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Negative index materials: at the frontier of macroscopic electromagnetism

Matériaux d'indice négatif : à la frontière de l'électromagnétisme macroscopique

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Abstract. The notions of negative refraction and negative index, introduced by V. Veselago more than 50 years ago, have appeared beyond the frontiers of macroscopic electromagnetism and purely formal during 30 years, until the work of J. Pendry in the late 1990s. Since then, the negative index materials and the metamaterials displayed extraordinary properties and spectacular effects which have tested the domain of validity of macroscopic electromagnetism. In this article, several of these properties and phenomena are reviewed. First, mechanisms underlying the negative index and negative refraction are briefly presented. Then, it is shown that the frame of the time-harmonic Maxwell's equations cannot describe the behavior of electromagnetic waves in the situations of the perfect flat lens and corner reflector due to the presence of essential spectrum at the perfect -1 index frequency. More generally, it is shown that simple corner structures filled with frequency dispersive permittivity have a whole interval of essential spectrum associated with an analog of "black hole" phenomenon. Finally, arguments are provided to support that, in passive media, the imaginary part of the magnetic permeability can take positive and negative values. These arguments are notably based on the exact expression, for all frequency and wave vector, of the spatially-dispersive effective permittivity tensor of a multilayered structure.

Résumé. Les notions de réfraction négative et d'indice négatif, imaginées par V. Veselago il y a plus de 50 ans, ont semblé au-delà des frontières de l'électromagnétisme macroscopique et sont restées purement formelles pendant 30 ans, jusqu'aux travaux de J. Pendry à la fin des années 1990. Depuis lors, les matériaux à indice négatif et les métamatériaux ont montré des propriétés extraordinaires et des effets spectaculaires qui ont mis à l'épreuve le domaine de validité de l'électromagnétisme macroscopique. Dans cet article, plusieurs de ces propriétés et phénomènes sont passés en revue. Tout d'abord, les mécanismes sous-jacents aux indices négatifs et à la réfraction négative sont brièvement présentés. Ensuite, il est montré que le cadre des équations de Maxwell harmoniques en temps ne peut pas décrire le comportement des ondes électromagnétiques dans les situations de la lentille plate et du réflecteur en coin parfaits en raison de la présence de spectre essentiel à la fréquence où l'indice prend la valeur -1. Plus généralement, il est montré que de simples structures en coin remplies d'une permittivité dispersive en fréquence ont un intervalle entier de spectre essentiel associé à un analogue du phénomène de « trou noir ». Enfin, des arguments sont fournis pour soutenir que, dans les milieux passifs, la partie imaginaire de la perméabilité magnétique peut prendre des valeurs positives et négatives. Ces arguments reposent notamment sur l'expression exacte, pour toutes les fréquences et tous les vecteurs d'onde, du tenseur de permittivité effective avec dispersion spatiale d'une structure multicouche.

Keywords. Negative index, Metamaterials, Frequency dispersion, Corner mode, Spatial dispersion, Passivity, Permeability.

Mots-clés. Indice négatif, Métamatériaux, Dispersion en fréquence, Mode de coin, Dispersion spatiale, Passivité, Perméabilité.

1. Introduction

The notion of negative index of refraction has been introduced more than 50 years ago by V. Veselago [1]. The refraction at an interface separating two media with positive and negative refractive indices is subject to the usual Snell–Descartes law:

$$n_1 \sin\phi_1 = n_2 \sin\phi_2. \tag{1}$$

Consequently, if the refractive indices n_1 and n_2 of the two media have opposite sign, e.g. $n_1 > 0$ and $n_2 < 0$, then the refraction angles ϕ_1 and ϕ_2 have also opposite sign, so that the ray is negatively refracted at the interface (see Figure 1). In macroscopic electromagnetism, media with negative refractive index can be modelled by magnetodielectric materials with simultenously negative values of the dielectric permittivity ε and magnetic permeability μ [1]. In such media, the wave vector $\boldsymbol{k}$ has opposite direction from the Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$, and the triplet formed by the electric field $\boldsymbol{E}$, the induction field $\boldsymbol{H}$ and the wave vector $\boldsymbol{k}$ is left-handed. Thus V. Veselago also coined a medium with negative refractive index a “left-handed material” [1].

Since no material can be found in nature with simultaneously negative values of the permittivity ε and permeability μ, the notion of negative refractive index has appeared beyond the frontiers of macroscopic electromagnetism and thus remained purely formal for thirty years, until the work of J. Pendry in 1999. In [2], J. Pendry *et al.* showed that *“microstructures built from nonmagnetic conducting sheets exhibit an effective magnetic permeability μ_{eff}, which can be tuned to values not accessible in naturally occurring materials”*, paving the way towards artificial magnetism, negative index materials and, more generally, metamaterials with extraordinary properties. Since then, the notion of negative index material has opened a vast range of possibilities and has tested the domain of validity of macroscopic electromagnetism.

In this paper, a brief overview of the electromagnetic negative index materials is presented through the mechanisms underlying the negative index of refraction, the negative index and the proposal of the perfect −1 index lens. The fundamental role of frequency dispersion in negative index materials and metamaterials is shown. Then, the spectral properties of corner structures with frequency dispersive permittivity are analyzed and an analog of “black hole” phenomenon is discussed. Finally, the key role of spatial dispersion (or non-locality) in effective permeability and metamaterials is highlighted. In particular the question on the sign of the imaginary part of the permeability in passive media is addressed. The new phenomena and questions brought by these topics within the frame of the macroscopic electromagnetism will be discussed.

2. Mechanisms underlying negative index materials

Media with negative refractive index have appeared unavailable since no natural medium may have simultaneously permittivity ε and permeability μ with real part taking negative values. Indeed, negative values of the permittivity occur in metals at frequencies around the visible range while, in the same range, the values of the permeability must be restricted around that of vaccum permeability [3]. The range of possible macroscopic electromagnetic responses has been first extended with the works on the so-called *bounds* on the effective parameters of composite

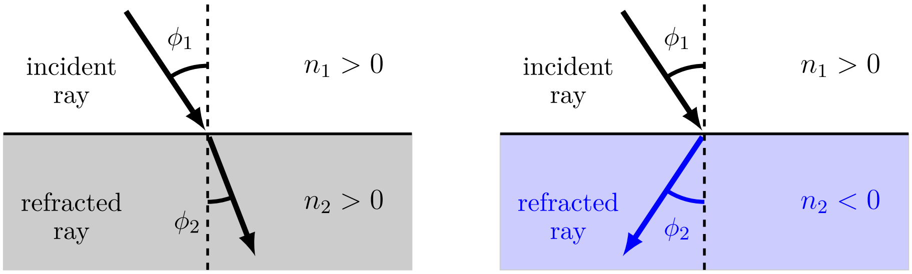

Figure 1. Refraction at an interface separating two media with positive refractive indices (left) and with positive and negative refractive indices (right).

materials, for instance on the effective permeability [4] and permittivity [5–8]. For given permittivity constants and volume fractions of the components constituting a composite, such *bounds* characterize the set of possible macroscopic responses and identify the microstructure producing the extreme effective parameters in this set, see the book of G. Milton [9] for an extensive presentation of the *bounds* of composites. These works on bounds offered new possibilities in terms of achievable values of permittivity and anisotropy. These works have been however restricted to the quasistatic regime in the frame of classical homogenization [10], where the effective parameters result from an averaging process. In this frame, the range of frequencies with negative values of permittivity cannot be significantly extended and, moreover, the effective permeability remains equal to the vacuum permeability as soon as the components constituting the composite are non-magnetic, leaving the negative refractive index unachievable in theory and in practice.

The fundamental steps that led to the negative indices have been completed thanks to the works of J. Pendry and his colleagues. Back in 1996, they proposed three-dimensional network structures made of thin metallic wires and showed theoretically, numerically and experimentally that such structures exhibit metallic behavior with low plasma frequency in the range of GHz [11, 12]. In such structures, the plasma frequency of the original metal ω_p, which is proportional to the ratio $\sqrt{N/m_{\mathrm{eff}}}$ of the electron density N and the electron effective mass m_{eff}, is made lower using two mechanisms: (i) the electron density N is reduced since the fraction of metal in the wires network is lower than in the bulk metal and (ii) the electron effective mass m_{eff} is enhanced by confining the electrons in the thin wires. With these mechanisms, the effective plasma frequency is strongly reduced and the metallic behavior encountered in the visible range is extended to the Ghz range, which allows effective permittivity with negative values in a new range of frequencies. Then, in 1999, these physicists proposed structures made of the so-called *split rings* that exhibit resonant effective magnetic permeability in the GHz range [2]. Here, the magnetic response is induced by loops of current in the rings. In addition, this magnetic response is enhanced by introducing a thin split which makes the *split ring* equivalent to a *LC* resonator, the capacitance C resulting from the thin split and the inductance L resulting from the ring. The resonance is essential since it enhances the effective magnetic response and thus offers the possibility to address negative values of the effective permeability. Finally, combining these conducting non-magnetic *split ring* resonators with thin wires, D. Smith *et al.* proposed a composite medium with simultaneously negative permittivity and permeability in the GHz range [13]: this work enabled the experimental demonstration in the Ghz range of a negative refractive index [14], the extraordinary electromagnetic property imagined by V. Veselago in 1968 [1].

It is stressed that, in this new kind of metallic composites proposed by J. Pendry and his colleagues, the microstructure induces resonances in the effective electric and magnetic responses, which makes the nature of the underlying mechanism different from the one encountered so far in classical homogenization and in the *bounds* of composites. Hence this new kind of composites offering extraordinary properties has been coined *metamaterials* in 2001 [15].

The implementation of metallic resonant composites operating at frequencies higher than Ghz appeared difficult and remains challenging, notably in the visible range, because of the requirements on the dimensions of the nanostructures and the presence of absorption in metals. However, in the visible and the near-infrared, purely dielectric periodic strutures, or photonic crystals [16, 17], have been exploited to obtain negative refraction at in interface separating such a structure and a homogeneous medium [18, 19]. In that case the resonance is not produced by the solely resonator itself (e.g. a *split ring*) but by the interaction between the dielectric particles periodically arranged. This resonant interaction requires that the distance between the particles be comparable to the wavelength, which results in severe limitations to consider a photonic crystal as an effectif homogeneous medium. Nevertheless, their ability to induce negative refraction in the visible range may have important consequences.

The mechanism leading to negative refraction with photonic crystals exploits the richness, in such periodic structures, of the dispersion law, i.e. the relationship $\omega(\boldsymbol{k})$ between the frequency ω and the Bloch wave vector $\boldsymbol{k}$. Indeed, the propagation of electromagnetic waves is governed by the group velocity $\boldsymbol{v}_g$ [20–22] defined as the gradient of the dispersion law: $\boldsymbol{v}_g = \partial_{\boldsymbol{k}}\omega(\boldsymbol{k})$. At an interface separating a homogeneous medium from a photonic crystal, the tangential component $\boldsymbol{k}^{\parallel}$ of the (Bloch) wave vector $\boldsymbol{k}$, parallel to the interface, is conserved according to the invariance of the periodic structure under the discrete set of lattice translations $\{\pm\boldsymbol{a}, \pm 2\boldsymbol{a}, \pm 3\boldsymbol{a}, \ldots\}$. Therefore, if the group velocity $\boldsymbol{v}_g$ has opposite direction from the wave vector $\boldsymbol{k}$, then this invariance of the tangential component $\boldsymbol{k}^{\parallel}$ of the wave vector results in the sign change of the tangential component $\boldsymbol{v}_g^{\parallel}$ at the interface. Such a situation, where the group velocity $\boldsymbol{v}_g$ and the Bloch wave vector $\boldsymbol{k}$ have opposite signs, can be realized thanks to the folding of the dispersion law in photonic crystals, as represented on Figure 2. Detailed analyses and numerical demonstrations of negative refraction of electromagnetic waves in photonic crystals can be found in [18, 19, 21], and experimental verification in [23].

