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The quantum Hall effect on $\mathbb{R}^4$

Henriette Elvang [a], Joseph Polchinski [b,*]

[a] *Department of Physics, University of California, Santa Barbara, CA 93106, USA*
[b] *Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030, USA*

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Abstract

Zhang and Hu have formulated an SU(2) quantum Hall system on the four-sphere, with interesting three-dimensional boundary dynamics including gapless states of nonzero helicity. In order to understand the local physics of their model we study the U(1) and SU(2) quantum Hall systems on flat $\mathbb{R}^4$, with flat boundary $\mathbb{R}^3$. In the U(1) case the boundary dynamics is essentially one-dimensional. The SU(2) theory can be formulated on $\mathbb{R}^4$ for any isospin I, but in order to obtain a flat boundary theory we must take $I \to \infty$ as in Zhang and Hu. The theory simplifies in the limit, the boundary becoming a collection of one-dimensional systems. We also discuss general constraints on the emergence of gravity from nongravitational field theories. ***To cite this article: H. Elvang, J. Polchinski, C. R. Physique 4 (2003).***

Résumé

L'effet Hall quantique sur $\mathbb{R}^4$. Zhang et Hu ont formulé un système de Hall quantique SU(2) sur la quatre-sphère, avec une dynamique de bord tridimensionnelle intéressante comprenant des états sans gap d'hélicité differente de zéro. Afin de comprendre la physique locale de leur modèle, nous étudions les systèmes de Hall quantique U(1) et SU(2) sur un $\mathbb{R}^4$ plat, avec pour bord un $\mathbb{R}^3$ plat. Dans le cas U(1) la dynamique de bord est essentiellement uni-dimensionnelle. La théorie SU(2) peut être formulée sur $\mathbb{R}^4$ pour n'importe quel isospin I, mais afin d'obtenir dans une théorie de bord plate nous devons prendre $I \to \infty$ comme Zhang et Hu. La théorie se simplifie dans cette limite, le bord devenant une collection de systèmes unidimensionnels. Nous discutons également des contraintes générales sur l'apparition de la gravité à partir de théories de champs non-gravitationnelles. ***Pour citer cet article : H. Elvang, J. Polchinski, C. R. Physique 4 (2003).***

1. Introduction

The two-dimensional quantum Hall effect (QHE) has been a rich and fascinating subject. The bulk has a mass gap, and so the low-lying excitations live on the one-dimensional edge. Many nontrivial phenomena of $(1+1)$-dimensional quantum field theory arise in the QHE edge dynamics.

Recently, Zhang and Hu have found a beautiful four-dimensional generalization of the QHE, with three-dimensional edge dynamics, based on fermions moving in a background SU(2) gauge field [1,2]. Their most striking result is the presence of

* Corresponding author.
E-mail addresses: elvang@physics.ucsb.edu (H. Elvang), joep@itp.ucsb.edu (J. Polchinski).

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gapless spin-two bosons in the edge theory, suggesting the emergence of gravity. The model as presently formulated is a free theory, so there is no gravitational force, and there are actually massless bosons of all helicities. However, it has been argued [1] that introducing interactions might plausibly remove the unwanted states while leaving a theory of gravity.

Our goal is to develop a better understanding of the *local* dynamics of the Zhang–Hu model, where most of the key physics issues should arise. The model is originally formulated with the spatial dimensions forming a four-sphere S^4. To expose the local physics one must take the infinite-radius limit while focusing on a patch with geometry $\mathbb{R}^4$. In the Zhang–Hu model this limit is nontrivial: the fermions couple to the background gauge field with isospin I, and one must take I to infinity along with the radius. We would like to understand better why this is necessary, and in what sense the limit exists. Further, if the limit does exist then we might hope that it allows for some simplification, so that the important aspects of the physics are clearer than in the formulation on S^4.

Let us mention in particular one puzzling feature of the Zhang–Hu model. The 'graviton' is a particle–hole state. It is argued in [1,2] that the particle–hole separation remains small at all times, even in the absence of interactions, so that one can think of the state as a single particle. However, the uncertainty principle normally forbids this. If the separation is initially finite, $|\delta\vec{x}| < \infty$, then the relative momentum of the particle and hole is uncertain, $|\delta\vec{p}| > 0$. But the velocity is in general a nontrivial function of the momentum, so that $|\delta\vec{v}| > 0$ as well, and then the separation will grow linearly in time. The one exception to this is for relativistic particles in one dimension, which move with velocity c independent of their momentum. This is the essence of bosonization: a noninteracting fermion–antifermion pair forms a bosonic excitation that remains localized. But in more than one dimension $\partial v^i/\partial p^j$ is nontrivial (in particular the direction of the velocity depends on that of the momentum), and there is no natural bosonization.

Our approach will be to formulate the quantum Hall effect directly on flat $\mathbb{R}^4$, making contact with the Zhang–Hu model only later. In Section 2 we consider the QHE based on gauge group $\mathrm{U}(1)$. We first review the two-dimensional theory and its edge dynamics. We then extend this to four dimensions in the obvious way, by introducing $\mathrm{U}(1)$ magnetic fields in two independent planes. We show that the edge dynamics is not truly three-dimensional. Rather, it corresponds to a one-dimensional system with an infinite number of fermion fields, with helicities $0, 1, 2, \ldots,$ or equivalently to parallel one-dimensional systems arrayed (fuzzily) in two transverse dimensions. Nevertheless, this system turns out to be a useful building block toward understanding the SU(2) system. By taking a particle and hole with different helicities, we obtain localized gapless particle–hole excitations of arbitrary helicity as claimed in [1]. We develop some of the properties of these states, and we find some curious aspects that may be an obstacle to a relativistic theory.

The failure of the $\mathrm{U}(1)$ example can be ascribed to insufficient spatial symmetry. The symmetry group is $\mathrm{U}(2)$, which is smaller than the spatial symmetry group (rotations plus translations) of $\mathbb{R}^3$. In Section 3 we show that by introducing an SU(2) gauge field as in [1], it is possible to retain an SO(4) symmetry that combines spatial rotations with gauge rotations. This reduces to the spatial symmetry group of $\mathbb{R}^3$ in the flat limit. We are able to formulate, and solve, this version of the QHE on flat $\mathbb{R}^4$ even for finite isospin I. However, the density of states in the lowest Landau level of our system is finite for finite I. A bubble of quantum Hall fluid thus has a maximum radius, so the edge theory lives on S^3 not $\mathbb{R}^3$. In order to take the limit of a large bubble of quantum Hall fluid, so that its edge becomes locally $\mathbb{R}^3$, we find it necessary to take $I \to \infty$ just as in [1].[1]

In Section 4 we simplify the system to the maximum extent possible by taking the $I \to \infty$ limit of our system at the beginning, before taking the size of the Hall bubble to be large. The result is a continuously infinite collection of four-dimensional $\mathrm{U}(1)$ systems, distinguished by the spatial orientation of the magnetic field. The corresponding edge theory is an infinite collection of one-dimensional theories, distinguished by their orientation in three dimensions.

