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3D Ising model: a view from the conformal bootstrap island

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3D Ising model: a view from the conformal bootstrap island

Modèle d'Ising en 3D, vu de l'île d'auto-amorçage conforme

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Abstract. We explain how the axioms of Conformal Field Theory are used to make predictions about critical exponents of continuous phase transitions in three dimensions, via a procedure called the conformal bootstrap. The method assumes conformal invariance of correlation functions, and imposes some relations between correlation functions of different orders. Numerical analysis shows that these conditions are incompatible unless the critical exponents take particular values, or more precisely that they must belong to a small island in the parameter space.

Résumé. Nous expliquerons comment utiliser les axiomes de la théorie conforme des champs pour faire des prédictions sur les exposants critiques des transitions de phase continues en trois dimensions, par une procédure dite "auto-amorçage conforme". La méthode consiste à faire l'hypothèse que les fonctions de corrélation ont l'invariance conforme, puis à imposer certaines relations entre les fonctions de corrélation d'ordre différent. L'analyse numérique montre alors que ces conditions ne sont compatibles que si les exposants critiques ont des valeurs particulières, plus précisément s'ils se trouvent dans un îlot étroit de l'espace des paramètres.

Keywords. Phase transitions, Critical exponents, Conformal symmetry, Ising model, Emergence.

Mots-clés. Transitions de phase, Exposants critiques, Symétrie conforme, Modèle d'Ising, Émergence.

1. Introduction

Physics has many *emergent laws*, which follow in a non-obvious way from more fundamental microscopic laws. Whenever this happens, we have two separate goals: to understand how the emergent law arises, and to explore its consequences.

One example is the Gibbs distribution of equilibrium statistical mechanics: the probability for the system in thermal equilibrium at temperature T to be found in a state n of energy E_n is proportional to $\exp(-E_n/T)$. One may be interested in deriving this emergent law from microscopic models of thermalization, or in exploring the myriad of its physical consequences.

This text, based on a recent talk for an audience of mathematical physicists, is about the "conformal field theory" (CFT), a set of emergent laws governing critical phenomena in equilibrium statistical mechanics (such as the liquid–vapor critical point or the Curie point of ferromagnets). CFT makes certain assumptions about the state of the system at a critical point. These assumptions can be given reasonable physical explanations, but for the purposes of this talk we will view them as axioms.

CFT is an "emergent law of second degree" with respect to the Gibbs distribution, by itself emergent. It sidesteps the Gibbs distribution similarly to how the Gibbs distribution sidesteps a thermalization model. Future work should derive the CFT axioms with mathematical rigor from the Gibbs distribution. Our goal here will be to explain the axioms and how they lead to concrete predictions for observable quantities through a procedure called the "conformal bootstrap".

CFT/bootstrap approach to critical phenomena is an alternative to the better-known Wilson's renormalization group (RG) theory. The RG is more directly related to the Gibbs distribution than CFT, although it too is not fully mathematically justified. The RG will not be treated here except for a few comments.

We will not give many references, which can be found in the recent review [1]. See also lecture notes [2–4]. An excellent set of recorded lectures is [5].

2. The first two CFT axioms

We will describe the axioms of Conformal Field Theory (CFT) on $\mathbb{R}^d$, $d \geqslant 3$. These axioms are well established in the physics literature. We will present them in a form hopefully more accessible to mathematicians. In particular, we will try to avoid (or at least explain) excessive physics jargon. Similar axioms, with additional bells and whistles, hold in $d = 2$ dimensions [6].[1]

Suppose we are given a collection of real-valued functions

$$\mathcal{T} = \{G_{i_1,\dots,i_n}(x_1,\dots,x_n)\}, \tag{1}$$

defined for $x_p \in \mathbb{R}^d$, $x_p \neq x_q$ ($p,q = 1,\dots,n$), where $n \geqslant 1$ and the indices i_p are non-negative integers.

Functions (1) are called "n-point correlators of fields $A_{i_1},\dots,A_{i_n}$" and are also denoted by

$$\langle A_{i_1}(x_1) A_{i_2}(x_2) \cdots A_{i_n}(x_n)\rangle. \tag{2}$$

The collection $\mathcal{T}$ is called a CFT if it satisfies certain axioms stated below. Different CFT's are just different collections of correlators satisfying those axioms.[2]

Note that the "field" A_i is just a label, a name, and (2) is just a notation for $G_{i_1,\dots,i_n}(x_1,\dots,x_n)$. The statistical average operation suggested by this notation does not have a direct meaning in the CFT axioms. It will be handy in the *interpretation* of the axioms (Section 4).

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[1]Our axioms should be viewed as a sketch of future complete axiomatics, which has not yet been written up in the mathematics literature. A different approach to axiomatize CFTs in $d \geqslant 3$ (akin to Segal's 2d CFT axiom) is in [7], but it makes the connection to concrete calculations less explicit. A nicer starting point is the recent mathematics paper [8] which develops 2d CFT from the conformal bootstrap perspective. It would be interesting to generalize it to $d \geqslant 3$.

[2]One also uses the term "Conformal Field Theory" in a meta-sense, as the study of all possible CFTs.

Axiom 1 (Simple properties). *Correlators have the following properties:*

(a) *They are invariant under permutation of any two fields:*[3]

$$\langle A_{i_1}(x) A_{i_2}(y) \cdots \rangle = \langle A_{i_2}(y) A_{i_1}(x) \cdots \rangle, \textit{etc.} \tag{3}$$

(b) *Index $i = 0$ is associated with the "unit field", replaced by 1 under the correlator sign:*

$$\langle A_0(x) \times \textit{anything} \rangle = \langle \textit{anything} \rangle. \tag{4}$$

(c) *The 1-point (1pt) correlators are given by*

$$\langle A_0(x) \rangle \equiv 1, \quad \langle A_i(x) \rangle \equiv 0 \quad (i \geqslant 1). \tag{5}$$

(d) *The 2pt correlators are given by*

$$\langle A_i(x) A_j(y) \rangle = \frac{\delta_{ij}}{|x - y|^{2\Delta_i}} \quad (x, y \in \mathbb{R}^d). \tag{6}$$

where δ_{ij} is the Kronecker symbol, and $\Delta_i \geqslant ((d-2)/2)$ $(i \geqslant 1)$ is a real number called "scaling dimension of field A_i". For the unit field we have $\Delta_0 = 0$.

