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L'EFFET HALL QUANTIQUE FRACTIONNAIRE
THE FRACTIONAL QUANTUM HALL EFFECT

Quantum transitions in bilayer states

Vincent Pasquier

Service de physique théorique, C.E. Saclay, 91191 Gif-sur-Yvette, France

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Abstract The possible phase transitions when two layers at filling factor $\nu_t = 1$ are gradually separated are studied in this article. In the bosonic case the system should undergo a pairing transition from a Fermi liquid to an incompressible state. In the Fermionic case, the state evolves from an incompressible $(1, 1, 1)$ state to a Fermi liquid. It is speculated that there is an intermediate phase involving charge two quasiparticles. ***To cite this article: V. Pasquier, C. R. Physique 3 (2002) 709–715.***

Hall effect / Fermi liquid / bilayer systems / Bose–Einstein condensate

Transition quantique entre états électroniques de systèmes bi-couche

Résumé J'étudie la transition de phase qui se produit lorsque deux couches dont le facteur de remplissage total vaut 1 sont séparées de façon graduelle. Dans le cas bosonique on s'attend à ce que le système subisse une transition de type BCS d'un liquide de Fermi vers un état incompressible. Dans le cas Fermionique l'état évolue à partir d'un état incompressible vers un liquide de Fermi. Je conjecture l'existence d'une phase intermédiaire pour laquelle la charge des quasiparicules est égale à deux. ***Pour citer cet article : V. Pasquier, C. R. Physique 3 (2002) 709–715.***

effet Hall / liquide de Fermi / système bicouches / condensat de Bose–Einstein

1. Introduction

The quantum Hall effect [1] is both a quantum and a macroscopic phenomena. Both aspects manifest themselves through the transport properties. The quantum character cannot be understood without invoking the splitting of levels $\hbar eB/m$ induced by the magnetic field B (m is the mass of the electron).

The relevant parameter which characterizes the system is its filling factor ν related to the electron density in units of magnetic flux (typically 10^{10} electrons per square cm for magnetic fields around 20 Tesla). For a small density $\nu \ll 1$ the electrons form a crystal due to the quenching of the kinetic energy. Experiments have shown that the system is a liquid which conducts the current up to quite small filling factors ($\nu \sim 1/7$). Moreover, the conductivity tensor

E-mail address: pasquier@spht.saclay.cea.fr (V. Pasquier).

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$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} \tag{1}$$

has very peculiar features: σ_{xy} is strictly constant and equal to $\nu e^2/h$ with ν a fractional filling factor for a wide variation of the magnetic field called the plateau region. It increases rapidly to reach a higher fractional value in between two plateaus. In the plateau regions σ_{xx} is strictly equal to zero and suddenly grows to reach large values in between the plateaus. Each plateau corresponds to a phase characterized by a specific wave function for the ground state. The system develops a gap responsible for the vanishing of the dissipative conductivity σ_{xx}. The transition region where the system switches between two plateaus is the quantum analogous of a continuous phase transition.

Here we investigate other types of transitions which occur when two electron (or bosonic) layers are separated from each other. In this case, the filling factor is kept fixed and the continuously varying parameter is the separation d between the two layers. The quantum transition results from the weakening of the interlayer interactions as they are separated. Two phases with a definite wave function can be identified when the layers are either very close or very far from each other.

We consider cases where the total filling factor is less than one and the dynamics is restricted to the lowest Landau level. The way particles organize is counter intuitive because their position is no longer a good quantum number. Instead, we must use the guiding center momentum P_x, P_y to localize them. In the symmetric gauge, for example, the expressions for P_x, P_y are given by:

