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(Physique mathématique, physique théorique/*Mathematical physics, theoretical physics*)

L'EFFET HALL QUANTIQUE FRACTIONNAIRE
THE FRACTIONAL QUANTUM HALL EFFECT

DOSSIER

Edge states tunneling in the fractional quantum Hall effect: integrability and transport

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Abstract This is a short review of nonperturbative techniques that have been used in the past 5 years to study transport out of equilibrium in low dimensional, strongly interacting systems of condensed matter physics. These techniques include massless factorized scattering, the generalization of the Landauer Büttiker approach to integrable quaisparticles, and duality. The case of tunneling between edges in the fractional quantum Hall effect is discussed in details. ***To cite this article: H. Saleur, C. R. Physique 3 (2002) 685–695.***

transport / shot noise / Keldysh / Yang–Baxter / duality

Effet tunnel entre états de bord de l'effet Hall quantique fractionnaire

Résumé Cet article présente un survol rapide des techniques non perturbatives qui ont été utilisées dans les 5 dernières années pour étudier le transport hors équilibre dans les systèmes de matière condensée avec fortes interactions. Ces techniques incluent la diffusion de particules de masse nulle, la généralisation de l'approche de Landauer Büttiker aux quasiparticules dans les systèmes intégrables, et la dualité. Le cas de l'effet tunnel entre états de bord de l'effet Hall quantique est discuté en detail. ***Pour citer cet article : H. Saleur, C. R. Physique 3 (2002) 685–695.***

transport / bruit de grenaille / Keldysh / Yang–Baxter

1. Introduction

Although the field of integrable systems is a remarkably lively one, it fair to say that it is often somewhat remote from experimental reality. Of course, integrable models are always related with physics: but they are usually considered as toy models where an interesting question can be investigated in detail, not models describing the exact situation encountered in the laboratory.

There have been, however, quite a few exceptions to this. For instance, the solution of the 2d Ising model [1] provided values for critical exponents that were observed in several phase transitions, and

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contributed tremendously to our understanding of critical phenomena. The most remarkable exception – and one that contributed very much to the development of the field – occurred in the context of the Kondo problem, where Andrei [2], and, independently, Wiegmann [3] showed that the sd spin $1/2$ model was integrable. Their approach gave exact results for thermodynamic quantities that could successfully be compared with experiment, and confirmed Wilson's numerical renormalization group calculations [4].

Having an 'exact solution' of the problem using a technique like integrability is not merely a luxury. In a field where interactions play a major role – and can, for instance, become extremely strong at low energy scales, such as in the Kondo problem – perturbative techniques cannot always provide the right answer, even qualitatively. The situation can become worse in problems where one is interested in transport out of equilibrium – like the shot noise in the quantum Hall effect, to be described below. The combination of the non-equilibrium and the interactions requires the use of sophisticated perturbative techniques – like the Keldysh formalism – which cannot, at the present time, be carried beyond the lowest orders. Numerical approaches are also difficult or impossible to use in such cases – simulations in real time suffer from well known sign problems, while simulations in imaginary time require continuation procedures, and are not in general adapted to situations out of equilibrium. In such dire circumstances, exact solutions are then one of the only ways to get useful information.

Impurity problems like the Kondo problem are certainly the most promising ones for bridging integrability and experiments: they exhibit very nontrivial physics, and yet are often manageable.

In the last few years, there has been a lot of interest in the properties of one-dimensional leads, where electrons are described by the Luttinger model, the simplest non-Fermi-liquid metal [5]. It was shown for instance in [6,7] that when an impurity is present in such a system, the current at $T=0$ behaves in a very different way from the free, Fermi liquid case, where it would be a continuous function of the impurity strength. In contrast, in the Luttinger liquid, the system becomes completely insulating at $T=0$ if the interactions are repulsive, while the defect simply heals if the interactions are attractive.

The Luttinger liquid model had been difficult to realize experimentally in the past, however. This is because in a one-dimensional conductor (such as a quasi-one-dimensional quantum wire, so thin that the transverse modes are frozen out at low temperature), random impurities occur in the fabrication. These impurities lead to localization due to backscattering processes between the excitations at the two Fermi points. In other words, the random impurities generate a mass gap for the fermions.