(e) *The set of scaling dimensions $\{\Delta_i\}$ is called the "spectrum". It is a discrete set without accumulation points (i.e. there are finitely many scaling dimensions below any $\Delta_* < \infty$).*

Axiom 2 (Conformal invariance). *Correlators are conformally invariant, in the sense that they satisfy the constraint*

$$G_{i_1,\ldots,i_n}(x_1,\ldots,x_n) = \left(\prod_{p=1}^{n} \lambda(x_p)^{\Delta_{i_p}} \right) G_{i_1,\ldots,i_n}(f(x_1),\ldots,f(x_n)), \tag{7}$$

or equivalently, using notation (2),

$$\langle A_{i_1}(x_1) \cdots A_{i_n}(x_n) \rangle = \left(\prod_{p=1}^{n} \lambda(x_p)^{\Delta_{i_p}} \right) \langle A_{i_1}(f(x_1)) \cdots A_{i_n}(f(x_n)) \rangle, \tag{8}$$

where $f(x)$ is an arbitrary conformal transformation of $\mathbb{R}^d$ and $\lambda(x) = |\partial f / \partial x|^{1/d}$ is its scale factor.

Recall that conformal transformations satisfy the constraint $\partial f^\mu / \partial x^\nu = \lambda(x) R^\mu{}_\nu(x)$ where $R^\mu{}_\nu(x) \in SO(d)$. For $d \geqslant 3$, these transformations form a group $SO(d+1,1)$.

Remark 1. Conformal transformations of $\mathbb{R}^d$ may send points to infinity, and should be thought more properly as acting on $\mathbb{R}^d \cup \{\infty\}$, the d-dimensional analogue of the Riemann sphere. To treat the point at infinity on equal footing with the other points, one can put $\mathbb{R}^d \cup \{\infty\}$ in one-to-one correspondence with the d-dimensional unit sphere S^d via the stereographic projection. This subtlety will be glossed over here.

2.1. *Basic consequences of conformal invariance*

We will state without proof a few basic consequences of the above axioms. One can check that the 2pt correlators given in Axiom 1(d) are consistent with Axiom 2. Note that the same scaling dimension Δ_i has to appear in all n-point correlators involving the field A_i. The 3pt correlators are fixed by Axiom 2 up to an overall factor:

$$\langle A_i(x_1) A_j(x_2) A_k(x_3) \rangle = \frac{c_{ijk}}{x_{12}^{\Delta_i + \Delta_j - \Delta_k} x_{13}^{\Delta_i + \Delta_k - \Delta_j} x_{23}^{\Delta_j + \Delta_k - \Delta_i}}, \tag{9}$$

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[3]One can also consider CFTs with fields having fermionic statistics, whose correlators change sign under permutations. Such CFTs are important e.g. for describing quantum critical points of many-electron systems. Here we only consider commuting fields for simplicity.

where c_{ijk} is totally symmetric by Axiom 1(a), and we denoted $x_{ij} = |x_i - x_j|$. For 4pt correlators Axiom 2 implies the following functional form:

$$\langle A_i(x_1) A_j(x_2) A_k(x_3) A_l(x_4) \rangle = \left(\frac{x_{24}}{x_{14}} \right)^{\Delta_i - \Delta_j} \left(\frac{x_{14}}{x_{13}} \right)^{\Delta_k - \Delta_l} \frac{g_{ijkl}(u,v)}{x_{12}^{\Delta_i + \Delta_j} x_{34}^{\Delta_k + \Delta_l}}, \tag{10}$$

where $g_{ijkl}(u,v)$ is a function of conformally invariant cross-ratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = u|_{1 \leftrightarrow 3} = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}. \tag{11}$$

By Axiom 1(a) functions g_{ijkl} with permuted indices are all related, e.g. permutation $1 \leftrightarrow 3$ generates the constraint:

$$u^{-\frac{\Delta_i + \Delta_j}{2}} g_{ijkl}(u,v) = v^{-\frac{\Delta_k + \Delta_j}{2}} g_{kjil}(v,u), \text{etc.} \tag{12}$$

2.2. *Primaries and descendants*

Group-theoretically, the transformation

$$A(x) \to \lambda(x)^\Delta A(f(x)) \tag{13}$$

is an irreducible representation π_Δ of the conformal group on scalar functions $A : \mathbb{R}^d \to \mathbb{R}$. Equation (7) means that the correlators $G_{i_1,\dots,i_n}$ belong to the invariant subspace of the tensor product representation $\otimes_{p=1}^n \pi_{\Delta_{i_p}}$ (so they can be called "invariant tensors").

We formulated Axioms 1, 2, 3 for the fields transforming as (13), called "scalar fields". These axioms can and should be extended to allow for fields with tensor indices. First of all, we have to add fields $\partial^\alpha A_i(x)$ which are partial derivatives (of arbitrary order) of the fields A_i. Their correlators are defined as derivatives of the original ones:

$$\langle \partial^\alpha A_i(x) \cdots \rangle := \partial_x^\alpha \langle A_i(x) \cdots \rangle. \tag{14}$$

This is, in a sense, just a convenient notation. The basic fields $A_i(x)$ whose correlators transform as (13) are called "primaries", while their derivatives "descendants". Transformation rules for correlators of descendants can be obtained by differentiating (13).