Section 5 is somewhat independent from the rest, an essay about emergent gravity. We explain why we do not believe that this is possible in the Zhang–Hu approach, and contrast this with the AdS/CFT duality which is an example of emergent gravity. We also relate this to the more familiar phenomenon of emergent gauge symmetry.

Reference [3] considers both $\mathrm{U}(1)$ and SU(2) magnetic fields on CP^2, so the discussion in our Section 2.2 would govern the local and edge dynamics of the $\mathrm{U}(1)$ case. The references in [4] develop the Zhang–Hu idea in other directions; it may be interesting to consider the local limits of these.

[1] There is another case in which the number of lowest Landau level states is infinite but the local density diverges at large radius, which is also unsatisfactory for going to the $\mathbb{R}^3$ limit.

2. The U(1) QHE on $\mathbb{R}^2$ and $\mathbb{R}^4$

2.1. The U(1) QHE in two dimensions

2.1.1. The bulk

We first review the physics of charged fermions in a constant magnetic field in two dimensions. For simplicity the fermions are spinless. We use units $\hbar = e/c = 1$, so the covariant derivative is $D_a = \partial_a - \mathrm{i}A_a$. The spatial dimensions are indexed $a,\ b$; since these are spatial indices, there is no distinction between upper and lower. We work in the gauge

$$A_1 = -\frac{B}{2}x_2, \qquad A_2 = \frac{B}{2}x_1. \tag{1}$$

The Hamiltonian is

$$H = -\frac{1}{2m}D_aD_a = \frac{1}{2m}\left(-\partial_a\partial_a + \frac{B^2}{4}x_ax_a - BL_{12}\right) = \frac{|B|(n+1) - BL_{12}}{2m}. \tag{2}$$

Here n is the total number of oscillator excitations and

$$L_{ab} = -\mathrm{i}(x_a\partial_b - x_b\partial_a). \tag{3}$$

For $B > 0$ the lowest Landau level (LLL) consists of all states with $L_{12} = n$; these have the minimum energy $B/2m$. It is convenient to work with complex coordinates,

$$z = \frac{1}{2}(x_1 + \mathrm{i}x_2), \qquad \partial_z = \partial_1 - \mathrm{i}\partial_2, \qquad D_z \ = \ \partial_z - B\bar{z}, \qquad D_{\bar{z}} = \partial_{\bar{z}} + Bz. \tag{4}$$

The Hamiltonian is then

$$H = \frac{B}{2m} - \frac{1}{2m}D_zD_{\bar{z}}. \tag{5}$$

The second term is nonnegative and for $B > 0$ the LLL states satisfy $D_{\bar{z}}\psi = 0$, implying that

$$\psi = f(z)\exp(-Bz\bar{z}) \tag{6}$$

with $f(z)$ analytic. The case $B < 0$ is given by $z \leftrightarrow \bar{z}$, so without loss of generality we take B positive in the remainder of this section.

The system is translationally invariant, and so there exist magnetic translation operators Π_a having the property

$$[\Pi_a, D_b] = 0. \tag{7}$$

In the gauge (1) these are simply given by $\Pi_a = -\mathrm{i}(\partial_a + \mathrm{i}A_a)$. There are two convenient bases for the LLL. The first are the eigenstates of L_{12},

$$f(z) \propto z^l, \qquad n = L_{12} = l. \tag{8}$$

The second are the eigenstates of Π_1,

$$f(z) \propto \exp\bigl(Bz^2 + 2ip_1z\bigr), \qquad \Pi_1 = p_1. \tag{9}$$

In the latter case, $|\psi|$ is independent of x_1 and Gaussian in x_2.

2.1.2. The edge

To produce a localized bubble one adds a confining potential to the Hamiltonian (we also add a constant so that the LLL energy is zero):

$$H' = H - \frac{B}{2m} + V, \quad V = \frac{\kappa}{2}x_ax_a, \tag{10}$$

with κ a positive constant. Now take the limit $m \to 0$. In this limit all excited states go to infinite energy and so only the LLL states mix under V; we can write

$$H' = V \quad \text{(between LLL states)}. \tag{11}$$

By rotational invariance, V is diagonal in the L_{12} basis, and therefore so is the Hamiltonian

$$\langle l|x_ax_a|l'\rangle = \frac{2}{B}(l+1)\delta_{ll'}, \qquad \langle l|H'|l'\rangle = \frac{\kappa}{B}(l+1)\delta_{ll'}. \tag{12}$$

The second-quantized Hamiltonian is

$$\mathbf{H}' = \frac{\kappa}{B} \sum_{l=0}^{\infty} (l+1) c_l^\dagger c_l. \tag{13}$$

With D fermions the ground state has levels $l = 0, 1, \ldots, D-1$ filled, forming a bubble of radius $r_0 = \sqrt{2D/B}$. The number of states per area is

$$\rho = \frac{D}{\pi r_0^2} = \frac{B}{2\pi}, \tag{14}$$

independent of D. Low-lying excitations involve fermions and holes with l close to D, which by Eq. (12) are near the edge. The level spacing κ/B corresponds to a massless field with velocity $v = r_0\kappa/B$. This is the same velocity that one gets by balancing the Lorentz force against that from the confining potential.

We are interested in the limit of an infinite bubble, where the edge S^1 becomes the real line $\mathbb{R}$. Take r_0 to infinity while holding B and v fixed, and focus on a point on the edge, say $x_a = (0, -r_0)$. By translation invariance we can take this point to be the origin, and in the limit the potential linearizes, $V = -vBx_2$. Then

$$H' = -vBx_2 = v\Pi_1 \quad \text{(between LLL states)}. \tag{15}$$

The last equality follows from $\Pi_1 + Bx_2 = -\mathrm{i}(D_z + D_{\bar z})/2$, since D_z $(D_{\bar z})$ gives zero acting to the left (right). Equivalently, it reflects the noncommutativity in the lowest Landau level, $[x_1, x_2] = -\mathrm{i}/B$. The Hamiltonian (15) describes fermions moving to the left with velocity v. The second quantized description is

$$\mathbf{H}' = -\mathrm{i}v \int_{-\infty}^{\infty} \mathrm{d}x_1\, \Psi^\dagger \partial_1 \Psi. \tag{16}$$

2.2. The U(1) QHE in four dimensions

2.2.1. The bulk

The most direct extension of the QHE to four dimensions is to introduce constant U(1) magnetic fields in two independent planes,

$$F_{12} = F_{34} = B. \tag{17}$$

The Hamiltonian is

$$H = -\frac{1}{2m} D_a D_a = \frac{1}{2m}\left[-\partial_a\partial_a + \frac{B^2}{4} x_a x_a - B(L_{12} + L_{34})\right], \tag{18}$$

where now a runs $1, \ldots, 4$. This is just two copies of the previous system. In particular, we can introduce two complex coordinates z_α,

$$z_1 = \frac{1}{2}(x_1 + \mathrm{i}x_2), \qquad z_2 = \frac{1}{2}(x_3 + \mathrm{i}x_4), \tag{19}$$

and the lowest Landau level consists of all states of the form

$$\psi = f(z_1, z_2) \exp\bigl(-Bz^\dagger \cdot z\bigr), \tag{20}$$

where $z^\dagger \cdot z = \overline{z_1} z_1 + \overline{z_2} z_2$. The background can be written

$$F_{\alpha\bar\beta} = 2\mathrm{i}B\delta_{\alpha\bar\beta}, \qquad F_{\alpha\beta} = F_{\bar\alpha\bar\beta} = 0. \tag{21}$$

In this form there is a manifest U(2) symmetry,

$$z_\alpha \to M_{\alpha\beta} z_\beta \tag{22}$$

for any 2×2 unitary matrix M. There are also translational symmetries in the four dimensions.