The discovery of metamaterials and of their extraordinary properties stimulated the development of new homogenization techniques and effective medium theories, beyond the classical homogenization operating in the quasistatic limit, i.e. where the size of the microstructure tends to zero [10]. Indeed, classical homogenization results in an averaging process which cannot report properties like artificial magnetism and negative refractive index from purely dielectric constituents. For instance, the analysis of metamaterials with negative permittivity and permeability [24] has shown that the effective parameters of such structures are not quasistatic. Hence, in addition to the seminal works of J. Pendry [2, 11, 12], several new techniques have been proposed in applied mathematics and theoretical physics, extending the notion and validity of homogenization and of effective medium theory to new situations, see reference [25] for a review in 2009. The classical two-scale homogenization technique [26] has been extended to high-contrast inclusions [27] and led to the prediction of effective permeability. The retrieval method, based on the extraction of constitutive parameters from Fresnel reflexion and transmission coefficients, has been investigated for layered metamaterials [28, 29]. The classical quasistatic limit as been also overcome in the case of periodic metamaterials made of dielectric meta-atoms, by an approach relating the macroscopic fields to the microscopic fields averaged over the Floquet unit cell [30–32], which can be considered as an extension to periodic arrays of meta-atoms of the classical derivation of macroscopic Maxwell's equations [33]. Also, perturbative expansions with respect to the frequency have been proposed: when starting from the quasistatic limit [34], it has been shown that the first order in frequency reports magnetoelectric coupling while the second order in frequency reports effective magnetism (the higher orders bringing refined corrections to all these parameters), a mechanism similar to the expansion on the wave vector [3, 35]; and when

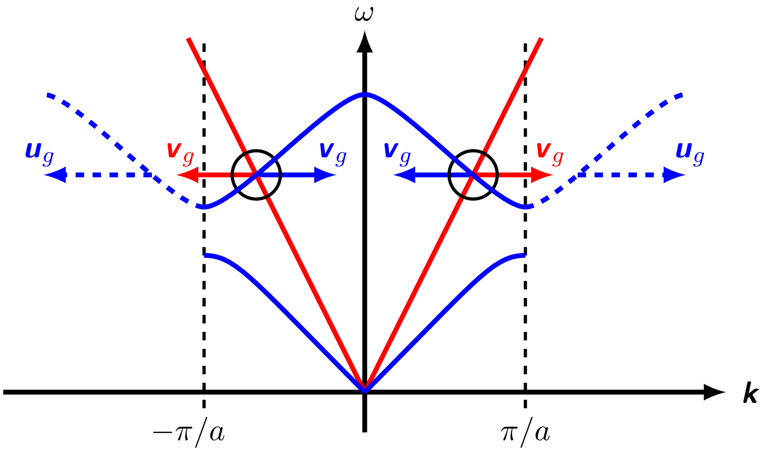

Figure 2. Effect of the folding of the dispersion law on group velocity. The red cone represents the dispersion law in a homogeneous medium with positive index: the group velocity $\boldsymbol{v}_g$ and the wave vector $\boldsymbol{k}$ in abscissa point in the same direction. The blue curve represents the dispersion law in a photonic crystals: if the dispersion law is unfolded (dotted blue curve) then the group velocity $\boldsymbol{u}_g$ and the wave vector $\boldsymbol{k}$ both point in the same direction; if the dispersion law is folded (continuous blue curve) then the group velocity $\boldsymbol{v}_g$ and the wave vector $\boldsymbol{k}$ point in opposite directions. At the couple $(\omega, \boldsymbol{k})$ indicated by the black circle, the folded dispersion law must be considered and the photonic crystal generates negative refraction.

starting from higher bands, it has been shown [36, 37] that the structure can be homogenized using the two-scale homogenization, leading to the notion of high frequency homogenization.

These non-asymptotic techniques revealed the importance of the effect of the physical boundaries of metamaterials and periodic structures [38–45]. They also highlighted the crucial role of the non-locality or *spatial dispersion* [29–32, 35, 42, 46, 47] in metamaterials and negative refractive index structures. In general, the modelling of metamaterials and periodic composites with techniques beyond the classical quasistatic limit, results unavoidably in the definition of effective parameters depending on $(\omega, \boldsymbol{k})$, the frequency (*frequency dispersion*) and the wave vector (*spatial dispersion*). Frequency and spatial dispersions are inherent to metamaterials and negative refractive media, which generated numerous questions and investigations on the *causality principle* and *passivity* of effective parameters [48–53]. In the next sections, these questions related to the dispersion are addressed.

3. The perfect lens and the spectral properties of frequency dispersive structures with negative permittivity

The most spectacular devices based on metamaterials are probably the perfect lens [54] and the invisibility cloak [55–57] proposed by J. Pendry. These propositions generated numerous interesting discussions and investigations in the community of classical electrodynamics. For the invisibility, the possibility to perfectly hide an obstacle implies that the solution to Maxwell's equations is strictly the same outside the invisibility cloak, independently of the obstacle inside the cloak, to that one would have in the absence of scattering object and cloak (so in free space). As a consequence, if the invisibility cloak is causal and passive, then the perfect invisibility can occur only at isolated frequencies [58]. Indeed, let $\boldsymbol{E}(\boldsymbol{x}, \omega)$ and $\boldsymbol{E}_0(\boldsymbol{x}, \omega)$ be the time-harmonic electric fields oscillating at the frequency ω in the presence of the cloak, respectively with and without the obstacle. For perfect invisibility, these two electric fields are equal for position vector $\boldsymbol{x}$ outside the cloak: $\boldsymbol{E}(\boldsymbol{x},\omega) = \boldsymbol{E}_0(\boldsymbol{x},\omega)$. As for causal and passive media these electric fields are analytic functions of the frequency (as soon as ω has positive imaginary part) [59], they must be equal either for isolated frequencies ω, or for all real frequencies. Perfect invisibility is therefore only achieved at isolated frequencies and is impossible over a frequency interval. These arguments show that a causal and passive invisibility cloak must be a frequency dispersive structure.

Frequency dispersion is also an important dimension of negative index materials and the perfect lens. In 1968, V. Veselago introduced the notion of negative index of refraction and showed that a simple plate of such a medium with thickness d "can focus at a point the radiation from a point source located at a distance $l < d$" [1]. In 2000, J. Pendry extended this flat lens to negative index material including evanescent waves, and concluded that it makes a perfect lens with infinite resolution [54], beyond the diffraction limit. This perfect lens and the many potential applications have been debated in the literature with intense discussions about the infinite resolution and the underlying arguments [60–63], the divergence of the field [64], the causality principle [64, 65], and even about the existence of negative index [65, 66]. The difficulty comes from the presence of a singularity in the Green's function and the solution of the time-harmonic Maxwell's equations at the frequency ω_1 of the perfect -1 index, where the relative permittivity and permeability take simultaneously the value -1: $\varepsilon(\omega_1) = -\varepsilon_0$ and $\mu(\omega_1) = -\mu_0$, where ε_0 and μ_0 are respectively the permittivity and the permeability in vacuum. This singularity is unusual since, in mathematics, it appears at the frequency ω_1 which is an eigenvalue with infinite degeneracy of the operator associated to the Maxwell's equations [67], i.e. at a frequency in the *essential spectrum* of Maxwell's equations. In physics, the strategy may be to consider the *low absorption limit*: a small absorption is introduced, e.g. $\varepsilon^{(\gamma)} = -\varepsilon_0 + \mathrm{i}\gamma$ and $\mu^{(\gamma)} = -\mu_0 + \mathrm{i}\gamma$ with $\gamma > 0$, so that the time-harmonic Maxwell's equations are well posed for $\gamma > 0$ [63], and then the limit $\gamma \downarrow 0$ is taken. However, the solution of time-harmonic Maxwell's equations does not converge when the absorption γ tends to zero. Therefore the *low absorption limit* fails in the situation of the flat lens at the frequency ω_1 of the perfect -1 index. In other words, one can conclude that the solution to the time-harmonic Maxwell's equations does not exist at the frequency ω_1 in the case of the perfect lens. Such situations where the time-harmonic Maxwell's equations have no solutions have been also uncountered with active (or gain) media [68–70].

The absence of solutions to the time-harmonic Maxwell's equations generated difficulties to analyze the behavior of the perfect flat lens, to the point of even questioning the possibility and the existence of perfect negative index media. The solution to all these difficulties lies in rigorously taking into account the frequency dispersion.

It has been noticed by V. Veselago in his seminal article [1] that the permittivity and the permeability must depend on frequency in negative index media. This requirement, which is a consequence of the *causality principle* and the *passivity*, can be established from the generalized expression of the Kramers–Kronig relations [59, 67, 71, 72] corresponding to the Herglotz–Nevanlinna representation theorem [73]. For a complex frequency ω with positive imaginary part, $\operatorname{Im}\omega > 0$, this generalized Kramers–Kronig expression of the permittivity is [67]

$$\varepsilon(\boldsymbol{x},\omega) = \varepsilon_0 - \int_{\mathbb{R}} \mathrm{d}\nu \, \frac{\sigma(\boldsymbol{x},\nu)}{\omega^2 - \nu^2}, \quad \sigma(\boldsymbol{x},\nu) = \operatorname{Im} \frac{\nu\varepsilon(\boldsymbol{x},\nu)}{\pi} \geq 0, \tag{2}$$

where the relation $\sigma(\boldsymbol{x},\nu) \geq 0$ is a consequence of the passivity [67]. Notice that the quantity $\sigma(\boldsymbol{x},\nu)$ is a generalized function of ν and may contain Dirac contributions (for instance in the non-aborptive case [67, 73]). This passivity requirement for real frequency ν can be extended to complex frequencies ω with positive imaginary part, since the imaginary part of the integral multiplied by ω in the expression above is positive:

$$\operatorname{Im}\omega\varepsilon(\boldsymbol{x},\omega) \geq \operatorname{Im}\omega\varepsilon_0. \tag{3}$$

Let ω_1 be a real frequency at which the imaginary part of the permittivity vanishes. Then $\sigma(\boldsymbol{x},\omega_1) = 0$ and the integral in the expression

$$\omega_1\varepsilon(\boldsymbol{x},\omega_1) = \omega_1\varepsilon_0 - \int_{\mathbb{R}} \mathrm{d}\nu \frac{\sigma(\boldsymbol{x},\nu)}{\omega_1 - \nu}, \tag{4}$$

is well-defined and real. Considering the derivative of this equation and using that $\sigma(\boldsymbol{x},\nu) \geq 0$, the following well-known inequality [1,3] is obtained: if $\operatorname{Im}\varepsilon(\boldsymbol{x},\omega_1) = 0$, then

$$\operatorname{Re}\frac{\partial\omega\varepsilon}{\partial\omega}(\boldsymbol{x},\omega_1) \geq \varepsilon_0 \quad \Longleftrightarrow \quad \operatorname{Re}\varepsilon(\boldsymbol{x},\omega_1) \geq \varepsilon_0 - \operatorname{Re}\frac{\partial\varepsilon}{\partial\omega}(\boldsymbol{x},\omega_1). \tag{5}$$

This inequality means that, if the permittivity $\varepsilon(\boldsymbol{x},\omega_1)$ takes at the frequency ω_1 a real value less than the vacuum permittivity ε_0, then the derivative of the permittivity with respect to the frequency cannot vanish at the frequency ω_1. This corresponds precisely to the case of negative index materials and, more generally, to the situations offered by metamaterials for which the effective permittivity (and possibly the effective permeability) takes negative values or values below ε_0. Therefore, the frequency dispersion must be considered in negative index media and in metamaterials (for instance for effective refractive index below unity, also called ultra-refraction). Otherwise, the absence of frequency dispersion introduces contradictions with the causality principle or the passivity requirement.

