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Short paper / Note

# The Meyer's estimate of solutions to Zaremba problem for second-order elliptic equations in divergent form

*L'estimation de Meyer pour les solutions au problème de Zaremba pour les équations elliptiques du second ordre sous forme divergente*

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*Dedicated to our wives.  
Blessed is the man that hath a wise wife...  
(Sirach 25:11, translated from the Orthodox Bible)*

**Abstract.** In this paper we obtain an estimate for the increased integrability of the gradient of the solution to the Zaremba problem for divergent elliptic operator in a bounded domain with nontrivial capacity of the Dirichlet boundary conditions.

**Résumé.** Dans cet article, nous obtenons une estimation de l'intégrabilité accrue du gradient de la solution du problème de Zaremba pour un opérateur elliptique divergent dans un domaine borné avec une capacité non triviale des conditions aux limites de Dirichlet.

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**Keywords.** Meyers estimates, Mixed problem, Embedding theorems, Capacity, Rapidly alternating type of boundary conditions.

**Mots-clés.** Estimations de Meyers, Problème mixte, Théorèmes d'intégration, Capacité, Type de conditions aux limites à alternance rapide.

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## 1. Introduction

In this paper we estimate solutions to the Zaremba problem for elliptic equations in bounded Lipschitz domain  $D \in \mathbb{R}^n$ , where  $n > 1$ , of the form

$$\mathcal{L}u := \operatorname{div}(a(x)\nabla u) \quad (1)$$

with uniformly elliptic measurable and symmetric matrix  $a(x) = \{a_{ij}(x)\}$ , i.e.  $a_{ij} = a_{ji}$  and

$$\alpha^{-1}|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \alpha|\xi|^2 \quad \text{for almost all } x \in D \text{ and for all } \xi \in \mathbb{R}^n. \quad (2)$$

Below we assume that the set  $F \subset \partial D$  is closed (possibly disconnected and having nontrivial microstructure,  $n - 2 < \dim F \leq n - 1$ ) and denote  $G = \partial D \setminus F$ . Consider the Zaremba problem

$$\mathcal{L}u = \operatorname{div} f \quad \text{in } D, \quad u = 0 \quad \text{on } F, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } G, \quad (3)$$

where  $\partial u / \partial \nu$  is an outer conormal derivative of  $u$ , and the components of the vector-function  $f = (f_1, \dots, f_n)$  are functions from  $L_2(D)$ . We define a solution to problem (3) in the following way. Denote by  $W_2^1(D, F)$  the completion of the set of infinitely differentiable in the closure of  $D$  functions, vanishing in the vicinity of  $F$ , by the norm

$$\|u\|_{W_2^1(D, F)} = \left( \int_D u^2 dx + \int_D |\nabla u|^2 dx \right)^{1/2}.$$

The function  $u \in W_2^1(D, F)$  is a solution to problem (3) if the following integral identity holds

$$\int_D a \nabla u \cdot \nabla \varphi dx = \int_D f \cdot \nabla \varphi dx \quad (4)$$

for all functions  $\varphi \in W_2^1(D, F)$ .

Note that for the Laplace equation in the case of sufficiently smooth boundary of the domain  $D$  and the boundary of the Neumann data  $G$ , classical solvability of problem (3) is proved in [1] by the potential theory methods. One of the first papers on properties of solutions to the Zaremba problem for nondivergent elliptic equations with regular coefficients is [2]. In this paper, in particular, the author discovered that at the junction of the Dirichlet and Neumann data the smoothness of solutions is lost.

In the case of homogeneous Dirichlet problem for (3) with right-hand side  $f \in L_p(D)$ , where  $p > 2$ , the increased integrability of the gradient of solutions to divergent uniformly elliptic equations with measurable coefficients on the plane follows from the results of [3]. Later, the same problem in multidimensional domain with sufficiently regular boundary was considered in [4].

Our interest is in the increased integrability of the gradient of a solution to problem (3). The condition on the structure of the support of the Dirichlet data  $F$  plays the key role. For

the formulation of the result we need the notion of capacity. Let us define the capacity  $C_p(K)$ ,  $1 < p < n$ , for the compact set  $K \subset \mathbb{R}^n$  by the identity

$$C_p(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla \varphi|^p dx : \varphi \in C_0^\infty(\mathbb{R}^n), \varphi \geq 1 \text{ on } K \right\}.$$