Fortunately, there is another possiblity: the edge excitations at the boundary of samples prepared in a fractional quantum Hall state should be extremely clean realizations of the (repulsive) Luttinger non-Fermi liquids, as was observed by Wen [8,9]. In contrast to quantum wires, these are stable systems because for $1/\nu$ an odd integer (here, ν denotes the filling fraction), the excitations only move in one direction on a given edge. Since the right and left edges are far apart from each other, backscattering processes due to random impurities in the bulk cannot localize those extended edge states. Moreover, the Luttinger interaction parameter is universally related to the filling fraction ν of the quantum Hall state in the bulk sample by a topological argument based on the underlying Chern–Simons theory, and does therefore not renormalize. The edge states should thus provide an extremely clean experimental realization of the Luttinger model.

I now describe an experimental set-up (see Fig. 1), that has allowed the detailed study of conductance in the presence of a single, tunable impurity [10,11].

A fractional quantum Hall state with filling fraction $\nu=1/3$ is prepared in the bulk of a quantum Hall bar which is long in the x-direction and short in the y-direction. This means that the bulk quantum Hall state is prepared in a Hall insulator state (longitudinal conductivity $\sigma_{xx}=0$), and that the (bulk) Hall resistivity is on the $\nu=1/3$ plateau where $\sigma_{xy}=(1/3)e^2/h$. This is achieved by adjusting the applied magnetic field, perpendicular to the plane of the bar. Since the plateau is broad, the applied magnetic field can be varied over a significant range without affecting the filling of $\nu=1/3$. Next, a gate voltage V_g is applied perpendicular to the long side of the bar, i.e. in the y direction at $x=0$. This has the effect of bringing the right and left moving edges close to each other near $x=0$, forming a *point contact*. Away from the contact

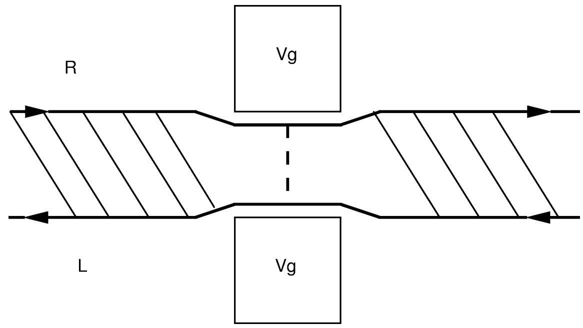

Figure 1. Schematic representation of a point contact, in which the R and L edges of a Hall fluid are brought together by a gate, inducing tunneling of charge between the two edges. The problem is mathematically equivalent to an impurity in a Luttinger liquid of coupling constant $g = \nu$, ν the filling fraction.

there is no backscattering (i.e., no tunneling of charge carriers) because the edges are widely separated, but now charge carriers can hop from one edge to the other at the point contact.

The right-moving (upper) edge of the Hall bar can now be connected to the battery on the left such that the charge carriers are injected into the right-moving lead of the Hall bar with an equilibrium thermal distribution at chemical potential μ_R. Similarly, the left-moving carriers (propagating in the lower edge) are injected from the right, with a thermal distribution at chemical potential μ_L. The difference of chemical potentials of the injected charge carriers is the driving voltage $V = \mu_R - \mu_L$. If $V > 0$, there are more carriers injected from the left than from the right, and a 'source-drain' current flows from the left to the right, along the x-direction of the Hall bar. In the absence of the point contact, the driving voltage places the right and left edges at different potentials (in the y-direction, perpendicular to the current flow), implying that the ratio of source-drain current to the driving voltage V is the Hall conductance $G = \nu e^2/h$ (both in linear response and at finite driving voltage V). When the point-contact interaction is included, at a finite driving voltage, more of the right moving carriers injected from the left are backscattered than those injected from the right, resulting in a loss of charge carriers from the source-drain current. In this case one can write the total source-drain current as $I(V) = I_0(V) + I_B(V)$, where I_0 is the current without point contact, and $I_B(V)$ is the (negative) backscattering current, quantifying the loss of current due to backscattering at the point contact.

This backscattering current is the main quantity of interest in the problem; it can be experimentally measured, and the question arose a few years ago, of whether it could also be computed in closed form, by using techniques of integrability similar to those developed in the context of the Kondo problem. This question led to a wealth of interesting developments, which we would like to summarize briefly here.

2. Field theoretic formulation

To start, we should emphasize that the questions of interest here all deal with transport properties, many of them out of equilibrium. In contrast, it is only static, thermodynamic quantities that had been computed in the Kondo problem, so new theoretical progress was necessary, before any useful results could be obtained. We will try to explain the ideas behind this progress in the following.