A canonical approach for frequency dispersion has been established in 1998 by A. Tip with the *auxiliary field formalism* [71]. This formalism has been originally introduced to define a proper frame for macroscopic Maxwell's equations in absorptive and frequency dispersive dieletric media, for their quantized version [71, 74], and for the generalization of the density of states and the description of the atomic decay in absorptive and frequency dispersive structures [71, 75]. This formalism is based on the introduction of auxiliary fields so that macroscopic Maxwell's equations can be written equivalently as a unitary time evolution equation involving both electromagnetic and auxiliary fields: the new augmented system satisfies an overall energy conservation and the frequency dependence of the permittivity is transferred to the auxiliary fields. In other words, this general technique transforms a time-dependent and non self-adjoint dissipative operator into a time-independent and self-adjoint augmented operator. In 2005, A. Figotin and J. Schenker have shown that this *auxiliary field formalism* introduced by A. Tip is precisely the unique minimal self-adjoint extension of the dissipative Maxwell's equations [76]. This canonical formalism has been extended to magnetodielectric materials in order to describe frequency dispersive negative index materials [67]. It has been shown that the time evolution of a system comprising a perfect -1 index material, i.e. with a frequency ω_1 at which $\varepsilon(\boldsymbol{x},\omega_1) = -\varepsilon_0$ and $\mu(\boldsymbol{x},\omega_1) = -\mu_0$ (for $\boldsymbol{x}$ in the -1 index material), is well-defined since the electromagnetic energy remains finite at all times as soon as this is the case at the initial time: hence the compatibility of the existence of perfect negative index materials with causality principle and passivity has been unambiguously established using the canonical extension of Maxwell's equations [67].

In the case of the flat lens with perfect -1 index at the frequency ω_1, the Green's function has a pole at the frequency ω_1 [67, 77, 78] and the time-harmonic Maxwell's equations has no solution at the oscillating frequency ω_1: the time-harmonic frame fails in the case of the flat lens with perfect -1 index (or perfect negative index). However, according to the canonical frame of the *auxiliary field formalism*, the solution to (time-dependent) Maxwell's equations is well-defined at all time if it is the response to an external current source [67] as in Figure 3. The long-time behavior of such a solution can be considered for a current source turned on at an initial

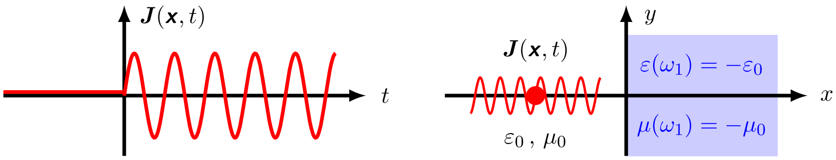

Figure 3. An external current source $\boldsymbol{J}(\boldsymbol{x},t)$ swithed on at the intial time $t = 0$ and then oscillating at the frequency ω_1. This source is located at the vicinity of a plane interface separating the vacuum from a medium with perfect -1 index at the frequency ω_1.

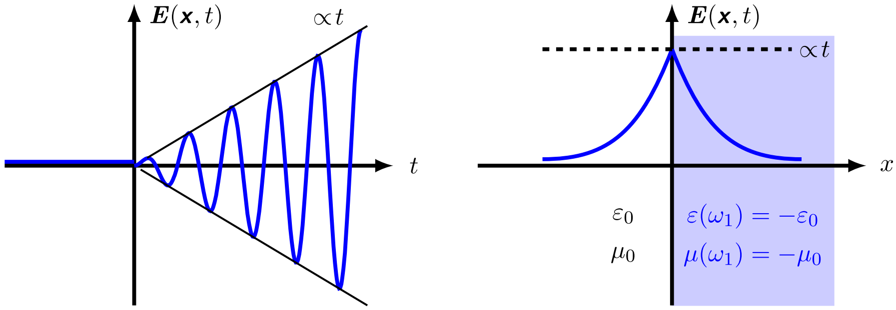

Figure 4. The reponse of the external current source swithed on at the intial time $t = 0$ and then oscillating at the frequency ω_1. The amplitude of the evanescent waves at the plane interface separating the vacuum from the perfect -1 index medium (see right panel) is linearly increasing with time (see left panel).

time and then oscillating with the operating frequency ω_1 [67, 77, 78]. In the case of a single plane interface separating a perfect -1 index medium and vacuum ("single interface" case), it has been shown that the evanescent components of this time-dependent solution have their amplitude increasing linearly with time [77–79], see Figure 4. Consequently, this solution does not converge for long times to the solution to the corresponding time-harmonic problem: the *limiting amplitude principle* is not valid in this case [79]. The situation is similar in the case of the perfect flat lens (two plane interfaces delimiting a -1 index layer from vacuum, or "two interfaces" case), leading to the conclusion that the image of a point source by the perfect -1 flat lens is not a point image [77,78]. An analyzis based on the calculation of the spectral projector [79] provided the complete characterization of the spectral properties in the "single interface" case. In particular the presence of *essential spectrum* in Maxwell's equations has been highligthed at the -1 frequency ω_1 which is an eigenvalue with infinite degeneracy [67, 79].

It turns out that the extraordinary property of the perfect -1 index and the induced phenomena in the perfect flat lens are related to the presence of *essential spectrum* in Maxwell's equations. Thus the complete characterization of the spectral properties of frequency dispersive and negative index structures appears to be an important issue. For instance, a perfect corner reflector made of two orthogonal planes delimiting positive and negative index media makes a cavity that traps light and where the density of states appears to be infinite [80–83]. This infinite density of states has been related to the existence of an infinite number of modes at the -1 index frequency [80, 83], i.e. the -1 index frequency is also included in the essential spectrum as an eigenvalue with infinite degeneracy in this case of the perfect corner reflector. Next, further investigations have shown that two dimensional Maxwell's systems with corners delimiting a medium with positive permittivity (e.g. vacuum) from a medium with negative permittivity (and—or—a

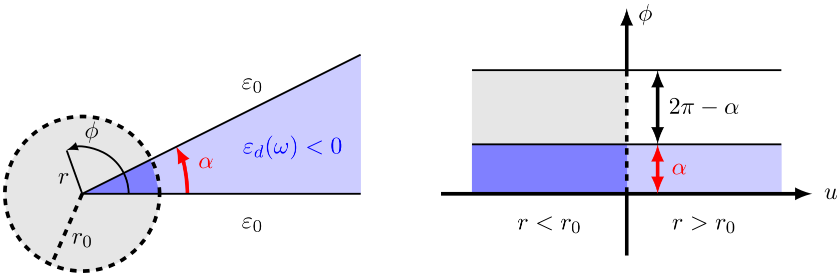

Figure 5. Left: a two-dimensional corner structure of angle α delimiting a frequency dispersive medium of permittivity $\varepsilon_d(\omega)$. Right: the 2π-periodic one-dimensional layered structure obtained after the change of variable $r \longmapsto u = \ln(r/r_0)$ and the assumption considering that the modes are localized in the vicinity of the corner.

negative permeability) bring up *essential spectrum* for an interval of negative values of the permittivity around $-\varepsilon_0$ (or around the permittivity ratio -1) [84–87]: for example, in the case of a 90 degrees corner, there is *essential spectrum* for the permittivity interval $[-3\varepsilon_0, -\varepsilon_0/3]$. This *essential spectrum* is associated with an analog of "black hole" phenomenon occurring in the vicinity of a corner which behaves like an unbouded domain. Such unusual effect and spectral properties, originally reported in the case of the negative index perfect corner reflectors, appears to be omnipresent in Maxwell's systems with corners delimiting a frequency dispersive medium [88]. Indeed, let the permittivity $\varepsilon_d(\omega)$ of the frequency dispersive medium be given by the Drude–Lorentz model:

$$\varepsilon_d(\omega) = \varepsilon_0 - \varepsilon_0 \frac{\Omega^2}{\omega^2 + \mathrm{i}\gamma\omega - \nu^2}, \tag{6}$$

where Ω, ν and γ are positive real constants. Then, there always exists a complex frequency ω_1 at which $\varepsilon_d(\omega_1) = -\varepsilon_0$: $\omega_1 = -\mathrm{i}\gamma/2 \pm \sqrt{\nu^2 + \Omega^2/2 - \gamma^2/4}$. And, for example, in the case of a 90 degrees corner, the permittivity interval $[-3\varepsilon_0, -\varepsilon_0/3]$ is spanned for the following range of complex frequencies

$$\left[-\mathrm{i}\gamma/2 \pm \sqrt{\nu^2 + \Omega^2/4 - \gamma^2/4}, -\mathrm{i}\gamma/2 \pm \sqrt{\nu^2 + 3\Omega^2/4 - \gamma^2/4}\right]. \tag{7}$$

If the permittivity is given by a more general expression, for instance a finite sum of Drude–Lorentz contributions, then the number of segments in the complex plane of frequencies, generally curved, increases like the degree of the polynomials involved in the permittivity expression. Hence the intervals of *essential spectrum* appear unavoidable in frequency dispersive systems with corners.