The second extension is a bit less trivial. We should generalize (13), allowing for fields with values in a finite-dimensional vector space V, $\dim V > 1$, transforming under the conformal group via

$$A(x) \to \lambda(x)^\Delta \rho(R(x)) A(f(x)), \tag{15}$$

where ρ is an irreducible representation of SO(d) acting in V. Such fields are called "primary spinning fields". One example is $V = \{$symmetric traceless rank-l tensors$\}$. Correlators of spinning fields then take values in the tensor product $\otimes_{p=1}^n V_{i_p}$ and satisfy a conformal invariance constraint similar to (7) but with factors of $\rho_{i_p}(R(x_p))$ in the l.h.s. (Derivatives of spinning fields are then also added as in (14).) Adding spinning fields would complicate the notation a bit. We will neglect them here, although practical conformal bootstrap computations always allow for their presence.

3. The OPE axiom

The last "OPE axiom" will relate different correlators, and in particular correlators with different n. This is unlike the previous axioms which involved one n-point correlator at a time.[4]

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[4] Except the rather trivial Axiom 1(b).
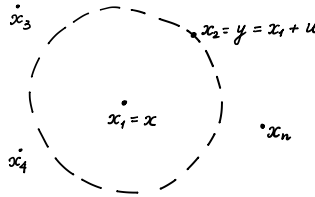
Figure 1. The OPE expansion applies when all points x_p ($p \geqslant 3$) lie further from x_1 than x_2.

Suppose we are given two collections of real numbers

$$\{\lambda_{ijk}\}, \{s_{ijk}^{(r)}, r \geqslant 1\}, \tag{16}$$

where i, j, k run over the field indices (non-negative integers). With these numbers as coefficients, "Operator Product Expansion" (OPE) is constructed as a set of formal equalities (one for each pair of fields A_i and A_j):

$$A_i(x) A_j(y) = \sum_{k=0}^{\infty} \frac{\lambda_{ijk}}{|u|^{\Delta_i + \Delta_j - \Delta_k}}$$
$$\times [A_k(x) + s_{ijk}^{(1)} u_\mu \partial_x^\mu A_k(x) + (s_{ijk}^{(2)} u_\mu u_\nu + s_{ijk}^{(3)} u^2 \delta_{\mu\nu}) \partial_x^\mu \partial_x^\nu A_k(x) + \cdots], \tag{17}$$

where $u = y - x$. Using the OPE for the first pair of fields inside the n-point correlator (2) with $n \geqslant 2$, we get a set of candidate identities among correlators:

$$\langle A_i(x) A_j(y) \mathbf{\Pi} \rangle = \sum_{k=0}^{\infty} \frac{\lambda_{ijk}}{|u|^{\Delta_i + \Delta_j - \Delta_k}} [\langle A_k(x) \mathbf{\Pi} \rangle + s_{ijk}^{(1)} u_\mu \partial_x^\mu \langle A_k(x) \mathbf{\Pi} \rangle + \cdots], \tag{18}$$

where we denoted $i_1 = i$, $i_2 = j$, $x_1 = x$, $x_2 = y$, and $\mathbf{\Pi} = \Pi_{p=3}^n A_{i_p}(x_p)$ is the product of all other fields in the correlator. In the l.h.s. we have an n-point correlator, while in the r.h.s. we have an infinite series of $(n-1)$-point correlators and derivatives thereof.

The OPE axiom gives a condition for when the candidate identity (18) is a true identity.

Axiom 3 (OPE). *There exists a set of coefficients* (16)*, such that* (18) *holds as a true relation between correlators (the series in the r.h.s. converges absolutely to the l.h.s.) as long as* $|x_p - x| > |u|$ *for all* $p \geqslant 3$ *(see Figure 1).*

What can be said about coefficients (16) which make this axiom work? To see this, let us apply (18) to a 3pt correlator. Because the 2pt correlators vanish for non-identical fields (δ_{ij} in Axiom 1(d)), the sum in the r.h.s. collapses to the single k value, and we get:

$$\langle A_i(x) A_j(y) A_k(x_3) \rangle = \frac{\lambda_{ijk}}{|u|^{\Delta_i + \Delta_j - \Delta_k}} \left[\frac{1}{|x - x_3|^{2\Delta_k}} + s_{ijk}^{(1)} u_\mu \partial_x^\mu \frac{1}{|x - x_3|^{2\Delta_k}} + \cdots \right], \tag{19}$$

where we used that $\langle A_k(x) A_k(x_3) \rangle = 1/|x - x_3|^{2\Delta_k}$. On the other hand, we already know that the 3pt correlator in the l.h.s. has form (9) by Axiom 2. Let us then expand (9) for small y and match with (19). From the leading term we find $\lambda_{ijk} = c_{ijk}$.[5] Relative to this overall normalization, the subleading terms on the l.h.s. are fixed, and this allows to determine $s_{ijk}^{(r)}$ uniquely as rational functions of Δ_i, Δ_j, Δ_k and d. We conclude that all coefficients (16) can be uniquely determined by demanding that the OPE axiom works for the 3pt correlators. Furthermore, the axiom says that the same set of coefficients should then also work for any n-point correlators.

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[5]In particular we learn that λ_{ijk} has to be symmetric, just as c_{ijk}. Note also that by putting $A_k = A_0 = 1$ in (9) and by using Axiom 1(d), we get $\lambda_{ij0} = c_{ij0} = \delta_{ij}$.

Definition 1. A conformal field theory (CFT) $\mathcal{T}$ in $d \geqslant 3$ dimensions is a collection of correlators (1) satisfying Axioms 1–3.