2.2.2. The boundary

The confining potential

$$V=\frac{\kappa x_a x_a}{2}=2\kappa z^{\dagger}\cdot z \tag{23}$$

gives two copies of the two-dimensional system (10). For example,

$$\langle l_1 l_2|V|l_1' l_2'\rangle=\frac{\kappa}{B}(l_1+l_2+2)\delta_{l_1 l_1'}\delta_{l_2 l_2'}, \tag{24}$$

where l_1 and l_2 are the eigenvalues of L_{12} and L_{34}. This potential preserves the $\mathrm{U}(2)$ symmetry (22) while breaking the translational symmetries.

Now let us go to the linearized limit,

$$V=u_a x_a. \tag{25}$$

By a $\mathrm{U}(2)$ rotation we can take $(u_1+\mathrm{i}u_2, u_3+\mathrm{i}u_4)$ to $(0,-ivB)$ so that the confining force is in the 4-direction. This corresponds to looking at a point on the sphere that is tangent to the 1–2–3 plane. Then

$$H'=-vBx_4=vP_3 \quad \text{(between LLL states)}. \tag{26}$$

We thus have two copies of the two-dimensional system. The first, in the 1–2 plane, has no potential and so an infinitely degenerate ground state. The second, in the 3–4 plane, has a linear potential and one-dimensional edge dynamics. We can use the L_{12} basis for the first and the P_3 basis for the second, so that there is an infinite number of one-particle states ψ_{l_1,p_3} with given momentum p_3.

The second-quantized description thus involves an infinite number of fields,

$$\mathbf{H}'=-\mathrm{i}v\int_{-\infty}^{\infty}\mathrm{d}x_3\sum_{l=0}^{\infty}\Psi_l^{\dagger}\partial_3\Psi_l. \tag{27}$$

Here $l\equiv l_1$ is the helicity, the eigenvalue of the rotation L_{12} around the direction of motion. Alternatively,

$$\mathbf{H}'=-\mathrm{i}v\int \mathrm{d}^3x\,\Psi^{\dagger}(\vec{x})\partial_3\Psi(\vec{x}), \tag{28}$$

but with the 1–2 plane noncommutative, $[x_1,x_2]=-\mathrm{i}/B$. The boundary theory is not truly three-dimensional, but rather one-dimensional with an infinite number of fields. We can understand this in terms of the symmetries of the system. We have noted that the confining potential (23) leaves a $\mathrm{U}(2)$ spatial symmetry. In the linear limit (25) the four symmetry generators become the translations in the 1-, 2-, and 3-directions and the rotation in the 1–2 plane. We are missing the additional two rotational symmetries of $\mathbb{R}^3$, which would rotate the 3-direction into the other two and so require fields moving in all directions.

2.2.3. Particle–hole states

Although the $\mathrm{U}(1)$ system is not truly three-dimensional, it is a useful warmup for the SU(2) system, and so we develop some of the properties of its particle–hole states. We focus on the two-body wavefunction

$$\psi(x,x')=\langle 0|\Psi(x)\Psi^{\dagger}(x')|\Sigma\rangle, \tag{29}$$

where $|\Sigma\rangle$ is a particle–hole state.

One basis for the particle–hole states is

$$\psi(x,x')=\psi_{l_1,p_3}(x)\psi^{*}_{-l_1',-p_3'}(x'), \tag{30}$$

taking the particle and hole each to have definite 3-momentum and definite helicity. The total quantum numbers for the pair are then $P_3=p_3+p_3'$ and $L_{12}=l_1+l_1'$. In particular there is an infinite number of ways to get $L_{12}=\pm 2$.

The total particle–hole momenta are $\Pi_a=\Pi_a^{\mathrm{p}}+\Pi_a^{\mathrm{h}}$ with $\Pi_a^{\mathrm{p}}=-\mathrm{i}\partial_a+A_a(x)$ and $\Pi_a^{\mathrm{h}}=-\mathrm{i}\partial_a'-A_a(x')$. Note that unlike the separate particle and hole momenta, the total momenta commute, $[\Pi_a,\Pi_b]=0$. Thus we can take for example a basis that are eigenstates of Π_1, Π_2, Π_3^{p}, and Π_3^{h} with respective eigenvalues P_1, P_2, p_3, and p_3'. One finds

$$\begin{aligned}\psi_{P_1,P_2,p_3,p_3'}(x,x') \propto \exp\bigl\{&-B\bigl(z^{\dagger}\cdot z+z'^{\dagger}\cdot z'-2z_1\overline{z_1'}-z_2^2-\overline{z_2'}^2\bigr)\\ &+\mathrm{i}(P_1-\mathrm{i}P_2)z_1+\mathrm{i}(P_1+\mathrm{i}P_2)\overline{z_1'}+2\mathrm{i}p_3z_2+2\mathrm{i}p_3'\overline{z_2'}\bigr\}.\end{aligned} \tag{31}$$

In the 1–2 plane these are Gaussian in the separation and plane waves in the center of mass. In the 3–4 plane they are plane waves in x_3 and x_3' and Gaussian in x_4 and x_4'.

The states (30) and (31) are both nonseparating: the particle and hole move in the 3-direction with fixed velocity, while in the 1–2 plane they are confined by the magnetic force as argued in [1]. The loophole in the argument given in the introduction is that the velocity here is $v_a = v\delta_{a3}$, *independent* of the momentum: bosonization is possible because the dynamics is one-dimensional.

To obtain a relativistic theory we should retain only states where the momentum is proportional to the velocity. The states with this property are the momentum eigenstates (31) such that $P_1 = P_2 = 0$. Note however from their explicit form that all these states have helicity identically zero: they are invariant under simultaneous rotation of z_1 and z_1'. This is an obstacle to a relativistic theory with spin.

