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Exotic nuclei/Les noyaux exotiques

Pairing and quartetting in exotic nuclei

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Abstract

A review is given of pair correlations in nuclei with an emphasis on the symmetry character of the superfluid solution which depends on (i) the isospin of the nucleus and (ii) the relative strength of the $T = 0$ and $T = 1$ pairing forces. The most general SO(8) model which accommodates neutrons and protons as well as $T = 0$ and $T = 1$ pairing, is solvable in three limits: only $T = 0$ pairing, only $T = 1$ pairing and equal strengths in the two channels. In these limits, the superfluid ground-state solution of $N = Z$ nuclei exhibits a quartet structure. The competition between superfluidity and magicity is discussed with reference to integrable models. ***To cite this article: P. Van Isacker, C. R. Physique 4 (2003).***

Résumé

Appariement dans les noyaux exotiques. Cette contribution passe en revue les différentes solutions superfluides qui existent dans le noyau. Une attention particulière est consacrée à leur charactère de symétrie qui dépend (i) de l'isospin du noyau et (ii) du rapport des forces d'appariement $T = 0$ et $T = 1$. Un modèle général avec neutrons et protons et avec appariement $T = 0$ et $T = 1$ est analytiquement soluble dans trois limites : sans appariement $T = 0$, sans appariement $T = 1$ et avec forces d'appariement égales. Les solutions analytiques démontrent l'existence d'une solution superfluide avec une structure en quartettes pour les noyaux $N = Z$. La compétition entre fluidité et magicité est examinée dans le contexte d'un modèle intégrable. ***Pour citer cet article : P. Van Isacker, C. R. Physique 4 (2003).***

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Mots-clés : Physique nucléaire ; Corrélations ; Appariement ; Aggrégats

1. Pairing correlations in nuclei

The discussion of pairing correlations in nuclei has been inspired traditionally by the treatment of superfluidity in condensed matter [1,2]. The superfluid phase in these systems is characterised by the presence of a large number of identical bosons in a single quantum state, which is called the condensate. In superconductors the bosons are pairs of electrons with opposite momenta that form at the Fermi surface; in superconducting nuclei they are pairs of valence nucleons with opposite angular momenta. The approximations made in BCS theory are less appropriate in nuclei since the number of nucleons is comparatively small. On the other hand, pairing phenomena are potentially more interesting in nuclei since two types of nucleons exist, neutrons and protons. This gives rise to a richer symmetry structure and a more complex condensate. The different symmetries of the superfluid solution are reviewed in Section 2 under the assumption that the valence nucleons (either of one type or of two different types) occupy a set of degenerate single-particle levels. Section 3 takes a closer look at the structure of the condensate in

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the presence of neutrons and protons, and shows that it has a quartet structure. The effect of lifting the single-particle degeneracy is discussed in Section 4. Some closing remarks are made in Section 5.

2. Symmetry structure of superfluid solutions

2.1. SU(2) *superfluidity*

The most basic situation with regard to pairing in nuclei is encountered when n identical nucleons occupy a set of degenerate single-particle states. For convenience in the subsequent discussion, ls coupling is used (with $s = \frac{1}{2}$) to label a single-particle state. (Similar results are obtained in jj coupling.) A nucleon creation operator is then denoted as $a^{\dagger}_{lm_l sm_s}$. The nucleons are assumed to interact through a pairing force of the form

$$V_{\mathrm{SU}(2)} = -g_0 S^0_+ S^0_-, \tag{1}$$

with

$$S^0_+ = \sqrt{\frac{1}{2}} \sum_l \sqrt{2l+1} \left(a^{\dagger}_{ls} \times a^{\dagger}_{ls}\right)^{(00)}_{00}, \qquad S^0_- = \left(S^0_+\right)^{\dagger}. \tag{2}$$