To proceed, let us now write the Hamiltonian of the problem. As argued by Wen, each of the two edges is a chiral Luttinger liquid. In the Luttinger Hamiltonian, there is a four fermion interaction, but it can be handled easily by using bosonization: putting the contributions of the two edges together, one can then describe the problem without impurity by a free, non-chiral boson. The point-contact interaction induces backscattering between the two edges. Since the tunneling takes place within the quantum Hall fluid, Laughlin quasiparticles of charge νe can tunnel, and this is in fact the most relevant process. In addition, higher order processes involving tunneling of multi quasiparticles, or electrons, are also possible, but are less relevant. In fact, in the case $\nu = 1/3$, all the other processes are irrelevant in the renormalization group sense, and we will not worry about them in what follows: this means we will only be able to discuss universal properties, characteristic of the scaling regime. This is an important restriction of this 'exact' solution, that can, however, be lifted in some cases.

In the bosonized Hamiltonian, the backscattering term is thus of the form $e^{ic(\phi_L-\phi_R)} + cc$, where c is a normalization constant that has to be adjusted carefully, to make sure that the tunneling particles have charge νe. With all normalizations right, and with an additional folding (that is actually crucial for the exact solution), the final hamiltonian is then

$$\mathcal{H} = \frac{1}{2}\int_{-\infty}^{0} \mathrm{d}x \left[(\partial_x \phi)^2 + (\Pi)^2\right] + 2\lambda \cos\sqrt{2\pi\nu}\,\phi(0), \tag{2.1}$$

where ν is the filling fraction, and $\lambda \propto V_g$.

The interaction is a relevant term: that is, in a renormalization group transformation, one has, b being the rescaling factor:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}b} = (1-\nu)\lambda + \mathrm{O}\big(\lambda^3\big). \tag{2.2}$$

(In particle physics language, $\mathrm{d}\lambda/\mathrm{d}b = -\beta(\lambda)$, so our relevant operator corresponds to a negative beta-function, i.e. an asymptotically free theory.) This means that at a large gate voltage, or, equivalently, at low temperature (since then, typical excitations have low energies, so the barrier appears high to them), the point contact will essentially split the system in half, and no current will flow through. In contrast, at a small gate voltage, or, equivalently, at high temperature, the point contact will essentially be invisible, and the current will just be I_0. The interesting question is: what happens in between – can we compute and measure the corresponding cross-over function?

For this latter question, let us stress again that we are interested in the universal, or scaling, regime, which is the only case where things will not depend in an complicated way on the microscopic details of the gate, and other experimental parameters. Ideally, what should be done in an experiment, [10] is first sweep through values of the gate voltage, the conductance signal showing a number of resonance peaks, which sharpen as the temperature is lowered. These resonance peaks occur for particular values $V_g = V_g^*$ of the gate voltage, due to tunneling through localized states in the vicinity of the point contact. Ideally, on resonance, the source-drain conductance is equal to the Hall conductance without point contact, i.e. $G_{\text{resonance}} = \nu e^2/h$. This value is independent of temperature, on resonance. Now, measuring for instance the linear response conductance as a function of the gate voltage near the resonance, i.e. as a function of $\delta V_g \equiv V_g - V_g^*$, at a number of different temperatures T, one gets resonance curves, one for each temperature. These peak at $\delta V_g = 0$. Finally, these conductance curves should collapse, in the limit of very small T and δV_g, onto a single universal curve when plotted as a function of $\delta V_g/T^{1-\nu}$. This is what is accessible using ideas of integrability.

3. The tunneling current

The Hamiltonian (2.1) is a very basic object that appears in a variety of other contexts, such as dissipative quantum mechanics, or quantum optics. It is nowadays referred to as the boundary sine-Gordon model, and is integrable [12]. In the following, we will write a few equations that are true whatever the value of the parameter ν in this model. To avoid confusion with the physical case of interest here, $\nu = 1/3$, we shall use the notation $\nu = g$.

Integrability can be used and formulated in a variety of different ways. A most useful conceptual progress in this sort of problem has been to think directly in terms of renormalized quantum field theories, instead of thinking of the 'bare models', as was done, e.g., for the Kondo problem. In that context, integrability appears within conformal perturbation theory [13] and is usually much easier to spot – this is how, in his pioneering work, Zamolodchikov [14] showed that the Ising model at T_c with a magnetic field is integrable in the continuum limit, although it is well known that the standard regularized square lattice version is *not* integrable. Also, excitations of the physical theory have a more direct physical meaning, and can be handled reasonably easily to compute transport properties.