References [1,2] identify *extreme dipole states* (*EDS*), which are the candidate graviton states. These have an analog in the U(1) model. To make contact with the notation of [2] we start with the spherically symmetric potential (23). The EDS are eigenstates of the SU(2) part of the unitary symmetry (22). Call this symmetry K_{1i} where $i = 1, 2, 3$, and the total for a particle–hole pair is $T_{1i} = K_{1i} + K'_{1i}$. Let the particle have total harmonic oscillator level n and the hole total level n'. The LLL states are sums of monomials of degree n in z_α and of degree n' in $\overline{z_\beta}'$, times an invariant Gaussian, so $k_1 = n/2$ and $k_1' = n'/2$. Then $t_1 \geqslant (n - n')/2$, and the EDS are defined to saturate this inequality, $t_1 = (n - n')/2$. One readily finds that these states are of the form

$$\psi_m^{\mathrm{EDS}}(x, x') \propto z_1^m z_2^{n-n'-m} (z'^\dagger \cdot z)^{n'} \exp\{-B(z^\dagger \cdot z + z'^\dagger \cdot z')\}. \tag{32}$$

To make contact with the basis (31) we must expand near the boundary,

$$(z_1, z_2) = (\tilde{z}_1, \tilde{z}_2) + (0, -\mathrm{i}r_0/2). \tag{33}$$

Also, because the vector potential is translation-invariant only up to a gauge transformation we must transform to

$$\tilde{\psi} = U\psi, \qquad U = \mathrm{e}^{\mathrm{i}Br_0(x_3' - x_3)/2}. \tag{34}$$

This is determined by $H\{\tilde{z}, \partial_{\tilde{z}}\} = UH\{z, \partial_z\}U^{-1}$. The tilded wavefunction in the tilded coordinates is to be compared (dropping the tildes) to the wavefunctions (31) obtained directly near the origin.

From the discussion in Section 2.1 it follows that as $r_0 \to \infty$, states of fixed energy relative to the Fermi level have

$$n = Br_0^2/2 + r_0 q, \qquad n' = Br_0^2/2 - r_0 q' \tag{35}$$

with q and q' fixed. Taking the limit of the states (32) with this scaling gives

$$\tilde{\psi}_m^{\mathrm{EDS}} \to z_1^m \psi_{0,0,q,q'}. \tag{36}$$

Thus for $m = 0$ the EDS is the zero-helicity plane wave state encountered above, while for positive m we obtain a non-normalizable state of helicity m. We conclude that the EDS of nonzero helicity are not good states in the $\mathbb{R}^3$ limit. We can also understand this as follows. One finds that $UT_{1i}\{z, \partial_z\}U^{-1} = -\mathrm{i}r_0\Pi_i/2$, so that the EDS condition linearizes to $(\Pi_1^2 + \Pi_2^2)\psi = 0$. The only normalizable solutions again have $P_1 = P_2 = 0$, but multiplying by a power of z_1 gives a nonnormalizable solution. Thus we can characterize the EDS with $m \neq 0$ as states of definite helicity and definite momentum-squared, but indefinite momentum. One can generalize the EDS to $t_1 = s + (n - n')/2$ with fixed s. This introduces an extra power of $\overline{z_1}'^s$ in the flat limit, allowing negative helicities but still nonnormalizable.

The energy of a particle–hole state is $E = v(n - n')/r_0 = v(q + q') = vP_3$. The EDS states thus have a relativistic dispersion relation $E^2 = v^2P^2$. Note that the non-EDS states are all tachyonic (in the sense of their momenta, not their velocities): $E^2 = v^2P_3^2 < v^2P^2$. This is a further obstacle to obtaining a relativistic theory.

3. The SU(2) QHE on $\mathbb{R}^4$

3.1. The model

By extending to an SU(2) magnetic field it is possible to obtain a larger spatial symmetry [1]. Consider the configuration

$$F_{23}^1 = F_{14}^1 = F_{31}^2 = F_{24}^2 = F_{12}^3 = F_{34}^3 = B. \tag{37}$$

In other words, $F_{ab}^i = B\eta_{ab}^i$ where

$$\eta_{ab}^i = \varepsilon_{iab4} + \delta_{ia}\delta_{4b} - \delta_{ib}\delta_{4a} \tag{38}$$

is the 't Hooft symbol. Note that a, b run $1, \ldots, 4$ and i, j run $1, 2, 3$.

Let us analyze the symmetries of this configuration. First use the separation of SO(4) into two commuting SO(3) algebras,

$$K_{1i}^{(0)} = -\frac{1}{4}\tilde{\eta}^i_{ab} L_{ab} = \frac{1}{2}(L_i + L_{4i}), \qquad K_{2i}^{(0)} = \frac{1}{4}\eta^i_{ab} L_{ab} = \frac{1}{2}(L_i - L_{4i}), \tag{39}$$

where

$$\tilde{\eta}^i_{ab} = -\varepsilon_{iab4} + \delta_{ia}\delta_{4b} - \delta_{ib}\delta_{4a} \tag{40}$$

is the parity-reflected 't Hooft symbol. We follow the notation of [1,2]. We can similarly separate the field strength

$$G^i_{1j} = -\frac{1}{4}\tilde{\eta}^j_{ab} F^i_{ab}, \qquad G^i_{2j} = \frac{1}{4}\eta^j_{ab} F^i_{ab}. \tag{41}$$

Then G^i_{1j} is invariant under $K_2^{(0)}$, while it transforms as a vector of $K_1^{(0)}$ on its j index. Similarly G^i_{2j} is invariant under $K_1^{(0)}$, while it transforms as a vector of $K_2^{(0)}$ on its j index. Also, each is a vector of isospin I on its i index. In this notation the configuration (37) is

$$G^i_{1j} = 0, \quad G^i_{2j} = B\delta^i_j/2. \tag{42}$$

It follows that this is invariant under $K_1^{(0)}$ and under simultaneous rotation by $K_2^{(0)}$ and by I. Thus we define [1]

$$K_{1i} = K_{1i}^{(0)}, \qquad K_{2i} = K_{2i}^{(0)} + I_i, \tag{43}$$

which are the symmetries of this configuration; here I_i is the $(2I+1)$-dimensional representation of $\mathrm{SU}(2)$. The generators (43) form an $\mathrm{SO}(3) \times \mathrm{SO}(3) = \mathrm{SO}(4)$ algebra, all generators of which act nontrivially on space. The generators K_{2i} have also an action on the SU(2) isospin indices.

The actual model that we will study is slightly different from the above but has the same symmetries. That is, we will take the vector potential

$$A^i_a = -\frac{B}{2}\eta^i_{ab} x_b. \tag{44}$$

In the corresponding field strength,

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \varepsilon_{ijk} A^j_a A^k_b, \tag{45}$$

the linear terms reproduce the earlier configuration (37), but the quadratic term is nontrivial and of order x^2. We take the potential to be simple, rather than the field strength, because it is this that appears in the Hamiltonian.

The configuration (44) is invariant under SO(4) rotations but it is clearly not translationally invariant because of the $\mathrm{O}(x^2)$ terms in the field strength. However, the confining potential that is to be added breaks these same translation symmetries.