The notation S refers to the fact that these are nucleon pairs coupled to $L = 0$ while the superscript 0 in $S^0_{\pm}$ refers to spin $S = 0$. Since the nucleons are identical, only one pair state is allowed by the Pauli principle, that is, the spin-antiparallel state for either neutrons or protons (see Fig. 1). Therefore, this pairing mode is called *spin singlet*. The Hamiltonian (1) has an SU(2) dynamical symmetry and by virtue of this symmetry can be solved analytically [3]. The ground state has a superfluid structure of the form (for even and odd n, respectively)

$$\left(S^0_+\right)^{n/2}|\mathrm{o}\rangle, \qquad a^{\dagger}_{lm_l sm_s}\left(S^0_+\right)^{(n-1)/2}|\mathrm{o}\rangle, \tag{3}$$

where $|\mathrm{o}\rangle$ represents the vacuum (i.e., the doubly-magic core nucleus). The conserved quantum number that emerges from these considerations is *seniority* [4], the number of nucleons not in pairs coupled to $L = 0$. The superfluid solution of the pairing Hamiltonian (1) leads naturally to three characteristic features of semi-magic nuclei: the constant excitation energy (independent of n) of the first-excited 2^+ state in even–even isotopes, the linear variation of pair removal or two-nucleon separation energies as a function of n and the odd–even staggering in the nuclear binding energy.

2.2. SO(5) *superfluidity*

This type of superfluidity arises for a system of neutrons and protons. It is assumed that the pairing interaction is isospin invariant which implies that it is the same in the three possible $T = 1$ channels, neutron–neutron, neutron–proton and proton–proton, and that (1) can be rewritten as

$$V_{\mathrm{SO}(5)} = -g_0 S^{01}_+ \cdot S^{01}_-, \tag{4}$$

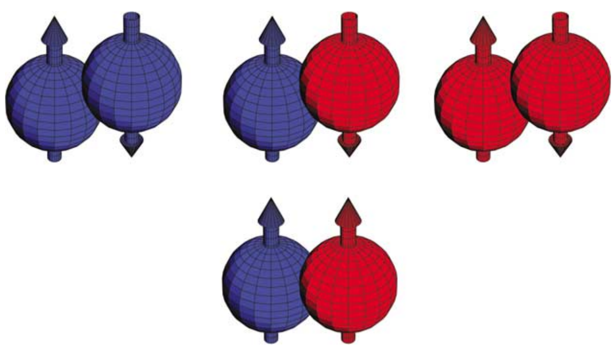

Fig. 1. The different types of pairs made up by neutrons (blue) and protons (red), with antiparallel (top) and parallel (bottom) spins.

where the dot indicates a scalar product in isospin. In terms of the nucleon operators $a^\dagger_{lm_lsm_stm_t}$, which now carry also isospin indices (with $t=\frac{1}{2}$), the pair operators are

$$S^{01}_{+,\mu}=\sqrt{\frac{1}{2}}\sum_l\sqrt{2l+1}\big(a^\dagger_{lst}\times a^\dagger_{lst}\big)^{(001)}_{00\mu},\qquad S^{01}_{-,\mu}=\big(S^{01}_{+,\mu}\big)^\dagger,\tag{5}$$

where the superscripts 0 and 1 in $S^{01}_{\pm,\mu}$ refer to spin $S=0$ and isospin $T=1$. The index μ (isospin projection) distinguishes neutron–neutron ($\mu=+1$), neutron–proton ($\mu=0$) and proton–proton ($\mu=-1$) pairs. There are thus three different pairs with $S=0$ and $T=1$ (top line in Fig. 1) but they are trivially related through isospin symmetry. The dynamical symmetry of the Hamiltonian (4) is SO(5) which makes the problem analytically solvable although in a much more laborious way [5,6] than in SU(2). The quantum number, besides seniority, that emerges from this analysis is *reduced isospin* [7], which is the isospin of the nucleons not in pairs coupled to $L=0$. As a consequence of the neutron–proton quadrupole interaction which breaks seniority and is responsible for deformation, SO(5) superfluidity has not found widespread application in nuclei.

