

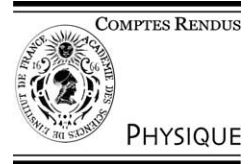


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Bose–Einstein condensates: recent advances in collective effects/Avancées récentes sur les effets collectifs dans les condensats de Bose–Einstein

Kinetic theory and Bose–Einstein condensation

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Abstract

In this article we discuss the role played by kinetic theory in describing the non-equilibrium dynamics of dilute systems of weakly interacting bosons. We illustrate how a simple kinetic equation for the time evolution of the spectral particle density can be derived from the spatially homogeneous Gross–Pitaevskii equation. This kinetic equation agrees with the usual Boltzmann–Nordheim equation of quantum kinetic theory in the long wavelength limit where the occupation numbers are expected to be large. The stationary solutions of the Gross–Pitaevskii kinetic equation are described. These include both thermodynamic equilibrium spectra and finite flux Kolmogorov–Zakharov spectra. These latter spectra are intrinsically nonequilibrium states and are expected to be relevant in the transfer of particles to low momenta in the initial stage of the condensation process. This is illustrated by some computations of a solution of the kinetic equation beginning with initial conditions far from equilibrium. The solution generates a flux of particles from large to small momenta which results in a singularity at zero momentum within finite time. We interpret this singularity as incipient condensate formation. We then present some numerical results on the post-singularity dynamics and the approach to equilibrium. Contrary to our original expectations we do not observe the Kolmogorov–Zakharov spectrum during the period of condensate growth. In the closing sections we address the issue of the connection between the Gross–Pitaevskii and Boltzmann–Nordheim kinetic equations. We argue that the two equations have differing regimes of applicability in momentum space, matching in an intermediate range. We make some suggestions of how this matching can be modeled in practice. *To cite this article: C. Connaughton, Y. Pomeau, C. R. Physique 5 (2004).*

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Résumé

La théorie cinétique et les condensats de Bose–Einstein. Dans cet article nous discutons le rôle joué par la théorie cinétique dans la dynamique hors équilibre d'un gaz de Bose dilué avec des interactions faibles. Nous montrons comment arriver à une équation cinétique assez simple pour l'évolution temporelle de la densité spectrale des particules en commençant avec l'équation de Gross–Pitaevskii pour le cas spatialement homogène. Cette équation cinétique est en accord avec l'équation de Boltzmann–Nordheim de la théorie cinétique quantique dans la limite où les longueurs d'onde sont grandes et on prévoit que les nombres de particules soient grands aussi. Les spectres stationnaires de l'équation cinétique de Gross–Pitaevskii sont décrits. Ils comprennent des spectres d'équilibre thermodynamique et des spectres qui supportent des flux finis dit spectres de Kolmogorov–Zakharov. Ces derniers spectres sont essentiellement des objets hors-équilibre et on suppose qu'ils sont importants dans l'apport des particules au moments faibles dans les premières étapes du processus de condensation. Ce point est illustré avec des calculs numériques de l'équation cinétique qui commencent avec des données qui sont loins de l'équilibre. La solution crée un flux de particules des moments élevés aux petits pour qu'une singularité se déclenche en temps fini au moment nul. Nous interprétons cette singularité comme le commencement du condensat. Ensuite quelques calculs numériques de la dynamique après le temps de singularité et l'approche à l'équilibre sont présentés. Contre nos intuitions le spectre de Kolmogorov–Zakharov n'est pas observé pendant la croissance du condensat. Enfin nous discutons les liens entre la théorie cinétique de Gross–Pitaevskii et celle de Boltzmann–Nordheim. Nous proposons que les deux équations ont des domaines d'applicabilité différents dans l'espace des

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moments et qu'il faut faire un raccordement dans un domaine intermédiaire. Nous présentons nos idées pour construire un modèle pratique qui pourrait faire cela. *Pour citer cet article : C. Connaughton, Y. Pomeau, C. R. Physique 5 (2004).*
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1. Introduction

The phenomenon of Bose–Einstein condensation occurs when a large number, N_c , of particles of a Bose gas enter the same single particle quantum state such that the fraction N_c/N remains finite in the thermodynamic limit, $N \rightarrow \infty$, N/V finite. In the case of a uniform gas, the particles condense in the single particle state having zero momentum. Predicted by Bose and Einstein in 1924 [1,2], this phenomenon has generated intense interest in the past decade following the extensive experimental progress which has been made in condensing atomic gases since the initial experiments in 1995 [3,4]. See [5] for a review.