Here and throughout,  $B_r^{x_0}$  is the open ball with radius  $r$  centered in the point  $x_0$ , and  $\text{mes}_{n-1}(E)$  is the  $(n-1)$ -measure of the set  $E \subset \partial D$ . Taking  $p = 2n/(n+2)$  as  $n > 2$  and  $p = 3/2$  as  $n = 2$  we assume one of the following conditions to be satisfied: for an arbitrary point  $x_0 \in F$  as  $r \leq r_0$  either the inequality

$$C_p(F \cap \overline{B}_r^{x_0}) \geq c_0 r^{n-p}, \quad (5)$$

or the inequality

$$\text{mes}_{n-1}(F \cap \overline{B}_r^{x_0}) \geq c_0 r^{n-1} \quad (6)$$

holds true, where the positive constant  $c_0$  does not depend on  $x_0$  and  $r$ . The condition (6) is stronger than (5), but it is more visual. Note that under one of these conditions the function  $v \in W_2^1(D, F)$  satisfies the Friedrichs inequality

$$\int_D v^2 dx \leq C \int_D |\nabla v|^2 dx, \quad (7)$$

which leads to the unique solvability of the problem (3) because of the Lax–Milgram theorem [5]. If the condition (6) holds, the inequality (7) is well known, since  $\text{mes}_{n-1}(F) > 0$ , and for the condition (5) this inequality follows, for instance, from the results of the monograph [6]. Let us explain this point in more detail. Denote by  $\mathcal{Q}_d$  the open cube with edge length  $d$  and faces parallel to the coordinate axes. Assume that the Lipschitz domain  $D$  has the diameter  $d$  and  $D \subset \mathcal{Q}_d$ . Let us define the capacity  $C_p(K, \mathcal{Q}_{2d})$  of the compact  $K \subset \mathcal{Q}_d$  with respect to the cube  $\mathcal{Q}_{2d}$ , by the formula

$$C_p(K, \mathcal{Q}_{2d}) = \inf \left\{ \int_{\mathcal{Q}_{2d}} |\nabla \varphi|^p dx : \varphi \in C_0^\infty(\mathcal{Q}_{2d}), \varphi \geq 1 \text{ on } K \right\}. \quad (8)$$

The theorem from [6, Section 14.1.2] and the comments to the results of the Chapter 14 about Lipschitz domains lead, in particular, to the inequality

$$\int_D v^2 dx \leq \frac{C(n, D)d^n}{C_2(F, \mathcal{Q}_{2d})} \int_D |\nabla v|^2 dx, \quad (9)$$

for functions  $v \in W_2^1(D, F)$ . Then we use the condition (5). First of all note that for  $1 < p < 2$  the definition of the capacity  $C_p(K, \mathcal{Q}_{2d})$  and the Hölder inequality gives the estimate

$$C_p(K, \mathcal{Q}_{2d}) \leq |\mathcal{Q}_{2d}|^{(2-p)/2} C_2^{p/2}(K, \mathcal{Q}_{2d}), \quad (10)$$

where  $|\mathcal{Q}_{2d}|$  is the  $n$ -dimensional measure of the cube  $\mathcal{Q}_{2d}$ . Now we use the fact for  $1 < p < n$  (see Proposition 4 from [7]) there exists a positive constant  $\gamma(n, p) \geq 1$  such that

$$C_p(K) \leq C_p(K, \mathcal{Q}_d) \leq \gamma C_p(K). \quad (11)$$

The condition (5) for  $1 < p < 2$  leads to  $C_p(F) > 0$ . Consequently  $C_2(F, \mathcal{Q}_{2d}) > 0$  because of (10) and (11). Using (9), we get the Friedrichs inequality (7).

## 2. Main result

To formulate the main result we give the definition of Lipschitz domain  $D$  in more detail.

A domain  $D$  will be called a Lipschitz domain, if for any point  $x_0 \in \partial D$  there exists an open cube  $Q$  centered in  $x_0$ , faces of which are parallel to the coordinate axes, the length of the cube edges are independent of  $x_0$ , and in some Cartesian coordinate system with origin in  $x_0$  the set  $Q \cap \partial D$  is a graph of the Lipschitz function  $x_n = g(x_1, \dots, x_{n-1})$  with the Lipschitz constant independent

of  $x_0$ . Denote the length of the edge of such cubes by  $2R_0$ , and the Lipschitz constant of the respective functions  $g$  by  $L$ . For definiteness, we assume that the set  $Q \cap D$  is located above the graph of the function  $g$ , and the constant  $r_0$  from the conditions (5) and (6) is less than or equal to the constant  $R_0$ .