Remark 2. We included in our axioms the conditions $\Delta_i \geqslant (d-2)/2$ for $i \geqslant 1$ in Axiom 1(d), and $\lambda_{ijk} \in \mathbb{R}$ in (16). By this, we are restricting our discussion to a subclass of conformal field theories called reflection positive (or unitary). Many statistical physics systems at criticality (such as the Ising model or $O(N)$ models) are known to be described by unitary CFTs.

3.1. *CFT data*

The spectrum Δ_i and the OPE coefficients λ_{ijk} comprise the "dataset" of a CFT $\mathcal{T}$:

$$\mathrm{Data}(\mathcal{T}) = \{\Delta_i, \lambda_{ijk}\}. \tag{20}$$

As discussed, $\mathrm{Data}(\mathcal{T})$ is in one-to-one correspondence with the 2pt and 3pt correlators of $\mathcal{T}$. Moreover, knowing $\mathrm{Data}(\mathcal{T})$ we can reconstruct all n-point correlators, for an arbitrarily high n. Indeed, from $\mathrm{Data}(\mathcal{T})$ we can construct the OPE (the coefficients $s_{ijk}^{(r)}$ are not included in $\mathrm{Data}(\mathcal{T})$ since they are uniquely determined by Δ's and d). Then, we can recursively reduce any n-point correlator to lower-point ones, until we get to the known 2pt and 3pt correlators.[6]

We thus see that the dataset $\mathrm{Data}(\mathcal{T})$ encodes full information about the CFT $\mathcal{T}$. Below we will describe a program of classifying CFTs by classifying their data sets. But first let us discuss the interpretation of the CFT axioms.

4. Interpretation

Notation $\langle A_{i_1}(x_1) A_{i_2}(x_2) \cdots A_{i_n}(x_n) \rangle$ for $G_{i_1,\dots,i_n}(x_1,\dots,x_n)$ acquires a meaning in the interpretation of the CFT axioms, as correlation functions of statistical systems at their critical points. CFT calculations are then interpreted as predictions for the critical exponents of statistical physics models. Although the CFT calculations based on the axioms are completely rigorous, the interpretation step is at present non-rigorous. Hopefully it will be justified in the future.

Let us discuss how this works for the 3d Ising model: a lattice model with the Hamiltonian $H = -\sum_{\langle xy \rangle} S_x S_y$ where $S_x = \pm 1$ are spins on a cubic lattice, with the nearest-neighbor ferromagnetic interaction.

The "3d Ising CFT" is a CFT in $d = 3$ describing the critical point of this model, and of any other model in the same universality class. Just as the lattice Ising model, this CFT has a global $\mathbb{Z}_2$ invariance with all fields divided into $\mathbb{Z}_2$-even and $\mathbb{Z}_2$-odd.[7] It contains a $\mathbb{Z}_2$-odd scalar primary field denoted $\sigma(x)$, whose correlators

$$\langle \sigma(x_1) \sigma(x_2) \cdots \sigma(x_n) \rangle \tag{21}$$

are interpreted as the 3d Ising model spin correlation functions

$$\langle S_{x_1} S_{x_2} \cdots S_{x_n} \rangle \tag{22}$$

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[6] We should take care that the OPE is used for a pair of fields at positions x_1, x_2 verifying conditions of Axiom 3, so that it converges. This is the case if x_2 is the unique position with the minimal distance from x_1. There are degenerate configurations when such a pair cannot be found, because each point has two or more nearest neighbors at equal distance (e.g. the vertices of a regular polygon). It is then always possible to apply a small conformal transformation which moves points to a non-degenerate configuration. In the new configuration the OPE converges and we can compute the value of the correlator. We then conformal-transform back to the original configuration. This way we can compute correlators in any configuration of non-coincident points.

[7] We have not included the notion of global symmetry in the CFT axioms, but this extension is straightforward. It just means that all fields transform in finite-dimensional irreducible representations of a compact global symmetry group G, forming a direct product with the conformal group. All correlators are invariant tensors of G, and the OPE respects this additional symmetry.

computed at the critical temperature $T = T_c$, at distances $|x_p - x_q|$ much larger than the lattice spacing. While $\langle \cdots \rangle$ in (21) is just a notation, in (22) it is a true average with respect to the Gibbs distribution, in the thermodynamic limit. By Axiom 2, correlator (21) is conformally invariant, and thus in particular scale invariant, scale transformations being a part of the conformal group. This means (Δ_σ is the scaling dimension of σ):

$$\langle \sigma(\lambda x_1) \sigma(\lambda x_2) \cdots \sigma(\lambda x_n) \rangle = \lambda^{-n\Delta_\sigma} \langle \sigma(x_1) \sigma(x_2) \cdots \sigma(x_n) \rangle. \tag{23}$$

On the other hand, Equation (22) clearly does not have such an exact scale invariance, already because it is defined on a lattice. The precise statement of agreement at large distances is[8]

$$\lim_{|x_p - x_q| \to \infty} \frac{\langle S_{x_1} S_{x_2} \cdots S_{x_n} \rangle}{\langle \sigma(x_1) \sigma(x_2) \cdots \sigma(x_n) \rangle} = C^n, \tag{24}$$

where C is some constant, which is n-independent but non-universal (e.g. it would change if we add next-to-nearest interactions to the lattice model, which does not change the universality class).

Other 3d Ising CFT fields will correspond to other lattice-scale operators. E.g. we can consider the product of two nearby spins (separated in an arbitrary direction)

$$E_x = S_x S_{x+1} - \langle S_x S_{x+1} \rangle, \tag{25}$$

where $\langle S_x S_{x+1} \rangle$ is subtracted so that $\langle E_x \rangle = 0$. The 3d Ising CFT contains a $\mathbb{Z}_2$-even scalar primary $\varepsilon(x)$ whose correlators describe long-distance limits of the E_x correlators, similarly to (24).

