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Bose–Einstein condensates: recent advances in collective effects/Avancées récentes sur les effets collectifs dans les condensats de Bose–Einstein

Bose–Einstein condensation in random potentials

Olivier Lenoble [a], Leonid A. Pastur [b], Valentin A. Zagrebnov [a]

[a] *Université de la Méditerranée (Aix-Marseille II), centre de physique théorique, CNRS-Luminy, case 907, 13288 Marseille cedex 09, France*
[b] *Centre mathématique de Jussieu, Université Paris 7, case 7012, 75251 Paris cedex 05, France*

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Abstract

We present a rigorous study of the perfect Bose-gas in the presence of a homogeneous ergodic random potential. It is demonstrated that the Lifshitz tail behaviour of the one-particle spectrum reduces the critical dimensionality of the (generalized) Bose–Einstein Condensation (BEC) to $d = 1$. To tackle the Off-Diagonal Long-Range Order (ODLRO) we introduce the *space average* one-body reduced density matrix. For a one-dimensional Poisson-type random potential we prove that randomness enhances the exponential decay of this matrix in domain free of the BEC. ***To cite this article: O. Lenoble et al., C. R. Physique 5 (2004).***

Résumé

Condensation de Bose–Einstein dans des potentiels aléatoires. Nous présentons une étude rigoureuse du gaz de Bose parfait en présence d'un potentiel aléatoire statistiquement homogène. Nous démontrons que le comportement des ailes de Lifshitz pour le spectre d'énergie à une particule, réduit à $d = 1$ la dimensionalité critique de la transition de Bose–Einstein. Pour étudier les corrélations non diagonales à longue portée, nous introduisons une *moyenne spatiale* de la matrice densité réduite à un corps. En l'absence de condensat et pour un potentiel aléatoire undimensionnel de type Poissonnien, nous montrons que la décroissance exponentielle de la matrice densité est plus rapide. ***Pour citer cet article : O. Lenoble et al., C. R. Physique 5 (2004).***

Keywords: Bose–Einstein condensation; Perfect Bose-gas

Mots-clés : Condensation de Bose–Einstein ; Gaz de Bose parfait

1. Introduction

The Bose–Einstein condensation (BEC) of the ideal Bose gas has been actively studied in recent years. In particular, it is of interest to consider the BEC in random media (see, e.g., [1,2]). In a simple-minded approach, where one just adds a random external field to the Hamiltonian of the ideal Bose gas, the problem reduces to the Schrödinger equation with random potential. It is known (see, e.g., [3]) that in this case the BEC possesses certain peculiarities, related to the fact that the ground state and nearby states are not extended; thus the effective density of particles is infinite in the thermodynamic limit. Hence, unlike the Fermi case, where the one body theory with a random Schrödinger operator describes sufficiently well the quantum motion in random media, in the Bose case the repulsive interaction between particles seems to be an important part of the Hamiltonian.

E-mail addresses: lenoble@cpt.univ-mrs.fr (O. Lenoble), pastur@math.jussieu.fr (L.A. Pastur), zagrebnov@cpt.univ-mrs.fr (V.A. Zagrebnov).

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Despite that, a rigorous study of the BEC of the *perfect* Bose-gas based on the random Schrödinger operator, is indispensable as a first step to understanding this phenomenon in random media.

Let $\Lambda_1 \subset \mathbb{R}^d$ be an open bounded connected domain with unit volume $|\Lambda_1| = 1$ and with a smooth boundary $\partial\Lambda_1$, containing the origin. For any $L > 0$ we can define the domain

$$\Lambda_L := \{x \in \mathbb{R}^d\colon\ L^{-1}x \in \Lambda_1\}, \tag{1}$$

by the isotropic dilation, and we have for its volume $|\Lambda_L| = L^d$.

We denote by $\mathcal{H}_L := L^2(\Lambda_L)$ the Hilbert space of one-particle wave functions corresponding to the quantum problem with the free single-particle Hamiltonian (for $\hbar = 1$, $m = 1$)

$$t_L := \left(-\frac{1}{2}\Delta\right)^D_{\Lambda_L}. \tag{2}$$

Here t_L is the self-adjoint operator, defined by $(-\Delta/2)$ in Λ_L, and by the Dirichlet boundary condition on $\partial\Lambda_L$.

It is known (see, e.g., [4]) that the operator t_L has a discrete spectrum $\sigma(t_L) = \{E_k(L)\}_{k\geqslant 1}$, consisting of isolated eigenvalues $E_k(L)$:

$$t_L\psi_k^D = E_k(L)\psi_k^D, \quad k \geqslant 1, \tag{3}$$

of finite multiplicity. Here $\{\psi_k^D \in \mathcal{H}_L\}_{k\geqslant 1}$ are the corresponding normalized eigenfunctions. The eigenvalues can be ordered such that

$$0 < E_1(L) < E_2(L) \leqslant E_3(L) \leqslant \cdots. \tag{4}$$

Remark 1. The rate of increasing of eigenvalues (4) is such that the operator t_L generates a Gibbs semigroup, i.e., $\exp(-\beta t_L) \in \text{Tr-class}(\mathcal{H}_L)$ for any $\beta > 0$. Moreover, it is known (see, e.g., [5]) that for $\beta > 0$ the *single-particle partition function*

$$\phi_L(\beta) := \frac{1}{|\Lambda_L|}\mathrm{Tr}_{\mathcal{H}_L}\,\mathrm{e}^{-\beta t_L} = \frac{1}{|\Lambda_L|}\sum_{k\geqslant 1}\mathrm{e}^{-\beta E_k(L)} \tag{5}$$

is (uniformly) bounded:

$$\phi_L(\beta) \leqslant \frac{1}{(2\pi\beta)^{d/2}}, \tag{6}$$

and the limit $\lim_{L\to\infty}\phi_L(\beta)$ exists.

Since for any $L < \infty$ the spectrum $\sigma(t_L)$ is discrete, bounded from below, and consists of isolated eigenvalues of finite multiplicity, one can introduce the following object.

Definition 1.1. The finite-volume *integrated density of state* (IDS) of t_L is the specific (by a unit volume) eigenvalue counting function

$$\mathcal{N}_L(E) := \frac{1}{|\Lambda_L|}\max\{k\colon\ E_k(L) < E\}. \tag{7}$$

This allows us to rewrite (5) in the form:

$$\phi_L(\beta) = \int_0^\infty \mathcal{N}_L(\mathrm{d}E)\,\mathrm{e}^{-\beta E}, \tag{8}$$

where $\mathcal{N}_L(\mathrm{d}E)$ is the positive measure, corresponding to the monotonous increasing functions (7). In other words, for any $L > 0$ the single-particle partition function (5) is the *Laplace–Stieltjes transform* of the measure $\mathcal{N}_L(\mathrm{d}E)$.