Indeed, a convenient approach to compute transport properties is to try to remain as close as possible to the free electrons picture. To do so, one describes the spectrum of (2.1) with massless quasiparticles interacting through their factorized S matrix. These quasiparticles are simply obtained by taking the high energy limit of the bulk sine-Gordon spectrum: there are thus kinks, antikinks, and breathers. Moreover, because one is in the high energy limit, these quasiparticles are massless: they are either right or left moving, with dispersion relation $e = \pm p$. In the following, we shall parametrize the energy by $e = \mu_j\, \mathrm{e}^{\theta}$. Here, μ_j is a parameter that has the dimension of a mass, but it is not really a mass, in the sense that the theory has no gap. What really matters is the ratios of the parameters μ_j for the various types, labelled by j, of particles. The mass parameter of the (anti) kinks will simply be denoted by μ.

It is important to realize that the massless quasiparticles we just introduced have essentially no meaning in the original physical problem, since they are excitations of the *folded problem*. In terms of the physical electrons, or the Laughlin quasiparticles, they are hopelessly complicated, nonlocal objects! However, the transformations between the various possible bases in the Hilbert space of the problem do preserve the charge: a kink is the quantum particle associated with classical solitons, and it does carry an electric charge, equal to the electron charge in this model. Similarly an antikink is the quantum version of an antisoliton, and carries a charge equal to *minus* the electronic charge. Breathers are bound states of kinks and antikinks, and do not carry any charge.

Quasiparticles do provide a convenient way of exactly handling the excitations of the problem and computing its physical properties, as we shall now see.

Indeed, the kink, antikink, and breathers are only weakly interacting. More precisely, they have a nontrivial scattering, but it is given by a factorized S matrix, solution of the Yang–Baxter equation. In the following, we will restrict ourselves to the case where g is the inverse of an integer, where this scattering is purely diagonal: therefore, the only effect of the interactions is that wave functions pick up a nontrivial, rapidity dependent phase, when two particles, both L or both R moving, are exchanged (L and R particles simply do not see each other). The key advantage of this approach, as compared, say, to using plane waves to describe the free boson excitations, is that the particles scatter in a simple way on the impurity – more accurately, in the folded version of the problem, at the boundary: they simply bounce back after picking up a phase, and, for the kinks and antikinks, can also switch charge in the process. Integrability has thus reduced the complicated problem we started with to a much simpler situation: we have a half line, with a gas of kinks, antikinks and breathers that go through each other with simple phase shifts, and also bounce back on the boundary. Computing the tunneling current is now an easy matter.

To start, one needs to determine the statistical distribution of these quasiparticles in the bulk, at temperature T, and with a voltage V, that acts as a chemical potential for kinks and antikinks. This is easily done using the technique called thermodynamic Bethe ansatz [15]. The S matrix enters the problem through the quantization condition (simply expressing the total phase picked up when going around the system)

$$\exp\bigl(\mathrm{i}\mu_j\, \mathrm{e}^{\theta_j} L\bigr) \prod_{k\neq j} S(\theta_j - \theta_k) = 1.$$

This means that particles cannot coexist in the system independently of one another; rather, their rapidities are all correlated. The quantization condition is technically more complicated, but fundamentally equivalent, to the usual quantization for free fermions, $\exp(\mathrm{i}\mu\, \mathrm{e}^{\theta_j} L) = 1$. In both cases, the next step is to write the energy and entropies, and to minimize the grand potential. In the free fermions case, this results in the well known facts that the density of allowed states is

$$n_j = \rho_j + \rho_j^h = \frac{1}{2\pi} \frac{\mathrm{d}\varepsilon_j}{\mathrm{d}\theta},$$

while the filling fractions are

$$f_j = \frac{1}{1+e^{\varepsilon_j/T}}, \quad f_\pm = \frac{1}{1+e^{(\varepsilon_\pm \mp V/2)/T}}, \quad \varepsilon = \mu\, e^\theta.$$