More generally, we expect to have a CFT associated with every universality class of continuous phase transitions. This CFT will share global symmetry ($\mathbb{Z}_2$, $O(N)$, etc) with the universality class, and its scaling dimensions will determine the critical exponents. It has not been proven yet, starting from the lattice models or in any other way, that all these CFTs actually exist. This is the non-rigorous part of the CFT game.

CFT fields A_i and their scaling dimensions Δ_i also have counterparts in the RG approach to critical phenomena [9].[9] Namely, they correspond to the eigenvectors and the eigenvalues of RG transformation linearized near a fixed point describing a continuous phase transition. Fields of scaling dimension $\Delta_i < d$ ($\Delta_i > d$) correspond to the relevant (irrelevant) deformations of the fixed point. This dictionary is not needed for the actual CFT calculations, but only for interpreting the results.

We expect that the above-mentioned fields σ and ε are the only two relevant fields of the 3d Ising CFT. This follows from the experimental fact that the critical point of the 3d Ising model is in the same universality class as the liquid–vapor critical point, which is reached by tuning two parameters (pressure and temperature).

5. Conformal bootstrap program

5.1. *Consistency*

Conformal bootstrap program attempts to classify CFTs by classifying their datasets. That this may be possible was first suggested by Polyakov [10].

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[8]Equivalently, one can consider a sequence of lattice models with a smaller and smaller lattice spacing a, and take the limit $a \to 0$ while keeping x_a fixed.

[9]On the other hand, the OPE coefficients λ_{ijk} do not feature prominently in the RG approach.

$$\overset{\bullet}{x_1} \qquad \overset{\bullet}{x_2} \; \overset{\bullet}{x_3} \qquad\qquad \overset{\bullet}{x_4}$$

Figure 2. The 4pt correlator for this configuration of points can be reduced to 3pt correlators using the OPE (18) with (x, y) being one of the following pairs: (x_1, x_2), (x_2, x_3), (x_3, x_2), (x_4, x_3).

We call a dataset $\mathcal{D} = \{\Delta_i, \lambda_{ijk}\}$ "consistent" if it is a dataset of some CFT: $\mathcal{D} = \mathrm{Data}(\mathcal{T})$. Ideally, we would like to have a list of all consistent data sets:[10]

$$\{\mathrm{Data}(\mathcal{T}_1), \mathrm{Data}(\mathcal{T}_2), \ldots\}, \tag{26}$$

but it is not currently known how to generate such a list. The following question is less ambitious but still very interesting:

$$\textit{Q1: Given a trial dataset } \mathcal{D}, \textit{ decide if it is inconsistent.} \tag{27}$$

It turns out that this has an algorithmic answer. This will allow progress on classification by ruling out inconsistent data sets (rather than by constructing consistent ones).

The idea is straightforward: given a trial dataset $\mathcal{D} = \{\Delta_i, \lambda_{ijk}\}$, we will try to construct all correlators, looking for some inconsistency with the axioms.

The first step is to construct the 2pt and 3pt correlators. These are simply given by explicit formulas from Axiom 1(d) and (9) with $c_{ijk} = \lambda_{ijk}$. So far no room for inconsistency.

Then we proceed to construct the 4pt correlators. For this we consider the OPE series reducing them to the 3pt correlators. All information needed to write down these series is contained in Δ_i and λ_{ijk}. But now we need to check a couple of things. First, do these series converge where Axiom 3 says they should? For this, the trial OPE coefficients λ_{ijk} should not grow too fast as a function of k for fixed i, j. The required growth condition can be shown to take a relatively simple form:

$$\sum_{k=0}^{\infty} (4\rho)^{\Delta_k} \lambda_{ijk}^2 < \infty \quad \forall \rho < 1. \tag{28}$$

Second, there are several ways to reduce a 4pt correlator to 3pt correlators via the OPE, and they all should agree in the overlapping regions of convergence. See Figure 2 for an example. This condition is called "crossing", and it is not automatically satisfied.[11] Assuming that it also holds, we can define the 4pt correlators as the sum of OPE series.

We then proceed to higher n-point correlators. Similarly to $n = 4$, they are reduced to $(n-1)$-point correlators via the OPE, and we need to check convergence and crossing. It turns out that crossing for $n \geqslant 5$ is automatically satisfied once we impose crossing for all 4pt correlators. On the other hand, the general convergence condition is stronger than (28), and it can be expressed as follows. Consider an infinite matrix consisting of OPE coefficients λ_{ijk} with a fixed j and arbitrary i, k:

$$(M^{(j)})_{ik} = \lambda_{ijk}. \tag{29}$$

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[10]We are not giving full details necessary to make this statement precise. One important subclass of CFTs are "local CFTs", which roughly correspond to critical points of lattice models with finite-range interactions. It is expected that most local CFTs are isolated. One exception are CFTs with "exactly marginal" fields of dimension $\Delta = d$, which form finite-dimensional continuous families. A folk conjecture says that exactly marginal fields in $d \geqslant 3$ require supersymmetry, which makes this exception non-generic.

[11]4pt crossing constraints were first discussed in Refs. [10, 11]. The word "crossing" comes from an analogy with relativistic Quantum Field Theory. There, the $2 \to 2$ scattering amplitude $\mathcal{M}(p_1, p_2 \to p_3, p_4)$ is invariant under "crossing transformations", when one incoming particle is moved ("crosses") into the group of outgoing particles, while one outgoing particle crosses in the opposite direction.

Then any such $M^{(j)}$, viewed as an operator from i-indexed sequences to k-indexed sequences, should be bounded with respect to a certain weighted ℓ_2 norm (with Δ-depending weights).[12] Equation (28) follows from this general condition when we apply the operator to a sequence consisting of a single nonzero element, and demand that the result have a finite norm.

