contact of the teeth.

In this study, a new dynamic worm drive model is proposed, based on previous research on other types of gears. A single-stage spur gear model established by Chaari et al. [15]. Two-stage gear model proposed by Beyaoui et al. [16]. Farhat et al. [17] proposed a numerical model of a single-stage gear box operating under variable stationary regime. They studied the interaction between gear mesh and bearings excitation in case of variable speed and load. Hmida et al. [18] adopted the model of Chaari et al. with the addition of an elastic couple. Instead of the elastic couple, a clutch has been used by Walha et al. [19] in a helical gear model. Bevel gears were modelled by Driss et al. [20]. Planetary gears were modelled by various researchers such as
Hammami et al. [21], Bouslema et al. [22], and Feki et al. [23]. With the same approach, the new dynamic model is composed of two blocks and it has fourteen degrees of freedom. It takes into account the geometry of the worm drive, the bearings of the two blocks, and the inertias of gears, motor, and receiver. The equation of motion is obtained through Lagrange formalism after computing the deflection of contact of gear teeth. In this study, the gear mesh stiffness is modelled using an approximative approach. Advanced gear mesh stiffness with finite element modelling is used in the study of Fernandez Del Rincon et al. [24]. Newmark method is used to solve this equation. Through this model, modal analysis is carried out. This model also allows for the analyses of the modal energetic distributions. The analysis of the dynamic coefficient follows the study of the dynamic overload.

2. Dynamic model of worm gearbox

Figure 1 shows a kinematic chain of the worm reducer. The first block is composed of a motor, a bearing block, and a worm connected by a transmission shaft. The second block is composed of a receiver, a bearing block, and a worm gear that are connected also by a transmission shaft. Lagrange formalism is used to solve the equation of motion after taking into consideration the displacement in the line of action.

The motor and the receiver are deemed rigid bodies. The shafts are supposedly light weighted compared to other components. Their torsional and axial stiffness are $k_{\theta i}$ ($i = 1, 2$) and $k_{zi}$ ($i = 1, 2$), respectively. Gear meshing and bearings are modelled using linear springs.

The translations of the two blocks are $x_i, y_i, z_i$ ($i = 1, 2$) and the rotations are $\phi_i, \psi_i, \theta_{1i}, \theta_{2i}$ ($i = 1, 2$).
$m_i$ is the weight of the worm when $i = 1$. $m_i$ is the weight of the worm gear when $i = 2$. It is measured using the following equation
\[ m_i = \pi b_i r_i^2 p_i \quad (i = 1, 2), \]
where $b_i$ is the width of the worm gear when $i = 1$ and $b_i$ is also the length of the worm when $i = 2$. $p_i$ represents the material density. $r_i$ is the pitch radius of the corresponding gear ($i = 1, 2$).

$I_{11}$ is the inertia of the receiver. $I_{12}$ is the inertia of the motor. $I_{21}$ is the inertia of the worm gear. $I_{22}$ is the inertia of the receiver.

### 2.1. Calculation of transmission error

The two points $M_1$ and $M_2$ belong to the active flank of the worm and the worm gear, respectively. Their relative displacement against a fixed benchmark $R(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$ is calculated using the following:
\[ \delta(l, t) = \overrightarrow{U}_1^R(M_1) \cdot \overrightarrow{n_1} + \overrightarrow{U}_2^R(M_2) \cdot \overrightarrow{n_2} \]  
(2)

$\overrightarrow{U}_j(M_i)$: the displacement of $M_j$ ($i = 1, 2$),

$\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ are the normal unitary outgoing vectors of $M_i$ with $\overrightarrow{n_2} = -\overrightarrow{n_1}$.

In the action plan, $l_i$ is the distance separating $M_i$ from the middle of the line of action.

The gear pair is supposedly rigid. Both of the gears are rotating in the positive sense (counterclockwise). By taking into consideration the above mentioned conditions, it becomes possible to write the following:
\[ \delta(l, t) = \left\{ \overrightarrow{U}_1^R(O_1) + \omega_1^R \wedge \overrightarrow{O_1} M_1 \cdot \overrightarrow{n_1} - \overrightarrow{U}_2^R(O_2) - \omega_2^R \wedge \overrightarrow{O_2} M_2 \cdot \overrightarrow{n_1} \right\}. \]

The displacement torsors of $O_1$ and $O_2$ are expressed against a fixed benchmark $R(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$.

\[ \delta(\tau_1^R) = \begin{cases} \overrightarrow{U}_1^R(O_1) = x_1 \overrightarrow{x} + y_1 \overrightarrow{y} + z_1 \overrightarrow{z} \\ \omega_1^R = \phi_1 \overrightarrow{x} + \psi_1 \overrightarrow{y} + \theta_{12} \overrightarrow{z} \end{cases} \]

(4)

\[ \delta(\tau_2^R) = \begin{cases} \overrightarrow{U}_2^R(O_2) = x_2 \overrightarrow{x} + y_2 \overrightarrow{y} + z_2 \overrightarrow{z} \\ \omega_2^R = \phi_2 \overrightarrow{x} + \psi_2 \overrightarrow{y} + \theta_{21} \overrightarrow{z} \end{cases} \]

(5)

The deflexion between the worm and the worm gear is calculated. Both are rotating in the positive sense (Figure 2).