The main arguments exhibiting the presence of *essential spectrum* can be the following [85]: let α in $]0, 2\pi[$ be the angle of a two-dimensional corner filled with a dispersive mediumof permittivity $\varepsilon_d(\omega)$ and $\boldsymbol{x} = (r, \phi)$ the considered cylindrical coordinates (see Figure 5). The permittivity of the system is independent of the radial variable r: $\varepsilon(\boldsymbol{x}, \omega) = \varepsilon(\phi, \omega)$, $\varepsilon(\phi, \omega) = \varepsilon_d(\omega)$ for an azimuthal variable ϕ in $[0, \alpha]$ and $\varepsilon(\phi, \omega) = \varepsilon_0$ for ϕ in $[\alpha, 2\pi]$. In the time-harmonic regime, the magnetic field component $H(r, \phi, \omega)$ of the transverse magnetic waves is the solution to the Helmholtz equation

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial H}{\partial r} + \frac{\varepsilon}{r^2}\frac{\partial}{\partial \phi}\frac{1}{\varepsilon}\frac{\partial H}{\partial \phi} + \omega^2 \varepsilon \mu_0 H = 0, \tag{8}$$

where the dependence on (r,ϕ,ω) has been omitted. Then the change of variable $r \longmapsto u = \ln(r/r_0)$ is performed in this Helmholtz equation (see Figure 5). The magnetic field component $\tilde{H}(u,\phi,\omega) = H(r_0 e^u,\phi,\omega)$ is now solution to

$$\frac{\partial^2 \tilde{H}}{\partial^2 u} + \varepsilon \frac{\partial}{\partial \phi} \frac{1}{\varepsilon} \frac{\partial \tilde{H}}{\partial \phi} = -r_0^2 e^{2u} \omega^2 \varepsilon \mu_0 \tilde{H}, \tag{9}$$

where the dependence on (u,ϕ,ω) has been omitted. Assuming that the modes generated by the corner are localized in the circle of radius r_0, i.e. $\tilde{H}(u,\phi,\omega) \approx 0$ for u positive, and choosing the radius r_0 small enough, then the right hand side in (9) can be neglected and set to zero. Next, a Fourier decomposition $\tilde{H}(u,\phi,\omega) \longmapsto \hat{H}(k,\phi,\omega)$ is applied, and the resulting equation is exactly that for a periodic one-dimensional layered structure (see Figure 5):

$$\varepsilon \frac{\partial}{\partial \phi} \frac{1}{\varepsilon} \frac{\partial \hat{H}}{\partial \phi} - k^2 \hat{H} = 0. \tag{10}$$

Hence, the existence of a mode 2π-periodic with respect to the azimuthal variable ϕ is subject to the following condition [89]:

$$\cosh[k\alpha]\cosh[k(2\pi-\alpha)] + \frac{1}{2}\left[\frac{\varepsilon_0}{\varepsilon_d(\omega)} + \frac{\varepsilon_d(\omega)}{\varepsilon_0}\right]\sinh[k\alpha]\sinh[k(2\pi-\alpha)] = 1. \tag{11}$$

For $k = 0$ the equality is achieved but the solution is trivial (constant) and yields vanishing electric field. Hence the existence of a corner mode is subject to a solution for $k \neq 0$. The function on the left hand side is made of two terms: the first term with the cosh functions starts from 1 at $k = 0$ and then is growing to $+\infty$; thus the second term with the sinh functions must decrease towards $-\infty$, which requires a real negative value for $\varepsilon_d(\omega)$. Since the factor in front of the sinh functions has absolute value greater than 1 (except in the case $\varepsilon_d(\omega) = -\varepsilon_0$ where it equals 1), the sum of the two terms in the left hand side tends to $-\infty$ for large values of k. Therefore, to obtain a solution $k \neq 0$ to (11), it is enough that the second derivative at $k = 0$ of the function on the left hand side be positive. This second derivative is

$$\alpha(2\pi-\alpha)\left[\frac{\alpha}{2\pi-\alpha} + \frac{2\pi-\alpha}{\alpha} + \frac{\varepsilon_0}{\varepsilon_d(\omega)} + \frac{\varepsilon_d(\omega)}{\varepsilon_0}\right], \tag{12}$$

which is positive if and only if

$$\frac{\varepsilon_d(\omega)}{\varepsilon_0} \in [-I_\alpha, -1/I_\alpha], \quad I_\alpha = \max\left\{\frac{\alpha}{2\pi-\alpha}, \frac{2\pi-\alpha}{\alpha}\right\}. \tag{13}$$

Notice that, for $\alpha = \pi$, i.e. when the corner becomes a plane interface, the number $I_\alpha = 1$ and the interval reduces to the point -1. In that case, one can check that the condition (11) is achieved for all k if $\alpha = \pi$ and $\varepsilon_d(\omega) = -\varepsilon_0$. The radial dependence of the corner modes is given by

$$r \longmapsto \exp[\mathrm{i}k \ln r/r_0] \tag{14}$$

which is oscillating with spatial frequency tending to infinity when $r \longrightarrow 0$. As a result, the electric field, deduced from the derivative, has a singularity like $1/r$ and then is not square integrable, i.e. is not finite energy. This behavior, represented on Figure 6, is different from the previous results reported in the textbooks [90] where only dielectric materials with positive permittivity and conducting materials have been considered, leading to the strongest singularity in $1/\sqrt{r}$ and to finite energy fields [90, Sections 5.2, 5.3 and 9.7.5]. In the present case of negative permittivity, the corner modes have infinite energy and are then "generalized eigenvectors" associated with the *essential spectrum* corresponding to the frequencies ω such that the ratio $\varepsilon_d(\omega)/\varepsilon_0$ is in the interval $[-I_\alpha, -1/I_\alpha]$ [88]. The radial dependence of these corner modes, with oscillations with spatial period tending to zero, makes an analog of "black hole" phenomenon occuring at the corner (see Figure 6). Indeed, the modes appear to propagate infinitely slowly and to accumulate energy when approaching the corner as if they were trapped by the corner which would behave

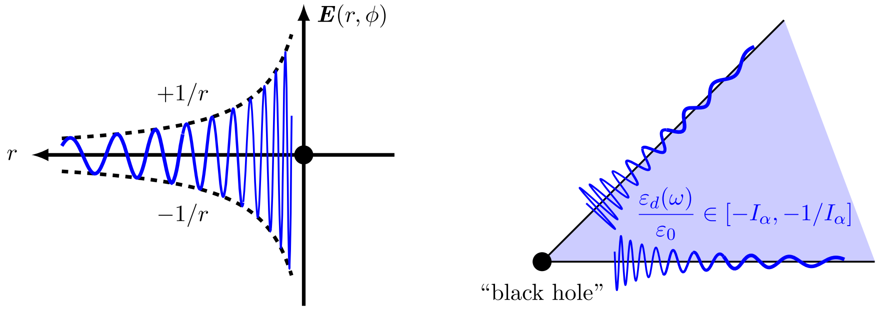

Figure 6. Left: the radial dependence of the electric field of a corner mode with amplitude increasing like $1/r$ and spatial frequency tending to infinity like $\ln(1/r)$ when $r \longrightarrow 0$. Right: a representation of the electric field of a corner mode at the vicinity of the corner.

like a semi-infinite open space. Finally, notice that it can be shown that there is no *essential spectrum* associated with the corner of angle α outside the interval $[-I_\alpha, -1/I_\alpha]$ using a "T-coercivity" argument [84–87].

This new and extraordinary behavior of corner modes exhibited in systems with negative permittivity (and—or—negative permeability) raises numerous challenging questions in applied mathematics (e.g. three-dimensional corners [81]), in physics with the analog of "black hole" phenomenon and in numerical modelling. In particular, it is stressed that the presence of the *essential spectrum* implies difficulties in the computation of modes of dispersive structures, with the lack of convergence in a frequency domain around the *essential spectrum* where the permittivity takes real negative values [91–93]. This unavoidable perturbation of numerical computation represents a challenging task in the method of quasi-normal modes expansion [92,93]. The introduction of perfectly matched layers at the corners [94] could be a promising way to address this task.

The perfect flat lens [54] and its generalization such as the perfect corner reflector [80,82,83] highlighted situations where, at the perfect -1 index frequency ω_1, the frame of the time-harmonic Maxwell's equations has no solutions and thus appears inappropriate to describe the behavior of electromagnetic waves. Such In addition, the presence of *essential spectrum* has been identified at the perfect -1 index frequency ω_1 which is an eigenvalue with infinite degeneracy for the perfect flat lens and corner reflector. Recently, it has been shown that more conventional structures like frequency dispersive corners also display *essential spectrum* [84–87] for a whole interval of frequencies for which there is no solution to the time-harmonic Maxwell's equations. However, it has been shown that the *auxiliary field formalism* introduced by A. Tip [71] provides a canonical approach for all these extraordinary situations with perfect negative index [67], metamaterials and negative permittivity where the frequency dispersion plays a vital role. In particular, it is stressed that this *auxiliary field formalism* offers the possibility to analyze rigorously a negative permittivity corner which makes an analog of "black hole" phenomenon.

4. Spatial dispersion and the imaginary part of the effective permeability

The modelling of metamaterials and negative index materials highlighted the role of spatial dispersion (or non-locality) [29–32,35,42,46,47] in composites displaying effective permeability. It makes sense since, in usual bulk materials, the magnetic properties can be derived from the electric permittivity tensor $\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega)$ depending on the frequency ω and the wave vector $\boldsymbol{k}$ [3,35,95]. Indeed, consider the Maxwell's equations in a homogeneous and isotropic magnetodielectric medium with permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$. For monochromatic plane-waves with space-time dependence in $\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)]$ these equations become

$$\boldsymbol{k}\times\boldsymbol{E}=\omega\mu(\omega)\boldsymbol{H},\quad \boldsymbol{k}\times\boldsymbol{H}=-\omega\varepsilon(\omega)\boldsymbol{E}, \tag{15}$$

and, after eliminating the field $\boldsymbol{H}$,

$$\boldsymbol{k}\times\frac{1}{\omega\mu(\omega)}\boldsymbol{k}\times\boldsymbol{E}+\omega\varepsilon(\omega)\boldsymbol{E}=\boldsymbol{0}. \tag{16}$$

This last equation can be written

$$\boldsymbol{k}\times\frac{1}{\omega\mu_0}\boldsymbol{k}\times\boldsymbol{E}+\boldsymbol{k}\times\left[\frac{1}{\omega\mu(\omega)}-\frac{1}{\omega\mu_0}\right]\boldsymbol{k}\times\boldsymbol{E}+\omega\varepsilon(\omega)\boldsymbol{E}=\boldsymbol{0}. \tag{17}$$

Thus, defining the dielectric tensor

$$\omega\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega)=\omega\varepsilon(\omega)-\left[\frac{1}{\omega\mu(\omega)}-\frac{1}{\omega\mu_0}\right]\left(k^2-\boldsymbol{k}\boldsymbol{k}\right), \tag{18}$$

where $k^2=\boldsymbol{k}\cdot\boldsymbol{k}$ and $\boldsymbol{k}\boldsymbol{k}$ is the rank-two tensor acting as $\boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{E}=(\boldsymbol{k}\cdot\boldsymbol{E})\boldsymbol{k}$, the Equation (16) is equivalent to

$$\boldsymbol{k}\times\frac{1}{\omega\mu_0}\boldsymbol{k}\times\boldsymbol{E}+\omega\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega)\cdot\boldsymbol{E}=\boldsymbol{0} \tag{19}$$

for isotropic permeability $\mu(\omega)$. Hence a magnetodielectric medium can be described by only the electric permittivity tensor $\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega)$. Conversely, the permeability $\mu(\omega)$ can be derived from the permittivity tensor. Let P_{lg} and $P_{\mathrm{tr}}=1-P_{\mathrm{lg}}$ be the orthogonal projections on the subspaces parallel (or *longitudinal*) and perpendicular (or *transverse*) to the vector $\boldsymbol{k}$:

$$P_{\mathrm{lg}}=\frac{\boldsymbol{k}\boldsymbol{k}}{k^2},\quad P_{\mathrm{tr}}=1-\frac{\boldsymbol{k}\boldsymbol{k}}{k^2}. \tag{20}$$

For isotropic permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$, the permittivity tensor (18) can be decomposed on these subspaces

$$\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega)=\varepsilon_{\mathrm{lg}}(\boldsymbol{k},\omega)P_{\mathrm{lg}}+\varepsilon_{\mathrm{tr}}(\boldsymbol{k},\omega)P_{\mathrm{tr}}, \tag{21}$$

where $\varepsilon_{\mathrm{lg}}(\boldsymbol{k},\omega)=\varepsilon(\omega)$ and $\varepsilon_{\mathrm{tr}}(\boldsymbol{k},\omega)=\varepsilon(\omega)-[1/\mu(\omega)-1/\mu_0]\,k^2/\omega^2$ are the longitunal and transverse components of the tensor. Hence, the permeability can be retrieved from the permittivity tensor according to the well-known relation [3, 35]

$$\frac{1}{\omega\mu(\omega)}=\frac{1}{\omega\mu_0}+\lim_{k\to 0}\frac{\omega\varepsilon_{\mathrm{lg}}(\boldsymbol{k},\omega)-\omega\varepsilon_{\mathrm{tr}}(\boldsymbol{k},\omega)}{k^2}. \tag{22}$$

Notice that this relation between the permeability and the permittivity tensor with spatial dispersion is established for bulk infinite media which are not delimited by boundaries.