Proposition 1.2. *There exists non-negative measure $\mathcal{N}^{(0)}(\mathrm{d}E)$ on $\mathbb{R}_+ := \{x \in \mathbb{R}\colon\ x \geqslant 0\}$ such that we have the weak convergence*

$$\lim_{L\to\infty}\mathcal{N}_L(\mathrm{d}E) = \mathcal{N}^{(0)}(\mathrm{d}E), \tag{9}$$

i.e., the convergence of $\mathcal{N}_L(E)$ to $\mathcal{N}^{(0)}(E)$ in all continuity points of the limiting IDS $\mathcal{N}^{(0)}$. Besides, we have for any $\beta > 0$:

$$\phi(\beta) := \lim_{L\to\infty} \int_0^\infty \mathcal{N}_L(\mathrm{d}E)\,\mathrm{e}^{-\beta E} = \int_0^\infty \mathcal{N}^{(0)}(\mathrm{d}E)\,\mathrm{e}^{-\beta E}. \tag{10}$$

The proof of the Proposition 1.2 is based on general properties of the Laplace transform of probability distributions [6] and on the Feynman–Kac formula. It was shown that this proof can be generalized to the case of the single-particle Hamiltonian in the presence of *random ergodic potentials*, see [7].

The existence of the IDS $\mathcal{N}^{(0)}$ and its properties are important for understanding of the thermodynamic behaviour of the *free* Bose-gas (*perfect* gas without external potential) and, in particular, of the Bose–Einstein condensation [6]. In the next sections we discuss this first for the free Bose-gas and then for the perfect Bose-gas in random ergodic potentials.

Remark 2. The limiting IDS of operator (2) can be calculated explicitly. The famous Weyl theorem (see, e.g., [4]) implies that

$$\lim_{E\to\infty} E^{-d/2}\mathcal{N}_{L=1}(E) = C_d, \tag{11}$$

where $(C_d)^{-1} = (2\pi)^{d/2}\Gamma(1+d/2)$. This, together with scaling property of the single-particle Hamiltonian (2) and definition of the IDS (7), imply the relation:

$$\mathcal{N}_L(E) = L^{-d}\mathcal{N}_{L=1}\big(L^2E\big),$$

which yields the limiting IDS for the operator (2):

$$\mathcal{N}^{(0)}(E) = \lim_{L\to\infty} \mathcal{N}_L(E) = C_d E^{d/2}. \tag{12}$$

2. Condensation of the free Bose-gas

Now we can turn to the many-body problem of non-interacting bosons without external potential (*free* Bose-gas) in container Λ_L.

We define the corresponding *grand-canonical Gibbs distribution* $P_{\beta,\mu}(\cdot)$ on the *probability space* of (infinite) sequences of non-negative integer numbers $(\mathbb{N}_+)^{\mathbb{N}}$, $\mathbb{N}_+ := \mathbb{N}\cup\{0\}$. They correspond to configurations $\underline{n} \in (\mathbb{N}_+)^{\mathbb{N}}$ of the boson one-particle quantum-state occupation numbers:

$$\underline{n} = \big\{n_k := n\big(\psi_k^D\big)\big\}_{k\geqslant 1}, \quad n_k = 0, 1, 2, \ldots, \tag{13}$$

assuming that only a finite number of terms of the sequence $\{n_k\}_{k\geqslant 1}$ is non-zero. Then for a given temperature $\beta^{-1} > 0$ and chemical potential $\mu < 0$ the grand-canonical Gibbs distribution has the form:

$$P_{\beta,\mu}(\underline{n}) := \big\{\Xi_L(\beta,\mu)\big\}^{-1} \exp\big\{-\beta\big(T_L(\underline{n}) - \mu N_L(\underline{n})\big)\big\}. \tag{14}$$

Here the random variable

$$T_L(\underline{n}) := \sum_{k\geqslant 1} E_k(L) n_k \tag{15}$$

is the *kinetic-energy*, whereas

$$N_L(\underline{n}) := \sum_{k\geqslant 1} n_k \tag{16}$$

is the *total number of particles* in the configuration $\underline{n}$. The normalizing factor $\Xi_L(\beta,\mu)$ is the grand-canonical partition function for the free bosons:

$$\Xi_L(\beta,\mu) = \sum_{\underline{n}\in\mathbb{N}^{\mathbb{N}_+}} \mathrm{e}^{-\beta(T_L(\underline{n})-\mu N_L(\underline{n}))} = \prod_{k\geqslant 1}\big\{1 - \mathrm{e}^{-\beta(E_k(L)-\mu)}\big\}^{-1}, \tag{17}$$

which exists for all $\mu < E_1(L)$. The definition of the *finite-volume* IDS (7) implies that the corresponding pressure $p_L(\beta,\mu)$ and the grand-canonical mean-value of the total particle-density $\rho_L(\beta,\mu) := \partial_\mu p_L(\beta,\mu)$ take the form:

$$p_L(\beta,\mu) = -\frac{1}{\beta}\int_0^\infty \mathcal{N}_L(\mathrm{d}E)\ln\bigl(1-\mathrm{e}^{-\beta(E-\mu)}\bigr), \tag{18}$$

$$\rho_L(\beta,\mu) = \int_0^\infty \mathcal{N}_L(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta(E-\mu)}-1}. \tag{19}$$

Then, by Proposition 1.2, the limiting pressure and density of the free Bose-gas exist for all $\mu \in (-\infty, 0)$:

$$p(\beta,\mu) = -\frac{1}{\beta}\int_0^\infty \mathcal{N}^{(0)}(\mathrm{d}E)\ln\bigl(1-\mathrm{e}^{-\beta(E-\mu)}\bigr), \tag{20}$$

$$\rho(\beta,\mu) = \int_0^\infty \mathcal{N}^{(0)}(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta(E-\mu)}-1}. \tag{21}$$

By definition $p(\beta,0) := \lim_{\mu\uparrow 0} p(\beta,\mu)$ and the *critical density* is defined as

$$\rho_c(\beta) := \lim_{\mu\uparrow 0}\rho(\beta,\mu), \tag{22}$$

if it is finite. We put $\rho_c(\beta) = \infty$ otherwise.

Remark 3. If $\rho_c(\beta) < \infty$, then there is a room for the phenomenon known as the Bose–Einstein Condensation (BEC) in the free Bose-gas. By virtue of (12) and (22) the boundedness of the critical density is equivalent to the following property of the IDS:

$$\int_0^\infty \mathrm{d}E\,\frac{\mathcal{N}^{(0)}(E)}{(\mathrm{e}^{\beta E}-1)^2} < \infty, \tag{23}$$

for any $\beta > 0$. Therefore, one gets $\rho_c(\beta) = \infty$, for $d \leqslant 2$, and $\rho_c(\beta) < \infty$ for $d > 2$. Moreover,

$$\rho_c(\beta) = \lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty \mathcal{N}^{(0)}(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta E}-1} = \int_0^\infty \mathrm{d}E\,\frac{\mathcal{N}^{(0)}(E)}{(\mathrm{e}^{\beta E}-1)^2}. \tag{24}$$

Now we can formulate a mathematical statement concerning the existence of the BEC in the free Bose gas [6]:

Proposition 2.1. *Let $\rho_c(\beta) < \infty$. For given $\beta > 0$, $\rho > 0$ and $L > 0$ we denote by $\mu_L(\beta,\rho)$ the unique root of equation*

$$\rho = \rho_L(\beta,\mu), \tag{25}$$

see (19). *Then*

(a) *for $\rho < \rho_c(\beta)$ the limit*

$$\lim_{L\to\infty}\mu_L(\beta,\rho) = \mu(\beta,\rho) < 0 \tag{26}$$

exists, and it is a unique negative root of the equation (*see* (21)):

$$\rho = \rho(\beta,\mu); \tag{27}$$

(b) *for $\rho \geqslant \rho_c(\beta)$ the limit*

$$\lim_{L\to\infty}\mu_L(\beta,\rho) = 0, \tag{28}$$

exists, and it is zero;

(c) *the BEC manifests itself in the following form*:

$$\rho_0(\beta,\rho) := \lim_{\varepsilon\downarrow 0}\lim_{L\to\infty}\int_0^\varepsilon \mathcal{N}_L(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta(E-\mu_L(\beta,\rho))}-1} = \rho - \rho_c(\beta) > 0, \tag{29}$$

where $\rho_0(\beta,\rho)$ is the BEC density.