$$P_x = p_x - \frac{qy}{2}, \qquad P_y = p_y + \frac{qx}{2}. \tag{2}$$

These guiding center coordinates do not commute: $[P_x, P_y] = -\mathrm{i}q$ where $q = eB/\hbar c$ is the charge of the particle times the magnetic field. As a result one cannot localize a particle better that over a cell of area $2\pi l^2$ with $l^2 = q^{-1}$. We can imagine that the effect of the magnetic field is to divide the space into cells, each of which corresponds to a quantum state. The precise definition of the filling factor ν is the number of electrons per cell. Note that the mass m is an irrelevant parameter which only appears in the level splitting and disappears from the dynamics. As a result, all the relevant parameters are solely due to the interactions. In principle it is a degenerate perturbation problem where the effective Hamiltonian is obtained by projecting the interaction potential V in the lowest level. If we denote by P this projector, the effective Hamiltonian is given by:

$$H = PVP. \tag{3}$$

Essentially, the effect of the projection is to replace the coordinates of the potential by the guiding center coordinates. Therefore H is a true operator (it has nondiagonal matrix elements) acting in the LLL Hilbert space.

In the fractional Hall effect, the plateaus can be explained through a careful study of the dynamics induced by (3). The aim here is to analyze similar phenomena in the bilayer systems. The systems are made of two layers and switch from one phase to the other as the separation between the layers is increased. A transition is expected to occur for $d \sim l$.

The case of electron bilayers of current experimental [2] and theoretical [3–5] interest are considered. The approach followed is very closed to that of Kim et al. [4] (especially their second section) although some of the conclusions are in better agreement with the recent proposal of Nomura and Yoshioka [5]. Bosonic bilayers are also studied, which are technically easier to understand which are potentially observable in the context of rotating Bose condensate.

2. Exciton in electron bilayers

When two parallel electron systems are sufficiently close to each other, interlayer Coulomb interactions can produce collective states which have no counterpart in the individual 2D system. An interesting case occurs when the total electron density equals the degeneracy of a single spin resolved Landau Level. In the balanced case, the filling factor of each layer is $\nu = 1/2$. If the separation between the layers d is large, they behave independently and are described by a gapless Fermi liquid [6]. No quantized Hall effect is seen. On the other hand, as d is reduced, the system undergoes a quantum phase transition to an incompressible state described by the filling factor $\nu_T = 1/2 + 1/2 = 1$. In this section I only discuss the incompressible state obtained when the layers are very close $d < l$ and leave the description of the compressible state and the transition to a later section.

Consider the case where both layers are on top of each other. Electrons in one layer are pseudospin up while those in the other layer are pseudospin down. The system must be in a ferromagnetic state if we assume that the effect of the interactions can be reduced to a short range repulsive potential. In the symmetric gauge the spatial part of the wave function for N_e electrons is then equal to a Vandermonde determinant, the so-called (1, 1, 1) state in Halperin's terminology [7]:

$$\Psi_{1,1,1} = \prod_{i<j} \left(z_i^\uparrow - z_j^\uparrow\right)\left(z_i^\downarrow - z_j^\downarrow\right)\left(z_i^\uparrow - z_j^\downarrow\right). \tag{4}$$

It is the unique wave function at filling factor one which vanishes when any two electrons are at the same position. The Pauli principle then forces the pseudospin part to be symmetric and therefore the pseudospin takes its maximum value $N_e/2$. If both layers are exactly half filled, one has $N_\uparrow = N_\downarrow = N_e$ so that the z component of the pseudospin $N_\uparrow - N_\downarrow$ is equal to zero and the pseudospin points in the $x{-}y$ easy-plane. The natural excitations are spin waves with a quadratic dispersion relation characteristic of a ferromagnet and not a linearly dispersing Goldstone collective mode. Said differently, the groundstate is a condensate of excitons obtained by acting with $(S^-)^{N_\downarrow}$ on the state with all electrons in the top layer.