The normal outgoing vector from the worm is expressed as
\[ \overrightarrow{n_1} = Q_1 Q_2 \cos \beta + \sin \beta \overrightarrow{x_1} \]

(6)

$Q_1 Q_2$ is the vector that represents the direction of movement
\[ Q_1 Q_2 = -\sin(\gamma - \alpha) \overrightarrow{y_1} + \cos(\gamma - \alpha) \overrightarrow{z_1} \]

(7)

$O_1 M_1$ positions $M_1$ in the worm using the following:
\[ O_1 M_1 = O_1 Q_1 + Q_1 P_1 + P_1 M_1 \]

(8)

\[ O_1 Q_1 = R_{b_1} y_1 - Q_1 Q_2 \sin(\alpha - \gamma) \overrightarrow{z_1} \quad \text{with} \quad Q_1 Q_2 = \frac{R_{b_1}}{\cos(\alpha - \gamma)} \]

(9)

$R_{b_1}$ is the base radius of the worm and $R_{b_2}$ is the base radius of the worm gear.

\[ \frac{Q_1 P_1}{P_1 M_1} = \frac{p_1 Q_1 Q_2}{l_1 \cos \beta x_1 - l_1 \sin \beta Q_1 Q_2} \]

(10)
Figure 2. Modelling of the teeth contact of the gears.

$\vec{\omega}_1$ is the rotation vector in $(\vec{x}_1, \vec{y}_1, \vec{z}_1)$. The translation of $M_1$ is expressed as follows:

$$\vec{U}^R_1(M_1) = \vec{U}(O_1) + \vec{\omega}_1 \wedge \overrightarrow{O_1 M_1}$$

$$\vec{U}^R_1(M_1) = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} \phi_1 \\ \psi_1 \\ \theta_{12} \end{pmatrix} \wedge \begin{pmatrix} l_1 \cos \beta \\ Rb_1 + \sin(\gamma - \alpha)(l_1 \sin \beta - p_1) \\ -Rb_1 \tan(\alpha - \gamma) + (p_1 - l_1 \sin \beta) \cos(\gamma - \alpha) \end{pmatrix}.$$  

On the other hand, $M_2$ of the worm gear is positioned by $\overrightarrow{O_2 M_2}$

$$O_2 M_2 = O_2 Q_2 + Q_2 P_2 + P_2 M_2$$

$$O_2 Q_2 = Rb_2 \cos(\alpha - \gamma)\vec{x}_2 - Rb_2 \sin(\alpha - \gamma)\vec{y}_2$$

$$Q_2 P_2 = -p_2 Q_1 Q_2 = p_2 \sin(\gamma - \alpha)\vec{x}_2 - p_2 \cos(\gamma - \alpha)\vec{y}_2$$

$$P_2 M_2 = l_2 \cos \beta \vec{z}_2 - l_2 \sin \beta \vec{Q}_1 Q_2.$$  

The translation of $M_2$ in $(\vec{x}_2, \vec{y}_2, \vec{z}_2)$ is

$$\vec{U}^R_2(M_2) = \vec{U}(O_2) + \vec{\omega}_2 \wedge \overrightarrow{O_2 M_2}$$

$$\vec{U}^R_2(M_2) = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} \phi_2 \\ \psi_2 \\ \theta_{21} \end{pmatrix} \wedge \begin{pmatrix} -Rb_2 \cos(\alpha - \gamma) + \sin(\gamma - \alpha)(l_2 \sin \beta + p_2) \\ Rb_2 \sin(\alpha - \gamma) - (p_2 + l_2 \sin \beta) \cos(\gamma - \alpha) \\ l_2 \cos \beta \end{pmatrix}.$$  

The deflexion $\delta(l, t)$ is

$$\delta(l, t) = \vec{U}^R_1(M_1) \cdot \vec{n}_1 + \vec{U}^R_2(M_2) \cdot \vec{n}_2$$

$$= t_1 (x_2 - y_1) + t_2 (y_2 - z_1) + t_3 (z_2 - x_1) + t_4 \phi_1 + t_5 \psi_1 + t_6 \theta_{12} + t_7 \phi_2 + t_8 \psi_2 + t_9 \theta_{21}.$$  