Remark 4. In fact Proposition 2.1 establishes a so-called *generalized* BEC [6]. This is in contrast to the conventional BEC, which corresponds to *macroscopic occupation* of only *one* (ground-state) level:

$$\rho_0(\beta,\rho)=\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\frac{1}{\mathrm{e}^{\beta(E_1(L)-\mu_L(\beta,\rho))}-1}. \tag{30}$$

The macroscopic occupation of *only* the ground-state level needs finer properties of the spectrum $\{E_k(L)\}_{k\geqslant 1}$.

Rewriting Eq. (25) in the form

$$\rho=\frac{1}{|\Lambda_L|}\frac{1}{\mathrm{e}^{\beta(E_1(L)-\mu_L(\beta,\rho))}-1}+\frac{1}{|\Lambda_L|}\sum_{k\geqslant 2}\frac{1}{\mathrm{e}^{\beta(E_k(L)-\mu_L(\beta,\rho))}-1}, \tag{31}$$

one finds that to realize the case (30) it is necessary that the differences $\lambda_k(L):=E_k(L)-E_1(L)$, $k\geqslant 2$, go to zero *slower* than $|\Lambda_L|^{-1}$. Then $E_1(L)-\mu_L(\beta,\rho)$ is of the order $|\Lambda_L|^{-1}$, that gives the one-level BEC condensation (30). This kind of condensation occurs, for example, when Λ_L is the three-dimensional cube.

Remark 5. In fact a sufficient condition for macroscopic occupation of *only* the ground-state level is rather simple: $E_2(L)/(E_2(L)-E_1(L))<c<\infty$, $d\geqslant 3$, see [8]. More delicate is the question about existence of the *second critical density* $\rho_m(\beta)\geqslant\rho_c(\beta)$ such that macroscopic occupation of single-levels states is only possible if $\rho>\rho_m(\beta)$, see [9,6,10].

On the other hand, if the differences $\lambda_k(L):=E_k(L)-E_1(L)$, $k\geqslant 2$, are of the order $|\Lambda_L|^{-1}$, then there are (*infinitely*) many *macroscopically* occupied levels

$$\rho_0^{(k)}(\beta,\rho):=\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\frac{1}{\mathrm{e}^{\beta(E_k(L)-\mu_L(\beta,\rho))}-1}\neq 0 \tag{32}$$

in addition to the ground-state.

Remark 6. If the number of these levels is infinite, this is (according to the nomenclature proposed by van den Berg–Lewis–Pulé, see [11,9,6]) the type II generalised BEC. (The type I is reserved for the case of finite number of levels.) The most unusual is the type III generalised BEC, when there are no macroscopically occupied levels but the limit in (29) is non-zero. It happens when the differences λ_k, $k\geqslant 2$, go to zero faster than $|\Lambda_L|^{-1}$.

The simplest model manifesting the *type III* generalised BEC is the free Bose-gas in the *anisotropic* prisms $\Lambda_L=|\Lambda_L|^{\alpha_1}\times\cdots\times|\Lambda_L|^{\alpha_d}$, with $\alpha_1\geqslant\alpha_2\geqslant\cdots\geqslant\alpha_d$, $\alpha_1+\alpha_2+\cdots+\alpha_d=1$ and $\alpha_1>1/2$, see [11].

For detailed discussion of the generalised BEC and for examples other than free Bose-gas, including some models of interacting Bose-gases, see [9,8,12–14] and reviews [6,10,15].

3. Random Schrödinger operator and perfect Bose-gas

We discuss first the single-particle Schrödinger operator with random potential.

Definition 3.1. Random potential $v^{(\cdot)}(\cdot):\Omega\times\mathbb{R}^d\to\mathbb{R}$, $x\mapsto v^\omega(x)$ is a random field on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, such that:

(a) v^ω is homogeneous and ergodic with respect to the group $\{\tau_x\}_{x\in\mathbb{R}^d}$ of probability-preserving transformations ($\mathbb{R}^d$-translations) on $(\Omega,\mathcal{F},\mathbb{P})$;
(b) v^ω is non-negative: $\inf_{x\in\mathbb{R}^d}\{v^\omega(x)\}\geqslant 0$.

By $\mathbb{E}\{\cdot\}:=\int_\Omega\mathbb{P}(\mathrm{d}\omega)\{\cdot\}$ we denote the expectation with respect to the probability measure in $(\Omega,\mathcal{F},\mathbb{P})$.

Definition 3.2. Random Schrödinger operator corresponding to a random potential v^ω is a family of random operators $\{h^\omega\}_{\omega\in\Omega}$:

$$h^\omega:=t+v^\omega, \tag{33}$$

where t is the self-adjoint operator $(-\Delta/2)$, acting in $L^2(\mathbb{R}^d)$.

Notice that assumptions (a) and (b) of Definition 3.1 guarantee that there exists a subset $\Omega_0 \subset \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that operator (33) is *essentially self-adjoint* on domain $\mathcal{C}_0^\infty(\mathbb{R}^d)$ for every $\omega \in \Omega_0$ (see, e.g., [7, Chapter I.2]).

To discuss the properties of the perfect Bose-gas in the presence of the random potential one has (as in the preceding section) to start with the bounded domain $\Lambda_L \subset \mathbb{R}^d$. Below we suppose that $\{\Lambda_L\}_{L>0}$ are cubes centered at the origin. Then (see [7, Chapter II.5], or [16]) one has the following statement:

Proposition 3.3. *If a random potential verifies the conditions* (a) *and* (b) *of Definition* 3.1*, then*:

(1) *The restriction of the random Schrödinger operator $\{h^\omega\}_{\omega\in\Omega_0}$ to domain Λ_L subjected to the Dirichlet boundary condition on $\partial\Lambda_L$ is the set of self-adjoint operators*

$$h_L^\omega := \left(-\frac{1}{2}\Delta + v^\omega\right)_{\Lambda_L}^D \tag{34}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$.

(2) *The spectrum $\sigma(h_L^\omega)$ is almost-surely discrete, bounded from below, and consists of isolated eigenvalues $\{E_k^\omega(L) > 0\}_{k\geqslant 1}$ of finite multiplicity.*

(3) *For $\mathbb{P}$-almost all ω operators* (34) *are generators of the Gibbs semigroups, i.e.,* $\exp(-\beta h_L^\omega) \in \text{Tr-class}(\mathcal{H}_L)$ *for any $\beta > 0$.*

Remark 7. (a) In fact the proposition is valid for some wider class of self-adjoint boundary conditions on $\partial\Lambda_L$ [7].