**Theorem 1.** *If  $f \in L_{2+\delta_0}(D)$  with  $\delta_0 > 0$ , then there exist positive constants  $\delta(n, \delta_0) < \delta_0$  and  $C$  such that for the solution to problem (3) the estimate*

$$\int_D |\nabla u|^{2+\delta} dx \leq C \int_D |f|^{2+\delta} dx \tag{12}$$

holds, where  $C$  depends only on  $\delta_0$ , the dimension  $n$ , the ellipticity constant  $\alpha$  from (2), the constant  $c_0$  from (5) and (6), and also the constants  $L$  and  $R_0$  from the definition of the Lipschitz domain  $D$ .

### 3. Proof

The proof of the Theorem is based on the internal estimates of increased summation and on the estimates of increased summation in the vicinity of the boundary for the gradient of solutions to problem (3). First, we establish the estimate for the gradient of solutions to problem (3) in the vicinity of the boundary of the domain  $D$ . Here we use the technique of the straightening of the boundary  $\partial D$ . Assuming  $Q_{R_0} = \{x : |x_i| < R_0, i = 1, \dots, n\}$ , we consider an arbitrary point  $x_0 \in \partial D$  and such local Cartesian system with origin in  $x_0$ , that the part of the boundary  $\partial D$  lying in the cube  $Q_{R_0}$ , is defined in this coordinate system by the equation  $x_n = g(x')$ , where  $x' = (x_1, \dots, x_{n-1})$ , and  $g$  is a Lipschitz function with the Lipschitz constant  $L$ . Moreover the domain  $D_{R_0} = Q_{R_0} \cap D$  is situated on the set of points  $x_n > g(x')$ . Next, in  $Q_{R_0}$  we pass to the new coordinate system by nondegenerate transformation

$$y' = x', \quad y_n = x_n - g(x'). \tag{13}$$

It is clear that the part of the boundary  $Q_{R_0} \cap \partial D$  transforms to a hyperplane

$$P_{R_0} = \{y : |y_i| < R_0, i = 1, \dots, n-1, y_n = 0\}.$$

Also it is easy to see that the image of the domain  $Q_{R_0}$  contains the cube

$$K_{R_0} = \{y : |y_i| < (1 + \sqrt{n-1}L)^{-1}R_0, i = 1, \dots, n\}. \tag{14}$$

In addition in the semi-cube  $K_{R_0}^+ = K_{R_0} \cap \{y : y_n > 0\}$  which is contained in the image of the domain  $D \cap Q_{R_0}$ , problem (3) has the form

$$\tilde{\mathcal{L}}u = \operatorname{div} \tilde{f} \quad \text{in } K_{R_0}^+, \quad u = 0 \quad \text{on } \tilde{F}_{R_0}, \quad \frac{\partial u}{\partial \tilde{\nu}} = 0 \quad \text{on } \tilde{G}_{R_0}. \tag{15}$$

We keep the same notation for its solution. Here

$$\tilde{\mathcal{L}}u := \operatorname{div}(b(y)\nabla u) \tag{16}$$

is the elliptic operator with symmetric matrix  $b(y) = \{b_{ij}(y)\}$  satisfying

$$\beta^{-1}|\xi|^2 \leq \sum_{i,j=1}^n b_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2 \quad \text{for almost all } y \in K_{R_0}^+ \text{ and for all } \xi \in \mathbb{R}^n,$$

where the constant  $\beta$  depends only on the constant  $\alpha$  from (2) and the Lipschitz constant  $L$  of the function  $g$ . The right-hand side of the equation has the form

$$\begin{aligned} \tilde{f}(y) &= (\tilde{f}_1(y), \dots, \tilde{f}_n(y)), \quad \text{where } \tilde{f}_i(y) = f_i(y', y_n + g(y')) \text{ for } i = 1, \dots, n-1, \\ \tilde{f}_n(y) &= \sum_{i=1}^{n-1} \frac{\partial g(y')}{\partial y_i} f_i(y', y_n + g(y')) + f_n(y', y_n + g(y')). \end{aligned} \tag{17}$$

The sets  $\tilde{F}_{R_0}$  and  $\tilde{G}_{R_0}$  satisfy  $\tilde{F}_{R_0} = \tilde{F} \cap P_{R_0} \cap K_{R_0}$  and  $\tilde{G}_{R_0} = \tilde{G} \cap P_{R_0} \cap K_{R_0}$ , where  $\tilde{F}, \tilde{G}$  are the images of the sets  $F \cap Q_{R_0}$  and  $G \cap Q_{R_0}$  respectively, and  $\partial u / \partial \tilde{\nu}$  is an outer conormal derivative of the function  $u$ , connected with the operator (16).