The theoretical description of BEC relies either on kinetic theory, the so-called Boltzmann–Nordheim kinetic equation or on the Gross–Pitaevskii equation, an equation for the time–space dependant amplitude of the condensate.

The Boltzmann–Nordheim (B–N) kinetic equation,

$$\frac{\partial n_{\mathbf{p}_1}}{\partial t} = \int W_{\mathbf{p}_1 \mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3} [n_{\mathbf{k}_2} n_{\mathbf{k}_3} (1 + n_{\mathbf{p}_1})(1 + n_{\mathbf{k}_1}) - n_{\mathbf{p}_1} n_{\mathbf{k}_1} (1 + n_{\mathbf{k}_2})(1 + n_{\mathbf{k}_3})] \\ \times \delta(\mathbf{p}_1 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \delta(p_1^2 + k_1^2 - k_2^2 - k_3^2) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \quad (1)$$

was derived by Nordheim [6] for the time evolution of the spectral density, $n_{\mathbf{k}}$, of a statistically homogeneous ensemble of interacting quantum particles soon after the development of quantum statistical mechanics in the early twentieth century. In the above equation

$$W_{\mathbf{p}_1 \mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3} = \frac{1}{m\hbar^3} (|f_{\mathbf{p}_1 - \mathbf{k}_1}|^2 + |f_{\mathbf{k}_1 - \mathbf{p}_1}|^2), \quad (2)$$

where f is the scattering length, m is the particle mass and \hbar is Planck's constant. In this article boldface type represents a vector in d -dimensional space. We shall typically use \mathbf{k} to represent a dummy variable which is integrated over and \mathbf{p} to represent one which is not. The distribution is normalised such that the particle density is given by

$$N/V = \frac{1}{\hbar^d} \int d\mathbf{k}_1 n_{\mathbf{k}_1}. \quad (3)$$

On the other hand, the Gross–Pitaevskii (G–P) equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\mathbf{x})\Psi + g|\Psi|^2\Psi, \quad (4)$$

first derived in the 1960s [7,8], is a mean equation for the space–time dependent amplitude of the condensate. The nonlinear interaction coefficient, g , is

$$g = \frac{4\pi\hbar^2 f}{m}. \quad (5)$$

Here δ is the effective scattering length of the two-body potential in the s -wave approximation. This approximation, inherent in the G–P equation, is valid provided that f is much smaller than the average distance between atoms, that is to say the gas is dilute. $V(\mathbf{x})$ represents the external potential – the trap in the case of real experiments. For an extensive review of the role of the G–P equation in the theory of BEC see [9]. In the absence of an external potential, we are considering a spatially homogeneous gas and the Gross–Pitaevskii equation becomes the nonlinear Schrödinger equation.

The outline of this article is as follows. In Section 2 we derive a kinetic theory for the G–P equation which describes the time evolution of the average spectral particle density. We refer to the resulting kinetic equation, Eq. (41), as the G–P kinetic equation to distinguish it from the B–N equation. This kinetic equation implies that $\langle \Psi \rangle = 0$, meaning it describes the regime where only short waves are relevant. It is the same kinetic equation as that used for a long time to describe the statistics of optical turbulence, see for example [10], and other physical systems governed by the nonlinear Schrödinger equation. Following this derivation, we address the issue of stationary solutions to the kinetic equation in Section 3. We particularly emphasise the