(b) For bounded random potentials the random Schrödinger operator (34) takes the form:

$$h_L^\omega := \left(-\frac{1}{2}\Delta\right)_{\Lambda_L}^D + v^\omega. \tag{35}$$

In fact one can consider first (35), then extend the results to more general cases (e.g., for δ-interaction) by taking limits [7].

Notice that Proposition 3.3 and Definition 1.1 motivate definition of

$$\mathcal{N}_L^\omega(E) := \frac{1}{|\Lambda_L|} \max\bigl\{k\colon\ E_k^\omega(L) < E\bigr\}, \quad \omega \in \Omega. \tag{36}$$

This is (*random*) finite-volume IDS corresponding to the random Schrödinger operators (34) or (35).

Now we are in position to discuss the Bose–Einstein condensation of the perfect Bose-gas in presence of a random potential.

First, using the random measures generated by IDS (36), one gets for the corresponding finite-volume pressure and the total particle density (cf. (18) and (19)):

$$p_L^\omega(\beta,\mu) = -\frac{1}{\beta}\int_0^\infty \mathcal{N}_L^\omega(\mathrm{d}E)\ln\bigl(1 - \mathrm{e}^{-\beta(E-\mu)}\bigr), \tag{37}$$

$$\rho_L^\omega(\beta,\mu) = \int_0^\infty \mathcal{N}_L^\omega(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta(E-\mu)} - 1}, \tag{38}$$

for all $\beta > 0$, $\mu < 0$, and any realization $\omega \in \Omega$.

To take the thermodynamic limit we use the following fact concerning the *weak* convergence of random IDS measures [7].

Proposition 3.4. *Let h_L^ω be defined in* (34)*, where the random potential verifies the conditions* (a) *and* (b) *of Definition* 3.1*. Then there exist a nonrandom measure $\mathcal{N}(\mathrm{d}E)$ and a set $\Omega_0 \subset \mathcal{F}$ of full probability, $\mathbb{P}(\Omega_0) = 1$, such that for every $\omega \in \Omega_0$ the convergence*

$$\lim_{L\to\infty} \mathcal{N}_L^\omega(\mathrm{d}E) = \mathcal{N}(\mathrm{d}E) \tag{39}$$

holds on the whole spectral axis except the (*at most countable*) *set of discontinuity points of $\mathcal{N}$. We have also*

$$\mathcal{N}(\mathrm{d}E) = \mathbb{E}\bigl\{\mathcal{E}_{h^\omega}(\mathrm{d}E; 0, 0)\bigr\}. \tag{40}$$

Here $\mathcal{E}_{h^\omega}(\mathrm{d}E; x, y)$ is the kernel of the spectral decomposition measure of the random Schrödinger operator h^ω, see Definition 3.2*. Moreover, the spectrum $\sigma(h^\omega)$ of h^ω is almost-sure* (*a.s.*) *nonrandom and coincides with the support of $\mathcal{N}$: $\sigma(h^\omega) = \operatorname{supp}\mathcal{N}$.*

Remark 8. In fact the nonrandomness (*self-averaging*) of the limiting IDS is known under much weaker conditions on random potential. Besides, the limiting IDS is independent of the boundary conditions for a sufficiently wide class of them, see [7,16] and a recent review [17].

Let $\inf_{x\in\mathbb{R}^d}\{v^\omega(x)\}=0$ in condition (b) of Definition 3.1. Suppose for simplicity that the (*nonrandom*) lower edge E_0 of the spectrum $\sigma(h^\omega)$ is *zero*. This yields

Corollary 3.5. *Let the random potential verify the conditions* (a) *and* (b) *of Definition* 3.1. *Then for* $\mathbb{P}$*-almost every* $\omega\in\Omega$ (*a.s.*) *the limits*

$$\text{a.s.-}\lim_{L\to\infty} p_L^\omega(\beta,\mu) = -\frac{1}{\beta}\int_0^\infty \mathcal{N}(\mathrm{d}E)\ln\bigl(1-\mathrm{e}^{-\beta(E-\mu)}\bigr) \equiv p(\beta,\mu), \tag{41}$$

$$\text{a.s.-}\lim_{L\to\infty} \rho_L^\omega(\beta,\mu) = \int_0^\infty \mathcal{N}(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta(E-\mu)}-1} \equiv \rho(\beta,\mu), \tag{42}$$

exist for all $\beta>0$ *and* $\mu<E_0=0$. *Moreover, the convergence is uniform in* μ *on compacts in* $(-\infty,0)$.

It follows from Corollary 3.5 and (42) that the definition of critical density for the Bose-gas in the random potential is the same as in the nonrandom case (22):

$$\rho_c(\beta) := \lim_{\mu\uparrow 0}\int_0^\infty \mathcal{N}(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta(E-\mu)}-1}. \tag{43}$$

Again the crucial for the existence of the BEC is the boundedness of the critical density, which, in its turn, is determined by the behaviour of the IDS $\mathcal{N}(E)$ near the lower edge E_0 of the spectrum (see Remark 3). If this edge is *zero*, then we have to know the asymptotic form of IDS for $E\to+0$. A favorable for $\rho_c(\beta)<\infty$ is the case of the IDS decreasing at least as $\mathrm{O}(E^{1+\varepsilon})$, $\varepsilon>0$. Random Schrödinger operators supply examples of this kind even for $d\leqslant 2$. Below we consider one of them, verifying conditions (a) and (b) of Definition 3.1.

Example 1. Let $u(x)\geqslant 0$, $x\in\mathbb{R}^d$, be continuous function with a compact support. We call it a local single-impurity potential. Let $\{\mu_\tau^\omega(\mathrm{d}x)\}_{\omega\in\Omega}$ be random Poisson measure on $\mathbb{R}^d$ with intensity $\tau>0$, i.e.,

$$\mathbb{P}\bigl(\{\omega\in\Omega\colon\ \mu_\tau^\omega(\Lambda)=n\}\bigr) = \frac{(\tau|\Lambda|)^n}{n!}\,\mathrm{e}^{-\tau|\Lambda|},\quad n\in\mathbb{N}_+=\mathbb{N}\cup\{0\}, \tag{44}$$

for any bounded Borel set $\Lambda\subset\mathbb{R}^d$. Then the non-negative random potential generated by the Poisson distributed local impurities has realizations

$$v^\omega(x) := \int_{\mathbb{R}^d}\mu_\tau^\omega(\mathrm{d}y)u(x-y) = \sum_j u\bigl(x-y_j^\omega\bigr), \tag{45}$$

where impurity positions $\{y_j^\omega\}\subset\mathbb{R}^d$ are the atoms of the random Poisson measure. Since the expectation $\mathbb{E}(\mu_\tau^\omega(\Lambda))=\tau|\Lambda|$, the parameter τ is concentration of impurities in $\mathbb{R}^d$.