Consider the even continuation of the function  $u$ , satisfying (15), with respect to the hyperplane  $\{y : y_n = 0\}$ . We keep the same notation for the continued function. This function satisfies the following relation:

$$\tilde{\mathcal{L}}_1 u = \operatorname{div} h \quad \text{in } K_{R_0} \setminus \tilde{F}_{R_0}, \quad u = 0 \quad \text{on } \tilde{F}_{R_0}. \tag{18}$$

Here

$$\tilde{\mathcal{L}}_1 u := \operatorname{div}(c(y)\nabla u),$$

the positive definite matrix  $c = \{c_{ij}(y)\}$  is such that the elements  $c_{jn}(y) = c_{jn}(y)$  for  $j \neq n$  are odd continuations of the elements  $b_{jn}(y)$  from (16), and all others elements  $c_{ij}(y)$  are even continuations of  $b_{ij}(y)$ . The vector-functions  $h = (h_1, \dots, h_n)$  in (18) are defined in an analogous way: the components  $h_i(y)$  for  $i = 1, \dots, n - 1$  are even continuations of the components  $\tilde{f}_i(y)$  from (15), and  $h_n(y)$  is odd continuation of  $\tilde{f}_n(y)$ . It is clear that the solution to problem (18) is a function  $u \in W_2^1(K_{R_0})$ , satisfying the integral identity (see (4))

$$\int_{K_{R_0}} c(y)\nabla u \cdot \nabla \varphi \, dy = \int_{K_{R_0}} h \cdot \nabla \varphi \, dy \tag{19}$$

for all test-functions  $\varphi \in W_2^1(K_{R_0}, F_{R_0})$ . Here  $W_2^1(K_{R_0}, F_{R_0})$  is the closure of the set of infinitely differentiable functions in the closure of  $K_{R_0}$ , vanishing in the vicinity of  $\partial K_{R_0}$  and  $F_{R_0}$ , by the norm

$$\|u\|_{W_2^1(K_{R_0}, F_{R_0})} = \left( \int_{K_{R_0}} u^2 \, dx + \int_{K_{R_0}} |\nabla u|^2 \, dx \right)^{1/2}.$$

Denote by  $Q_R^{y_0}$  the open cube centered in  $y_0$ , the edge length equal to  $2R$ , and faces parallel to the coordinate axes. Below we assume that

$$y_0 \in K_{R_0/2} \setminus \partial K_{R_0/2}, \quad \text{where } R \leq \frac{1}{2} \operatorname{dist}(y_0, \partial K_{R_0/2}).$$

and denote

$$\int_{Q_R^{y_0}} f \, dx = \frac{1}{|Q_R^{y_0}|} \int_{Q_R^{y_0}} f \, dx,$$

where  $|Q_R^{y_0}|$  is the  $n$ -dimensional measure of the cube  $Q_R^{y_0}$ .

By means of the conditions (5) or (6), the appropriate choice of the test-function in (19), the embedding theorem (see [6, Section 14.1.2] and the estimate of the Proposition 4 from [6, Section 13.1.1]), and also the Poincaré–Sobolev inequality with  $p$  from (5), we obtain

$$\left( \int_{Q_R^{y_0}} |\nabla u|^2 \, dy \right)^{1/2} \leq C(n, \alpha, c_0, L) \left( \left( \int_{Q_{2R}^{y_0}} |\nabla u|^p \, dy \right)^{1/p} + \left( \int_{Q_{2R}^{y_0}} |h|^2 \, dy \right)^{1/2} \right).$$

From this estimate (holds true for all considered cubes  $Q_R^{y_0}$ ) and the generalized Gehring Lemma (see [8], [9], and also [10, Chapter 7]), bearing in mind the edge length of the cube  $K_{R_0}$  (see (14)), we get under the condition  $h \in L_{2+\delta_0}(K_{R_0})$  with  $\delta_0 > 0$ , the estimate

$$\int_{K_{R_0/4}} |\nabla u|^{2+\delta} \, dy \leq C(n, \alpha, \delta_0, c_0, L, R_0) \int_{K_{R_0/2}} |h|^{2+\delta} \, dy$$

with positive constant  $\delta = \delta(n, \delta_0)$ . This estimate can be rewritten because of the evenness of the function  $u$  with respect to the hyperplane  $\{y : y_n = 0\}$ , in the form (see (15))

$$\int_{K_{R_0/4}^+} |\nabla u|^{2+\delta} \, dy \leq C(n, \alpha, \delta_0, c_0, L, R_0) \int_{K_{R_0/2}^+} |\tilde{f}|^{2+\delta} \, dy. \tag{20}$$