non-equilibrium stationary states which carry fluxes of energy and particles through momentum space. The spectra associated with these non-equilibrium stationary states are called Kolmogorov–Zakharov (K–Z) spectra. They are an analogue of the Kolmogorov–Obukhov spectrum of hydrodynamic turbulence which is responsible for energy transport in far from equilibrium fluids. Of particular interest in the G–P kinetic theory is the inverse cascade which is responsible for the transfer of particles from high to low momenta required for BEC to occur in a system which is initially away from thermodynamic equilibrium. In Section 4 we look at some time dependent solutions of the kinetic theory. Using self-similarity arguments and numerical simulations we show that the inverse cascade can generate a finite flux of particles to the zero momentum state starting from a continuous particle distribution within finite time. This can be seen in some sense as corresponding to the onset of condensation proper. In Section 5 we present some preliminary numerical computations of a system of kinetic equations which describe the post- t^* evolution. Surprisingly we do not see the K–Z scaling in the post- t^* regime but rather a rapid approach to a spectrum very close to thermodynamic equilibrium. In the penultimate section we address the subtle question of the connection between the B–N and G–P kinetic theories. We argue that the B–N kinetic theory should be relevant for large momenta and the G–P equation relevant for the low momenta where condensation occurs. The two should match in some intermediate range of scales. We argue that this matching should be realised by thinking of the K–Z spectrum of the B–N equation with its flux of particles as a source of particles for the G–P equation. We outline some arguments of how this might be achievable in practice by adding a source term to the G–P equation which models the effect of the particle cascade from larger momenta. We close with some comments about the unanswered questions which we feel are important to a fuller understanding of the relationship between kinetic theories and BEC.

2. Kinetic theory for the Gross–Pitaevskii equation

2.1. Hamiltonian description of G–P

Let us consider now the spatially uniform case where the external potential, $V(\mathbf{x})$, is absent. The G–P equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + g|\Psi|^2 \Psi, \quad (6)$$

follows from the variational principle

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{\delta H}{\delta \Psi^*}, \quad (7)$$

where the Hamiltonian,

$$H = \int d\mathbf{x} \left(\frac{\hbar^2}{2m} |\nabla \Psi|^2 + \frac{g}{2} |\Psi|^4 \right), \quad (8)$$

measures the total energy of the condensate. In addition to conserving, H , Eq. (6) also conserves the total particle number,

$$N = \int d\mathbf{x} |\Psi|^2. \quad (9)$$

2.2. Wave and condensate solutions

The trivial solution of (6) corresponds to a spatially uniform condensate,

$$\Psi(\mathbf{x}, t) = |\Psi_0| e^{-(ig/\hbar)|\Psi_0|^2 t + \alpha}, \quad (10)$$

where $|\Psi_0|$ is an arbitrary condensate amplitude and α is an arbitrary phase.

The other obvious type of solutions of (6) are the wave-packet solutions to the linearised equation. These should be relevant when $g \ll 1$ corresponding to the dilute, weakly interacting regime. These are best expressed through their Fourier transform

$$\Psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{d/2}} \int a_{\mathbf{k}}(0) e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega_{\mathbf{k}} t} d\mathbf{k}, \quad (11)$$

where d is the number of spatial dimensions and

$$\omega_{\mathbf{k}} = \frac{\hbar}{2m} k^2. \quad (12)$$

We note from (12) that linear waves on a Bose–Einstein condensate are highly dispersive. In the weakly interacting regime, these waves and their interactions play an important part in the dynamics which will be examined in more detail in this article.

2.3. Fourier space description

Since we are dealing with waves, it is convenient to work in Fourier space. We adopt the following notation for the Fourier transform pair

$$\begin{aligned} A_{\mathbf{k}}(t) &= \frac{1}{(2\pi)^{d/2}} \int \Psi(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}, \\ \Psi(\mathbf{x}, t) &= \frac{1}{(2\pi)^{d/2}} \int A_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}. \end{aligned} \quad (13)$$

In Fourier transformed variables, the Hamiltonian is

$$H = \int \omega_{\mathbf{k}} A_{\mathbf{k}} A_{\mathbf{k}}^* d\mathbf{k} + \frac{1}{2} \frac{g}{(2\pi)^d} \int A_{\mathbf{k}} A_{\mathbf{k}_1} A_{\mathbf{k}_2}^* A_{\mathbf{k}_3}^* \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \quad (14)$$

and Hamilton's equations, (7), give

$$\frac{\partial A_{\mathbf{p}}}{\partial t} + i\omega_{\mathbf{p}} A_{\mathbf{p}} = -\frac{ig}{(2\pi)^d} \int A_{\mathbf{k}_1}^* A_{\mathbf{k}_2} A_{\mathbf{k}_3} \delta(\mathbf{p} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \quad (15)$$