To summarize, consistent datasets are those which satisfy the general convergence and the 4pt crossing conditions. The convergence condition is the less interesting of the two. Below we will focus on the 4pt crossing, which will allow us to put constraints on the fields of low scaling dimension.

5.2. *Conformal blocks and 4pt crossing*

Here we will describe how to put 4pt crossing constraint into a more explicit form, by expanding 4pt correlators in a basis of special functions called conformal blocks.

Recall that conformally invariant 4pt correlators have form (10). When we compute the 4pt correlator in the r.h.s. of (10) using the OPE, we should get something consistent with this formula. Let us see how this happens. Applying the OPE to the first pair of fields, we get an expression of the form:

$$\langle A_i(x_1) A_j(x_2) A_k(x_3) A_l(x_4) \rangle = \sum_m \frac{\lambda_{ijm}}{x_{12}^{\Delta_i + \Delta_j - \Delta_m}} [\langle A_m(x_1) A_k(x_3) A_l(x_4) \rangle + \cdots], \tag{30}$$

where … denotes terms proportional to $s_{ijm}^{(r)}$ times derivatives acting on the 3pt correlator $\langle A_m A_k A_l \rangle$, which is in turn given by λ_{mkl} times an x-dependent function which can be read off from (9). It can be shown that by doing all derivatives and infinite sums over r, the r.h.s. of (30) takes the form:

$$\left(\frac{x_{24}}{x_{14}} \right)^{\Delta_i - \Delta_j} \left(\frac{x_{14}}{x_{13}} \right)^{\Delta_k - \Delta_l} \frac{1}{x_{12}^{\Delta_i + \Delta_j} x_{34}^{\Delta_k + \Delta_l}} \sum_m \lambda_{ijm} \lambda_{mkl} G_{\Delta_m}(u, v). \tag{31}$$

The functions $G_{\Delta_m}(u, v)$ appearing here are called "conformal block". These functions are fixed by conformal symmetry. They depends on the exchanged scaling dimension Δ_m, and on the space dimension d.[13] Notably, they do not depend on the OPE coefficients λ_{ijk} whose product appears as a prefactor in (31).

Theory of conformal blocks is huge and it's not possible to do it justice in this text. It has connections to representation theory, orthogonal polynomials, and integrable quantum mechanics. There are no fully general closed form expressions of conformal blocks in terms of the classical special functions. Fortunately, they admit rapidly convergent power series expansions which allow efficient numerical evaluation. This is what is used in practical applications.

The conformal block is simple only for the exchanged unit field: $A_m = A_0 = 1$, when we have:

$$G_0(u, v) = 1, \quad \lambda_{ij0} = \delta_{ij}, \quad \lambda_{0kl} = \delta_{kl}, \tag{32}$$

where we also gave the OPE coefficients for this case (see footnote 5).

Comparing (31) with (10) we see that they are consistent if we identify:

$$g_{ijkl}(u, v) = \sum_m \lambda_{ijm} \lambda_{mkl} G_{\Delta_m}(u, v). \tag{33}$$

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[12] This is related to something called "radial quantization", which we do not describe in this text. This convergence condition for higher n-point corelators has not been discussed in detail in the literature.

[13] They also depend on the external dimension differences $\Delta_i - \Delta_j$, $\Delta_k - \Delta_l$ but we will omit this from the notation. In a full treatment involving spinning fields, the conformal blocks also depend on the spin of the fields.

This gives a compact formula to compute the 4pt correlators in terms of the CFT data. We can also obtain a compact expression for the 4pt crossing constraints, by substituting (33) into (12):

$$u^{-\frac{\Delta_i+\Delta_j}{2}}\sum_m \lambda_{ijm}\lambda_{mkl}G_{\Delta_m}(u,v) = v^{-\frac{\Delta_k+\Delta_j}{2}}\sum_m \lambda_{kjm}\lambda_{mil}G_{\Delta_m}(v,u). \tag{34}$$

As (12), this corresponds to the permutation $x_1 \leftrightarrow x_3$. Constraints corresponding to other permutations take a rather similar form. They should also be considered, although we will not discuss them here explicitly.

Now, we can test a trial dataset $\mathscr{D}$ for consistency, by checking (34) for all possible choices of i, j, k, l, in the region of overlapping convergence. This region is not empty. E.g., let us fix points $x_{1,3,4}$ so that x_4 is far away from $x_{1,3}$. Then the l.h.s. should converge within the set $\{x_2 : x_{12} < x_{13}\}$, and the r.h.s. in $\{x_2 : x_{23} < x_{13}\}$. These two balls have a nontrivial overlap.

5.3. *Partially specified datasets*

In Sections 5.1 and 5.2, we gave an answer to the consistency question (27). Unfortunately, the described procedure is not by itself practically useful, since it assumes that the trial dataset $\mathscr{D}$ is fully specified, which includes infinitely many parameters (all scaling dimensions and OPE coefficients). To correct for this, let us define the notion of a "partially specified trial dataset", which is a list $\mathscr{L}$ of finitely many assumptions on scaling dimensions and OPE coefficients. We say that $\mathscr{L}$ is consistent if there is at least one CFT $\mathscr{T}$ whose dataset Data($\mathscr{T}$) satisfies the assumptions. The following is then a more practical version of question (27):

> Q2: Given a partially specified trial dataset $\mathscr{L}$, decide if it is inconsistent. (35)

Although this looks like a much harder question than (27), it turns out that this question can also be answered, based on (34), using numerical algorithms. This was first shown by Rattazzi, Tonni, Vichi and the author [12] and led to the rapid development of the numerical conformal bootstrap in the last 10 years. We will explain how this work on an example in the next section.