Remark 9. The random potential (45) is obviously homogeneous and ergodic (even strongly mixing), i.e., it verifies the conditions (a) and (b) of Definition 3.1. Less trivial, see, e.g., [18,19] and [7, Chapter II.5], is that:

(a) The almost-sure nonrandom spectrum $\sigma(h^\omega)=\mathbb{R}_+$. This means, in particular, that

$$\text{a.s.-}\lim_{L\to\infty} E_1^\omega(L)=0. \tag{46}$$

In other words, the lower edge of the spectrum of the random operator h^ω is $E_0=0$, i.e., it coincides with the lower edge of the spectrum of the nonrandom operator t, see (33).

(b) The asymptotic behaviour of $\mathcal{N}(E)$ as $E \to 0$ has the form (the Lifshitz tail):

$$\ln \mathcal{N}(E) \sim -\tau \left(\frac{c_d}{E}\right)^{d/2}, \quad E \downarrow 0, \tag{47}$$

with $c_d > 0$. Recall that in the nonrandom case (12) one has: $\mathcal{N}(E) \sim E^{d/2}$, $E \downarrow 0$.

4. Bose–Einstein Condensation in random potentials

The suppression (47) of the IDS in the vicinity of the lower edge of the spectrum $E_0 = 0$ (*the Lifshitz tail* (47)) makes the critical density (43) *finite* for all $d \geqslant 1$. This means that the presence of a random potential may change the mechanism and the nature of condensation of the perfect Bose-gas. To make this clear, first we need an analogue of Proposition 2.1 for the random Schrödinger operators.

Theorem 4.1. *Let $\rho_L^\omega(\beta,\mu)$ be defined as in* (38) *and $\rho_c(\beta)$ by* (43)*. Assume that the lower edge of the spectrum E_0 of the random Schrödinger operator is* 0*, and that $\rho_c(\beta) < \infty$. For given $\beta > 0$, $\rho > 0$ and $L > 0$ denote by $\mu_L^\omega(\beta,\rho)$ the unique root of equation*

$$\rho = \rho_L^\omega(\beta,\mu), \tag{48}$$

for a realization $\omega \in \Omega$, see (38)*. Then*

(a) *for $\rho < \rho_c(\beta)$ the limit*

$$\text{a.s.-}\lim_{L\to\infty} \mu_L^\omega(\beta,\rho) = \mu(\beta,\rho) < 0 \tag{49}$$

is the unique root of equation:

$$\rho = \rho(\beta,\mu), \tag{50}$$

see (42);

(b) *for $\rho \geqslant \rho_c(\beta)$ the limit*

$$\text{a.s.-}\lim_{L\to\infty} \mu_L^\omega(\beta,\rho) = 0, \tag{51}$$

and the almost-sure nonrandom BEC manifests itself in the following form:

$$\rho_0(\beta,\rho) := \lim_{\varepsilon\downarrow 0}\left\{\text{a.s.-}\lim_{L\to\infty}\int_0^\varepsilon \mathcal{N}_L^\omega(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta(E-\mu_L^\omega(\beta,\rho))}-1}\right\} = \rho - \rho_c(\beta) > 0, \tag{52}$$

where $\rho_0(\beta,\rho)$ is the BEC density.

Proof. Since the critical density is bounded, we have by (42) and (43)

$$\rho_c(\beta) = \rho(\beta,\mu = 0). \tag{53}$$

Notice that Eq. (48) implies the following inequality for every $\omega \in \Omega$:

$$\rho = \frac{1}{|\Lambda_L|}\sum_{k\geqslant 1}\frac{1}{\mathrm{e}^{\beta(E_k^\omega(L)-\mu_L^\omega(\beta,\rho))}-1} \leqslant \frac{\phi_L^\omega(\beta)\,\mathrm{e}^{\beta\mu_L^\omega(\beta,\rho)}}{1-\mathrm{e}^{-\beta(E_1^\omega(L)-\mu_L^\omega(\beta,\rho))}}, \tag{54}$$

where

$$\phi_L^\omega(\beta) := \frac{1}{|\Lambda_L|}\sum_{k\geqslant 1}\mathrm{e}^{-\beta E_k^\omega(L)} = \int_0^\infty \mathcal{N}_L^\omega(\mathrm{d}E)\,\mathrm{e}^{-\beta E} \tag{55}$$

is the random single-particle partition function, cf. (5). On the other hand, for any $\omega \in \Omega$ solution of Eq. (48) is bounded from above:

$$\mu_L^\omega(\beta,\rho) < E_1^\omega(L). \tag{56}$$

Inequalities (54), (56) yield the estimate

$$\beta^{-1}\ln\frac{\rho}{\rho\,\mathrm{e}^{-\beta E_1^{\omega}(L)}+\phi_L^{\omega}(\beta)}\leqslant \mu_L^{\omega}(\beta,\rho)<E_{k=1}^{\omega}(L) \tag{57}$$

for any $\omega\in\Omega$ and $L>0$. Then by Proposition 3.4, Remark 9 and estimate (57) we obtain

$$\beta^{-1}\ln\frac{\rho}{\rho+\phi(\beta)}\leqslant \liminf_{L\to\infty}\mu_L^{\omega}(\beta,\rho)\leqslant \limsup_{L\to\infty}\mu_L^{\omega}(\beta,\rho)\leqslant 0 \tag{58}$$

for $\mathbb{P}$-almost all ω. We denote this set by Ω_0.

(a) By (58) it follows that for each $\omega\in\Omega_0$ the sequence $\{\mu_L^{\omega}(\beta,\rho)\}_{L>0}$ has at least one accumulation point: $\mu_*^{\omega}(\beta,\rho)$. Suppose that $\mu_*^{\omega}(\beta,\rho)=0$, and let $\{\mu_{L_j}^{\omega}(\beta,\rho)\}_{L_j>0}$ be a subsequence converging to this point. Since by (42) for $\rho<\rho_c(\beta)$ the unique solution $\mu(\beta,\rho)$ of Eq. (50) is strictly negative, by monotonicity of $\rho_L^{\omega}(\beta,\mu)$ as a function of μ, we get

$$\rho_L^{\omega}\big(\beta,\mu(\beta,\rho)/2\big)<\rho_L^{\omega}\big(\beta,\mu_{L_j}^{\omega}(\beta,\rho)\big)=\rho \tag{59}$$

for all L_j greater than some $\hat{L}$. On the other hand by Corollary 3.5 and by the same monotonicity we get $\lim_{L\to\infty}\rho_L^{\omega}(\beta,\mu(\beta,\rho)/2)>\rho$ that contradicts (59). Therefore, $\mu_*^{\omega}(\beta,\rho)<0$. Then by (59) and by uniform convergence in (42) this accumulation point is a solution of Eq. (50). Since for $\rho<\rho_c(\beta)$ this equation has a unique solution, we obtain $\mu_*^{\omega}(\beta,\rho)=\mu(\beta,\rho)<0$ for almost all ω.