Curiously, the configuration (37), in spite of its simple appearance, is not translationally invariant either. That is, there is no magnetic translation Π_a having the property

$$[\Pi_a, D_b] = 0 \tag{46}$$

for all $a,\ b$. Here the covariant derivative is

$$D_a = \partial_a - \mathrm{i}A^i_a I_i \equiv \partial_a - \mathrm{i}\mathbf{A}_a, \tag{47}$$

while

$$\Pi_a = -\mathrm{i}(\partial_a - \mathrm{i}\mathbf{V}_a) \tag{48}$$

is the combination of a translation in the a-direction with some infinitesimal gauge transformation $\mathbf{V}_a$. To show that there is no such symmetry, note first that the property (46), with the Jacobi identity, implies

$$\big[[\Pi_a, \Pi_b], [D_\rho, D_\sigma]\big] = 0 \quad \Rightarrow \quad [\mathbf{W}_{ab}, \mathbf{F}_{cd}] = 0. \tag{49}$$

Here

$$\mathbf{F}_{cd} = F^i_{cd} I_i = \partial_c \mathbf{A}_d - \partial_d \mathbf{A}_c - \mathrm{i}[\mathbf{A}_c, \mathbf{A}_d] \tag{50}$$

is the field strength in matrix notation, while $\mathbf{W}_{ab}$ is similarly constructed from $\mathbf{V}_a$. Since the $\mathbf{F}_{cd}$ span a complete set of SU(2) generators it follows that

$$\mathbf{W}_{ab} = 0 \quad \Rightarrow \quad \mathbf{V}_a = g\partial_a g^{-1} \tag{51}$$

for some $g(x)$ in $\mathrm{SU}(2)$. But then the definition (46) implies

$$\big[g\partial_a g^{-1}, D_b\big] = 0 \quad \Rightarrow \quad \big[\partial_a, g^{-1} D_b g\big] = \big[\partial_a, \partial_b - \mathrm{i}\mathbf{A}^g_b\big] = 0. \tag{52}$$

That is, there is a gauge in which the vector potential $\mathbf{A}_a^g$ is constant and so

$$\mathbf{F}_{cd}^g = -\mathrm{i}\left[\mathbf{A}_c^g, \mathbf{A}_d^g\right]. \tag{53}$$

Finally, let $c = 1$ and let d run over $2, 3, 4$. Then the left-hand side runs over a complete set of independent SU(2) generators, while the right cannot (its trace with $\mathbf{A}_c^g$ always vanishes).

Essentially, the naive translational invariance of the configuration (37) is broken by the action of parallel transport on the isospin index. It is interesting to compare this with the Zhang–Hu configuration [1] which has the larger symmetry SO(5). One can think of the gauge curvature in that configuration as conspiring with the curvature of the S^4 to allow the extra symmetries to exist. This is one reason why in that system the gauge field strength must go to zero as the radius of the S^4 goes to infinity, and so why the isospin must be taken to infinity to get a nontrivial limit. By keeping only SO(4) symmetry from the start it is possible to find a larger set of models on the flat $\mathbb{R}^4$.

However, there will ultimately be a penalty for the lack of translation invariance. In the usual QHE, the combination of translation invariance and localized states implies an infinitely degenerate LLL with a uniform density of states. This will not be the case here, and will necessitate taking the $I \to \infty$ limit.

3.2. *The spectrum*

The Hamiltonian for a spinless particle coupled to the vector potential (44) is

$$H = -\frac{1}{2m} D_a D_a + \frac{\kappa}{2} x_a x_a = H_1 + H_2, \tag{54}$$

where H_1 is the oscillator Hamiltonian

$$H_1 = \frac{1}{2m}\left(-\partial_a \partial_a + m^2\omega^2\right), \quad m^2\omega^2 = \frac{B^2}{4} I(I+1) + m\kappa, \tag{55}$$

and H_2 is the spin–isospin interaction

$$H_2 = -\frac{B}{m} K_2^{(0)} \cdot I = -\frac{B}{2m}\left(K_2 \cdot K_2 - I \cdot I - K_2^{(0)} \cdot K_2^{(0)}\right). \tag{56}$$

Note that we have introduced a harmonic potential from the start, since this entails no loss of symmetry. There is no change of variables that reverses the sign of B, and the physics will depend on the sign.

It is straightforward to diagonalize the Hamiltonian by addition of angular momenta. However, the reader who is interested in the $\mathbb{R}^3$ limit of the edge need not work through the detailed counting of states and enumeration of cases, but may jump to the next section, since in the limit the Hamiltonian becomes even simpler. The only result one needs from the remainder of this section is that in order to reach the $\mathbb{R}^3$ limit one must also take $I \to \infty$. Thus the $\mathbb{R}^3$ limit of our model coincides with the $\mathbb{R}^3$ limit of the Zhang–Hu model.

To diagonalize H consider first the oscillator part. With n excitations the oscillator energy is $E_1 = (n+2)\omega$. The raising operators

$$a_a^\dagger = -\partial_a + m\omega x_a \tag{57}$$

are vectors of SO(4), which can also be written as matrices

$$a_\alpha^{\dagger\beta} \equiv a_4^\dagger \delta_\alpha^\beta + i a_i^\dagger \left(\sigma^i\right)_\alpha^\beta. \tag{58}$$

These transform as spin-$\frac{1}{2}$ under both $K_1^{(0)}$ and $K_2^{(0)}$; the $K_1^{(0)}$ index is written as a subscript and the $K_2^{(0)}$ index as a superscript. At level n, the product of n $a_a^\dagger$'s gives an n-fold symmetric tensor; by subtracting traces this decomposes into irreducible representations

$$(n) \oplus (n-2) \oplus (n-4) \oplus \cdots \oplus \left\{(1) \text{ or } (0)\right\}, \tag{59}$$

where (r) denotes the rank r traceless symmetric tensor. In terms of the SO(3) $\times$ SO(3) quantum numbers $(k_1^{(0)}, k_2^{(0)})$, the representation (r) is $(\frac{1}{2}r, \frac{1}{2}r)$ and so at level n the states are

$$\left(\tfrac{1}{2}n, \tfrac{1}{2}n\right) \oplus \left(\tfrac{1}{2}n-1, \tfrac{1}{2}n-1\right) \oplus \left(\tfrac{1}{2}n-2, \tfrac{1}{2}n-2\right) \oplus \cdots \oplus \left\{\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \text{ or } (0,0)\right\}. \tag{60}$$

For each value $(\frac{1}{2}r, \frac{1}{2}r)$ the quantum numbers $k_{1,3}^{(0)}$ and $k_{2,3}^{(0)}$ run independently from $-\frac{1}{2}r$ to $+\frac{1}{2}r$. The total dimension is

$$(n+1)^2 + (n-1)^2 + (n-3)^2 + \cdots + \{4 \text{ or } 1\} = \frac{1}{6}(n+3)(n+2)(n+1). \tag{61}$$

The equality of $k_1^{(0)}$ and $k_2^{(0)}$ follows from the operator identity $K_1^{(0)} \cdot K_1^{(0)} = K_2^{(0)} \cdot K_2^{(0)}$. It is also evident from the explicit form of the states,

$$\left(\tfrac{1}{2}m, \tfrac{1}{2}m\right) = \left\{a^{\dagger}_{\alpha_{(1}}{}^{\beta_1} a^{\dagger}_{\alpha_2}{}^{\beta_2} \cdots a^{\dagger}_{\alpha_{m)}}{}^{\beta_m}\right\}\left\{a^{\dagger}_{\alpha_{[m+1}}{}^{\beta_{m+1}} a^{\dagger}_{\alpha_{m+2]}}{}^{\beta_{m+2}}\right\} \cdots \left\{a^{\dagger}_{\alpha_{[n-1}}{}^{\beta_{n-1}} a^{\dagger}_{\alpha_{n]}}{}^{\beta_n}\right\}|0\rangle, \tag{62}$$

where we symmetrize the first m α indices and antisymmetrize the rest in pairs: the β indices automatically have the same symmetry.