with a corresponding complex conjugated equation for $A_{\mathbf{p}}^*$. For notational reasons it is expedient to work in the interaction representation adopting the variables,

$$a_{\mathbf{p}}(t) = A_{\mathbf{p}}(t) e^{i\omega_{\mathbf{p}} t}. \quad (16)$$

The equation of motion for $a_{\mathbf{p}}(t)$ is

$$\frac{\partial a_{\mathbf{p}}}{\partial t} = -\frac{ig\varepsilon^2}{(2\pi)^d} \int e^{i\Omega_{pk_1, k_2 k_3} t} a_{\mathbf{k}_1}^* a_{\mathbf{k}_2} a_{\mathbf{k}_3} \delta_{pk_1, k_2 k_3} d\mathbf{k}_{123}. \quad (17)$$

Here we have introduced the following notation which we shall continue to use throughout the article to compact the formulae:

$$\begin{aligned} \Omega_{pk_1, k_2 k_3} &= \omega_{\mathbf{p}} + \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3}, \\ \delta_{pk_1, k_2 k_3} &= \delta(\mathbf{p} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3), \\ d\mathbf{k}_{123} &= d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3. \end{aligned}$$

In order to avoid doing a perturbation expansion in the dimensional parameter, g , we have introduced the formal dimensionless parameter, ε^2 , by making the replacement $A_{\mathbf{p}} \rightarrow \varepsilon A_{\mathbf{p}}$. The statement that ε is small is the statement that the nonlinear contribution to the energy is small with respect to the linear one.

2.4. Essentials of a statistical description

Since the full solution of (17) contains much redundant information about the phases and fast oscillations of almost linear waves, it is more sensible to look for a statistical description of the system. Given a random ensemble of initial conditions for the $A_{\mathbf{k}}$'s, can we compute the probability distribution of $A_{\mathbf{k}}$ at later times? For simplicity, we shall assume that the initial distribution is Gaussian although if we were willing to do a bit more work, this assumption could be weakened considerably. We are interested in the behaviour of the moments of $A_{\mathbf{k}}$ which we define as follows

$$\begin{aligned} M_2(\mathbf{p}_1; \mathbf{p}_2) &\equiv \langle A_{\mathbf{p}_1} A_{\mathbf{p}_2}^* \rangle, \\ M_4(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4) &\equiv \langle A_{\mathbf{p}_1} A_{\mathbf{p}_2} A_{\mathbf{p}_3}^* A_{\mathbf{p}_4}^* \rangle, \\ M_6(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3; \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6) &\equiv \langle A_{\mathbf{p}_1} A_{\mathbf{p}_2} A_{\mathbf{p}_3} A_{\mathbf{p}_4}^* A_{\mathbf{p}_5}^* A_{\mathbf{p}_6}^* \rangle \end{aligned}$$

and so on. We remark that the odd moments are zero because, in effect, we average over phases. We are particularly interested in the second moment, $M_2(\mathbf{p}_1, \mathbf{p}_2)$, since it is simply related to the Fourier transform of the particle distribution of the condensate.

It is a simple matter to derive the equations of motion for the moments from (17). We shall need the first two. By integrating these equations we obtain the following formal expressions for the moments.

$$M_2(\mathbf{p}_1; \mathbf{p}_2)(t) = \frac{ig\varepsilon^2}{(2\pi)^d} \int \left[\int_0^t e^{i\Omega_{k_2 k_3, p_2 k_1} \tau} M_4(\mathbf{p}_1, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)(\tau) d\tau \right] \delta_{p_2 k_1, k_2 k_3} d\mathbf{k}_{123} \\ - \frac{ig\varepsilon^2}{(2\pi)^d} \int \left[\int_0^t e^{i\Omega_{p_1 k_1, k_2 k_3} \tau} M_4(\mathbf{k}_2, \mathbf{k}_3, \mathbf{p}_2, \mathbf{k}_1)(\tau) d\tau \right] \delta_{p_1 k_1, k_2 k_3} d\mathbf{k}_{123}, \quad (18)$$