2.3. SO(8) *superfluidity*

For a neutron and a proton there exists a different paired state with *parallel* spins (bottom line of Fig. 1). The most general pairing interaction for a system of neutrons and protons thus involves a *spin-singlet* and a *spin-triplet* term,

$$V_{\mathrm{SO}(8)}=-g_0S^{01}_+\cdot S^{01}_- - g_1S^{10}_+\cdot S^{10}_-,\tag{6}$$

where the $S=1$, $T=0$ pair operators are defined as

$$S^{10}_{+,\mu}=\sqrt{\frac{1}{2}}\sum_l\sqrt{2l+1}\big(a^\dagger_{lst}\times a^\dagger_{lst}\big)^{(010)}_{0\mu0},\qquad S^{10}_{-,\mu}=\big(S^{10}_{+,\mu}\big)^\dagger.\tag{7}$$

The index μ is the spin projection in this case and distinguishes different spatial orientations of the $S=1$ pair. The pairing Hamiltonian (6) now involves two parameters g_0 and g_1, the strengths of the spin-singlet and spin-triplet interaction, respectively. While in the previous cases the single strength parameter g_0 just defines an overall scale, this is no longer so for SO(8). Specifically, superfluid solutions with an intrinsically different structure are obtained for different ratios g_0/g_1.

In general, the eigenproblem associated with the interaction (7) can only be solved numerically which, given the typical size of a nuclear shell-model space, can be a formidable task. However, for specific choices of g_0 and g_1 the solution of $V_{\mathrm{SO}(8)}$ can be obtained analytically [8]. The analysis reveals the existence of SO(8), which is the 'enveloping' algebra formed by the pair operators (5) and (7), their commutators, the commutators of these among themselves, and so on until a closed algebraic structure is obtained. Closure is attained with the number operator $\hat n$, the spin and isospin operators S_μ and T_μ and the Gamow–Teller-like operators $Y_{\mu\nu}$ which are vectors in spin and isospin, and these together form SO(8) [8,9]. (The explicit definition of these operators can be found, e.g., in [10].)

The symmetry character of the interaction (7) is obtained by studying the subalgebras of SO(8). Of relevance are the subalgebras $\mathrm{SO}_S(5)\equiv\{S^{10}_{\pm,\mu},\hat n,S_\mu\}$, $\mathrm{SO}_S(3)\equiv\{S_\mu\}$, $\mathrm{SO}_T(5)\equiv\{S^{01}_{\pm,\mu},\hat n,T_\mu\}$, $\mathrm{SO}_T(3)\equiv\{T_\mu\}$ and $\mathrm{SO}(6)\equiv\{S_\mu,T_\mu,Y_{\mu\nu}\}$, which can be placed in the following lattice of algebras:

$$\mathrm{SO}(8)\supset\left\{\begin{array}{l}\mathrm{SO}_S(5)\otimes\mathrm{SO}_T(3)\\ \mathrm{SO}(6)\\ \mathrm{SO}_T(5)\otimes\mathrm{SO}_S(3)\end{array}\right\}\supset\mathrm{SO}_S(3)\otimes\mathrm{SO}_T(3).\tag{8}$$

By use of the explicit form of the generators of SO(8) and its subalgebras, and their commutation relations [9], the following relations can be shown to hold:

$$\begin{aligned}S^{10}_+\cdot S^{10}_- &= \frac12C_2\big[\mathrm{SO}_S(5)\big]-\frac12C_2\big[\mathrm{SO}_S(3)\big]-\frac18(2\Omega-n)(2\Omega-n+6),\\ S^{10}_+\cdot S^{10}_- + S^{01}_+\cdot S^{01}_- &= \frac12C_2\big[\mathrm{SO}(8)\big]-\frac12C_2\big[\mathrm{SO}(6)\big]-\frac18(2\Omega-n)(2\Omega-n+12),\\ S^{01}_+\cdot S^{01}_- &= \frac12C_2\big[\mathrm{SO}_T(5)\big]-\frac12C_2\big[\mathrm{SO}_T(3)\big]-\frac18(2\Omega-n)(2\Omega-n+6),\end{aligned}\tag{9}$$