To diagonalize H_2, add $K_2^{(0)}$ and I to go to a basis of definite k_2. Then

$$E = (n+2)\omega - \frac{B}{2m}\left[k_2(k_2+1) - I(I+1) - k_1(k_1+1)\right]. \tag{63}$$

We have used $k_1 = k_1^{(0)} = k_2^{(0)}$. States are labeled by the quantum numbers

$$(n, k_1, k_{1,3}, k_2, k_{2,3}) \tag{64}$$

with the ranges

$$\begin{aligned} n &\in \{0, 1, 2, \ldots\}, \\ k_1 &\in \left\{\tfrac{1}{2}n, \tfrac{1}{2}n - 1, \ldots, \tfrac{1}{2} \text{ or } 0\right\}, \qquad k_{1,3} \in \{k_1, k_1 - 1, \ldots, -k_1\}, \\ k_2 &\in \left\{I + k_1, I + k_1 - 1, \ldots, |I - k_1|\right\}, \qquad k_{2,3} \in \{k_2, k_2 - 1, \ldots, -k_2\}. \end{aligned} \tag{65}$$

3.3. The lowest Landau level

Unlike the U(1) theory, the physics depends on the sign of B. Thus the analysis separates into two cases.

3.3.1. $B > 0$

For given k_1, the energy is minimized by taking k_2 to have its maximum value $k_1 + I$, so that

$$E = (n+2)\omega - Bk_1 I/m \quad (k_2 = k_1 + I). \tag{66}$$

For given n, this is minimized in turn by taking k_1 to have its maximum value $\frac{1}{2}n$, and so

$$E = 2\omega + n(\omega - BI/2m) \quad \left(k_1 = \tfrac{1}{2}n,\ k_2 = \tfrac{1}{2}n + I\right). \tag{67}$$

In order that this be independent of n, we must take $\omega = BI/2m$ and so the harmonic potential is $\kappa = -B^2 I/4m$. In contrast to the U(1) case, we need a harmonic potential to obtain a large degeneracy; this is due to the lack of translation invariance of the vector potential.

The LLL states, all with $E = 2\omega = BI/m$, are then

$$\text{I:} \quad \left(n, \tfrac{1}{2}n, k_{1,3}, \tfrac{1}{2}n + I, k_{2,3}\right), \quad n \in \{0, 1, 2, \ldots\}, \tag{68}$$

with degeneracy $(n+1)(n+2I+1)$ for given n.

3.3.2. $B < 0$

Now for given k_1, the energy is minimized by taking k_2 to have its minimum value $|k_1 - I|$, giving

$$E = \begin{cases} (n+2)\omega - |B|k_1(I+1)/m & (k_2 = I - k_1 \geqslant 0), \\ (n+2)\omega - |B|(k_1+1)I/m & (k_2 = k_1 - I \geqslant 0). \end{cases} \tag{69}$$

For given n and either sign of $I - k_1$, this is again minimized by taking k_1 to have its maximum value $\frac{1}{2}n$, and so

$$E = \begin{cases} 2\omega + n\left(\omega - |B|[I+1]/2m\right) & \left(k_1 = \tfrac{1}{2}n,\ k_2 = I - \tfrac{1}{2}n \geqslant 0\right), \\ 2\omega - |B|I/m + n\left(\omega - |B|I/2m\right) & \left(k_1 = \tfrac{1}{2}n,\ k_2 = \tfrac{1}{2}n - I \geqslant 0\right). \end{cases} \tag{70}$$

There are now two values of κ that give a large ground state degeneracy. For $\kappa = B^2(I+1)/4m$ so that $\omega = |B|[I+1]/2m$, the states with $n \leqslant 2I$ are degenerate and lie below those with $n > 2I$. For $\kappa = -B^2 I/4m$ so that $\omega = |B|I/2m$, the states with $n \geqslant 2I$ are degenerate and lie below those with $n < 2I$.

To summarize, for $\kappa = B^2(I+1)/4m$ the LLL states have $E = |B|[I+1]/m$ and quantum numbers

$$\text{II:} \quad \left(n, \tfrac{1}{2}n, k_{1,3}, I - \tfrac{1}{2}n, k_{2,3}\right), \quad n \in \{0, 1, 2, \ldots, 2I\}, \tag{71}$$

with degeneracy $(n+1)(2I-n+1)$ for given n. For $\kappa = -B^2 I/4m$ the LLL states have $E = |B|/m$ and quantum numbers

$$\text{III:} \quad \left(n, \tfrac{1}{2}n, k_{1,3}, \tfrac{1}{2}n - I, k_{2,3}\right), \quad n \in \{2I, 2I+1, \ldots\}, \tag{72}$$

with degeneracy $(n+1)(n-2I+1)$.

3.3.3. Discussion

The next step is to find the boundary theory, increasing the harmonic potential slightly so as to confine a finite bubble of fermions, and then taking the size of the bubble to infinity while focusing on a point on the boundary. We have three LLL systems to work with, labeled I, II, and III above.

However, none of these allows a straightforward limiting process. Consider the mean value of $x_a x_a = r^2$ in the LLL states. Since the LLL states have distinct SO(4) quantum numbers, r^2 is diagonal in the basis (64) and a short calculation gives

$$r^2 = \frac{n+2}{m\omega} \quad \text{(LLL)}. \tag{73}$$

The volume of the shell between n and $n+1$ is then

$$V = 2\pi^2 r^3 \frac{\delta r}{\delta n} \approx \frac{2\pi^2 r^2}{|B| I}. \tag{74}$$

We take $n, I \gg 1$ so that the levels are closely spaced. The number of states in the shell, divided by the volume V, is

$$\begin{aligned}
&\text{I:} && \rho = \frac{|B|^2 I^3}{2\pi^2}\left(1 + \frac{|B| r^2}{4}\right),\\
&\text{II:} && \rho = \frac{|B|^2 I^3}{2\pi^2}\left(1 - \frac{|B| r^2}{4}\right),\\
&\text{III:} && \rho = \frac{|B|^2 I^3}{2\pi^2}\left(-1 + \frac{|B| r^2}{4}\right).
\end{aligned} \tag{75}$$

The range of r is implicitly limited by the positivity of ρ. In all cases ρ is a nontrivial function of r. This is in contrast to the familiar Abelian case where the density is constant. The r-dependence would not be present if the LLL were translation invariant, but we have emphasized that this invariance is absent. If we try to make a boundary system on $\mathbb{R}^3$ by taking $r \to \infty$ in case I or III, the limit is singular because the local density of states diverges as r^2. In case II we do not even have this option: the LLL has a finite radius even in the absence of a confining potential.