$$M_4(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4)(t) = \frac{ig\varepsilon^2}{(2\pi)^d} \int \left[\int_0^t e^{i\Omega_{k_2 k_3, p_3 k_1} \tau} M_6(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_1, \mathbf{p}_4, \mathbf{k}_2, \mathbf{k}_3)(\tau) d\tau \right] \delta_{p_3 k_1, k_2 k_3} d\mathbf{k}_{123} \\ + \frac{ig\varepsilon^2}{(2\pi)^d} \int \left[\int_0^t e^{i\Omega_{k_2 k_3, p_4 k_1} \tau} M_6(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_1, \mathbf{p}_3, \mathbf{k}_2, \mathbf{k}_3)(\tau) d\tau \right] \delta_{p_4 k_1, k_2 k_3} d\mathbf{k}_{123} \\ - \frac{ig\varepsilon^2}{(2\pi)^d} \int \left[\int_0^t e^{i\Omega_{p_1 k_1, k_2 k_3} \tau} M_6(\mathbf{p}_2, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_1, \mathbf{p}_3, \mathbf{p}_4)(\tau) d\tau \right] \delta_{p_1 k_1, k_2 k_3} d\mathbf{k}_{123} \\ - \frac{ig\varepsilon^2}{(2\pi)^d} \int \left[\int_0^t e^{i\Omega_{p_2 k_1, k_2 k_3} \tau} M_6(\mathbf{p}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_1, \mathbf{p}_3, \mathbf{p}_4)(\tau) d\tau \right] \delta_{p_2 k_1, k_2 k_3} d\mathbf{k}_{123}. \quad (19)$$

Note that these expressions form an unclosed hierarchy of equations with the expression for each moment involving the moment of next order up.

2.5. Derivation of the kinetic equation from the dynamical equation

We shall look for an asymptotic expansion, or each moment as a power series in ε^2 ,

$$M_n(\mathbf{k}_1, \dots, \mathbf{k}_n) = \sum_{m=0}^{\infty} \varepsilon^{2m} M_n^{(2m)}(\mathbf{k}_1, \dots, \mathbf{k}_n), \quad (20)$$

for each n . We are particularly interested in $M_2(\mathbf{p}_1; \mathbf{p}_2)$:

$$M_2(\mathbf{p}_1; \mathbf{p}_2) = M_2^{(0)}(\mathbf{p}_1, \mathbf{p}_2) + \varepsilon^2 M_2^{(2)}(\mathbf{p}_1, \mathbf{p}_2) + \varepsilon^4 M_2^{(4)}(\mathbf{p}_1, \mathbf{p}_2) + \dots \quad (21)$$

We require that this series be asymptotic, meaning that

$$\lim_{\varepsilon^2 \rightarrow 0} \frac{1}{\varepsilon^{2M}} \left[M_n(\mathbf{k}_1, \dots, \mathbf{k}_n) - \sum_{m=0}^M \varepsilon^{2m} R_n^{(m)}(\mathbf{k}_1, \dots, \mathbf{k}_n) \right] = 0, \quad (22)$$

for any finite M . We now substitute the expansion, (20), into the expressions, (18), (19) . . . , and solve iteratively, order by order in ε^2 .

To zeroth order, each moment is constant and given by it's initial value. By assuming Gaussian initial conditions, we can close the hierarchy of moment equations by expressing higher order moments as a sum of products of pairs of second order ones using a Wick decomposition. This allows us to get at the results without recourse to the more complex asymptotic closure arguments [11,12] required in the case of non-Gaussian initial conditions.

At higher orders, the perturbation series includes 'secular' terms which grow as powers of t . For example, for a given moment, we might have the following schematic behaviour at order ε^2 :

$$M_n^{(2)}(t) = \widehat{M}_n^{(2)}(t) + t \widetilde{M}_n^{(2)}(t), \quad (23)$$

where both $\widehat{M}_n^{(2)}(t)$ and $\widetilde{M}_n^{(2)}(t)$ are $O(1)$ as $t \rightarrow \infty$. The presence of the secular term $\widetilde{M}_n^{(2)}(t)$ means that the expansion (20) for $M_n(t)$ is no longer valid for times greater than $1/\varepsilon^2$. This is because at this time, the second term becomes of order one. Since we are interested in the long time behaviour of the system, this is a problem.