6. Example: constraining the 3d Ising CFT

Let us fix two real numbers Δ_1, Δ_2 in the interval $[1/2, 3]$, and consider the following list of assumptions $\mathscr{L} = \mathscr{L}(\Delta_1, \Delta_2)$ about a 3d CFT:

- $\mathbb{Z}_2$ global symmetry;
- there is one field which is $\mathbb{Z}_2$-odd, one which is $\mathbb{Z}_2$-even, and they have scaling dimensions Δ_1 and Δ_2;
- all other fields have scaling dimensions $\Delta_i \geqslant 3$ i.e. are irrelevant.

As discussed in Section 4, the 3d Ising CFT satisfies $\mathscr{L}(\Delta_\sigma, \Delta_\varepsilon)$. Our strategy will be to exclude a large part of the (Δ_1, Δ_2)-plane by showing that $\mathscr{L}(\Delta_1, \Delta_2)$ is inconsistent there. This will imply that the scaling dimensions of the 3d Ising CFT must belong to the remaining part of the plane.

6.1. *One crossing constraint*

Consider first the 4pt crossing for $\langle A_1 A_1 A_1 A_1 \rangle$. Putting $i = j = k = l = 1$ in (34), we obtain:

$$u^{-\Delta_1}\sum_m p_m G_{\Delta_m}(u,v) = v^{-\Delta_1}\sum_m p_m G_{\Delta_m}(v,u), \quad p_m = \lambda_{11m}^2 \geqslant 0. \tag{36}$$

We know that $p_0 = G_0 = 1$ (see (32)), so isolating those terms we write this as

$$h(u,v) \equiv v^{-\Delta_1} - u^{-\Delta_1} = \sum_{m=2}^{\infty} p_m F_{\Delta_m}(u,v), \tag{37}$$

$$F_\Delta(u,v) := u^{-\Delta_1} G_\Delta(u,v) - v^{-\Delta_1} G_\Delta(v,u).$$

Note that F_Δ also depends on Δ_1. The sum starts from $m = 2$ because $\lambda_{111} = 0$ for the $\mathbb{Z}_2$-odd A_1.

Geometrically, Equation (37) means that h, viewed as a vector in a space of two-variable functions, belongs to a convex cone $\mathscr{C}$ generated by vectors F_{Δ_2} and F_Δ with $\Delta \geqslant 3$. We include all F_Δ with $\Delta \geqslant 3$ as generators of the cone since we don't know the exact values of Δ_m for $m \geqslant 3$, but only that $\Delta_m \geqslant 3$. Denote by $\mathscr{C}^*$ the dual convex cone, which is the set of all linear functionals α which are positive on all vectors generating the cone:

$$\alpha[F_{\Delta_2}] \geqslant 0, \quad \alpha[F_\Delta] \geqslant 0 \quad \forall \Delta \geqslant 3. \tag{38}$$

Suppose that there exists a functional $\alpha_0 \in \mathscr{C}^*$ such that

$$\alpha_0[h] < 0. \tag{39}$$

Then by acting with α_0 on (37) we get a contradiction. So, this equation cannot be satisfied for any nonnegative p_m. This is how one shows that the assumption $\mathscr{L}(\Delta_1, \Delta_2)$ is inconsistent: by exhibiting a functional α_0 which satisfies (38) and (39).

Numerically, one works with a finite dimensional space of functionals $\mathscr{A}(\Lambda)$ which are finite sums of partial derivatives at a particular point:

$$\alpha[f] = \sum_{m+n \leqslant \Lambda} \alpha_{m,n} \partial_u^m \partial_v^n f(u_0, v_0), \tag{40}$$

where Λ is a parameter, to be taken as large as possible to have the maximal constraining powers (within the available computer resources). One then minimizes $\alpha[h]$ over all $\alpha \in \mathscr{C}^* \cap \mathscr{A}(\Lambda)$, looking for a functional satisfying (39). This is a convex optimization problem (continuous linear programming), which can be solved by efficient numerical algorithms. If the minimum is negative, then we ruled out $\mathscr{L}(\Delta_1, \Delta_2)$. If it is positive, and cannot be made negative by increasing Λ, this would mean that $\mathscr{L}(\Delta_1, \Delta_2)$ is consistent with crossing for $\langle A_1 A_1 A_1 A_1 \rangle$.

With this procedure, Ref. [13] showed that the constraint $\mathscr{L}(\Delta_1, \Delta_2)$ is inconsistent in a significant portion of parameter space. Invoking an extra and so far unproven assumption, that the 3d Ising CFT lies at a singular boundary point of the consistent region (the so called "kink"), Refs. [13, 14] gave the first conformal bootstrap determination of $\Delta_\sigma, \Delta_\varepsilon$. Subsequent work has shown that the kink assumption is unnecessary, provided that one includes crossing constraints for the 4pt correlators $\langle A_1 A_1 A_2 A_2 \rangle$, $\langle A_2 A_2 A_2 A_2 \rangle$. We will now explain briefly how this was done.

6.2. *Several crossing constraints*

To increase the constraining power, a natural idea is to include crossing constraints for the other 4pt correlators of fields A_1 and A_2. While $\langle A_2 A_2 A_2 A_2 \rangle$ is completely analogous to $\langle A_1 A_1 A_1 A_1 \rangle$, one encounters a crucial difference when analyzing $\langle A_1 A_1 A_2 A_2 \rangle$. Namely, its conformal block expansion involves products of two different OPE coefficients $\lambda_{11m} \lambda_{22m}$. These products are not necessarily positive, because λ_{ijk} may have either sign. On the other hand, positivity of the coefficients $p_m = \lambda_{11m}^2$ played a crucial role in making the minimization problem of Section 6.1 convex. To overcome this obstacle, one analyzes all three correlators together, and considers the matrix

$$P_m = \begin{pmatrix} \lambda_{11m}^2 & \lambda_{11m} \lambda_{22m} \\ \lambda_{11m} \lambda_{22m} & \lambda_{22m}^2 \end{pmatrix}. \tag{41}$$