Although the two descriptions of magnetodielectric media seem to be equivalent, the introduction of spatial dispersion and the gathering, in the permittivity tensor, of all the magnetic and dielectric properties of the materials, break the symmetry between the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ in the Maxwell's equations. However, the symmetry between these fields and between the permittivity and permeability is widely used in textbooks of classical electrodynamics [3, 33, 90]. In particular, the Kramers–Kronig relations and the passivity requirement for the permittivity are generally also considered as valid for the permeability: for instance, it is generally considered that the imaginary of permittivity is positive [3, Section 80], $\operatorname{Im}\mu(\omega)\geq 0$, which becomes for all frequency ω with positive imaginary part

$$\operatorname{Im}\omega\mu(\omega)\geq\operatorname{Im}\omega\mu_0. \tag{23}$$

Nevertheless, it turns out that the behaviors of the permittivity and the permeability are different in the static regime [3, 33]: unlike the static permittivity $\varepsilon(0)$ (i.e. $\varepsilon(\omega)$ at the limit $\omega\to 0$), which always takes real value greater than ε_0, the static permeability $\mu(0)$ (i.e. $\mu(\omega)$ at the limit $\omega\to 0$) can take real values either greater (paramagnetic media) or less (diamagnetic media) than μ_0.

This alternative for the static permeability seems in contradiction with the causality principle and the passivity. Indeed, the generalized Kramers–Kronig relation (4) applied to the permeability implies

$$\mu(0) = \mu_0 + \frac{2}{\pi}\int_0^\infty \mathrm{d}\nu \, \frac{\operatorname{Im}\mu(\nu)}{\nu}, \tag{24}$$

which requires for $\mu(0)$ to be greater than μ_0 if $\operatorname{Im}\mu(\omega) \geq 0$ for all positive real frequency ω [48]. In the textbook [3], this contradiction is explained by the frequency range where the macroscopic magnetic permeability makes sense, which is limited to the relatively low frequencies. Consequently, it is specified in [3, Section 82] that the Kramers–Kronig relations like (4) and (24) must be modified for the magnetic permeability. Hence the requirement on the imaginary part of the permeability, $\operatorname{Im}\mu(\omega) \geq 0$, has been confirmed in reference [3].

The introduction of metamaterials and negative index materials with effective permeability led to revisit these statements. Indeed, it has been found that the effective parameters of metamaterials present anomalous dispersion: in [96], "*the resonant behavior of the effective magnetic permeability is accompanied by an antiresonant behavior of the effective permittivity*" and "*the imaginary parts of the effective permittivity and permeability are opposite in sign*". These numerical results first generated some controversy [97, 98], but next the anomalous dispersion and the possibility for negative imaginary part of the effective permeability have been confirmed by several studies, which generated a series of investigations. For instance, fundamental questions on the Poynting vector and the energy have been adressed [48,49], the validity of the Kramers–Kronig relations for the magnetic permeability has been examined [51], the validity of the causality principle and the physical meanining of the metamaterials constitutive parameters has been analyzed [50, 52].

Hereafter, the objective is to bring arguments supporting that the imaginary part of the magnetic permeability $\mu(\omega)$ in passive media can take both positive and negative values, i.e. that the relation (23) is not valid. In contrast, these arguments support that the Kramers–Kronig relations make sense for the permeability. Since these claims are in contradiction with the electrodynamics based on the symmetry between, in one hand, the electric field $\boldsymbol{E}$ and the permittivity $\varepsilon(\omega)$ and, in the other hand, the magnetic induction field $\boldsymbol{H}$ and the permeability $\mu(\omega)$, the description with the dielectric tensor $\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega)$ and spatial dispersion is considered.

First, a simple model with spatial dispersion is considered, the hydrodynamical model [99, 100], with the permittivity tensor expression

$$\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega) = \varepsilon_0 - \varepsilon_0 \frac{\Omega^2}{\omega^2 + \mathrm{i}\omega\gamma - \omega_0^2 - v^2 k^2} P_{\mathrm{lg}} - \varepsilon_0 \frac{\Omega^2}{\omega^2 + \mathrm{i}\omega\gamma - \omega_0^2} P_{\mathrm{tr}}. \tag{25}$$

Using the relation (22), it is possible to define from this model the magnetic permeability

$$\frac{1}{\omega\mu(\omega)} = \frac{1}{\omega\mu_0} - \omega\varepsilon_0 \frac{\Omega^2 v^2}{(\omega^2 + \mathrm{i}\omega\gamma - \omega_0^2)^2}, \tag{26}$$

and to obtain for the imaginary part at real frequency

$$\operatorname{Im}\frac{1}{\omega\mu(\omega)} = \varepsilon_0 \frac{2\gamma\Omega^2 v^2 \omega^2 (\omega^2 - \omega_0^2)}{[(\omega^2 - \omega_0^2)^2 + \omega^2\gamma^2]^2}. \tag{27}$$

This result clearly shows that the imaginary part of the obtained permeability can take both positive and negative values, depending on ω^2 is smaller or larger than ω_0^2. In addition, from the expression (26) of the permeability, the following identity, similar to the relation (24), is derived

$$\int_0^\infty \mathrm{d}\nu \operatorname{Im}\frac{1}{\nu\mu(\nu)} = \frac{\pi}{2}\left[\frac{1}{\mu(0)} - \frac{1}{\mu_0}\right] = 0, \tag{28}$$

where it has been used that $\mu(0) = \mu_0$ to obtain that the integral vanishes. This last identity confirms that the imaginary part of the permeability must take positive and negative values.

The same results can be obtained starting from a general permittivity tensor $\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega)$, provided the asymptotic behavior at large complex frequencies with positive imaginary part,

$$\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega) \underset{|\omega|\to\infty}{\longrightarrow} \varepsilon_0 - \frac{\Omega^2}{\omega^2}, \tag{29}$$

is independent of the wave vector $\boldsymbol{k}$ [101]. In combination with the analytic properties of the permittivity in the half plane of complex frequencies with positive imaginray parts, this asymptotic behavior implies the identity, or *sum rule*,

$$\int_0^\infty \mathrm{d}\nu \, \mathrm{Im}\, \nu \underline{\underline{\varepsilon}}(\boldsymbol{k},\nu) = \frac{\pi\Omega^2}{2}, \tag{30}$$

also independent of the wave vector $\boldsymbol{k}$ [101, 102]. Since the permeability (22) is defined from the term quadratic in $\boldsymbol{k}$ in the tensor $\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega)$, its imaginary part must be subject to [101, 102]

$$\int_0^\infty \mathrm{d}\nu \, \mathrm{Im}\, \frac{1}{\nu\mu(\nu)} = 0. \tag{31}$$

Thus, starting from a general permittivity tensor, it can be shown that the imaginary part of the permeability must take positive and negative values. It is stressed that these last arguments, implying that $\mu(0) = \mu_0$, exclude the existence of media with magnetic properties in the static regime, i.e. with $\mu(0) \neq \mu_0$. This limitation can be however overtaken considering a permittivity tensor $\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega)$ with an essential singularity at the point $(\boldsymbol{k},\omega) = (0,0)$ [101]. A simple example is the hydrodynamical Drude model, i.e. the expression (25) with ω_0 set to 0, which leads to the permeability

$$\frac{1}{\mu(\omega)} = \frac{1}{\mu_0} - \varepsilon_0 \frac{\Omega^2 \nu^2}{(\omega + \mathrm{i}\gamma)^2}, \tag{32}$$

corresponding to a diamagnetic medium with $\mu(0) < \mu_0$. Notice that, in that case, the imaginary part of $\omega \longmapsto \omega\mu(\omega)$ is negative for all real frequency ω, i.e. its sign does not change. A paramagnetic medium could be obtained by inverting the $(\boldsymbol{k},\omega)$-dependence of the longitudinal and transverse components of the permittivity tensor in the expression (25). In addition, it is stressed that another possibility to overpass the limitation (31) may be to consider an asymptotic behavior different from the one (29) considered in [101].

The possibility for the imaginary part of the permeability to take positive and negative values is now investigated through the effective permeability of a composite medium. Here, a stack of non-magnetic homogeneous layers is considered. Indeed, the simplicity of such a structure makes it possible to define exactely, using the retrieval method, an effective permeability for all frequencies and wave vectors. In addition, it has been shown that the effective parameters of a multilayered structure present the suitable properties to ensure the causality principle and the passivity requirement [53]. Hence, this composite medium is a good candidate to investigate the sign of the imaginary part of the effective permeability.