It is however, a problem with a well-known solution, known as the method of multiple scales. The secular terms can be removed in a consistent way by allowing the zeroth order solution to depend weakly on time. We postulate the existence 'multiple time-scales', $\mathcal{T}_{2m} = \varepsilon^{2m} t$, which we take to be independent, and replace

$$M_n^{(0)} \rightarrow M_n^{(0)}(t) = M_n^{(0)}(\mathcal{T}_2, \mathcal{T}_4, \dots). \quad (24)$$

Our iteration procedure will then generate additional terms which come from differentiating these weak time dependences. In our example we now find that at order ε^2 we would have

$$M_n^{(2)}(t) = \widehat{M}_n^{(2)}(t) + t \widetilde{M}_n^{(2)}(t) - t \frac{\partial M_n^{(0)}}{\partial T_2}. \quad (25)$$

We can now render the perturbation series valid for times larger than $1/\varepsilon^2$ by choosing the weak time dependence of $M_n^{(0)}$ to cancel the secular terms. Thus we arrive at the following consistency condition governing the large time behaviour

$$\frac{\partial M_n^{(0)}}{\partial T_2} = \widetilde{M}_n^{(2)}(t). \quad (26)$$

Of course, in our example, we have only removed secular terms which arise at order ε^2 . Further such terms will arise at order ε^4 which would invalidate the expansion for times larger than $1/\varepsilon^4$. Hence then multiple scale procedure must be applied order by order.

In the present article we shall focus on the second order moment where the multiple scale consistency condition yields the kinetic equation governing the long time dynamics of the spectral particle density. In this case we shall find that there are no secular terms at order ε^2 and the first nontrivial consistency condition arises at order ε^4 . We shall not go any further than this since the algebra involved quickly becomes difficult to manage.

Note that the assumption of statistical homogeneity implies

$$M_2(\mathbf{p}_1; \mathbf{p}_2) = m_2(\mathbf{p}_1) \delta(\mathbf{p}_1 - \mathbf{p}_2). \quad (27)$$

Here we have introduced $n(\mathbf{p}_1)$ to represent the spectral particle density. We now expand $M_2(\mathbf{p}_1; \mathbf{p}_2)$ in powers of ε^2 up to order ε^4 . Eqs. (18) and (19) then give

$$M_2^{(0)}(\mathbf{p}_1; \mathbf{p}_2)(t) = M_2^{(0)}(\mathbf{p}_1, \mathbf{p}_2)(0) \equiv n(\mathbf{p}_1) \delta_{p_1, p_2}, \quad (28)$$

$$\begin{aligned} M_2^{(2)}(\mathbf{p}_1; \mathbf{p}_2)(t) &= \frac{ig}{(2\pi)^d} \int \Delta[\Omega_{k_2 k_3, p_2 k_1}](t) M_4^{(0)}(\mathbf{p}_1, \mathbf{k}_1; \mathbf{k}_2, \mathbf{k}_3) \delta_{p_2 k_1, k_2 k_3} d\mathbf{k}_{123} \\ &\quad - \frac{ig}{(2\pi)^d} \int \Delta[\Omega_{p_1 k_1, k_2 k_3}](t) M_4^{(0)}(\mathbf{k}_2, \mathbf{k}_3; \mathbf{p}_2, \mathbf{k}_1) \delta_{p_1 k_1, k_2 k_3} d\mathbf{k}_{123}. \end{aligned} \quad (29)$$