C. R. Mécanique — 2021, 349, n° 2, 241-258
The constants of the deflexion are presented in Table 1.

### 2.2. The equation of motion

The expression of the kinetic energy of the two blocks is in the following form

\[
T = \frac{1}{2} m_1 (\ddot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2} m_2 \dot{\phi}_1^2 + \frac{1}{2} I_{11} \ddot{\theta}_{11}^2 + \frac{1}{2} I_{12} \ddot{\theta}_{12}^2 + \frac{1}{2} (I_{11x} + I_{12x}) \dot{\phi}_1^2 + \frac{1}{2} (I_{11y} + I_{12y}) \dot{\phi}_2^2 + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) + \frac{1}{2} I_{21} \dot{\theta}_{21}^2 + \frac{1}{2} I_{22} \dot{\theta}_{22}^2 + \frac{1}{2} (I_{21x} + I_{22x}) \dot{\phi}_2^2 + \frac{1}{2} (I_{21y} + I_{22y}) \dot{\phi}_2^2.
\]

The strain energy of the two blocks is the following form

\[
U = \frac{1}{2} k_{x1} x_1^2 + \frac{1}{2} k_{y1} y_1^2 + \frac{1}{2} k_{z1} z_1^2 + \frac{1}{2} k_{\phi1} \dot{\phi}_1^2 + \frac{1}{2} k_{\psi1} \dot{\psi}_1^2 + \frac{1}{2} k_{\theta1} (\theta_{11} - \theta_{12})^2 + \frac{1}{2} k_{x2} x_2^2 + \frac{1}{2} k_{y2} y_2^2 + \frac{1}{2} k_{z2} z_2^2 + \frac{1}{2} k_{\phi2} \dot{\phi}_2^2 + \frac{1}{2} k_{\psi2} \dot{\psi}_2^2 + \frac{1}{2} k_{\theta2} (\theta_{21} - \theta_{22})^2 + \frac{1}{2} k_m(t) \dot{\phi}_2^2(t).
\]

The differential equation system of the movement reaction is put in a matrix form

\[
M(\ddot{q}) + C(\dot{q}) + (|K_A| + |K(t)|)(\dot{q}) = F(t).
\]

With

\[
\{q\} = \{x_1, y_1, z_1, x_2, y_2, z_2, \phi_1, \psi_1, \phi_2, \psi_2, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}\}
\]

\[
[M] \text{ is the mass matrix}
\]

\[
[M] = \begin{bmatrix} M_L & 0 \\ 0 & M_A \end{bmatrix}
\]

\[
[M_L] \text{ is a matrix composed of the following weight terms}
\]

\[
[M_L] = \text{diag}(m_1, m_1, m_2, m_2)
\]

\[
[M_A] \text{ is a matrix composed of the following inertia terms}
\]

\[
[M_A] = \text{diag}(I_{11x} + I_{12x}, I_{11y} + I_{12y}, I_{21x} + I_{22x}, I_{21y} + I_{22y}, I_{11}, I_{12}, I_{21}, I_{22}).
\]

The matrix of the average stiffness is written as follows:

\[
[K_A] = \begin{bmatrix} K_p & 0 \\ 0 & K_q \end{bmatrix}
\]

\[K_p\] is composed of the stiffness of the bearing blocks. It is in the following form

\[
[K_p] = \text{diag}(k_{x1}, k_{y1}, k_{z1}, k_{x2}, k_{y2}, k_{z2})
\]
\[ [K_g] \text{ is composed of } [K_{g_1}] \text{ which is the tensional stiffness of the shaft. } [K_{g_1}] \text{ and } [K_{w_1}] \text{ are the stiffnesses of the bearings. } [K_g] \text{ is in the following form}
\]
\[
[K_g] =
\begin{bmatrix}
  k_{g_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & k_{w_1} & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & k_{g_2} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & k_{g_1} & -k_{g_1} & 0 & 0 \\
  0 & 0 & 0 & -k_{g_1} & k_{g_1} & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & k_{g_2} & -k_{g_2} \\
  0 & 0 & 0 & 0 & 0 & -k_{g_2} & k_{g_2}
\end{bmatrix}
\]

\[ [K(t)] \text{ is the matrix of the mesh stiffness. It is time-dependent (constants } t_{1...9} \text{ are given in Table 1).}
\]
\[
[K(t)] =
\begin{bmatrix}
  K_{11}(t) & K_{12}(t) \\
  K_{21}(t) & K_{22}(t)
\end{bmatrix}
\]