We make the inverse transformation to (13). It is easy to see that the preimage of the semi-cube  $K_{R_0/2}^+$  is contained in the set  $D_{R_0}$ , and the preimage of the semi-cube  $K_{R_0/4}^+$  contains the set  $D_{\theta R_0}$ , where  $\theta = \theta(n, L) > 0$ . Keeping in mind (17), by means of (20) we have

$$\int_{D_{\theta R_0}} |\nabla u|^{2+\delta} dx \leq C(n, \alpha, \delta_0, c_0, L, R_0) \int_{D_{R_0}} |f|^{2+\delta} dx.$$

Passing to the Cartesian system of coordinates with the origin in  $x_0 \in \partial D$ , which we used from the very beginning of the reasoning, we get

$$\int_{D \cap Q_{\theta R_0}^{x_0}} |\nabla u|^{2+\delta} dx \leq C(n, \alpha, \delta_0, c_0, L, R_0) \int_{D \cap Q_{R_0}^{x_0}} |f|^{2+\delta} dx.$$

Since  $x_0 \in \partial D$  is an arbitrary boundary point and the boundary  $\partial D$  is compact, one can find such finite cover of  $\partial D$ , that the closed set

$$\mathcal{D}_{\theta_1 R_0} = \{x \in D : \text{dist}(x, \partial D) \leq \theta_1 R_0\}, \quad \theta_1 = \theta_1(n, L) > 0$$

is contained in the union of the sets  $D \cap Q_{\theta R_0}^{x_i}$ , where  $x_i \in \partial D$ . Then, summing the inequalities

$$\int_{D \cap Q_{\theta R_0}^{x_i}} |\nabla u|^{2+\delta} dx \leq C(n, \alpha, \delta_0, c_0, L, R_0) \int_{D \cap Q_{R_0}^{x_i}} |f|^{2+\delta} dx,$$

we came to the estimate

$$\int_{\mathcal{D}_{\theta_1 R_0}} |\nabla u|^{2+\delta} dx \leq C(n, \alpha, \delta_0, c_0, L, R_0) \int_D |f|^{2+\delta} dx.$$

The internal estimate

$$\int_{D \setminus \mathcal{D}_{\theta_1 R_0}} |\nabla u|^{2+\delta} dx \leq C(n, \alpha, \delta_0, R_0) \int_D |f|^{2+\delta} dx$$

is well known and follows, for instance, from the paper [4]. As a result, combining the last two inequalities, we came to (12).

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## References

- [1] S. Zaremba, "On a mixed problem for Laplace's equation", *Uspehi Matem. Nauk (N.S.)* **1** (1946), no. 3–4(13–14), p. 125-146 (Russian). MR0025032.
- [2] G. Fichera, "Sul problema misto per le equazioni lineari alle derivate parziali del secondo ordine di tipo ellittico", *Rev. Roumaine Math. Pures Appl.* **9** (1964), p. 3-9 (Italian).
- [3] B. V. Bojarskii, "Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients", *Mat. Sb. N.S.* **43** (1957), no. 85, p. 451-503 (Russian). MR0106324.
- [4] N. G. Meyers, "An  $L^p$ -estimate for the gradient of solutions of second order elliptic divergence equations", *Ann. Scuola Norm.-Sci.* **17** (1963), no. 3, p. 189-206, 3-e série.
- [5] P. D. Lax, A. Milgram, "Parabolic equations", in *Contributions to the Theory of Partial Differential Equations*, Ann. Math. Studies, vol. 33, Princeton University Press, Princeton, 1954, p. 167-190.
- [6] V. Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, Springer, Berlin–New York, 2011.
- [7] V. G. Maz'ia, "On the continuity at a boundary point of solutions of quasilinear elliptic equations", *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* **13** (1970), p. 42-55 (Russian). MR 0274948 (43:706).
- [8] F. W. Gehring, "The  $L^p$ -integrability of the partial derivatives of a quasiconformal mapping", *Acta Math. Vietnamica* **130** (1973), p. 265-277.
- [9] M. Giaquinta, G. Modica, "Regularity results for some classes of higher order nonlinear elliptic systems", *J. Reine Angew. Math.* **311/312** (1979), p. 145-169.
- [10] I. V. Skrypnik, *Methods for Analysis of Nonlinear Elliptic Boundary Value Problems*, Translations of Math. Monographs, vol. 139, AMS, Providence, 1994.