where n is the nucleon number and $\Omega\equiv\sum_l(2l+1)$ is the orbital shell size (i.e., $\Omega=1,3,6,\ldots$ for the $s,p,sd,\ldots$ shell). This shows that the interaction (7) in the three cases (i) $g_0=0$, (ii) $g_1=0$ and (iii) $g_0=g_1$, can be written as a combination of Casimir operators of algebras belonging to a *chain of nested algebras* of the lattice (8). (They can be considered as dynamical

Table 1
Favoured irreducible representations of different pairing Hamiltonians for even–even (EE) odd–mass (OE) and odd–odd (OO) nuclei.

Nucleus	v	$T=0$ pairing		$T=0$ & $T=1$ pairing		$T=1$ pairing	
		$\mathrm{SO}_S(5)$	$\mathrm{SO}_S(3)$	SO(8)	SO(6)	$\mathrm{SO}_T(5)$	$\mathrm{SO}_T(3)$
EE	0	$(\Omega-T,\Omega_T)$	$s(\Omega_T)$	$(\Omega 000)$	$(T00)$	$(\Omega 0)$	T
OE	$\frac{1}{2}$	$(\Omega-T,\Omega_T)$	$s(\frac{1}{2})$	$(\Omega-\frac{1}{2},\frac{1}{2}\frac{1}{2}\frac{1}{2})$	$(T\frac{1}{2}\frac{1}{2})$	$(\Omega-\frac{1}{2},\frac{1}{2})$	T
OO $N=Z$	0	$(\Omega 0)$	$S=1$	$(\Omega 000)$	(100)	$(\Omega 0)$	$T=1$
OO $N\neq Z$	0	$(\Omega-T,\Omega_T)$	$s(\Omega_T-1)$	$(\Omega 000)$	$(T+1,00)$	$(\Omega 0)$	$T+1$
	2	$(\Omega-T,\Omega_T)$	$s(\Omega_T-1)$	$(\Omega-1,111)$	$(T10)$	$(\Omega-1,1)$	T

With $\Omega_T=\min(T,\Omega-T)$ and $s(x)=\max(\Omega_T+T-n/2, x \bmod 2)$.

symmetries of the SO(8) model in the sense explained, e.g., in [11,12].) As a consequence, in the three cases $g_0=0$, $g_1=0$ and $g_0=g_1$, eigenvalues are known analytically[1] as a sum of those of the different Casimir operators which are given by

$$\begin{aligned}
\langle C_2[\mathrm{SO}(8)]\rangle &= \omega_1(\omega_1+6)+\omega_2(\omega_2+4)+\omega_3(\omega_3+2)+\omega_4^2,\\
\langle C_2[\mathrm{SO}(6)]\rangle &= \sigma_1(\sigma_1+4)+\sigma_2(\sigma_2+2)+\sigma_3^2,\\
\langle C_2[\mathrm{SO}(5)]\rangle &= \upsilon_1(\upsilon_1+3)+\upsilon_2(\upsilon_2+1),\\
\langle C_2[\mathrm{SO}(3)]\rangle &= S(S+1) \text{ or } T(T+1).
\end{aligned} \tag{10}$$