\[ [K_{11}(t)] = k_m(t)
\]
\[
\begin{bmatrix}
  t_3^2 & t_1 t_3 & t_2 t_3 & -t_1^2 & -t_1 t_2 & -t_1 t_3 & 0 \\
  t_1 t_3 & t_1^2 & t_1 t_2 & -t_1 t_2 & -t_1^2 & -t_2 t_3 & 0 \\
  t_2 t_3 & t_1 t_2 & t_2^2 & -t_1 t_3 & -t_2 t_3 & -t_3^2 & 0 \\
  -t_1^2 & -t_1 t_2 & -t_1 t_3 & t_1 t_2 & t_1^2 & t_2 t_3 & t_3^2 \\
  -t_1 t_3 & -t_2 t_3 & -t_3^2 & t_1 t_3 & t_2 t_3 & t_3^2 & t_4 t_5 \\
  -t_1 t_3 & -t_2 t_3 & -t_3^2 & t_1 t_3 & t_2 t_3 & t_3^2 & t_4 t_5 \\
  -t_1 t_3 & -t_2 t_3 & -t_3^2 & t_1 t_3 & t_2 t_3 & t_3^2 & t_4 t_5 \\
  -t_1 t_3 & -t_2 t_3 & -t_3^2 & t_1 t_3 & t_2 t_3 & t_3^2 & t_4 t_5
\end{bmatrix}
\]

\[ [K_{12}(t)] = k_m(t)
\]
\[
\begin{bmatrix}
  -t_1 t_4 & -t_1 t_5 & -t_1 t_6 & -t_1 t_7 & 0 & -t_1 t_8 & -t_1 t_9 & 0 \\
  -t_2 t_4 & -t_2 t_5 & -t_2 t_6 & -t_2 t_7 & 0 & -t_2 t_8 & -t_2 t_9 & 0 \\
  -t_3 t_4 & -t_3 t_5 & -t_3 t_6 & -t_3 t_7 & 0 & -t_3 t_8 & -t_3 t_9 & 0 \\
  t_1 t_4 & t_1 t_5 & t_1 t_6 & t_1 t_7 & 0 & t_1 t_8 & t_1 t_9 & 0 \\
  t_2 t_4 & t_2 t_5 & t_2 t_6 & t_2 t_7 & 0 & t_2 t_8 & t_2 t_9 & 0 \\
  t_3 t_4 & t_3 t_5 & t_3 t_6 & t_3 t_7 & 0 & t_3 t_8 & t_3 t_9 & 0 \\
  t_4 t_5 & t_4 t_6 & t_4 t_7 & 0 & t_4 t_8 & t_4 t_9 & 0 \\
  t_5 t_6 & t_5 t_7 & t_5 t_8 & 0 & t_5 t_9 & 0 & 0 & 0 \\
  t_5 t_6 & t_5 t_7 & t_5 t_8 & 0 & t_5 t_9 & 0 & 0 & 0 \\
  t_5 t_6 & t_5 t_7 & t_5 t_8 & 0 & t_5 t_9 & 0 & 0 & 0 \\
  t_5 t_6 & t_5 t_7 & t_5 t_8 & 0 & t_5 t_9 & 0 & 0 & 0 \\
  t_5 t_6 & t_5 t_7 & t_5 t_8 & 0 & t_5 t_9 & 0 & 0 & 0 \\
  t_5 t_6 & t_5 t_7 & t_5 t_8 & 0 & t_5 t_9 & 0 & 0 & 0
\end{bmatrix}
\]

\[ [K_{21}(t)] = [K_{12}(t)]^T
\]

\[ [K_{22}(t)] = k_m(t)
\]
\[
\begin{bmatrix}
  t_3^2 & t_4 t_5 & t_4 t_6 & t_4 t_7 & 0 & t_4 t_8 & t_4 t_9 & 0 \\
  t_4 t_5 & t_5^2 & t_5 t_6 & t_5 t_7 & 0 & t_5 t_8 & t_5 t_9 & 0 \\
  t_4 t_6 & t_5 t_6 & t_6^2 & t_6 t_7 & 0 & t_6 t_8 & t_6 t_9 & 0 \\
  t_4 t_7 & t_5 t_7 & t_6 t_7 & t_7^2 & 0 & t_7 t_8 & t_7 t_9 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  t_4 t_8 & t_5 t_8 & t_6 t_8 & t_7 t_8 & 0 & t_8^2 & t_8 t_9 & 0 \\
  t_4 t_9 & t_5 t_9 & t_6 t_9 & t_7 t_9 & 0 & t_8 t_9 & t_9^2 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Figure 3 shows the geometrical parameters that the gear mesh stiffness \( k_m(t) \) depends on. \( L_{\text{max}} \) and \( L_{\text{min}} \) are written as follows:

\[ L_{\text{max}} = (A \cdot B + A \cdot b + a \cdot B + c) \cdot L_1 \]

\[ L_{\text{min}} = (A \cdot B + A \cdot b + a \cdot B + (a + b - 1)) \cdot L_1 \quad \text{if } (a + b) > 1 \]

\[ L_{\text{min}} = (A \cdot B + A \cdot b + a \cdot B) \cdot L_1 \quad \text{if } (a + b) < 1. \]

\( A \) and \( B \) are integers. \( a \) and \( b \) are decimal functions and \( c = a \) if \( a < b \) and \( c = b \) if \( a \geq b \). \( [C] \) is the damping matrix. It is applied following Rayleigh form

\[ [C] = \mu[K] + \lambda[M] \]

\( \mu \) and \( \lambda \) are constants of proportionality.