In the case we are considering here, the same formulas hold, but the parameter ε_j is not equal to the bare energy anymore. Rather, this 'pseudo-energy' is the solution of a complicated set of integral equations which can be written generically

$$\mu_j\, e^\theta = \varepsilon_j + T \sum_k \frac{K_{jk}}{2\pi} \star \ln\bigl(1 + e^{(\mu_k - \varepsilon_k)/T}\bigr), \tag{3.1}$$

where $\star$ denotes convolution, and the kernel K_{jk} is the logarithmic derivative of the S_{jk} matrix element. We do not need the exact expression of the kernels to appreciate the salient feature of these equations: at temperature T, the filling fractions of the various quasiparticles are not independent, but correlated via coupled integral equations. This has some striking consequences. For instance, the filling fraction of kinks or antikinks at rapidity $-\infty$ (i.e. at vanishing bare energy) is $f = g$. Therefore, except formally for $g = 1/2$ (which is a free fermion theory, where kink and antikink stand respectively for particle and hole) there is no symmetry between particles and holes. It is important to realize that the interactions would have other effects, in general, for other questions asked. For instance, in the case of free fermions, the total density $n = \rho + \rho^h$, $\rho = nf$, the *fluctuations* also depend on the ε_j through the well known formula $\overline{(\Delta\rho)^2} = nf(1-f)$. Such a formula does not hold in the present case: the fluctuations of the various species are correlated – their computation plays an important role in the DC noise at nonvanishing temperature and voltage, see below.

Next, we consider the role of the impurity. In the original version of the problem, we had L and R moving electrons that were backscattered: a formal way to think of this, is that there was a $U(1)$ charge, $Q = Q_R - Q_L$, that was not conserved ($Q_L + Q_R$ is of course always conserved, since no particles are either created or destroyed). After folding, this nonconservation still occurs: whenever a kink bounces back as a kink, or an antikink as an antikink, we have $\Delta Q = \pm 2$. The UV, high energy fixed point corresponds to Neumann (free) boundary conditions, where kinks always bounce back as antikinks (just like on a string with the extremity free), and the charge is conserved: there is no backscattering current, as expected. The IR, low energy fixed point, corresponds instead to Dirichlet (fixed) boundary conditions, where kinks always bounce back as kinks (like on a string with fixed extremity), and the charge is maximally nonconserved: the current is completely backscattered.

To finish the job, all what we need to know is the probability, for a given gate voltage, that an incident kink bounces back as a kink. This is a very technical question: to answer it, one needs in general to solve fully the boundary sine-Gordon model (with a bulk interaction), impose the various Yang–Baxter, boundary Yang–Baxter and crossing constraints, to get complex expressions with products of gamma functions. Fortunately, in the case we are interested in, and provided we only want the probabilities, and not the complete phase shifts, the answer is amazingly simple: one finds:

$$p_{++} = \frac{1}{e^{2(g^{-1}-1)(\theta-\theta_B)}}. \tag{3.2}$$

This probability should depend on the ratio of the energy of the incident particle to a typical energy scale associated with the impurity. By the renormalization group equations, the coupling λ (proportional itself to the change in gate voltage away from a resonance) defines an energy scale by $T_B \propto \lambda^{1/(1-g)}$. The probability should then depend only on $\mu\, e^\theta / T_B$. Parametrizing $T_B = \mu\, e^{\theta_B}$, we find that this probability is a function of the difference $\theta - \theta_B$, as indicated in (3.2).

Equipped with all this, the rest of the computation is straightforward, although slightly technical. One simply writes a Boltzmann equation to compute the backscattered current, using the key idea that the particles scatter one by one, with no particle production, on the boundary, much as in a free situation. After a few manipulations, the final expression is

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty}(f_+ - f_-)(1 - p_{++})n(\theta)\,\mathrm{d}\theta, \tag{3.3}$$

where $n(\theta)$ is the densite of allowed states at rapidity θ for kinks or antikinks (the two coincide). Note how formula (3.3) is similar to the well known Landauer Büttiker formulas written in the context of noninteracting electrons tunneling through barriers [16,17].