A stack of non-magnetic homogeneous layers of total thickness h with a plane of symmetry at mid-height is considered (see left panel in Figure 7). The space variable in the stacking direction is denoted by x. The multilayered structure is located between the planes $x = -h$ and $x = 0$ and is described by the frequency dispersive and isotropic permittivity $\varepsilon(x,\omega)$, the magnetic permeability being set to the permeability of vacuum μ_0. Outside the multilayered structure, i.e. for $x < -h$ and $x > 0$, the permittivity is set to ε_0. In practice, $\varepsilon(x,\omega)$ is piecewise constant with respect to z and, according to the symmetry of the structure, $\varepsilon(-x,\omega) = \varepsilon(x-h,\omega)$. The structure (and the permittivity) is invariant under translations in the plane parallel to the layers and thus a two-dimensional Fourier decomposition is performed in these tangential directions: the two-component wave vector resulting from this Fourier decomposition is denoted by $\boldsymbol{k}_\parallel$ and $k_\parallel = \sqrt{\boldsymbol{k}_\parallel \cdot \boldsymbol{k}_\parallel}$ is its norm. Then, the time-harmonic Maxwell's equations become a set of two independent scalar equations for the electric and magnetic fields components orthogonal

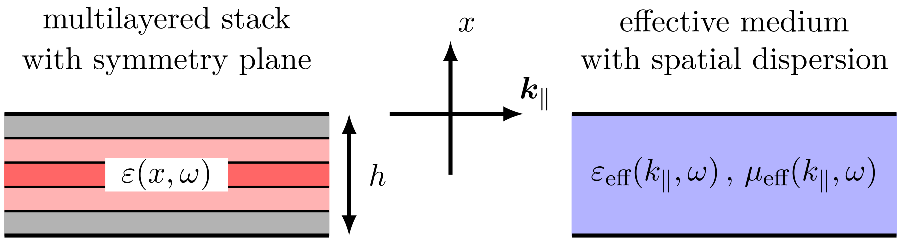

Figure 7. Left: the considered multilayered structure with a plane of symmetry described by the frequency dependent and isotropic permittivity $\varepsilon(x,\omega)$. Right: the equivalent effective medium described by the effective permittivity $\varepsilon_{\text{eff}}(k_\parallel,\omega)$ and permeability $\mu_{\text{eff}}(k_\parallel,\omega)$.

to both $\boldsymbol{k}_\parallel$ and the stacking direction. Let $U^e(x,k_\parallel,\omega)$ and $U^m(x,k_\parallel,\omega)$ be these components of the electric and magnetic fields: for $w = e, m$, the Maxwell's equations take the form

$$\frac{\partial}{\partial x}\frac{1}{\xi^w(x,\omega)}\frac{\partial}{\partial x}U^w(x,k_\parallel,\omega) + \frac{\omega^2\mu_0\varepsilon(x,\omega) - k_\parallel^2}{\xi^w(x,\omega)}U^w(x,k_\parallel,\omega) = 0, \tag{33}$$

where $\xi^e(x,\omega) = \mu_0$ and $\xi^m(x,\omega) = \varepsilon(x,\omega)$. The solutions to these equations can be determined from the 2×2 transfer matrices $T^e(k_\parallel,\omega)$ and $T^m(k_\parallel,\omega)$ relating the values of the fields' tangential components parallel to the layers at the planes $x = 0$ and $x = -h$ [41,53]. For $w = e, m$, the general expression of these transfer matrices is [53]

$$T^w(k_\parallel,\omega) = \begin{bmatrix} A^w(k_\parallel,\omega) & B^w(k_\parallel,\omega) \\ C^w(k_\parallel,\omega) & A^w(k_\parallel,\omega)\end{bmatrix}, \tag{34}$$

where the coefficents are analytic functions in the half plane of complex frequencies ω with positive imaginary part and are related by the identity

$$A^w(k_\parallel,\omega)^2 - B^w(k_\parallel,\omega)C^w(k_\parallel,\omega) = 1. \tag{35}$$

Each transfer matrix $T^e(k_\parallel,\omega)$ and $T^m(k_\parallel,\omega)$ is thus determined by two independent parameters. In particular, defining, for $w = e, m$,

$$\begin{aligned} k_\perp^w(k_\parallel,\omega) &= \frac{1}{\mathrm{i}h}\ln\left[A^w(k_\parallel,\omega) + \mathrm{i}\sqrt{1 - A^w(k_\parallel,\omega)^2}\right], \\ X^w(k_\parallel,\omega) &= \sqrt{\frac{B^w(k_\parallel,\omega)}{C^w(k_\parallel,\omega)}}, \end{aligned} \tag{36}$$

the transfer matrices (34) can be equivalently expressed as

$$T^w(k_\parallel,\omega) = \begin{bmatrix} \cos\left[k_\perp^w(k_\parallel,\omega)h\right] & \mathrm{i}\sin\left[k_\perp^w(k_\parallel,\omega)h\right]X^w(k_\parallel,\omega) \\ \mathrm{i}\sin\left[k_\perp^w(k_\parallel,\omega)h\right]/X^w(k_\parallel,\omega) & \cos\left[k_\perp^w(k_\parallel,\omega)h\right]\end{bmatrix}. \tag{37}$$

It is stressed that the imaginary part of the parameter $k_\perp^w(k_\parallel,\omega)$ cannot vanish for frequencies ω with postive imaginary part [53], otherwise this would allow the existence of Bloch modes in the one-dimensional system resulting from the periodic stacking of the multilayered structure [53, 103]. The sign of the imaginary part of $k_\perp^w(k_\parallel,\omega)$ can be chosen positive, i.e.

$$\operatorname{Im}\omega > 0 \quad\Longrightarrow\quad \operatorname{Im}k_\perp^w(k_\parallel,\omega) > 0, \tag{38}$$

which fix the sign of the square root in the definition (36) of the second parameter $X^w(k_\parallel,\omega)$. This remarkable property (38) ensures that the definition (36) preserves in the domain $\operatorname{Im}\omega > 0$ the analytic property of the parameters $k_\perp^w(k_\parallel,\omega)$ and $X^w(k_\parallel,\omega)$ since $A^w(k_\parallel,\omega)$ cannot take the values ± 1 and $B^w(k_\parallel,\omega)$ and $C^w(k_\parallel,\omega)$ cannot vanish.

The transfer matrices and thus the four independent parameters $k_\perp^w(k_\parallel,\omega)$ and $X^w(k_\parallel,\omega)$ fully determine the solution to Maxwell's equations outside the multilayered structure. For instance, the fields reflected and transmitted by the multilayered structure can be expressed from these four parameters. Thus, these parameters can be used to define an homogeneous effective medium that will lead to the same solutions to Maxwell's equations outside the multilayered structure. Notice that this procedure corresponds to the retrieval method [28, 29]. The homogeneous effective medium must be described by four effective parameters with the $(k_\parallel,\omega)$-dependence. According to the symmetry of the structure, let $\varepsilon_{\text{eff}}(k_\parallel,\omega)$ and $\mu_{\text{eff}}(k_\parallel,\omega)$ be the effective anisotropic permittivity and permeability defined by, for $w = e, m$,

$$\xi^w_{\text{eff}}(k_\parallel,\omega) = \begin{bmatrix} \xi^w_\parallel(k_\parallel,\omega) & 0 & 0 \\ 0 & \xi^w_\parallel(k_\parallel,\omega) & 0 \\ 0 & 0 & \xi^w_\perp(k_\parallel,\omega) \end{bmatrix}, \quad \xi^e = \mu, \quad \xi^m = \varepsilon, \tag{39}$$

where $\xi^w_\parallel(k_\parallel,\omega)$ are the components in the plane parallel to the layers and $\xi^w_\perp(k_\parallel,\omega)$ are the components in the stacking direction. In this effective medium, the Maxwell's equations for the components $U^e(x,k_\parallel,\omega)$ and $U^m(x,k_\parallel,\omega)$ of the electric and magnetic fields become

$$\frac{\partial^2}{\partial x^2} U^w(x,k_\parallel,\omega) + \left[\omega^2 \mu_\parallel(k_\parallel,\omega)\varepsilon_\parallel(k_\parallel,\omega) - k_\parallel^2 \frac{\xi^w_\parallel(k_\parallel,\omega)}{\xi^w_\perp(k_\parallel,\omega)}\right] U^w(x,k_\parallel,\omega) = 0, \tag{40}$$

where $\xi^e = \mu$ and $\xi^m = \varepsilon$. The transfer matrices $T^e_{\text{eff}}(k_\parallel,\omega)$ and $T^m_{\text{eff}}(k_\parallel,\omega)$ corresponding to a layer of the effective medium with thickness h are, for $w = e, m$,

$$T^w_{\text{eff}}(k_\parallel,\omega) = \begin{bmatrix} \cos\left[k^w_{\text{eff}}(k_\parallel,\omega)h\right] & \mathrm{i}\sin\left[k^w_{\text{eff}}(k_\parallel,\omega)h\right] X^w_{\text{eff}}(k_\parallel,\omega) \\ \mathrm{i}\sin\left[k^w_{\text{eff}}(k_\parallel,\omega)h\right] / X^w_{\text{eff}}(k_\parallel,\omega) & \cos\left[k^w_{\text{eff}}(k_\parallel,\omega)h\right] \end{bmatrix}, \tag{41}$$

where

$$\begin{aligned} k^w_{\text{eff}}(k_\parallel,\omega) &= \sqrt{\omega^2 \mu_\parallel(k_\parallel,\omega)\varepsilon_\parallel(k_\parallel,\omega) - k_\parallel^2 \xi^w_\parallel(k_\parallel,\omega)/\xi^w_\perp(k_\parallel,\omega)}, \\ X^w_{\text{eff}}(k_\parallel,\omega) &= \omega \xi^w_\parallel(k_\parallel,\omega)/k^w_{\text{eff}}(k_\parallel,\omega). \end{aligned} \tag{42}$$

As a final step, the identification of the transfer matrices of the multilayered structure with the ones of the effective medium provides the four equations $k^w_{\text{eff}}(k_\parallel,\omega) = k^w_\perp(k_\parallel,\omega)$ and $X^w_{\text{eff}}(k_\parallel,\omega) = X^w(k_\parallel,\omega)$, with $w = e, m$. These four equations define the following components of the effective permittivity and permeability:

$$\begin{aligned} \omega\varepsilon_\parallel(k_\parallel,\omega) &= k^m_\perp(k_\parallel,\omega) X^m(k_\parallel,\omega), \\ \frac{1}{\omega\varepsilon_\perp(k_\parallel,\omega)} &= \frac{k^e_\perp(k_\parallel,\omega)X^e(k_\parallel,\omega) - k^m_\perp(k_\parallel,\omega)/X^m(k_\parallel,\omega)}{k_\parallel^2}, \\ \omega\mu_\parallel(k_\parallel,\omega) &= k^e_\perp(k_\parallel,\omega) X^e(k_\parallel,\omega), \\ \frac{1}{\omega\mu_\perp(k_\parallel,\omega)} &= \frac{k^m_\perp(k_\parallel,\omega)X^m(k_\parallel,\omega) - k^e_\perp(k_\parallel,\omega)/X^e(k_\parallel,\omega)}{k_\parallel^2}. \end{aligned} \tag{43}$$

The components $\mu_\parallel(k_\parallel,\omega)$ and $\mu_\perp(k_\parallel,\omega)$ of the effective permeability $\mu_{\text{eff}}(k_\parallel,\omega)$ have been defined for a non vanishing parallel wave vector $\boldsymbol{k}_\parallel$ while the definition (22) is at the limit $k \longrightarrow 0$. However, it is possible to define a permeability depending upon the wave vector as [101]

$$\frac{1}{\omega\mu(\boldsymbol{k},\omega)} = \frac{1}{\omega\mu_0} + \frac{\omega\varepsilon_{\text{lg}}(\boldsymbol{k},\omega) - \omega\varepsilon_{\text{tr}}(\boldsymbol{k},\omega)}{k^2}, \tag{44}$$

which preserves the equivalence of the descriptions of an isotropic medium by $\varepsilon(\omega)$ and $\mu(\omega)$ and by the permittivity tensor $\underline{\underline{\varepsilon}}(\boldsymbol{k},\omega)$ with expression (18).