The vector \( [F_0] \) of the external static force is

\[ [F_0] = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \tau_m, 0, 0, \tau_r]^T. \]
Figure 3. Evolution of the length of the line of action.

Table 2. Characteristics of the worm drive

<table>
<thead>
<tr>
<th></th>
<th>Worm</th>
<th>Worm gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Number of starts</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Gear material</td>
<td>Steel (S45C)</td>
<td>Bronze (CAC702)</td>
</tr>
<tr>
<td>Weight of the blocks (kg)</td>
<td>3.24</td>
<td>2.27</td>
</tr>
<tr>
<td>Rotation speed (rpm)</td>
<td>1500</td>
<td>30</td>
</tr>
<tr>
<td>Stiffnesses of the bearings (N/m)</td>
<td>$k_{x1} = k_{y1} = k_{x2} = k_{y2} = 10^8$</td>
<td>$k_{z1} = k_{z2} = 2.1 \times 10^5$</td>
</tr>
<tr>
<td>Stiffnesses of the shafts (N/m)</td>
<td>$k_{z1} = k_{z2} = 2.1 \times 10^5$</td>
<td>$k_{x1} = k_{x2} = 4 \times 10^7$</td>
</tr>
<tr>
<td>Torsional stiffnesses of the bearings (N·rd/m)</td>
<td>$k_{\varphi1} = k_{\varphi2} = 4 \times 10^7$</td>
<td>$k_{\psi1} = k_{\psi2} = 8.4 \times 10^4$</td>
</tr>
<tr>
<td>Torsional stiffness of the shaft (N·rd/m)</td>
<td>$k_{\theta1} = k_{\theta2} = 8.4 \times 10^4$</td>
<td>$k_{\varphi1} = k_{\varphi2} = 4 \times 10^7$</td>
</tr>
<tr>
<td>Modulus (m)</td>
<td>$1 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>Normal pressure angle $\alpha_n$</td>
<td>$20^\circ$</td>
<td></td>
</tr>
<tr>
<td>Lead angle $\gamma$</td>
<td>$3.58^\circ$</td>
<td></td>
</tr>
<tr>
<td>Parameters of the evolution of the line of action</td>
<td>$A = 3, B = 1, c = a = 0.7, b = 0.014$</td>
<td></td>
</tr>
<tr>
<td>Parameters of the damping matrix</td>
<td>$\mu = 10^{-5}, \lambda = 0.05$</td>
<td>[25]</td>
</tr>
</tbody>
</table>

3. Numerical results

The modal analysis of worm gearbox is carried out and its dynamic behaviour is studied. Parameters of the said gearbox are presented in Table 2.

3.1. Modal analysis of the worm gear

In this subsection, natural frequencies and modes are identified and the distribution of the modal strain and kinetic energies are analysed.

3.1.1. Natural frequencies and natural modes

Natural frequencies are identified in Figure 4 that presents frequency response function (FRF) of some degrees of freedoms (DOF). Said DOF are $x_2, y_2, \psi_2$, and $\theta_{22}$. The choice of said DOF was
Figure 4. Frequency response function (FRF) of $x_2$, $y_2$, $\psi_2$, and $\theta_{22}$.

Figure 5. Examples of different mode shapes.

Table 3. Natural frequencies of the model

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Natural mode</th>
<th>Frequency (Hz)</th>
<th>Natural mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Frq_1 = 0$</td>
<td>Rotation</td>
<td>$Frq_8 = 5251$</td>
<td>Rotation</td>
</tr>
<tr>
<td>$Frq_2 = 340$</td>
<td>Rotation</td>
<td>$Frq_9 = 5717$</td>
<td>Rotation</td>
</tr>
<tr>
<td>$Frq_3 = 870$</td>
<td>Translation</td>
<td>$Frq_{10} = 5863$</td>
<td>Combined</td>
</tr>
<tr>
<td>$Frq_4 = 1040$</td>
<td>Translation</td>
<td>$Frq_{11} = 1.35 \times 10^4$</td>
<td>Combined</td>
</tr>
<tr>
<td>$Frq_5 = 1980$</td>
<td>Rotation</td>
<td>$Frq_{12} = 1.5 \times 10^4$</td>
<td>Combined</td>
</tr>
<tr>
<td>$Frq_6 = 3972$</td>
<td>Rotation</td>
<td>$Frq_{13} = 1.6 \times 10^4$</td>
<td>Combined</td>
</tr>
<tr>
<td>$Frq_7 = 4010$</td>
<td>Rotation</td>
<td>$Frq_{14} = 1.89 \times 10^4$</td>
<td>Combined</td>
</tr>
</tbody>
</table>

based on the clarity of the figure. Other DOF have more or less the same shape with different amplitudes. The exceptions were in $x_1$, $y_2$, and $z_1$ that showed the minimum appearance of peaks specially in low natural frequencies.