A few manipulations lead to the more manageable result for the linear conductance

$$G = \frac{e^2}{h}\frac{(g^{-1}-1)}{2}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{1}{1+\mathrm{e}^{\varepsilon/T}}\,\frac{1}{\cosh^2[(g^{-1}-1(\theta-\theta_B)]}, \tag{3.4}$$

where ε is the pseudoenergy for (anti)kinks. All one needs to know to get the exact values of G are the values of ε, which follow easily from a numerical solution of the TBA equations (3.1). The resulting curve is shown in Fig. 2, together with experimental results [10] and the results of Monte Carlo simulations [11], for $g = \nu = 1/3$. The agreement with the simulations is clearly very good. As far as the experimental data

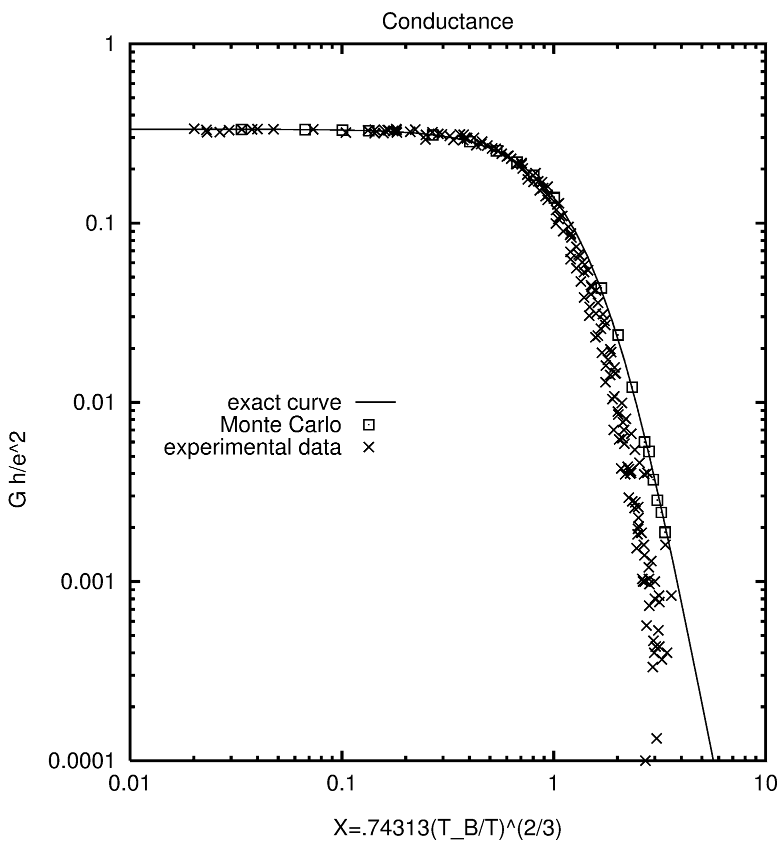

Figure 2. Comparison of field theoretic results with Monte Carlo simulations and experimental data for $g = \nu = 1/3$.

go, it is also very satisfactory, except in the strong backscattering regime. Recall however that the field theoretic prediction holds true only in the scaling limit: the experimental data are still quite scattered for low values of G, indicating that this limit is not reached yet – actually the scattering of the data is of the same order of magnitude as the discrepancy from the theoretical curve, as is reasonable to expect.

4. Noise

Although we have focussed mostly on the linear conductance, it should be clear that the formalism allows the computation of DC transport properties *out of equilibrium*, when $V \neq 0$. A particularly fascinating property in that category is, in fact, the shot noise. Indeed, recall that noise, in contrast with current, really does measure the charge of the carriers: since it is Laughlin quasi particles which tunnel in the weak backscattering limit, the problem we are discussing should therefore provide a way [18,19] to detect fractional charges in the laboratory!

More precisely, if we consider our problem at $T = 0$, to get rid of thermal fluctuations, we are left with a noise for the tunneling current due to the fluctuations in the scattering process: an incident kink having a probability p_{++} to bounce back as a kink, and $p_{+-} = 1 - p_{++}$ to bounce back as an antikink, the noise fluctuations for particles at rapidity θ depend on $p_{++} - p_{++}^2$, and all ingredients are therefore available, to compute $\langle I^2 \rangle$. In fact, one can show that the very simple fluctuation dissipation result holds [20]:

$$\left\langle I^2 \right\rangle = \frac{ge}{2(1-g)} \left[V \frac{\partial I}{\partial V} - I \right]. \tag{4.1}$$

This formula has the following interesting limiting behaviour. In the weak backscattering limit one finds the shot noise of noninteracting particles of charge ge:

$$\left\langle I^2 \right\rangle \approx ge(I_0 - I), \quad T_B \text{ small}, \tag{4.2}$$

while in the strong backscattering limit, one finds the shot noise of noninteracting particles of charge e (the electrons)

$$\left\langle I^2 \right\rangle \approx eI, \quad T_B \text{ large}. \tag{4.3}$$

The shot noise has been measured recently in a beautiful series of experiments at Saclay [21] and the Weizmann Institute [22], confirming the behaviour (4.2), and thus the existence of fractional charges.