(b) Now let $\rho\geqslant\rho_c(\beta)$ and suppose that the accumulation point is strictly negative: $\mu_*^{\omega}(\beta,\rho)<0$. Then again by the uniform convergence in (42) we get:

$$\rho=\lim_{L\to\infty}\rho_{L_j}^{\omega}\big(\beta,\mu_{L_j}(\beta,\rho)\big)=\rho\big(\beta,\mu_*^{\omega}(\beta,\rho)\big)\geqslant\rho_c(\beta). \tag{60}$$

Since $\mu_*^{\omega}(\beta,\rho)<0$, the monotonicity implies $\rho(\beta,\mu_*^{\omega}(\beta,\rho))<\rho(\beta,\mu=0)=\rho_c(\beta)$. By (58) and by contradiction with (60) we conclude that $\mu_*^{\omega}(\beta,\rho)=0$ and that

$$\text{a.s.-}\lim_{L\to\infty}\mu_L^{\omega}\big(\beta,\rho\geqslant\rho_c(\beta)\big)=0. \tag{61}$$

To prove the last statement (52) we rewrite Eq. (48) in the form:

$$\rho=\int_0^{\varepsilon}\mathcal{N}_L^{\omega}(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta(E-\mu_L^{\omega}(\beta,\rho))}-1}+\int_{\varepsilon}^{\infty}\mathcal{N}_L^{\omega}(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta(E-\mu_L^{\omega}(\beta,\rho))}-1}. \tag{62}$$

Then by (61) and by the uniform convergence in (42) we find:

$$\lim_{\varepsilon\downarrow 0}\left\{\text{a.s.-}\lim_{L\to\infty}\int_{\varepsilon}^{\infty}\mathcal{N}_L^{\omega}(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta(E-\mu_L^{\omega}(\beta,\rho))}-1}\right\}=\lim_{\varepsilon\downarrow 0}\int_{\varepsilon}^{\infty}\mathcal{N}(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta E}-1}=\rho_c(\beta). \tag{63}$$

Therefore, by virtue of (62) and (63) on gets that there is a macroscopic accumulation of bosons in an *infinitesimal* band in the vicinity of the lower edge of the spectrum $E_0=0$:

$$\lim_{\varepsilon\downarrow 0}\left\{\text{a.s.-}\lim_{L\to\infty}\int_0^{\varepsilon}\mathcal{N}_L^{\omega}(\mathrm{d}E)\frac{1}{\mathrm{e}^{\beta(E-\mu_L^{\omega}(\beta,\rho))}-1}\right\}=\rho-\rho_c(\beta). \tag{64}$$

This proves (52). □

Remark 10. Condensation established by (52) is a priori a generalized BEC. To prove the conventional type I condensation in the ground state, or the type II condensation, one needs detailed information about distribution of the level spacings distances in the neighbourhood of the lower edge of the single particle spectrum, see Remarks 5, 6.

Remark 11. In contrast to the nonrandom case (see Remark 3) the suppression of the IDS in the neighbourhood of the lower edge $E_0=0$ of the spectrum of the random Schrödinger operator (the Lifshitz tail (47)) makes the BEC (52) possible even for $d=1,2$. It is true for any non-zero concentration of the impurities τ, (47).

Remark 12. The one-dimensional case is instructive, since it makes evident another difference in the nature of the BEC in random and nonrandom cases. It is known that in general the whole spectrum of the one-dimensional random Schrödinger operators with weakly correlated random potential is pure point and the eigenfunctions are exponentially localized (see, e.g., [7, Chapter VI.15], and [20] for the Poisson potential of Example 1 for $d = 1$). If the lower edge of the spectrum is zero (as for the potential of Example 1), the perfect bosons condense in the 'package' (52) of localized states in the vicinity of $E = 0$. This is in contrast to the free perfect bosons: for $d > 2$ they have the (generalized) BEC into extended states (29).

In fact the Lifshitz tail for the IDS near edges of the spectrum is a fairly generic phenomenon for random Schrödinger operators [7].

Example 2. Take one-dimensional random potential corresponding to δ-localized impurities with amplitude $a > 0$:

$$v^{\omega}(x) := \int_{\mathbb{R}^d} \mu_{\tau}^{\omega}(\mathrm{d}y) a\delta(x-y) = a\sum_{j}\delta\bigl(x - y_j^{\omega}\bigr), \tag{65}$$

where $\{y_j^{\omega}\}_j \subset \mathbb{R}^1$ are the atoms of the random Poisson measure (44). This random potential is not included into the case (45), but behaviour of the IDS at edge of the spectrum is known [21,19], [7, Chapter III.6]:

$$\ln\mathcal{N}(E) = -\frac{\pi\tau}{\sqrt{2E}}\bigl(1 + \mathrm{O}\bigl(E^{1/2}\bigr)\bigr), \tag{66}$$

as $E \downarrow 0$. Hence, again the corresponding critical density $\rho_c(\beta) < \infty$, and one gets a generalized BEC proved in Theorem 4.1.

Notice that the Lifshitz tail (66) does not depend on the finite amplitude $a > 0$ in (65). In the limit $a \to +\infty$ the IDS is known explicitly for all values of E:

$$\mathcal{N}(E) = \tau\frac{\mathrm{e}^{-\pi\tau/\sqrt{2E}}}{1 - \mathrm{e}^{-\pi\tau/\sqrt{2E}}}. \tag{67}$$

The BEC of the one-dimensional perfect Bose-gas in the random potential corresponding to (67) was studied for the first time in [3].

5. Off-Diagonal Long-Range Order

Above, our criterion of the BEC was based on boundedeness of $\rho_c(\beta)$. It gives almost no information on the nature of condensation, see Remarks 10–12. More insight into this question may give so-called local observables, in particular, the *one-body reduced density matrix* [22,23].

For the free Bose-gas (3), (15), it has the form (see, e.g., [24]):

$$\rho_L(\beta,\mu;x,y) = \sum_{k\geqslant 1}\frac{1}{\mathrm{e}^{\beta(E_k(L)-\mu)} - 1}\overline{\psi_k^D(x)}\psi_k^D(y). \tag{68}$$

Its diagonal part is the *local* particle number density

$$\rho_L(\beta,\mu;x) := \rho_L(\beta,\mu;x,x) = \sum_{k\geqslant 1}\frac{1}{\mathrm{e}^{\beta(E_k(L)-\mu)} - 1}\bigl|\psi_k^D(x)\bigr|^2. \tag{69}$$

Then the *space average* density

$$\rho_L(\beta,\mu) := \frac{1}{|\Lambda_L|}\int_{\Lambda_L}\mathrm{d}x\rho_L(\beta,\mu;x) \tag{70}$$

coincides with (19).

Remark 13. The mathematical result, which makes a contact of the one-body reduced density matrix with BEC in the free Bose-gas is due to [24]. Take the origin $x = 0$ as the point of dilation of Λ_1. In this case the BEC is of the type I only in the ground state, see Remark 5. (For example, let Λ_1 be a cube in $\mathbb{R}^3$.) Then the limiting one-body reduced density matrix $\rho(\beta,\mu(\beta,\rho);x,y)=\lim_{L\to\infty}\rho_L(\beta,\mu(\beta,\rho);x,y)$ has the form:

$$\rho\big(\beta,\mu(\beta,\rho);x,y\big)=\begin{cases}\displaystyle\sum_{s=1}^{\infty}(2\pi\beta s)^{-d/2}\,\mathrm{e}^{s\beta\mu(\beta,\rho)-\|x-y\|^2/2\beta s}, & \rho<\rho_c(\beta),\\[2ex] \rho_0(\beta,\rho)\big|\psi^D_{k=1,L=1}(0)\big|^2+\displaystyle\sum_{s=1}^{\infty}(2\pi\beta s)^{-d/2}\,\mathrm{e}^{-\|x-y\|^2/2\beta s}, & \rho\geqslant\rho_c(\beta),\end{cases}\tag{71}$$

where $\rho_0(\beta,\rho)=\rho-\rho_c(\beta)$ is the condensate density and $\psi^D_{k=1,L=1}(0)$ is the ground state eigenfunction (3) in domain Λ_1 evaluated at the point of dilation $x=0$. The limit

$$\mathit{ODLRO}(\beta,\rho):=\lim_{\|x-y\|\to\infty}\rho\big(\beta,\mu(\beta,\rho);x,y\big)\tag{72}$$

is called the Off-Diagonal Long-Range Order (ODLRO).