This figure shows all the natural frequencies except $Frq_1$ (because $Frq_1 = 0$). All natural frequencies and their natural mode types are summarized in Table 3.

There are three types of natural modes: Translation, rotation, or a combination between both. Figure 5a is the rigid body mode. Figures 5b, c and d correspond respectively to a rotation, translation, and combined natural modes.
3.1.2. Modal energetic analysis

The modal strain energy and the modal kinetic energy for the mean value of mesh stiffness is written as the following expressions [26]. The total modal strain energy can be written as the sum of the strain energies of rotation and translation from each component. The total kinetic energy can also be written as the sum of the kinetic energies of rotation and translation from each component of the system

\[ T_\phi = \frac{1}{2} \varphi_k^T \mathbf{K} \varphi_k = \sum_{k=1}^{12} T_{\phi,k} + T_{\phi,w} \]  

\[ U_\varphi = \frac{1}{2} w_j^T \varphi_j^T M \varphi_j = \sum_{j=1}^{14} U_{\varphi,j}, \]

where \( j \) is the index of the corresponding DOF and \( k \) is the index of the corresponding stiffness. \( j \) and \( k \) are detailed in the following tables (Tables 4 and 5). \( T_{\phi,w} \) is the strain energy of the worm–worm gear meshing.

According to this modal energetic study, natural frequencies can be classified into two frequency bands: a band called “teeth modes” and a band called “bearing modes”.

Teeth modes are characterized by a dominant modal strain energy located in the gear meshing and a dominant modal movement located in the worm or in the worm gear. While, bearing mode is characterized by a dominant modal strain energy located in the bearings. Figure 6 shows the distribution of the modal strain energies.

The \( X \)-axis of Figure 6 is detailed in Table 6. Figure 6 also shows that the bearing modes are located in a frequency band that starts in \( Fr_{q_6} \) and ends in \( Fr_{q_9} \). The dominant strain energies present the torsional stiffness of the bearings about \( X \) and \( Y \) direction. In all the other frequencies, the dominant strain energy represents gear meshing. In the later frequencies, the modes are called teeth modes. Unlike all the other frequencies of the teeth mode, the strain energies of \( Fr_{q_2} \) and \( Fr_{q_5} \) are not dominant in the gear meshing. In \( Fr_{q_2} \), the dominant strain energy is divided between the torsional stiffness of the shaft of the second block about \( Z \) direction and gear meshing. In \( Fr_{q_5} \), the strain energy is dominant in the torsional stiffness of the shaft of the first block about \( Z \) direction.

The \( X \)-axis of Figure 7 is detailed in Table 7. The first block rotates following \( Z \) and translates following \( X \) or \( Y \). These motions do not have a dominant kinetic energy in any natural frequency. While each of the other 11 motions dominate in one or two natural frequencies each.

Table 8 summarizes the dominant modal movement and the dominant strain energy in low-frequency modes.

### 3.2. Dynamic behaviour of worm gearbox

In order to contribute to the study of the dynamic behaviour of worm gearbox, the dynamic response of said gearbox under ideal conditions is observed, and the evolution of its dynamic coefficient is evaluated.
Figure 6. The distribution of strain energy.
Figure 7. Kinetic energy distribution.
Table 6. X-axis of the strain energy distribution

<table>
<thead>
<tr>
<th></th>
<th>Translational stiffness of the bearing of the first block in X</th>
<th>8</th>
<th>Translational stiffness of the bearing of the first block about Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Translational stiffness of the bearing of the first block in Y</td>
<td>9</td>
<td>Torsional stiffness of the bearing of the second block about X direction</td>
</tr>
<tr>
<td>2</td>
<td>Translational stiffness of the shaft of the first block in Z</td>
<td>10</td>
<td>Torsional stiffness of the bearing of the second block about Y</td>
</tr>
<tr>
<td>3</td>
<td>Translational stiffness of the bearing of the second block in X</td>
<td>11</td>
<td>Torsional stiffness of the shaft of the first block about Z</td>
</tr>
<tr>
<td>4</td>
<td>Translational stiffness of the bearing of the second block in Y</td>
<td>12</td>
<td>Torsional stiffness of the shaft of the second block about Z</td>
</tr>
<tr>
<td>5</td>
<td>Translational stiffness of the shaft of the second block in Z</td>
<td>13</td>
<td>Gear meshing</td>
</tr>
<tr>
<td>6</td>
<td>Torsional stiffness of the bearing of the first block about X</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7. X-axis of the kinetic energy distribution