The measurement of shot noise in one of a possible series of experiments to try to better understand the ‘transmutation’ of electrons into Laughlin quasi particles. Another particularly intriguing question would be what happens when one tries to inject electrons directly in a $\nu = 1/3$ chiral edge. In that edge, the electrons are not stable, and will disintegrate into bunches of quasiparticles. The simplest process would involve three quasiparticles, but there could also be more of these, combined with quasiholes. Still another interesting questions concerns higher moments of the current distribution, and, ultimately, the whole probability distribution of the tunneling current [23]. All these questions can also be studied using the same integrable techniques.

5. Formal developments

The development of techniques to compute nonequilibrium transport properties exactly in these systems has given rise, as so often in physics, to a large variety of further theoretical advances.

The most notable of these concern *duality*. Indeed, it has been realized from the beginning that the quantum Hall tunneling set up should give rise to a qualitative duality between the strong and weak backscattering regimes. This can predicted simply by looking at the shape of the device in either limit,

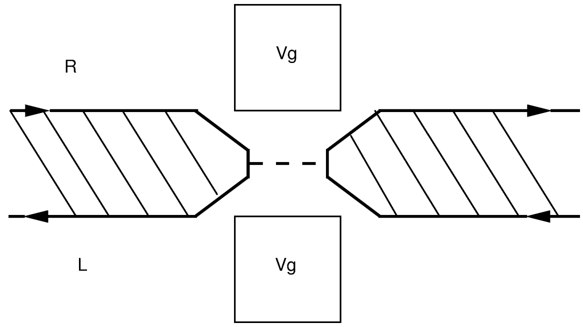

Figure 3. In the limit of very strong backscattering, the problem looks identical to Fig. 1, except that it is now electrons that tunnel.

which looks identical after a 90° rotation, up to the exchange between electrons and quasiparticles (see Fig. 3).

More precisely (that is, in the langage of quantum field theory) the weak and strong coupling fixed points – Neumann and Dirichlet boundary conditions for the free boson, respectively – are exchanged by the usual duality, $\phi \to \tilde{\phi}$. Moreover, while the perturbing operator near the UV fixed point is $\cos\sqrt{2\pi g}\phi$, it can be argued easily that the leading irrelevant operator near the IR fixed point is $\cos\sqrt{(2\pi)/g}\tilde{\phi}$: the two problems look similar up to the replacement of the field by its dual, and of g by $1/g$. In physical terms, the latter amounts to replacing Laughlin quasi particles by electrons, in agreement with the foregoing considerations about the noise. In general, this is all the duality one should be allowed to expect. The reason for this is slightly technical, but quite fundamental: the vicinity of the strong coupling fixed point is controlled by irrelevant operators, and the approach to this fixed point in the RG trajectory of interest is determined by an infinity of such operators, with exactly finely tuned coefficients – a more or less equivalent way to stress this difficulty is to stress that perturbation theory by irrelevant operators is not renormalizable, and requires the introduction of an infinity of counter terms. However, and very surprisingly, an exact duality was discovered in the problem at $T = 0$ [24]; later on, this duality was extended – based on a set of mathematical conjectures – to arbitrary T [25]: it can be written (in units of e^2/h):

$$I(\lambda, g, V, T) = gV - gI\left(\lambda_d, \frac{1}{g}, gV, T\right), \tag{5.1}$$

where V is the applied Hall voltage. The mathematical meaning of formula (5.1) is the following: if one knows how the current behaves at small coupling as a function of g, one is able right away to determine the analytical continuation of the perturbative series beyond the radius of convergence, by simply replacing g by $1/g$, λ by λ_d and V by gV in the formula.

The physical origin of the duality took, however, a little while to be understood [26]. It is absolutely true that the behaviour of the current near the strong coupling fixed point requires the knowledge of an infinity of counter terms. Because of the underlying integrability in the problem, it was actually possible to determine all these counter terms. They turned out to satisfy some remarkable properties: in particular (at least within an analytical regularization scheme) no other harmonic than the fundamental $\cos\sqrt{2\pi g}\tilde{\phi}$ appears (so the other counter terms involve only derivatives of the field $\tilde{\phi}$, and are like density density couplings), and the various counter terms are all commuting with one another. As a result, it was possible to show by using Keldysh perturbation theory that these counter terms simply do not affect the DC current, which is entirely determined by the leading irrelevant operator indeed, and duality (5.1) quickly follows (note that duality might not necessarily hold for other physical quantities).