It is stressed that the four functions $\varepsilon_\parallel(k_\parallel,\omega)$, $1/\varepsilon_\perp(k_\parallel,\omega)$, $\mu_\parallel(k_\parallel,\omega)$ and $1/\mu_\perp(k_\parallel,\omega)$ defining the effective parameters, are analytic with respect to the frequency ω in the upper half complex plane of ω with positive imaginary part. This is a consequence of the analytic properties of the parameters $k_\perp^w(k_\parallel,\omega)$ and $X^w(k_\parallel,\omega)$ which follow from the relation (38). Also, for the parameters $1/\varepsilon_\perp(k_\parallel,\omega)$ and $1/\mu_\perp(k_\parallel,\omega)$ the numerators vanish when $k_\parallel \longrightarrow 0$ since in that case the solutions of two equations for e and m waves are identical, which implies $k_\perp^e(0,\omega) = k_\perp^m(0,\omega)$ and $X^e(0,\omega) = 1/X^m(0,\omega)$. Hence the effective parameters have the analytic properties required by the causality principle and the derivation of the Kramers–Kronig relations.

The consequences of the passivity on the effective parameters should be derived from the relation $\operatorname{Im}\omega\varepsilon(x,\omega) \ge \operatorname{Im}\omega\varepsilon_0$. Let the multilayered structure be periodically stacked so that it fills all the semi-infinite space $x<0$: $\varepsilon(x,\omega) = \varepsilon(x-h,\omega)$ for $x<0$ and $\varepsilon(x,\omega) = \varepsilon_0$ for $x>0$. Let $\boldsymbol{e}_\perp$ be the unit vector in the stacking direction x and $\nabla(\boldsymbol{k}_\parallel)$ the differential operator $(\mathrm{i}\boldsymbol{k}_\parallel + \boldsymbol{e}_\perp \partial/\partial x)$ after the Fourier decomposition in the plane parallel to the layers. After the Fourier decomposition, the Helmhotz operator $\mathsf{H}(\omega)$ for the multilayered stack is

$$\mathsf{H}(\boldsymbol{k}_\parallel,\omega)\boldsymbol{E}(x) = -\nabla(\boldsymbol{k}_\parallel) \times \frac{1}{\omega\mu_0}\nabla(\boldsymbol{k}_\parallel) \times \boldsymbol{E}(x) + \omega\varepsilon(x,\omega)\boldsymbol{E}(x), \tag{45}$$

where the $(\boldsymbol{k}_\parallel,\omega)$-dependence of the electric field has been omitted. Let $\mathsf{H}_{\text{eff}}(\boldsymbol{k}_\parallel,\omega)$ be the Helmhotz operator of the effective structure which coincides with (45) except for the domain $x<0$ where $\varepsilon(x,\omega)$ and μ_0 are replaced by $\varepsilon_{\text{eff}}(k_\parallel,\omega)$ and $\mu_{\text{eff}}(k_\parallel,\omega)$. An electromagnetic source $\boldsymbol{J}(x)$ outside the composite is considered, i.e. $\boldsymbol{J}(x) = 0$ if $x<0$. The electric field generated by this source in presence of the multilayered structure is the solution to

$$\mathsf{H}(\boldsymbol{k}_\parallel,\omega)\boldsymbol{E}(x) = \boldsymbol{J}(x), \tag{46}$$

and the electric field generated by this source in presence of the effective structure is the solution to

$$\mathsf{H}_{\text{eff}}(\boldsymbol{k}_\parallel,\omega)\boldsymbol{E}_{\text{eff}}(x) = \boldsymbol{J}(x). \tag{47}$$

The two electric fields are identical outside the multilayered structure: $\boldsymbol{E}(x) = \boldsymbol{E}_{\text{eff}}(x)$ if $x>0$. Thus, defining the inner product by

$$\langle \boldsymbol{E}, \boldsymbol{J}\rangle = \int_{\mathbb{R}} \mathrm{d}x\, \overline{\boldsymbol{E}(x)}\cdot \boldsymbol{J}(x), \tag{48}$$

the identity $\langle \boldsymbol{E}_{\text{eff}}, \boldsymbol{J}\rangle = \langle \boldsymbol{E}, \boldsymbol{J}\rangle$ holds for all source $\boldsymbol{J}(x)$ vanishing for $x<0$, and takes the form

$$\langle \boldsymbol{E}_{\text{eff}}, \mathsf{H}_{\text{eff}}(\boldsymbol{k}_\parallel,\omega)\boldsymbol{E}_{\text{eff}}\rangle = \langle \boldsymbol{E}, \mathsf{H}(\boldsymbol{k}_\parallel,\omega)\boldsymbol{E}\rangle. \tag{49}$$

Notice that the integrals are well-defined as soon as the imaginary part of the frequency is strictly positive: $\operatorname{Im}\omega > 0$. In order to take the limit $\operatorname{Im}\omega \longrightarrow 0$, it is assumed that there is a material in the multilayered stack with absorption, i.e. $\operatorname{Im}\omega\varepsilon(x,\omega) > 0$ at some $x<0$. Then, considering the imaginary part, the identity (49) implies for real frequencies ω

$$\operatorname{Im}\langle \boldsymbol{E}_{\text{eff}}, \mathsf{H}_{\text{eff}}(\boldsymbol{k}_\parallel,\omega)\boldsymbol{E}_{\text{eff}}\rangle = \langle \boldsymbol{E}, \operatorname{Im}\omega\varepsilon(\omega)\,\boldsymbol{E}\rangle > 0. \tag{50}$$

Notice that the integrals over x in this relation (50) reduce to the domain $x<0$. Let P^e and $P^m = 1 - P^e$ be the orthogonal projections on the electric and magnetic waves:

$$P^e = 1 - \frac{\boldsymbol{k}_\parallel\boldsymbol{k}_\parallel}{k_\parallel^2} - \boldsymbol{e}_\perp\boldsymbol{e}_\perp, \quad P^m = \frac{\boldsymbol{k}_\parallel\boldsymbol{k}_\parallel}{k_\parallel^2} + \boldsymbol{e}_\perp\boldsymbol{e}_\perp, \tag{51}$$

where the rank-two tensors act as $\boldsymbol{k}_\parallel\boldsymbol{k}_\parallel\cdot\boldsymbol{E} = (\boldsymbol{k}_\parallel\cdot\boldsymbol{E})\boldsymbol{k}_\parallel$ and $\boldsymbol{e}_\perp\boldsymbol{e}_\perp\cdot\boldsymbol{E} = (\boldsymbol{e}_\perp\cdot\boldsymbol{E})\boldsymbol{e}_\perp$. For $x<0$ the electric field $\boldsymbol{E}_{\text{eff}}(x)$ is the solution to $\mathsf{H}_{\text{eff}}(\boldsymbol{k}_\parallel,\omega)\boldsymbol{E}_{\text{eff}}(x) = 0$ and thus, for $w = e, m$,

$$\nabla(\boldsymbol{k}_\parallel)\times P^w\cdot\boldsymbol{E}_{\text{eff}}(x) = \mathrm{i}\boldsymbol{k}^w\times\boldsymbol{E}_{\text{eff}}(x), \quad \boldsymbol{k}^w = \boldsymbol{k}_\parallel - k_\perp^w(k_\parallel,\omega)\boldsymbol{e}_\perp, \tag{52}$$

where the $(k_\parallel,\omega)$-dependence of the vectors $\boldsymbol{k}^w(k_\parallel,\omega)$ is omitted. Notice that the minus sign in front of the component along $\boldsymbol{e}_\perp$ has been chosen according to the condition (38) in order to ensure the exponential decrease of the transmitted field at the limit $x \longrightarrow -\infty$. Hence, for $x<0$, it is obtained

$$\begin{aligned}\mathsf{H}_{\text{eff}}(\boldsymbol{k}_\parallel,\omega)\boldsymbol{E}_{\text{eff}}(x) = {} & \omega\varepsilon_{\text{eff}}(k_\parallel,\omega)\boldsymbol{E}_{\text{eff}}(x)\\ & + \boldsymbol{k}^e \times \frac{1}{\omega\mu_{\text{eff}}(k_\parallel,\omega)}\,\boldsymbol{k}^e\times P^e\cdot\boldsymbol{E}_{\text{eff}}(x)\\ & + \boldsymbol{k}^m \times \frac{1}{\omega\mu_{\text{eff}}(k_\parallel,\omega)}\,\boldsymbol{k}^m\times P^m\cdot\boldsymbol{E}_{\text{eff}}(x).\end{aligned} \tag{53}$$

Considering, for $w=e,m$, the operation $\boldsymbol{k}^w\times$ as a rank-two antisymmetric tensor, the relation (50) implies that the rank-two tensor

$$\omega\varepsilon_{\text{eff}}(k_\parallel,\omega) + \boldsymbol{k}^e \times \frac{1}{\omega\mu_{\text{eff}}(k_\parallel,\omega)}\,\boldsymbol{k}^e\times P^e + \boldsymbol{k}^m \times \frac{1}{\omega\mu_{\text{eff}}(k_\parallel,\omega)}\,\boldsymbol{k}^m\times P^m \tag{54}$$

has positive imaginary part. Since $P^m\cdot\boldsymbol{k}^m = \boldsymbol{k}^m$ and $P^e\cdot\boldsymbol{k}^m = 0$, the contraction of the tensor above by $\boldsymbol{k}^m$ and its complex conjugate $\overline{\boldsymbol{k}}^m$ provides the relation

$$\operatorname{Im}\omega\overline{\boldsymbol{k}}^m\cdot\varepsilon_{\text{eff}}(k_\parallel,\omega)\cdot\boldsymbol{k}^m = k_\parallel^2\operatorname{Im}\omega\varepsilon_\parallel(k_\parallel,\omega) + |k_\perp^m|^2\operatorname{Im}\omega\varepsilon_\perp(k_\parallel,\omega) > 0. \tag{55}$$

Thus a condition forcing the imaginary part of the effective permittivity components to be positive is obtained. On the other hand, there is no condition on the imaginary part of the effective permeability. Indeed, some arguments lead to the conclusion that the imaginary part of the effective permeability $\omega\mu_{\text{eff}}(k_\parallel,\omega)$ must take positive and negative values.

As pointed out when it has been defined (43), the inverse effective permeability $1/\mu_{\text{eff}}(k_\parallel,\omega)$ is an analytic function in the upper half plane of complex frequencies ω with positive imaginary part (notice that $\omega\mu_\parallel(k_\parallel,\omega)$ cannot vanish as well as $k_\perp^e(k_\parallel,\omega)$ and $X^e(k_\parallel,\omega)$). This follows from the analytic properties of the permittivity $\varepsilon(x,\omega)$ of the multilayered structure and the contruction of the effective parameters. In addition, since the permittivity $\varepsilon(x,\omega)$ tends to that of vacuum ε_0 when $|\omega|\to\infty$, all the effective parameters tend as well to ε_0 and μ_0. Hence the relation (24) deduced from the Kramers–Kronig relations is true for the inverse effective permeability at $k_\parallel = 0$:

$$\frac{1}{\mu_{\text{eff}}(0,0)} = \frac{1}{\mu_0} + \frac{2}{\pi}\int_0^\infty \mathrm{d}\nu\,\operatorname{Im}\frac{1}{\nu\mu_{\text{eff}}(0,\nu)}. \tag{56}$$

And, more generally, the same relation is obtained when the wave vector $k_\parallel$ is set to $k_\parallel^2 = \omega^2\varepsilon_0\mu_0 u_\parallel^2$ with $u_\parallel^2 < 1$, which corresponds to an excitation at a fixed angle. Next, it is used that the quasistatic limit provides $\mu_{\text{eff}}(0,0) = \mu_0$ since the starting multilayered structure is non-magnetic: the relation (56) becomes

$$\int_0^\infty \mathrm{d}\nu\,\operatorname{Im}\frac{1}{\nu\mu_{\text{eff}}(0,\nu)} = 0. \tag{57}$$

Hence it can be concluded that the imaginary part of the effective permeability $\mu_{\text{eff}}(0,\nu)$ must take positive and negative values. Notice that, if the permittivity $\varepsilon(x,\omega)$ of the multilayered structure is well-defined for all frequency ω, then the inverse effective permeability $1/\mu_{\text{eff}}(k_\parallel,\omega)$ is also exactely and well-defined for all frequency ω. Thus, in the present case, contrary to the situation described in [3, Section 82], the Kramers–Kronig relations and the integrals (56) and (57) make sense.