<table>
<thead>
<tr>
<th></th>
<th>Translation of the first block following X</th>
<th>8</th>
<th>Rotation of the first block following Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Translation of the first block following Y</td>
<td>9</td>
<td>Rotation of the first block following X</td>
</tr>
<tr>
<td>2</td>
<td>Translation of the second block following X</td>
<td>10</td>
<td>Rotation of the first block following Y</td>
</tr>
<tr>
<td>3</td>
<td>Translation of the second block following X</td>
<td>11</td>
<td>Rotation of the motor</td>
</tr>
<tr>
<td>4</td>
<td>Translation of the second block following Y</td>
<td>12</td>
<td>Rotation of the worm</td>
</tr>
<tr>
<td>5</td>
<td>Translation of the second block following Z</td>
<td>13</td>
<td>Rotation of the worm gear</td>
</tr>
<tr>
<td>6</td>
<td>Rotation of the first block following X</td>
<td>14</td>
<td>Rotation of the receiver</td>
</tr>
</tbody>
</table>

3.2.1. Dynamic response of worm gearbox

Figure 8 presents the gear mesh stiffness of the worm drive. Solving the movement equation by using Newmark method provides the following results. Figure 9 shows the acceleration of the worm gear couple against a time frame.

The spectrum of the acceleration of the worm is presented in Figure 10. The first peak is the mesh frequency of the worm gear \(f_m = 25 \text{ Hz}\). The following frequencies are its multiples. When the harmonic of the mesh frequency coincides with the natural frequency, disturbances occur in the spectrum. The first two disturbances coincide with the first observable natural frequencies \(Fr_{q2}\) and \(Fr_{q3}\).

3.2.2. Dynamic overload

In the study of [27], it is mentioned that, in a modal analysis, the parametric instability occur when harmonics of the mesh frequency are close to particular combinations of the natural frequencies. The instability appears when the following expressions are resulted

\[lf_m \approx Fr_{qa} + Fr_{qb}, \tag{44}\]

It is to say

\[f_m \approx \frac{Fr_{qa} + Fr_{qb}}{l}, \tag{45}\]

for \(a\) and \(b\) are the index of a corresponding natural frequency and \(l\) is an integer. When \(l = 1\) and \(a = b\), it is called primary instability, when \(l = 2\) and \(a = b\), it is called secondary instability, and when \(l \neq 1\) and \(a \neq b\) it is called combination instability. Higher order instabilities can be found which is the case in this study. It is then important to do a load sweep on the mechanism.
and extract the dynamic coefficient of loaded gear teeth. This allows for the identification of the critical areas of the system where overloaded gear teeth are found. The dynamic coefficient $C_d$ is calculated using (46). $F_D$ is the dynamic load that can be calculated using (47).

**Figure 8.** (a) Multiple and (b) single gear mesh stiffness against a time frame.

**Figure 9.** The time-dependent acceleration of the worm.
Figure 10. Spectrum the acceleration of the worm.

\[ C_d = \frac{F_D}{F} \]  
\[ F_D = F + F_{d\text{RMS}} \]  
\[ F_d(t) = K_m(t) \ast \delta(t). \]

\( F \) is the static load applied by the receiver. It is considered that it is equal to nominal load that can be delivered by the motor. \( F_d \) is calculated using (48). It is the overload created by the dynamic components. \( F_{d\text{RMS}} \) is the RMS value of \( F_d \). RMS stands for root mean square.

The dynamic coefficient is the result of a series of simulations in a frequency range that varies from 0 to 130 Hz. In this frequency band, the variation of the loads in the bearings is observed.

The evolution of the dynamic coefficient is presented in Figure 11. It is clear that there is an increase in the dynamic coefficient in certain mesh frequencies. Since this dynamic coefficient is the ratio of the dynamic load divided by the static load and thus results in the teeth dynamic overload. This can be explained by resonance of the mesh frequency with the harmonic of a natural frequency.

The dynamic coefficient is significant for \( Frq_2/6 \) and \( Frq_2/5 \) (deduced from (45)) as shown in Figure 11. The later frequencies correspond to 2690 rpm and 3475 rpm, respectively. These speeds cause the overload to become critical. The manufacturer must take into consideration these cases in order to avoid having the system functioning in critical conditions. For the other frequencies presented in Figure 11 (\( Frq_2/11, Frq_2/8, Frq_2/4, Frq_4/11 \)), the increase in the dynamic coefficient is not too high, but it is better to not to overlook them. Overall, \( Frq_2 \) seems to
Table 8. Dominant motion and dominant strain energy in low frequencies