Since the both sums in (71) decay exponentially as $\|x-y\|\to\infty$, one gets nontrivial ODLRO only for $\rho>\rho_c(\beta)$. For example, when $\rho<\rho_c(\beta)$, i.e., $\mu(\beta,\rho)<0$, we get the following estimate:

$$\rho(\beta,\mu<0;x,y)\leqslant C_d\big(\|x-y\|,\beta,\mu\big)\,\mathrm{e}^{-\sqrt{2|\mu|}\|x-y\|}\big(1+\mathrm{O}\big(\|x-y\|^{-1}\big)\big),\tag{73}$$

with

$$C_d\big(\|x-y\|,\beta,\mu\big)=\frac{1}{(2\pi)^{(d-1)/2}\beta}\,\frac{(2|\mu|)^{(d-3)/4}}{\|x-y\|^{(d-1)/2}}.\tag{74}$$

Notice that non-zero value of the ODLRO in the case of the ground state BEC depends on the profile of the corresponding eigenfunction. That is why even non-zero BEC of the *type I* does not guarantee the ODLRO and casts a doubt on the common belief that the condensation is governed by asymptotic behaviour of the one-body reduced density matrix, see discussion in [24,25]. The ODLRO in the *anisotropic* prisms, i.e., in the case of the BEC of *type II or III*, is more complicated. It is related to the second critical density $\rho_m(\beta)\geqslant\rho_c(\beta)$, see [10] and Remark 5.

Remark 14. Generalization of the one-body reduced density matrix to the perfect Bose-gas in a random potential is straightforward:

$$\rho^{\omega}_L(\beta,\mu;x,y)=\sum_{k\geqslant1}\frac{1}{\mathrm{e}^{\beta(E^{\omega}_k(L)-\mu)}-1}\,\overline{\psi^{D,\omega}_{k,L}(x)}\,\psi^{D,\omega}_{k,L}(y),\tag{75}$$

where $\{\psi^{D,\omega}_{k,L}\}_{k\geqslant1}$ are eigenfunctions of operator (35). Then as in (69) the local particle density has the form:

$$\rho^{\omega}_L(\beta,\mu;x):=\sum_{k\geqslant1}\frac{1}{\mathrm{e}^{\beta(E^{\omega}_k(L)-\mu)}-1}\big|\psi^{D,\omega}_{k,L}(x)\big|^2.\tag{76}$$

Notice that the position dependence of (75) and (76) make them *non-self-averaging*. In contrast the space average density

$$\rho^{\omega}_L(\beta,\mu):=\frac{1}{|\Lambda_L|}\int\limits_{\Lambda_L}\mathrm{d}x\,\rho^{\omega}_L(\beta,\mu;x)\tag{77}$$

is self-averaging, since it simply coincides with (38).

This motivates us to introduce the *space average* one-body reduced density matrix

$$\tilde\rho^{\omega}_L(\beta,\mu;x,y):=\frac{1}{|\Lambda_L|}\int\limits_{\Lambda_L}\mathrm{d}a\,\rho^{\omega}_L(\beta,\mu;x+a,y+a)\tag{78}$$

as a measure of ODLRO for the Bose-gas in an ergodic random potential. It is assumed in (78) that the integrand is extended by zero if spacial arguments leave domain Λ_L.

Lemma 5.1. *Let the random potential verify the conditions of Proposition* 3.4. *Then for $\mathbb{P}$-almost all $\omega \in \Omega$ the limit*

$$\text{a.s.-}\lim_{L\to\infty} \tilde{\rho}_L^{\omega}(\beta,\mu;x,y) = \tilde{\rho}(\beta,\mu;x-y), \tag{79}$$

exists for any $\beta > 0$, $\mu < 0$, and for any fixed $x, y \in \mathbb{R}^d$, and it is nonrandom.

Proof. It essentially follows the idea, which goes back to [26]. Since $\mu < 0$, by virtue of definitions (75) and (78) we obtain

$$\tilde{\rho}_L^{\omega}(\beta,\mu;x,y) = \sum_{s=1}^{\infty} \mathrm{e}^{s\beta\mu} \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}a \big(\mathrm{e}^{-s\beta h_L^{\omega}}\big)(x+a, y+a). \tag{80}$$

Using the Feynman–Kac formula (see, e.g., [5]) we get the representation

$$\begin{aligned} &\frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}a \big(\mathrm{e}^{-s\beta h_L^{\omega}}\big)(x+a, y+a) \\ &\quad = \frac{1}{(2\pi s\beta)^{d/2}} \mathrm{e}^{-\|x-y\|^2/2s\beta} \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}a \int_{\Omega_{0,y-x}^{s\beta}} \mathrm{d}\nu^{s\beta}\big(\xi(\cdot)\big) \mathrm{e}^{-\int_0^{s\beta} \mathrm{d}t v^{\omega}(\xi(t)+x+a)} \chi_{\Lambda_L, s\beta}\big(\xi(\cdot)+x+a\big). \end{aligned} \tag{81}$$

Here we denote by $\mathrm{d}\nu^T(\cdot)$ the normalized Wiener measure on the set of trajectories

$$\Omega_{0,z}^T := \big\{\xi\colon \xi(0) = 0, \quad \xi(T) = z\big\}$$

and by $\chi_{\Lambda_L,T}(\eta(\cdot))$ the characteristic function of the set of Wiener trajectories $\{\eta([0,T]) \subset \Lambda_L\}$. Since for each $\xi(\cdot) \in \Omega_{0,y-x}^{s\beta}$ we have

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}a \chi_{\Lambda_L,s\beta}\big(\xi(\cdot)+x+a\big) = 1, \tag{82}$$

the non-negativity and ergodicity of random potential v^{ω} imply the limits

$$\begin{aligned} &\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}a \int_{\Omega_{0,y-x}^{s\beta}} \mathrm{d}\nu^{s\beta}\big(\xi(\cdot)\big) \mathrm{e}^{-\int_0^{s\beta} \mathrm{d}t v^{\omega}(\xi(t)+x+a)} \chi_{\Lambda_L,s\beta}\big(\xi(\cdot)+x+a\big) \\ &\quad = \lim_{L\to\infty} \int_{\Omega_{0,y-x}^{s\beta}} \mathrm{d}\nu^{s\beta}\big(\xi(\cdot)\big) \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}a\, \mathrm{e}^{-\int_0^{s\beta} \mathrm{d}t v^{\omega}(\xi(t)+x+a)} \\ &\quad = \int_{\Omega_{0,y-x}^{s\beta}} \mathrm{d}\nu^{s\beta}\big(\xi(\cdot)\big) \mathbb{E}\big(\mathrm{e}^{-\int_0^{s\beta} \mathrm{d}t v^{\omega}(\xi(t))}\big). \end{aligned} \tag{83}$$