Finally, it can be checked that the Kramers–Kronig relations and the resulting sum rules are consistent for the effective permittivity. According to the analytic properties of the inverse effective permittivity, the relation (56) is true for $\varepsilon_{\text{eff}}(k_\parallel,\omega)$:

$$\frac{1}{\varepsilon_{\text{eff}}(0,0)} = \frac{1}{\mu_0} + \frac{2}{\pi}\int_0^\infty \mathrm{d}\nu\,\operatorname{Im}\frac{1}{\nu\varepsilon_{\text{eff}}(0,\nu)}. \tag{58}$$

The value of the effective permittivity of the multilayered structure at the quasistatic limit is given by [34]

$$\varepsilon_{\parallel}(0,0) = \int_{-h}^{0} \mathrm{d}x\, \varepsilon(x,0), \qquad \frac{1}{\varepsilon_{\perp}(0,0)} = \int_{-h}^{0} \mathrm{d}x\, \frac{1}{\varepsilon(x,0)}. \tag{59}$$

Since the permittivity of the multilayered structure takes real values greater ε_0 at the static limit [3], then this is also the case for the effective permittivity: $\varepsilon_{\parallel}(0,0) > \varepsilon_0$ and $\varepsilon_{\perp}(0,0) > \varepsilon_0$. These relations are consistent with the sum rule (59) and the condition (55) on the imaginary part of the effective permittivity.

These arguments confirm the lack of symmetry between the permittivity $\varepsilon(\omega)$ and the permeability $\mu(\omega)$. In that case it is relevant to consider the Maxwell's equations with spatial dispersion. In this article, the exact and explicit expression of the effective parameters of a multilayered structure has been derived for all the frequencies ω and wave vector $\boldsymbol{k}_{\parallel}$. According to (53), the expression of the corresponding effective permittivity tensor with spatial dispersion takes the form

$$\omega \underline{\underline{\varepsilon}}_{\mathrm{eff}}(\boldsymbol{k},\omega) = \omega \mathscr{E}_{\mathrm{eff}}(\boldsymbol{k},\omega) + \boldsymbol{k} \times \left[\frac{1}{\omega \mathscr{U}_{\mathrm{eff}}(\boldsymbol{k},\omega)} - \frac{1}{\omega \mu_0} \right] \boldsymbol{k} \times, \tag{60}$$

where the "permittivity part" $\mathscr{E}_{\mathrm{eff}}(\boldsymbol{k},\omega)$ and the "permeability part" $\mathscr{U}_{\mathrm{eff}}(\boldsymbol{k},\omega)$ should be expressed from $\varepsilon_{\mathrm{eff}}(k_{\parallel},\omega)$ and $\mu_{\mathrm{eff}}(k_{\parallel},\omega)$. The coefficients of the tensor $\underline{\underline{\varepsilon}}_{\mathrm{eff}}(\boldsymbol{k},\omega)$ depending on $(\boldsymbol{k},\omega)$ are different from the effective parameters depending on $(k_{\parallel},\omega)$ in the rank-two tensor (54) because they do not take into account the dispersion laws for $w = e, m$: $\boldsymbol{k}(\omega) = \boldsymbol{k}_{\parallel} \pm k_{\perp}^{w}(k_{\parallel},\omega)\boldsymbol{e}_{\perp}$. Thus the following functions are introduced for $w = e, m$:

$$K^{w}(k,\omega) = \sqrt{k^2 - k_{\perp}^{w}(k_{\parallel},\omega)^2}, \tag{61}$$

where the sign of the square root will not play a role since all the coefficients and parameters used here depend on the square of $k_{\parallel}$ (the starting equations (33) and (40) depend on $k_{\parallel}^2$). Defining the "permittivity part" as

$$\mathscr{E}_{\mathrm{eff}}(\boldsymbol{k},\omega) = \varepsilon_{\mathrm{eff}}(K^{e}(k,\omega),\omega) P^{e} + \varepsilon_{\mathrm{eff}}(K^{m}(k,\omega),\omega) P^{m}, \tag{62}$$

it is obtained for $w = e, m$ that $K^{w}(k,\omega)$ equals $k_{\parallel}$ and $\mathscr{E}_{\mathrm{eff}}(\boldsymbol{k},\omega)P^{w}$ equals $\varepsilon_{\mathrm{eff}}(k_{\parallel},\omega)P^{w}$ when the dispersion law is complied at $\boldsymbol{k}(\omega) = \boldsymbol{k}_{\parallel} \pm k_{\perp}^{w}(k_{\parallel},\omega)\boldsymbol{e}_{\perp}$. Similarly, using that the rank-two antisymmetric tensor $\boldsymbol{k}\times$ acting on P^{m} gives P^{e} acting on $\boldsymbol{k}\times$, the "permeability part" of the tensor can be defined as

$$\mathscr{U}_{\mathrm{eff}}(\boldsymbol{k},\omega) = \mu_{\mathrm{eff}}(K^{e}(k,\omega),\omega) P^{m} + \mu_{\mathrm{eff}}(K^{m}(k,\omega),\omega) P^{e}. \tag{63}$$

Notice that the projections P^{e} and P^{m}, defined by (51), depend on the vector $\boldsymbol{k}_{\parallel}$ and, consequently, the "permittivity part" $\mathscr{E}_{\mathrm{eff}}(\boldsymbol{k},\omega)$ and the "permeability part" $\mathscr{U}_{\mathrm{eff}}(\boldsymbol{k},\omega)$ depend on the vector $\boldsymbol{k}$ although the effective parameters only depend on the norm $k_{\parallel}$. Finally, substituing the expressions (62) and (63) in (60), the effective permittivity tensor is given by

$$\omega \underline{\underline{\varepsilon}}_{\mathrm{eff}}(\boldsymbol{k},\omega) = \omega \underline{\underline{\varepsilon}}^{e}_{\mathrm{eff}}(\boldsymbol{k},\omega) P^{e} + \omega \underline{\underline{\varepsilon}}^{m}_{\mathrm{eff}}(\boldsymbol{k},\omega) P^{m}, \tag{64}$$

where, for $w = e, m$,

$$\omega \underline{\underline{\varepsilon}}^{w}_{\mathrm{eff}}(\boldsymbol{k},\omega) = \omega \varepsilon_{\mathrm{eff}}(K^{w}(k,\omega),\omega) + \boldsymbol{k} \times \left[\frac{1}{\omega \mu_{\mathrm{eff}}(K^{w}(k,\omega),\omega)} - \frac{1}{\omega \mu_0} \right] \boldsymbol{k} \times . \tag{65}$$

Hence the effective permittivity tensor $\underline{\underline{\varepsilon}}_{\mathrm{eff}}(\boldsymbol{k},\omega)$ has been constructed for all frequency ω and wave vector $\boldsymbol{k}$. This exact and explicit expression of an anisotropic permittivity tensor with spatial dispersion could be the starting point of further investigations on spatial dispersion in macroscopic electromagnetism. As a first result, it has been shown that the imaginary part of the effective permeability of a passive multilayered structure must take positive and negative values.

5. Conclusion

The advent of negative index materials opened questions that have tested the domain of validity of macroscopic electromagnetism. The existence of a negative index of refraction appeared as unreachable during more than 30 years until the introduction of microstructured resonant media and metamaterials. Then, considerable progress has been made in the engineering and the design of microstructured media reporting extraordinary properties. In this article, several mechanisms leading to negative index and negative refraction have been briefly reviewed: the original ideas developed by J. Pendry and his colleagues with the design of microstructured metallic media displaying electric and magnetic resonances; the exploitation of the richness of the dispersion law in dielectric photonic crystals to obtain negative refraction; and the development of numerous non-asymptotic homogenization techniques and effective medium modellings for composite media. All these advances over the last twenty years are now particularly exploited in the design of metasurfaces [104, 105] and topological insulators [106, 107]. They have also been extended to other classical waves equations in acoustics, mechanics and hydrodynamics [108].

Then, it has been seen how the extraordinary properties of negative index materials and metamaterials must be associated with frequency dispersion and spatial dispersion. In addition, it has been shown that the time-harmonic Maxwell's equations cannot describe properly systems like the perfect negative index flat lens or corner reflector. On the other hand, the introduction of the *auxiliary field formalism* provides a canonical approach to describe frequency dispersive negative index structures. It has been shown that the spectacular effects in the perfect flat lens and corner reflector are associated to the presence, at the prefect -1 index frequency, of essential spectrum in the Maxwell's equations. More generally, the presence of intervals of essential spectrum has been highligthed in corner structures at frequencies where frequency dispersive permittivity takes negative values. This essential spectrum generated by the corner is associated with an analog of "black hole" phenomenon, the corner behaving like an unbounded domain. This raises challenging and fascinating questions in applied mathematics (e.g. in the case of three-dimensional corners), in physics with the analog of "black hole" phenomenon and in numerical modelling for the computation of modes of dispersive structure (e.g. in the quasi-normal modes expansion). In particular, the canonical formalism for dissipative and frequency dispersive Maxwell's equations, the *auxiliary fields formalism*, offers the opportunity to analyze rigorously an analog of "black hole" phenomenon.

In the last section, it has been shown how the effective permittivity, which has been intensively analyzed for negative index materials and metamaterials, highlighted ambiguities in the passivity requirement and Kramers–Kronig relations for the permeability. In this article, several arguments have been reported to support that, in a passive medium, the imaginary part of the permeability can take positive and negative values. This statement is in contradiction with the usual presentation of macroscopic electromagnetism where the permittivity and the permeability are introduced in a symmetric way, and thus in passive media both have positive imaginary part. The approach considered here was to define the permeability from the permittivity with spatial dispersion, which breaks the symmetry between the permittivity and the permeability. The effective permeability of a passive and non-magnetic multilayered structure has been derived exactly for all frequency and wave vector: in that case, it has been shown that the effective permeability is subject to the Kramers–Kronig relations and has imaginary part taking positive and negative values. In addition, the full effective anisotropic permittivity tensor with spatial dispersion has been derived explicitely for all frequency and wave vector and could be the starting point of further investigations on spatial dispersion in macroscopic electromagnetism.

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