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Dominant modal movement</th>
<th>Dominant strain energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frq₁</td>
<td>Rotation of the worm gear</td>
<td>Gear meshing</td>
</tr>
<tr>
<td>Frq₂</td>
<td>Rotation of the receiver</td>
<td>Gear meshing</td>
</tr>
<tr>
<td>Frq₃</td>
<td>Translation of the shaft of the first block in Z direction</td>
<td>Gear meshing</td>
</tr>
<tr>
<td>Frq₄</td>
<td>Translation of the second block following X</td>
<td>Gear meshing</td>
</tr>
<tr>
<td>Frq₅</td>
<td>Translation of the second block following X</td>
<td>Torsional stiffness of the shaft of the first block following Z</td>
</tr>
<tr>
<td>Frq₆</td>
<td>Rotation of the first block following X</td>
<td>Torsional stiffness of the bearing of the first block following X</td>
</tr>
<tr>
<td>Frq₇</td>
<td>Rotation of the first block following Y</td>
<td>Torsional stiffness of the bearing of the first block following X</td>
</tr>
<tr>
<td>Frq₈</td>
<td>Rotation of the second block following X</td>
<td>Torsional stiffness of the bearing of the first block following Y</td>
</tr>
<tr>
<td>Frq₉</td>
<td>Rotation of the second block following Y</td>
<td>Torsional stiffness of the bearing of the first block following Y</td>
</tr>
<tr>
<td>Frq₁₀</td>
<td>Rotation of the worm</td>
<td>Gear meshing</td>
</tr>
<tr>
<td>Frq₁₁</td>
<td>Translation of the second block following Z</td>
<td>Gear meshing</td>
</tr>
<tr>
<td>Frq₁₂</td>
<td>Translation of the second block following Y</td>
<td>Gear meshing</td>
</tr>
<tr>
<td>Frq₁₃</td>
<td>Translation of the second block following Z</td>
<td>Gear meshing</td>
</tr>
<tr>
<td>Frq₁₄</td>
<td>Rotation of the worm</td>
<td>Gear meshing</td>
</tr>
</tbody>
</table>

Figure 11. Evolution of the dynamic coefficient.

be dominant for most of the overloads that appeared in the system. For this frequency, the strain energy was divided between the gear meshing and the torsional stiffness of the shaft of the first block about Z (Figure 6), which is a particular case. We call back that the normal case is having the dominant strain energy in the bearings for bearing modes and in gear meshing in teeth modes.
So, in order to avoid having problems in the system, this particular natural frequency must be taken into consideration.

4. Conclusion

A new dynamic worm gear model is proposed. This model takes into consideration the geometry and the contact conditions of the worm drive, bearings behaviour, and inertias. Modal analysis of worm gearbox is achieved using this model. Natural frequencies are identified through FRF. Natural modes are classified into rotational, translational, and combined modes. They are also classified according to their modal energetic distributions: (i) “teeth mode” is characterized by the dominant strain energy in the contact of the teeth of the worm gear and the dominant movement of gears and (ii) “bearing mode” is characterized by the dominant strain energy in the bearings.

The proposed model of the worm gear also allows for the study of the dynamic of this gearbox. This gearbox is running under ideal conditions. It depends on the state of contact of the teeth. The spectrum of the dynamic coefficient shows several peaks at some gear mesh frequencies. In addition, the dynamic coefficient evolution allows to conclude that the dynamic load increases when a gear mesh frequency coincides with certain natural frequency or some other frequencies that divides natural frequencies.

Nomenclature

- $k_{th}$: The torsional stiffness
- $k_{zl}$: The axial stiffness
- $k_{xi}, k_{yi}$: The radial stiffness
- $x_i, y_i, z_i$: The translations of the $i$th block
- $\phi_i, \psi_i \theta_i, \theta_{2i}$: The rotations of the $i$th block
- $m_i$: The weight of the $i$th block
- $b_i$: The width of the worm gear when $i = 1$ and the length of the worm gear when $i = 2$
- $r_i$: The radius of the $i$th gear
- $\rho_i$: The density of the $i$th gear
- $I_{11}$: The inertia of the motor
- $I_{12}$: The inertia of the worm
- $I_{21}$: The inertia of the worm gear
- $I_{22}$: The inertia of the receiver
- $\delta$: The deflexion between the worm and the worm gear
- $M_i$: The point belonging to the active flank of the $i$th gear
- $\overrightarrow{U_i(M_i)}$: The displacement of $M_i$
- $\overrightarrow{n_i}$: The normal unitary outgoing vectors of $M_i$
- $l_i$: The distance separating $M_i$ from the middle of the line of action
- $T$: Kinetic energy of the model
- $R_{b1}$: The base radius of the worm gear
- $R_b$: The base radius of the worm gear
- $U$: Strain energy of the model
- $W$: The work of the system
- $\tau_m$: The torque of the motor
- $\tau_r$: The torque of the receiver
- $M$: The mass matrix
- $C$: The damping matrix
- $K_A$: The average stiffness matrix
- $K(t)$: The mesh stiffness matrix
- $\mu, \lambda$: Constants of proportionality of the damping matrix
- $F_0$: The vector of external static forces
- $T_{\phi}$: The modal kinetic energy
- $U_{\phi}$: The modal strain energy
- $w_i$: The angular frequency of the $i$th block
- $C_d$: The dynamic coefficient
- $F$: The static load
- $F_D$: The dynamic force
- $F_d$: The overload created by the dynamic component
- $F_{dRMS}$: The root mean square value of $F_d$
References