Mathematically, the duality means that there is an exact instantons expansion in this problem. Indeed, there is still another way to understand the leading irrelevant operator $\cos\sqrt{(2\pi)/g}\tilde{\phi}$. Consider the action (2.1) at large coupling λ. To leading order, the field is localized in the minima of the potential $\phi(0) = n\pi/\sqrt{2\pi g}$, corresponding to the Dirichlet boundary conditions at the IR fixed point. The leading

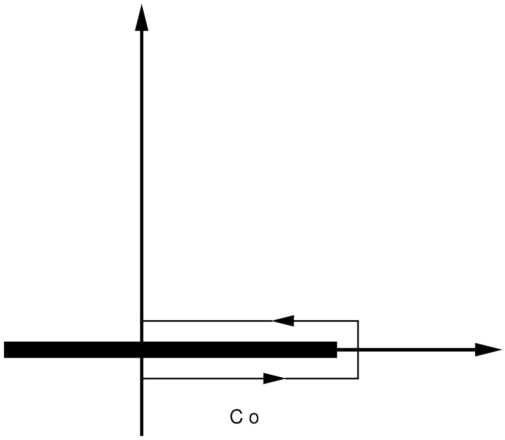

Figure 4. The contour to compute the current using the integral formula (5.2) starts at the origin just under the cut, wraps around the branch point on the positive real axis, and gets back to the origin right above the cut.

fluctuations around this fixed point are obtained by instantons connecting neighbouring minima, hence having charge ± 1. The action of these instantons can be evaluated, and, to leading order, their interaction turns out to be purely logarithmic: that is, they define a Coulomb-gas-like perturbative expansion around the strong coupling fixed point, that coincides with the one of the leading irrelevant operator $\cos\sqrt{(2\pi)/g}\tilde{\phi}$. The exact duality means that, for the current at least, this instanton expansion is actually exact!

Exact instanton expansions are a rare feast, usually associated with higher dimensional supersymmetric theories. There is no supersymmetry in the present problem, but integrability acts almost as powerfully, and it is perhaps not too surprising, that structures emerge, which bear a strange resemblance with the works of Seiberg and Witten [27,28]. For instance, it turns out that the current at $T = 0$ can be written in the remarkably simple closed form [29,30]:

$$\mathcal{I} = \frac{\mathrm{i}}{4u} \int_{C_0} \mathrm{d}x \frac{1}{\sqrt{x + x^g - u^2}}, \tag{5.2}$$

where $\mathcal{I} = I/(gV)$, $u \propto V/T_B$ is the only dimensionless variable in the problem, the curve C_0 starts at the origin, loops around the branch point on the positive real axis, and goes back to the origin (see Fig. 4). The duality reads then $\mathcal{I}(g,u) = 1 - \mathcal{I}(1/g,u)$, and follows now from a simple change of variables $x \to x^g$ in the integral.

The existence of this integral representation seems to be the tip of a yet quite unexplored iceberg. For instance, the underlying thermodynamic Bethe ansatz can be reformulated, at least at $T = 0$, as a system of monodromies. The current satisfies a differential equation of order n, if $g = 1/n$, n an integer, and the other solutions of this equation are related to the densities of various types of integrable particles in the ground state.

In what looks a priori like a different direction, the current appeared as a key ingredient in the analysis of conformal field theories as integrable quantum field theories that has been carried out in an impressive series of paper by the Russian school recently. Most noticeable maybe is the relation of this current with the 'quantum Q-operator' – the quantum analog of the famous Baxter's Q operator, and the discovery that this current is also related to the reflection coefficient for an integrable Schrödinger problem [31].

6. Conclusions

To conclude, tunneling between edges in the fractional quantum Hall effect provides a very interesting situation with a wealth of nonperturbative physical features, such as the shot noise of collective, fractionnally charged excitations, or the duality between Laughlin quasiparticles and electrons. Interactions *do* play an essential role here, and it is quite remarkable that methods derived from the Yang–Baxter equation are exactly what is needed to compute exactly several transport properties of experimental interest, precisely in a setting where traditional Fermi liquid methods would be unapplicable. As often happens, physics in turn leads to more formal developments: duality, quantum Q-operators and the like, that will certainly give rise to further important progress in our understanding of integrability.

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