In view of (80) this proves the lemma and gives for the right-hand side of (79) the explicit representation:

$$\tilde{\rho}(\beta,\mu;x-y) = \sum_{s=1}^{\infty} \frac{1}{(2\pi s\beta)^{d/2}} \mathrm{e}^{s\beta\mu - \|x-y\|^2/2s\beta} \int_{\Omega_{0,y-x}^{s\beta}} \mathrm{d}\nu^{s\beta}\big(\xi(\cdot)\big) \mathbb{E}\big(\mathrm{e}^{-\int_0^{s\beta} \mathrm{d}t v^{\omega}(\xi(t))}\big), \tag{84}$$

for $\mu < 0$. $\square$

Corollary 5.2. *For $\mu < 0$ the space average one-body reduced density matrix $\tilde{\rho}(\beta,\mu;x-y)$ of the perfect Bose-gas in a non-negative ergodic random potential verifies the inequalities*:

$$\rho(\beta,\mu-\tau\tilde{u};x-y) \leqslant \tilde{\rho}(\beta,\mu;x-y) \leqslant \rho(\beta,\mu;x-y),$$

where $\rho(\beta,\mu;x-y)$ is the one-body reduced density matrix of the free Bose-gas.

Proof. Indeed, by virtue of (71) and (84) for non-negative random potentials v^{ω} we get

$$\tilde{\rho}(\beta,\mu;x-y) \leqslant \rho(\beta,\mu;x-y). \tag{85}$$

On the other hand, by the Jensen inequality and (45) one gets

$$\mathbb{E}\big(\mathrm{e}^{-\int_0^{s\beta}\mathrm{d}t\, v^{\omega}(\xi(t))}\big) \geqslant \mathrm{e}^{-\int_0^{s\beta}\mathrm{d}t\,\mathbb{E}v^{\omega}(\xi(t))} = \mathrm{e}^{-s\beta\tau\tilde{u}}, \tag{86}$$

where we put $\tilde{u} := \int_{\mathbb{R}^1}\mathrm{d}x\, u(x)$. This implies the estimate of (84) from below:

$$\tilde{\rho}(\beta,\mu;x-y) \geqslant \rho(\beta,\mu-\tau\tilde{u};x-y). \quad \square \tag{87}$$

Since the BEC exists in the presence of random potential even for *dimension one*, below we reduce our study of the asymptotic behaviour of the one-body reduced density matrix only to this case. To make the upper bound estimate more precise than (85), we consider the case of the Poisson potential of Example 1 with $\operatorname{supp} u(x) = [-\delta/2, \delta/2]$.

Theorem 5.3. *Let $d = 1$ and let $\tilde{\gamma} := 1 - \mathrm{e}^{-\tilde{u}}$. Then*

$$\tilde{\rho}(\beta,\mu;x-y) \leqslant \rho(\beta,\mu;x-y)\,\mathrm{e}^{-\tau\tilde{\gamma}(|x-y|-\delta)} \tag{88}$$

$\mathbb{P}$-almost sure for any $\mu < 0$.

Proof. Choose $x < y$ and let $\chi_{[x,y]}$ be characteristic function of the interval $[x, y]$. We denote by $v^{\omega}_{x,y} := v^{\omega}\chi_{[x,y]}$, the restriction of the random potential to this interval. Then non-negativity of $u(x)$ implies (cf. (84))

$$\mathbb{E}\big(\mathrm{e}^{-\int_0^{s\beta}\mathrm{d}\tau\, v^{\omega}(\xi(\tau)+x)}\big) \leqslant \mathbb{E}\big(\mathrm{e}^{-\int_0^{s\beta}\mathrm{d}\tau\, v^{\omega}_{x,y}(\xi(\tau)+x)}\big). \tag{89}$$

Let $n^{\omega}(x+\delta/2, y-\delta/2) := \operatorname{card}\{\{y^{\omega}_j\}_j \subset [x+\delta/2, y-\delta/2]\}$ be (*random*) number of Poisson points in the interval $[x+\delta/2, y-\delta/2]$ for configuration $\omega \in \Omega$. Then

$$\mathbb{E}\big(\mathrm{e}^{-\int_0^{s\beta}\mathrm{d}\tau\, v^{\omega}_{x,y}(\xi(\tau)+x)}\big) \leqslant \mathbb{E}\big(\mathrm{e}^{-n^{\omega}(x+\delta/2,y-\delta/2)\tilde{u}}\big). \tag{90}$$

By virtue of the Poisson distribution (44) we get

$$\mathbb{E}\big(\mathrm{e}^{-n^{\omega}(x+\delta/2,y-\delta/2)\tilde{u}}\big) \leqslant \sum_{n=0}^{\infty}\frac{\{\tau(|x-y|-\delta)\}^n}{n!}\,\mathrm{e}^{-\tau(|x-y|-\delta)}\,\mathrm{e}^{-n\tilde{u}} = \mathrm{e}^{-\tau\tilde{\gamma}(|x-y|-\delta)}. \tag{91}$$

Therefore, by (84) and estimates (89)–(91) we obtain:

$$\tilde{\rho}(\beta,\mu;x-y) \leqslant \mathrm{e}^{-\tau\tilde{\gamma}(|x-y|-\delta)}\sum_{s=1}^{\infty}\frac{1}{(2\pi s\beta)^{1/2}}\,\mathrm{e}^{s\beta\mu-|x-y|^2/2s\beta}\int\limits_{\Omega^{s\beta}_{0,y-x}}\mathrm{d}\nu^{s\beta}\big(\xi(\tau)\big). \tag{92}$$

This gives (88), because the sum in the r.h.s. coincides with the one-body reduced density matrix for the free Bose-gas in one dimension, see (71) for $\mu < 0$, $d = 1$, and the Wiener measures are normalized. $\square$

Corollary 5.4. *Inequalities* (73) *and* (88) *show that the presence of random potential enhances the exponential decay of the one-body reduced density matrix by a supplementary exponential factor with the exponential proportional to the impurity concentration τ.*

Remark 15. Notice that for concentration $\tau \downarrow 0$ our lower (87) and upper (88) estimates of the space average one-body reduced density matrix (78) give a plausible result:

$$\lim_{\tau\downarrow 0}\tilde{\rho}(\beta,\mu;x-y) = \rho(\beta,\mu;x-y). \tag{93}$$

This convergence to the one-body reduced density matrix of the free Bose-gas bolsters our definition (78).

6. Conclusion

The present paper is essentially initiated by mathematical results about the random Schrödinger operator and about the BEC in the perfect Bose-gas. We show that the *self-averaging* of the pressure and of the mean particle density in the *thermodynamic limit* allows us to make rigorous the corresponding physical arguments (see, for example, [3]) concerning the BEC in random potentials. In fact the nonrandomness of the IDS of the random one-particle Schrödinger operator in this limit offers the means for rigorous analysis of the gas without interaction. On the other hand the *self-averaging* is a quite general property of random systems to give a basis for reexamination of some physical results in this direction for *interacting* boson models in random external potential, see, e.g., the recent paper [2].

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