Note that the density of states is constant in cases I and II in the limited range $r^2 \ll |B|^{-1}$. However, in order to take $r \to \infty$ we must take $B \to 0$, and then must also take $I \to \infty$ to get a nontrivial result. Equivalently, $r^2 \ll |B|^{-1}$ is $n \ll I$, so $n \to \infty$ implies $I \to \infty$. Thus, while we are able to formulate the SU(2) QHE on $\mathbb{R}^4$ for finite I, when we attempt to reach the boundary theory on $\mathbb{R}^3$ we are forced to take the same limit as in [1,2].

In fact, our case II is very similar to the Zhang–Hu model on S^4. In both cases the LLL has a finite number of states, and the SO(4) representations are the same,

$$(k_1, k_2) = \left(\tfrac{1}{2}n, I - \tfrac{1}{2}n\right), \quad n \in \{0, 1, \ldots, 2I\}. \tag{76}$$

The total degeneracy

$$\sum_{n=0}^{2I} (n+1)(2I-n+1) = \frac{1}{6}(2I+1)(2I+2)(2I+3) \tag{77}$$

is then the same. In the Zhang–Hu model the LLL is uniformly distributed on S^4. Roughly speaking, one can think of our case II as cutting this open at the north pole and spreading it out to form a ball on $\mathbb{R}^4$. Near the origin of $\mathbb{R}^4$, corresponding to the south pole of S^4, the Zhang–Hu system and ours match; this is the region of interest for reaching the limit of flat $\mathbb{R}^3$.

4. The $I \to \infty$ limit

4.1. The bulk

We have concluded that we must keep $I \gg n$ as $n \to \infty$. It is logical therefore to first take $I \to \infty$ at fixed n, and then $n \to \infty$. We have been unable to avoid the problem of an infinite-dimensional SU(2) representation, but at least we can make a virtue of necessity and take advantage of the simplifications that occur when $I \to \infty$. Also, this is more closely parallel to the usual QHE, where the Hamiltonian is held fixed (aside from scaling the confining potential) as the size of the bubble is taken to infinity. Note that there is another limiting process as well, taking $m \to 0$ to restrict to the LLL. This limit commutes with $I \to \infty$; for example, in either order the ratio ρ/I, where ρ is the density of LLL states, approaches the r-independent value $b^2/2\pi^2$. It is simplest to take the limits in the order $I \to \infty$, then $m \to 0$, and finally $n \to \infty$.

In order to obtain a nontrivial $I \to \infty$ limit of the Hamiltonian (54), we must hold fixed $b = BI$; in this same limit $\kappa \to 0$ and the Hamiltonian becomes

$$H = \frac{1}{2m}\left(-\partial_a\partial_a + \frac{b^2}{4}x_a x_a - 2b\,\vec{e}\cdot\vec{K}_2^{(0)}\right). \tag{78}$$

Here we have defined

$$e_i = \frac{I_i}{\sqrt{I(I+1)}}. \tag{79}$$

Since

$$[e_i, e_j] = \frac{\mathrm{i}\varepsilon_{ijk}e_k}{\sqrt{I(I+1)}}, \qquad \vec{e}\cdot\vec{e} = 1, \tag{80}$$

$\vec{e}$ becomes a *classical* unit vector as $I \to \infty$.

The Hamiltonian (78) is the same as the Abelian Hamiltonian (18), with the replacements

$$B \to b, \qquad L_{12} + L_{34} \to 2\vec{e}\cdot\vec{K}_2^{(0)} = e_i(L_i - L_{4i}). \tag{81}$$

In particular, for $\vec{e} = (0,0,1)$, $2\vec{e}\cdot\vec{K}_2^{(0)} = L_{12} + L_{34}$ and the Hamiltonians are identical. Thus we have a simple interpretation of this system in the $I \to \infty$ limit: it is an infinite number of copies of the $\mathrm{U}(1)$ quantum Hall system on $\mathbb{R}^4$, with the spatial orientation of the magnetic field indexed by the unit vector $\vec{e}$. Note that in the limit translation invariance on $\mathbb{R}^4$ is restored.

The LLL then consists of states with the appropriate analyticity

$$\psi(\vec{e}, x) = f(\vec{e}, z^1, z^2)\,\mathrm{e}^{-bx_a x_a/4}, \tag{82}$$

where now the coordinates z have an implicit dependence on $\vec{e}$,

$$z_1 = (u_i + \mathrm{i}v_i)x_i, \quad z_2 = e_i x_i + \mathrm{i}x_4. \tag{83}$$

Here $(\vec{e}, \vec{u}, \vec{v})$ form an orthonormal frame in three dimensions. One can see this by rotating to a frame where $\vec{e} = (0,0,1)$, where it reduces to the earlier $\mathrm{U}(1)$ analysis.[2] One can then verify that

$$H = \frac{b}{m} - \frac{1}{2m}D_\alpha D_{\bar\alpha}, \quad D_\alpha = \partial_\alpha - b\overline{z_\alpha},\ D_{\bar\alpha} = \partial_{\bar\alpha} + bz_\alpha. \tag{84}$$

4.2. The boundary

As in the $\mathrm{U}(1)$ case, the $r_0 \to \infty$ limit is equivalent to linearizing around the origin, introducing a potential $V = -vbx_4$. Between LLL states this becomes

$$H' = ve_i P_i. \tag{85}$$

Again, this is an infinite collection of $\mathrm{U}(1)$ systems, with all possible spatial orientations: the velocity of the boundary excitations is $v\vec{e}$. In second-quantized form one can write for example

$$\mathbf{H}' = -\mathrm{i}v\int \mathrm{d}^2e\,\mathrm{d}^3x\,\Psi^\dagger(\vec{e},\vec{x})\,\vec{e}\cdot\vec{\partial}\,\Psi(\vec{e},\vec{x}), \tag{86}$$

but where the space is noncommutative in the directions orthogonal to $\vec{e}$, $[x_i, x_j] = -\mathrm{i}\varepsilon_{ijk}e_k/b$.

As has been noted in various places, one can think of the $I \to \infty$ limit as a six-dimensional system with a five-dimensional boundary, elevating $\vec{e}$ to a coordinate. The space is then $\mathbb{R}^4 \times S^2$, and its boundary is $\mathbb{R}^3 \times S^2$. However, the boundary dynamics is still one-dimensional. The velocity is independent of the momentum – it depends only on the position on S^2, and is tangent to $\mathbb{R}^3$.

[2] Since the space of complex structures on $\mathbb{R}^4$ is part of the twistor construction, one could say that we are now considering a Fermi liquid on twistor space.