If one is interested in the properties of the superfluid ground state, the remaining task is the determination of the *favoured* representations of the different algebras, that is, of the labels ω_i, σ_i, υ_i, S, and/or T of the ground state. The physical meaning of these labels is as follows. The S and T are the total spin and total isospin. The $\mathrm{SO}_T(5)$ labels $(\upsilon_1\upsilon_2)$ are known from the SO(5) formalism for $T=1$ pairing [5,6]: $\upsilon_1=\Omega-\frac{1}{2}v$ and $\upsilon_2=t$, where v is the seniority and t is the reduced isospin. A similar formalism involving the algebra $\mathrm{SO}_S(5)$ can be developed for $T=0$ pairing by interchanging the role of S and T and leads to an associated seniority v and the concept of *reduced spin* s which is the spin of the nucleons not in pairs coupled to $L=0$. The SO(6) labels $(\sigma_1\sigma_2\sigma_3)$ characterise a supermultiplet [13] and they are equivalent to SU(4) labels. Finally, the SO(8) labels $(\omega_1\omega_2\omega_3\omega_4)$ are known from the solution of the full pairing problem [9]. Specifically, $\omega_1=\Omega-\frac{1}{2}v$ and the remaining three labels are the *reduced supermultiplet labels* (i.e., the supermultiplet labels of the nucleons not in pairs coupled to $L=0$).

With use of the above physical interpretation the results for the favoured representations shown in Table 1 can be obtained. Although the derivation of these favoured representations may seem complex, the final result is simple and analytic. To find the expectation value of a given pairing Hamiltonian (with $g_0=0$, $g_1=0$ or $g_0=g_1$) in the ground state, it suffices to apply the appropriate formula in (9) and to calculate the expectation values (10) with the labels as given in Table 1.

The favoured representation in even–even nuclei always corresponds to seniority $v=0$ while in odd–mass nuclei it has $v=\frac{1}{2}$. In odd–odd nuclei the favoured seniority is not unique: it is always $v=0$ for $N=Z$ but otherwise can be either $v=0$ or $v=2$, depending on n and Ω.

Table 1 summarises the properties of the superfluid ground state. From it can be deduced the familiar features of spin-singlet superfluidity in semi-magic nuclei but it equally well reveals the spin-aligned structure typical of spin-triplet superfluidity in ^{3}He [14]. Since a realistic shell-model Hamiltonian has values $g_0\approx g_1$ [15], the solution obtained for $g_0=g_1$ should be the correct starting point for nuclei. Its quartet properties are discussed in the next section.

3. Quartetting and singlet + triplet pairing

A very special kind of SO(8) solution occurs for the ground state of $N=Z$ nuclei. For example, in the SO(6) limit of the SO(8) model, which corresponds to equal $T=0$ and $T=1$ pairing strengths, the exact ground-state solution can be written as [16]

$$\left(S_+^{10}\cdot S_+^{10}-S_+^{01}\cdot S_+^{01}\right)^{n/4}|\mathrm{o}\rangle. \tag{11}$$

This shows that the superfluid solution acquires a quartet structure in the sense that it reduces to a condensate of bosons which each correspond to four nucleons. Since the boson in (11) is a scalar in spin and isospin, it can be thought of as an α particle; its orbital character, however, might be different from that of an actual α particle. A quartet structure is also present in the two

[1] In fact, the case $g_0=-g_1$ is also solvable but it has little physical relevance.

SO(5) limits of the SO(8) model, which yields a ground-state wave function of the type (11) with either the first or the second term suppressed. Thus, a reasonable *ansatz* for the $N = Z$ ground-state wave function of the SO(8) pairing interaction (6) with arbitrary strengths g_0 and g_1 is

$$\left(\cos\theta\, S_+^{10}\cdot S_+^{10} - \sin\theta\, S_+^{01}\cdot S_+^{01}\right)^{n/4}|\mathrm{o}\rangle, \tag{12}$$

where θ is a parameter which depends on the ratio g_0/g_1.

The justification for the use of (12) as a trial state is that, for the specific values $\theta = 0, \pi/4, \pi/2$, it gives the *exact* ground-state wave function in the three limits of the SO(8) model. Outside these limits the parameter θ must be determined either by minimising the expectation value of $V_{\mathrm{SO(8)}}$ in the trial state [16] or by maximising the overlap with the exact wave function [17]. Both procedures lead to essentially identical values for θ and to a quartet trial state which is very close to the exact wave function (deviations of only a fraction of a percent [17]). Similar results are obtained for excited states.