Crucially, this matrix is positive semidefinite: $P_m \succcurlyeq 0$. This condition is convex, and provides a good substitute for the simple positivity in the bootstrap problems involving multiple correlators. The resulting problem is that of continuous semidefinite programming, and it can still be attacked by efficient numerical algorithms. This was realized and carried out in Refs. [15–17] which found a consistent "island" near $\Delta_1 \approx 0.5181489(10)$, $\Delta_2 \approx 1.412625(10)$. The 3d Ising CFT point $(\Delta_\sigma, \Delta_\varepsilon)$ must live somewhere in this tiny island (Figure 3).
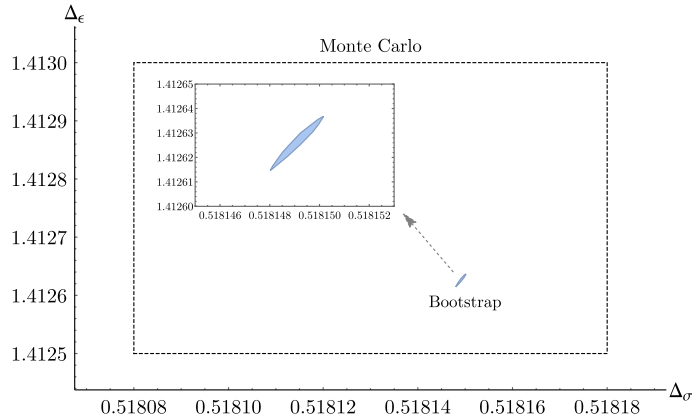
Figure 3. The bootstrap island to which the 3d Ising CFT must belong [17]. Also shown is a Monte Carlo result [18] for the same scaling dimensions. This plot used $\Lambda = 43$ in (40).

The scaling dimensions $\Delta_\sigma, \Delta_\epsilon$ determine the main critical exponents of the 3d Ising model $\alpha, \beta, \gamma, \delta, \eta, \nu$. In what follows we will focus on η and ν, given by

$$\eta = 2\Delta_\sigma - 1, \tag{42}$$
$$\nu = 1/(d - \Delta_\varepsilon). \tag{43}$$

Equation (43) deserves a comment, because it expresses an off-critical quantity (the exponent ν describing behavior of the correlation length close to the critical point) via a critical theory parameter Δ_ε. This is an example of how CFT can make predictions about small deviations from the critical theory, which arise at short distances from relevant perturbations, and at large distances from the irrelevant ones. Such predictions are done via a technique called "conformal perturbation theory," which we have not explained. Equation (44) below is another simple example.

Another important quantity is the "correction to scaling" exponent ω. It appears in the rate $\sim 1/r^\omega$ at which the limit in (24) is achieved, assuming that all distances $|x_p - x_q| \sim r$ are of the same order. It also appears in the subleading singularities of all quantities exhibiting powerlaw behavior near the critical point (e.g. the specific heat). While describing deviations from criticality, ω like ν can be expressed in terms of a purely critical parameter:

$$\omega = \Delta_3 - d, \tag{44}$$

where Δ_3 is the scaling dimension of the leading irrelevant $\mathbb{Z}_2$-even scalar operator. The conformal bootstrap determines Δ_3 (and hence ω) by scanning the island in Figure 3 and reconstructing the spectrum which provides a solution to the 4pt crossing [19].

In Table 1 we report the values of the critical exponents ν, η, ω according to the conformal bootstrap, Monte Carlo simulations and RG calculations. We also include some experimental measurements of ν and η.[14] The conformal bootstrap predictions are the most precise, and they are in a good agreement with the Monte Carlo and RG. There is also reasonable agreement between the theory and the experiment, although the experimental accuracy is not amazing.

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[14]Since ω parametrizes subleading powers, it is harder to measure, and we are not aware of any published result.

Table 1. Some representative theoretical and experimental determinations of the 3d Ising critical exponents. See [20], Section 3.2, for more references

Ref	Year	Method/Experiment	ν	η	ω
[17, 19]	2016	Conformal bootstrap	0.629971(4)	0.036298(2)	0.82968(23)
[18]	2010	Monte Carlo	0.63002(10)	0.03627(10)	0.832(6)
[21]	1998	RG	0.6304(13)	0.0335(25)	0.799(11)
[22]	1989	Binary fluid	0.628(8)	0.0300(15)	
[23]	2009	Binary fluid	0.629(3)	0.032(13)	
[24]	1994	Binary mixture	0.623(13)	0.039(4)	
[25]	2000	Liquid–vapor	0.62(3)		
[26]	1998	Liquid–vapor		0.042(6)	
[27]	1987	Uniaxial antiferromagnet	0.64(1)		

7. Conclusions

Conformal bootstrap calculations provide predictions for observable physical quantities from the CFT axioms. Agreement of these predictions with alternative theoretical determinations and the experiment increase our belief in the validity of the axioms.

Feynman [28] called the Gibbs distribution the "summit of statistical mechanics", the entire subject being either the "climb-up" to derive it, or the "slide-down" when it is applied. Echoing Feynman, we may call the CFT a summit of the theory of critical phenomena, the conformal bootstrap being the way to slide down. To climb up would be to prove the validity of the interpretation of the CFT axioms described in Section 4. Unfortunately, relatively little rigorous work has been done in the way of climbing up.[15] One should also not forget a second major peak in the same mountain range: the Renormalization Group.

Acknowledgements

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[15]The last 20 years, starting with Smirnov [29], have seen significant progress in showing rigorously conformal invariance of specific 2d models. This program is still far from establishing conformal invariance of a generic critical theory, let alone the full scope of the CFT axioms, such as the existence of a complete set of local operators and the OPE. We are also not aware of any relevant mathematical work in $d \geqslant 3$.

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