This description can be refined using an excitonic picture. The excitons can be introduced by starting from a situation where the pseudospin up layer is filled and the other layer is empty. Suppose one electron is removed from the top layer to be put in the down layer. In this simple situation one has a hole in the up-layer interacting with an electron in the down-layer. The dynamics can be solved using the model Hamiltonian (3) and it can be shown that the electron and the hole form a bound state [8]. The main point of the following discussion is to show that the bound state wave function is independent of the the interacting potential and has a group theoretical interpretation [9]. Particles in the lowest Landau level organize into representations of a deformation of the displacement group generated by $P_x,\ P_y$ (2) and the angular momentum $L = xp_y - yp_x$ obeying the relations:

$$[P_x, P_y] = -\mathrm{i}q, \qquad [L, P_x] = \mathrm{i}P_y, \qquad [L, P_y] = -\mathrm{i}P_x. \tag{5}$$

They are characterized by the charge $\pm q$ of the particle (in (5) the charge q must be replaced by $-q$ when we consider a hole) which play a similar role as the angular momentum for the rotations (the Casimir operator is $P_x^2 + P_y^2 - 2qL$). The wave function of the bound state is the analogous of a Clebsh–Gordon coefficient which couples the two representations of charge q and $-q$ into an irreducible representation of charge zero. The exciton having a zero charge, it does not feel the exterior magnetic field and its guiding center coordinates $P_x,\ P_y$ can be diagonalized simultaneously. The bound state is a dipole oriented perpendicularly to its momentum $\mathbf{P}$ of length Pl^2. The dispersion relation can be computed in terms of the interaction V and is quadratic at low momentum $\varepsilon(p) \sim Vp^2l^2/2$. Excitons behave as effective bosons interacting with the Hamiltonian [10]:

$$H = \frac{1}{2\Omega} \sum_{a,b} \int V_{ab}(x-y)\rho_a(x)\rho_b(y)\,\mathrm{d}^2x\,\mathrm{d}^2y. \tag{6}$$

Here a, b is a layer index and the Hamiltonian takes into account the fact that the attraction between different layers is weaker than the repulsion in the same layer: $V_{\uparrow,\downarrow} < V_{\uparrow,\uparrow}$. Note that (6) is nothing but the second quantized rewriting of (3). When the two layers are on top of each other ($d = 0$), the SU(2) symmetry is recovered and the excitons interact weakly, which explains why the dispersion relation is quadratic in the momentum. The first effect of the separation is to introduce a repulsion between the excitons. If we model them by a slightly non-ideal Bose gas, the dispersion relation is parameterized by the repulsion pseudopotential U_0 equal to zero for $d = 0$ and increasing with d. As a result the exciton behaves like a Goldstone boson with sound velocity $u \sim \sqrt{U_0/ml^2}$ increasing with the separation. This goes with a smooth decreasing of the total spin as seen in [5]. The apparent contradiction between the absence of a gap and the observed incompressibility is due to the fact that the exciton is neutral and does not interfere with the charge gap responsible for the Hall effect.

When the separation d is so large that $V_{\uparrow,\downarrow} = 0$ the fluid is no more incompressible and the Bose-gas picture is no longer correct. Instead, each layer can be modeled by a gas of neutral fermionic dipoles [11, 12,10,13,14]. The problem is then to understand how the transition between the gas of bosonic excitons and the two uncorrelated Fermi liquids occurs.

3. Boson bilayers

Before discussing this quantum transition, I wish to draw an analogy with the reversed phenomena that occurs when two bosonic layers are moved away from each other. A physical context could be two Bose-condensates in a rotating trap gradually separated from each other. To use a language adapted to the Hall effect, I treat the rotation as if it were a magnetic field which means that the rotation frequency is equal to the harmonic trap frequency [15,16]. The magnetic length is then defined in terms of this critical frequency.