In summary, the α-like condensate (12) provides an excellent approximation to the $N = Z$ ground state of the pairing Hamiltonian (6) for any combination of g_0 and g_1. The important (and as yet unanswered) question is now: To what extent does this quartet structure survive other terms that are present in a realistic shell-model Hamiltonian, in particular, possible single-particle splittings? With that question in mind, we now turn to the discussion of superfluidity in the presence of a single-particle mean field.

4. Superfluidity versus magicity

The shell structure of the atomic nucleus can be deduced from various observables such as [18] excitation energies in even–even nuclei, nucleon separation energies, nuclear level densities and (interaction) cross sections. Of these observables, the first gives the most direct information on shell structure and is available for a large number of nuclei. Fig. 2 shows, as a function of neutron number N and proton number Z, the quantity $E_{\mathrm{x}}(2_1^+)\ A^{1/3}$ where $E_{\mathrm{x}}(2_1^+)$ is the energy of the first-excited 2^+ state and $A = N + Z$ is the mass number of the nucleus. The figure shows this quantity for all even–even nuclei with $N, Z \geqslant 8$ for which $E_{\mathrm{x}}(2_1^+)$ is known experimentally [19]. This excitation energy is multiplied with $A^{1/3}$ to account for the gradual decrease of the strength of the nucleon interaction with mass number, and the resulting quantity is normalised to one. If one discards the light nuclei (say $N, Z < 28$) where the situation is made complex by very specific and detailed shell effects, the figure shows immediately the occurrence of shell stabilisation for the magic numbers 28, 40, 50, 82 and 126. It also reveals three types of nuclear structure (the so-called tripartite division of nuclei [20]): deformed nuclei (red) with a very low-lying 2_1^+ level, vibrational nuclei (yellow to green) where this level occurs at higher energy and semi-magic nuclei (green to blue) where it is higher still. (This tripartite division is confirmed by a similar plot of the ratio $E_{\mathrm{x}}(4_1^+)/E_{\mathrm{x}}(2_1^+)$ which is a more sensitive indicator of nuclear structural character.)