For particle–hole states to have a finite value of $T_{2i} = K_{2i} + K'_{2i}$ as $I \to \infty$, it is necessary to take $\vec{e}^{\,\mathrm{p}} = -\vec{e}^{\,\mathrm{h}} \equiv \vec{e}$. A basis of such states, analogous to the plane wave basis (31), would then be

$$|\vec{e}, \vec{P}_\perp, \vec{e}\cdot\vec{p}, \vec{e}\cdot\vec{p}\,'\rangle, \tag{87}$$

where $\perp$ denotes the two dimensions orthogonal to $\vec{e}$; one should note that $\vec{e}\cdot\vec{p}$ and $\vec{e}\cdot\vec{p}\,'$ are always positive. The T_1 eigenstates are obtained as in the $\mathrm{U}(1)$ case, while the T_2 eigenstates correspond to appropriate superpositions of different values of $\vec{e}$, since T_2 rotates $\vec{e}$.

From the point of view of obtaining a relativistic theory with spin, the same problems as discussed in Section 2.2.3 for the $\mathrm{U}(1)$ case arise here. To obtain a relativistic theory we need in some way to truncate the one-particle spectrum to states in which $\vec{P}$ is parallel to $\vec{e}$.[3] However, the only such states have zero helicity. The extreme dipole states of nonzero helicity are nonnormalizable. The states with $\vec{P}$ not parallel to $\vec{e}$ are all tachyonic, not in their velocities but in the sense that $P^2 > E^2/v^2$. Since the energy of a state is $E = ve\cdot P$, the states with $\vec{e} \parallel \vec{P}$ are actually the *highest* energy states with given $\vec{P}$.

Because of the effective one-dimensionality of the edge theory it is likely that one can solve various four-fermion interactions by means of bosonization, though the $I \to \infty$ limit is somewhat subtle because $\delta(0)$ appears in various expressions, from the $\vec{e}$ dependence. For now we just note that the most obvious effect of interactions is to allow the relativistic states with $E = vP$ to decay to tachyonic states with $E < vP$, which would be a problem for obtaining a relativistic theory.

5. Discussion

We first summarize our conclusions. On $\mathbb{R}^4$ we have formulated the $\mathrm{U}(1)$ and $\mathrm{SU}(2)$ quantum Hall systems, with arbitrary $\mathrm{SU}(2)$ isospin I. In the former case the boundary theory is effectively one-dimensional. In the latter case it is necessary to take $I \to \infty$ in order to obtain a boundary theory, and the result is essentially an infinite collection of one-dimensional theories.

As claimed in [1,2], even in the free theory there are localized gapless particle–hole excitations with arbitrary helicity. Taking the flat limit as we have done clarifies the nature of these states. We have noted some specific difficulties with obtaining a relativistic theory – the absence of nonzero-helicity states with $\vec{v} \parallel \vec{P}$, and the existence of tachyonic states. However, independent of the relativistic application, the QHE on $\mathbb{R}^4$ is a rich and interesting system. We believe that for analyzing any local issues the limiting form that we have obtained in Section 4 is the appropriate starting point. In particular it will be possible to solve certain four-fermion interactions.

We now discuss some general aspects of the emergence of gravity from nongravitational field theories, aside from the specific details noted above. Let us suppose that it is possible to add interactions to the Zhang–Hu model in such a way that the low energy fixed point becomes Poincaré invariant; likely this would require a certain degree of fine tuning. Then as noted in [1], Weinberg's theorem [5] would require that the low energy interactions of massless helicity-two states take the form of general relativity, if these states are present and if their interactions are nontrivial at zero momentum transfer. The Fierz–Pauli theorem [6] (regarding the impossibility of coupling massless higher-spin states to conserved currents) would then require that the states of helicity greater than two decouple.

However, under the same conditions the Weinberg–Witten theorem [7] would require that the helicity-two states actually be absent from the low energy spectrum. The conditions for the Weinberg–Witten theorem are quite general – Poincaré invariance and the existence of a conserved energy-momentum tensor – so it is difficult to see how the theorems of [5] could operate without the Weinberg–Witten theorem as well. (Note that the energy-momentum tensor in four spatial dimensions reduces to an energy-momentum tensor in the three-dimensional boundary theory by integrating over x_4.) Thus it appears that an interacting theory of gravity cannot arise in this way.

One can perhaps understand this heuristically as follows. An important feature of gravity is that there are no local observables: to say where a measurement is made one must specify a process of parallel transport. This is an essential feature of general relativity. The Zhang–Hu model, like any ordinary nongravitational quantum field theory, does have local observables. This would be evaded if all local operators decoupled from the low energy physics,[4] but this is not possible for the energy momentum tensor which must have a nonzero expectation value in any state of nonzero energy. From this point of view it might make more sense to look for a theory of quantum gravity in the zero energy states of the LLL without confining potential, rather than the edge states with the potential. Note however the complete change of interpretation: time is no longer associated with Hamiltonian evolution, rather it must emerge 'holographically' from correlations in the states.[5]

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[3] The states that must be removed were termed 'incoherent fermionic excitations' in [1].

[4] This possibility was also noted by C. Johnson.

[5] A more sophisticated obstacle to emergent gravity, pointed out by S. Shenker, is the holographic principle. There is strong reason to believe that in quantum gravity the maximum entropy in a given volume is proportional to the surface area. If there is an underlying nongravitational QFT one expects the entropy to be proportional to the volume.

In perturbative string theory one invokes Weinberg's theorem to predict that the low energy amplitudes will be those of general relativity, and this is borne out by explicit calculation [9]. This does not conflict with the Weinberg–Witten theorem because string theory has no local observables – Weinberg's theorem uses only properties of the S-matrix,[6] whereas the Weinberg–Witten theorem assumes existence of an energy-momentum tensor.

There is in fact a well-known example of emergent gravity: the AdS/CFT duality [10]. On the CFT side there is a supersymmetric gauge theory without gravity, but at large N and large 't Hooft coupling the effective description is in terms of quantum gravity, string theory actually. The important point is that not only does gravity emerge, but spacetime as well. Only the boundary of the gravitational theory is locally realized in the gauge theory, so there are no local bulk observables. The local observables of the gauge theory become boundary data in the gravitational theory [11]. Note that the bulk diffeomorphism invariance is invisible in the gauge theory; the $\mathrm{SU}(N)$ gauge invariance is a different gauge symmetry, which acts as a local internal symmetry, not a local spacetime symmetry, on the boundary.

This emergence of diffeomorphism invariance from 'nothing' is analogous to what happens in the various examples of the emergence of gauge symmetries: in coset field theories [12], in lattice models [13], and in the magnetic duals to supersymmetric gauge theories [14]. The essential point is that gauge symmetry and diffeomorphism invariance are just redundancies of description. In the examples where they emerge, one begins with nonredundant variables and discovers that redundant variables are needed to give a local description of the long-distance physics. In general relativity, the spacetime coordinates are themselves part of the redundant description. Thus it appears that, as in the AdS/CFT example, the emergence of general relativity requires the emergence of spacetime itself.

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