The physical problem consists of two kinds of bosons in a magnetic field at filling factor $\nu = 1$. We imagine that the particle index is a layer index. At zero separation the interaction between particles in different layers is the same as the interaction between particles in the same layer. By analogy with the quantum Hall state at $\nu = 1/2$, we expect the system to be described by a Fermi liquid state. We then separate the two layers which are exactly at half filling. When they are sufficiently far away that particles between different layers do not interact any more, one is left with a two copies of a $\nu = 1/2$ bosonic system which are incompressible states. Therefore, we expect that a transition will occur at some separation where the dissipative conductivity suddenly vanishes as the incompressible state builds up. This is exactly the reverse situation as with electrons; one goes from a compressible Fermi liquid state to an incompressible boson condensate as the two layers are separated from each other.

Let us first consider the zero separation state which should correspond to a Fermi liquid state. The picture developed for the Fermi liquid state at $\nu = 1$ [10] is in terms of neutral fermionic dipoles consisting of the charge e boson and a fermionic hole with a charge $-e$. Energetically, the system would like to have a Slater determinant wave function (4). This wave function is, however, in conflict with the bosonic statistics and the neutral dipole Fermi liquid is the less costly manner in which it adjusts itself to satisfy the correct statistics. It can be written in a product form [17]:

$$\Psi(z_i) = P\left\{\text{Fermi liquid} \prod_{i<j} z_i - z_j\right\}. \tag{7}$$

As explained earlier, the projection P is the origin of the dipole interpretation of the Fermionic quasiparticles. Each Fermi liquid quasiparticle is a dipole made of a boson correlated to a hole in the Slater determinant factor. The difference between the dipoles and the bilayer excitons is their fermionic statistics.

Another possibility to satisfy the Bose statistics is the Pfaffian state [18]:

$$\Psi(z_i) = \mathrm{Pf}\left\{\frac{1}{z_i - z_j}\right\} \prod_{i<j} z_i - z_j. \tag{8}$$

The Pfaffian factor being antisymmetric guarantees that the total wave function is symmetric. Each denominator $1/z_i - z_j$ in the Pfaffian removes the correlation hole between particles i and j. Thus, the

Pfaffian induces a pairing between the particle and can be thought as a kind of BCS wave function where the composite Fermions are in a p-paired $L = -1$ state. Unlike the Fermi liquid, this state is incompressible.

In the bilayer case the bosons carry a spin index which specifies in which layer they lie. The Fermi liquid takes advantage of this to reduce its energy by putting two dipoles with up and down spin in the same momentum state thus reducing the Fermi momentum by a factor $\sqrt{2}$ with respect to the spinless case. The system is in the paramagnetic state. The Pfaffian state on the other hand is ferromagnetic and can be obtained by acting with $(S^-)^{N_\downarrow}$ on the state with all bosons in the top layer without energy gain. As a result the Fermi liquid is probably energetically favored with respect to the Pfaffian in this bilayer situation.

In the large separation limit the $\nu = 1/2$ bosonic state of one layers is a $\nu = 1/2$ Laughlin type wave function

$$\Psi(z_i) = \prod_{i<j}^{N/2} (z_i - z_j)^2 \tag{9}$$

which is legitimate for bosons. This state is incompressible and minimizes the energy of a single layer. To understand how the transition from a Fermi liquid to this kind of state occurs it is useful to rewrite the product $(2, 2, 0)$ of the two $\nu = 1/2$ factors (9) as a paired state:

$$\Psi(z_i) = \mathrm{Det}\left\{\frac{1}{z_i^\uparrow - z_j^\downarrow}\right\} \prod_{i<j}^{N} z_i - z_j \tag{10}$$

which can be shown using the Cauchy identity. This rewriting clearly shows that the large separation limit can be understood as a pairing between the bosons of the top layer with those of the bottom layer. The pairing factor $1/z_i^\uparrow - z_j^\downarrow$ annihilates a correlation hole between these two bosons and carries an angular momentum $L = -1$.

If the state at zero separation is the Pfaffian state (8), it can be continuously deformed [19] into the state (9) without undergoing a phase transition. For this one needs to multiply the matrix element in (8) by a factor $1 + \mu\sigma_i\sigma_j$ and let μ vary between 0 and -1.

The pairing instability also follows in the dipole approach [4].