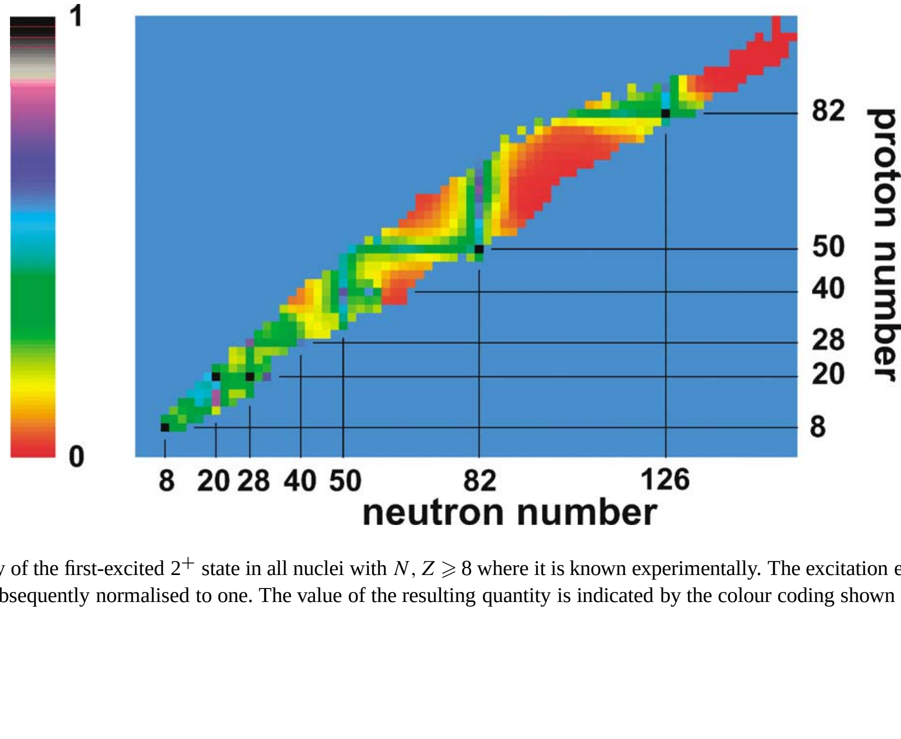

Fig. 2. The energy of the first-excited 2^+ state in all nuclei with $N, Z \geqslant 8$ where it is known experimentally. The excitation energy is multiplied with $A^{1/3}$ and subsequently normalised to one. The value of the resulting quantity is indicated by the colour coding shown on the left.

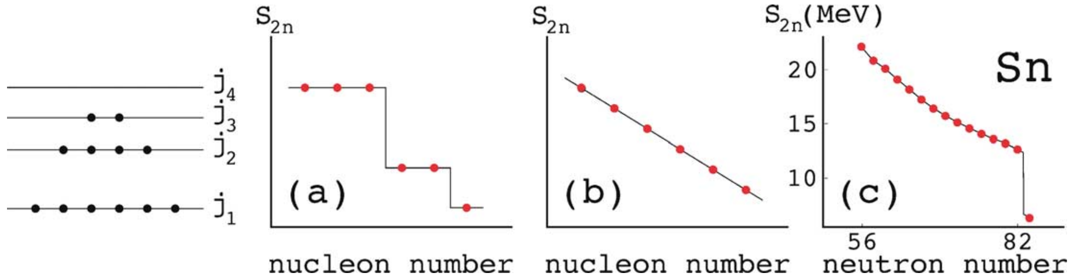

Fig. 3. The two-nucleon separation energy S_{2n} as an indicator of SU(2) superfluidity. If there are no pairing correlations among the identical nucleons occupying the levels shown on the left, the separation energy, as a function of nucleon number, behaves as in (a). Superfluidity leads to the behaviour shown in (b). The measured two-neutron separation energies in (c) show that the superfluid solution is appropriate for the tin isotopes with active neutrons in the 50–82 shell.

The third class of nuclei is the realm of the traditional SU(2) pairing models. Within this class, nuclei display an additional variety in structure which is indicative of the presence or absence of pairing correlations among the identical valence nucleons. One way to probe these correlations is from the energy of the first 2^+ state. A more sensitive test of SU(2) pairing correlations is from the two-nucleon separation energy S_{2n}, the energy required to extract two nucleons from a nucleus (see Fig. 3). For a system of identical nucleons which occupy a set of non-degenerate single-particle levels as shown on the left of the figure, the complete absence of pairing correlations leads to a jagged behaviour of S_{2n} as a function of n (see Fig. 3(a)). The other extreme, strong pairing correlations among nucleons distributed over closely spaced single-particle levels, is represented in Fig. 3(b). The latter behaviour derives from the superfluid character of the even–even ground state (3) which leads to the following expression for the two-nucleon separation energy:

$$S_{2n} = \frac{1}{2} g_0 (\Omega - 2n), \tag{13}$$

that is, a smooth decrease as the nucleon number increases. Fig. 3(c) shows the two-neutron separation energies measured in the tin isotopes, as a function of neutron number. As far as the 50–82 shell is concerned, the data agree with the superfluid solution. At $N = 82$ a large jump in S_{2n} is observed. This indicates that superfluidity is confined to the 50–82 shell and does not carry into the next shell. In consequence, the isotope ^{132}Sn is doubly magic.

The competition between magicity (from large single-particle gaps) and superfluidity (from strong pairing correlations) thus emerges as the defining feature of semi-magic nuclei [21]. Traditionally, the nuclear pairing problem with non-degenerate single-particle levels has been treated with the BCS technique imported from condensed-matter physics. Nevertheless, in view of the non-conservation of particle number, this might be a questionable approximation in nuclei. An exact method to solve the problem of particles distributed over non-degenerate levels interacting through a pairing force has been known since long [22] but, surprisingly, passed almost unnoticed despite its enormous potential impact. Only recently Richardson's work has been properly recognised as well as generalised to other classes of integrable pairing models [23].