The dipoles at the Fermi surface have a length $k_f l^2$ and an orientation perpendicular to their momentum. For obvious geometrical reasons a dipole with momentum $\mathbf{k}$ tends to bind with a dipole $-\mathbf{k}$. When the repulsion between bosons of the two layers decreases, this strengthens the binding between dipoles with opposite spins and very plausibly induces the pairing instability in the p-channel.

To conclude this section, two scenarios are possible in the case of bosonic bilayer systems. In the first one the system is incompressible at all separations and it is in the Pfaffian state at zero separation. In the second more probable one the state is a Fermi liquid at zero separation and undergoes a pairing transition to an incompressible state as the separation is increased.

4. Transition in electron bilayers

I now return to the transition in the $\nu_t = 1$ Fermionic layers. The problem is more difficult and this section is speculative.

The intuition gained in the bosonic bilayer case was that layer separation induced attraction between the bosons in different layers at $d \neq 0$ which resulted in the disappearance of the correlation hole between them. What made life easy was that the quasiparticles relevant at $d = 0$ were Fermions and the pairing mechanism was reminiscent of a BCS transition. In the present case, excitons are bosons and we have seen that the effect of the separation is to repel them. Therefore we abandon the exciton picture and try to model the transition as a pairing mechanism between the electrons directly. This is possible if we multiply the wave function (4) by a symmetric factor which does not spoil its polynomial character nor modifies the filling factor. It suggests multiplying the wave function by a Permanent Factor [20,21]:

$$\Psi(z_i) = \mathrm{Per}\left\{\frac{1}{z_i^{\uparrow} - z_j^{\downarrow}}\right\}\Psi_{1,1,1} = \mathrm{Det}\left\{\frac{1}{(z_i^{\uparrow} - z_j^{\downarrow})^2}\right\}\prod_{i,j}\left(z_i^{\uparrow} - z_j^{\downarrow}\right)^2. \tag{11}$$

The second equality results from Borchart identity [22]. The first writing exhibits it can be obtained as a paired wave function from the state $(1,1,1)$ (4). The second writing represents the state as a paired state built on the $(0,0,2)$ bosonic Laughlin state. Note that the weak pairing state with the square factor removed in the determinant yields back the $(1,1,1)$ state [4]. Although the two states (4) and (11) are both incompressible, the transition should have consequences in drag experiments [23]. In (4) electrons of the first layer are bound to holes in the second layer whereas in (11) they form pairs with the electrons of the second layer in agreement with the conclusions of [5].

This cannot be the complete story however since at large separation the Fermi liquid states are built as in (7) on a $(2,2,0)$ incompressible state, not a $(0,0,2)$ one as in (11). A possible precursor to the Fermi liquid state is a product of two Pfaffian states. The main difference between the Pfaffian states and our trial state (11) is that in the Pfaffian the electrons are paired inside one single layer whereas in (11) the pairs involve two electrons in different layers. It is possible that this repairing occurs in a continuous way. The Pfaffian incompressible state then undergoes a second phase transition towards a Fermi liquid state. Although this scenario with two phase transitions is neither economical nor easy to formalize precisely, it is difficult to rule out an intermediate phase involving paired quasiparticles. An experimental compelling evidence of this possibility would be to observe charge two carriers in this intermediate phase.

5. Concluding remarks

The two layer systems clearly exhibit quantum phase transitions mediated by interactions. Such transitions are now well studied in the electron context and it would be very interesting to see them in bosonic systems. A first step would however be to clearly identify a fractional Hall regime in rotating Bose condensates and the most promising direction seems to me the analogous of the Jain series at filling factors $\nu = p/p+1$ terminating in a Fermi liquid state at $\nu = 1$. The transition discussed here would be a second step. In the fractional Hall regime, the message of this essay is to stress that the most likely transition between a Fermi liquid state and an incompressible state is through a pairing mechanism which may hopefully be seen experimentally.

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