In Richardson's model the pairing interaction (1) is supplemented with a single-particle term:

$$H_{\mathrm{SU}(2)} = \sum_l \varepsilon_l \hat{n}_l - g_0 S_+^0 S_-^0, \tag{14}$$

where $\hat{n}_l$ is the operator that counts the number of nucleons in orbit l and ε_l is the single-particle energy of that orbit. The solvability of the Hamiltonian (14) arises as a result of the symmetry $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \cdots$ where each SU(2) algebra pertains to a specific l. Whether the solution of (14) can be called superfluid depends on the differences $\varepsilon_l - \varepsilon_{l'}$ in relation to the strength g_0. Nevertheless, the solution is known in closed form for all possible choices of ε_l. For example, for an even number of particles n the Hamiltonian (14) has a ground state of the form (up to a normalisation factor)

$$\prod_{\alpha=1}^{n/2} \left(\sum_l \frac{1}{2\varepsilon_l - e_\alpha} S_+^0(l) \right) |\mathrm{o}\rangle, \tag{15}$$

where $S_+^0(l)$ is an S pair in orbit l,

$$S_+^0(l) = \sqrt{2l+1} \left(a_{ls}^\dagger \times a_{ls}^\dagger \right)_{00}^{(00)}. \tag{16}$$

The e_α are the solutions of the $n/2$ non-linear equations

$$1 - g_0 \sum_l \frac{2l+1}{2\varepsilon_l - e_\alpha} - 4 g_0 \sum_{\beta(\neq\alpha)} \frac{1}{e_\beta - e_\alpha} = 0, \quad \alpha = 1, 2, \ldots, \frac{n}{2}. \tag{17}$$

A characteristic feature of the ground state (15) is that it no longer consists of a superposition of *identical* bosons since the coefficients $(2\varepsilon_l - e_\alpha)^{-1}$ vary as α runs from 1 to $n/2$. This represents a departure from the concept of superfluidity.

It is clear that one may attempt to generalise Richardson's approach to the systems discussed in Section 2, that is, to SO(5) and SO(8) superfluidity with non-degenerate single-particle levels giving rise to $SO(5) \times SO(5) \times \cdots$ and $SO(8) \times SO(8) \times \cdots$ models. Although this has been achieved to some extent, much remains to be studied about the ensuing solutions.

5. Final remarks

Over the years overwhelming evidence has been gathered which confirms the existence of a nuclear superfluid phase in semi-magic nuclei. The theoretical reasons for its existence can be traced back to the nature of the $T = 1$ interaction among identical nucleons which, to a good approximation, conserves SU(2) symmetry and the associated seniority quantum number. The breaking of seniority as a consequence of single-particle splittings generated by the nuclear mean field, represents a departure from the superfluid solution. This departure is well understood in terms of an analytically solvable model. The typical situation in semi-magic nuclei is that single-particle splittings within a major shell are sufficiently small for superfluidity to prevail but that single-particle splittings across major shells are sufficiently large to destroy it.

Once the valence shell contains both neutrons and protons, the character of the nucleon–nucleon interaction drastically changes and induces a severe breaking of the seniority quantum number. As a result, one cannot expect a ground state of the superfluid type (12) but, at best, hope for a large component of it. A related question is the value of the variational parameter θ in the trial state (12). Since $g_0 \approx g_1$ one would naively expect $\theta \approx \pi/4$ and thus a sizeable contribution of the first term. This would be of particular interest since it corresponds to the elusive spin-triplet superfluidity. Several studies have shown, however, that the size of this component is adversely affected by the spin-orbit splitting (see, e.g., [24]) and possibly also by other terms in the shell-model Hamiltonian. The question whether and, if so, how this component can be probed experimentally is thus still open.

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