$$\begin{aligned} M_2^{(4)}(\mathbf{p}_1; \mathbf{p}_2)(t) &= \frac{g^2}{(2\pi)^{2d}} \int E[\Omega_{p_1 k_2 k_3 k_4, p_2 k_1 k_5 k_6}; \Omega_{k_2 k_3, p_2 k_1}](t) M_6^{(0)}(\mathbf{k}_1, \mathbf{k}_5, \mathbf{k}_6; \mathbf{k}_4, \mathbf{k}_2, \mathbf{k}_3) \delta_{p_2 k_1, k_2 k_3} \delta_{p_1 k_4, k_5 k_6} d\mathbf{k}_{123456} \\ &\quad + \frac{g^2}{(2\pi)^{2d}} \int E[\Omega_{p_1 k_1 k_3 k_4, p_2 k_2 k_5 k_6}; \Omega_{p_1 k_1, k_2 k_3}](t) M_6^{(0)}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4; \mathbf{k}_1, \mathbf{k}_5, \mathbf{k}_6) \delta_{p_1 k_1, k_2 k_3} \delta_{p_2 k_4, k_5 k_6} d\mathbf{k}_{123456} \\ &\quad + \frac{g^2}{(2\pi)^{2d}} \int E[\Omega_{p_1 k_5 k_6, k_2 k_3 k_4}; \Omega_{p_1 k_1, k_2 k_3}](t) M_6^{(0)}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4; \mathbf{p}_1, \mathbf{k}_5, \mathbf{k}_6) \delta_{p_1 k_1, k_2 k_3} \delta_{k_1 k_4, k_5 k_6} d\mathbf{k}_{123456} \\ &\quad - \frac{g^2}{(2\pi)^{2d}} \int E[\Omega_{p_1 k_1 k_4, k_2 k_5 k_6}; \Omega_{p_1 k_1, k_2 k_3}](t) M_6^{(0)}(\mathbf{k}_2, \mathbf{k}_5, \mathbf{k}_6; \mathbf{p}_4, \mathbf{p}_2, \mathbf{k}_1) \delta_{p_1 k_1, k_2 k_3} \delta_{k_3 k_4, k_5 k_6} d\mathbf{k}_{123456} \\ &\quad - \frac{g^2}{(2\pi)^{2d}} \int E[\Omega_{p_1 k_1 k_4, k_3 k_5 k_6}; \Omega_{p_1 k_1, k_2 k_3}](t) M_6^{(0)}(\mathbf{k}_3, \mathbf{k}_5, \mathbf{k}_6; \mathbf{k}_4, \mathbf{p}_2, \mathbf{k}_1) \delta_{p_1 k_1, k_2 k_3} \delta_{k_2 k_4, k_5 k_6} d\mathbf{k}_{123456} \\ &\quad - \frac{g^2}{(2\pi)^{2d}} \int E[\Omega_{k_2 k_5 k_6, p_2 k_1 k_4}; \Omega_{k_2 k_3, p_2 k_1}](t) M_6^{(0)}(\mathbf{p}_1, \mathbf{k}_1, \mathbf{k}_4; \mathbf{k}_2, \mathbf{k}_5, \mathbf{k}_6) \delta_{p_2 k_1, k_2 k_3} \delta_{k_3 k_4, k_5 k_6} d\mathbf{k}_{123456} \\ &\quad - \frac{g^2}{(2\pi)^{2d}} \int E[\Omega_{k_3 k_5 k_6, p_2 k_1 k_4}; \Omega_{k_2 k_3, p_2 k_1}](t) M_6^{(0)}(\mathbf{p}_1, \mathbf{k}_1, \mathbf{k}_4; \mathbf{k}_3, \mathbf{k}_5, \mathbf{k}_6) \delta_{p_2 k_1, k_2 k_3} \delta_{k_2 k_4, k_5 k_6} d\mathbf{k}_{123456} \\ &\quad + \frac{g^2}{(2\pi)^{2d}} \int E[\Omega_{k_2 k_3 k_4, p_2 k_5 k_6}; \Omega_{k_2 k_3, p_2 k_1}](t) M_6^{(0)}(\mathbf{p}_1, \mathbf{k}_5, \mathbf{k}_6; \mathbf{k}_4, \mathbf{k}_2, \mathbf{k}_3) \delta_{p_2 k_1, k_2 k_3} \delta_{k_1 k_4, k_5 k_6} d\mathbf{k}_{123456}. \end{aligned} \quad (30)$$

Note that the time dependence of the above terms has been localised into the following two integrals

$$\Delta[x](t) = \int_0^t e^{ix\tau} d\tau, \quad (31)$$

$$E[x + y; y](t) = \int_0^t \Delta[x](\tau) e^{iy\tau} d\tau. \quad (32)$$

Their behaviour for large times will determine the secular terms arising in the expansion.

The really tedious step in the analysis is to split the $M_n^{(0)}$ appearing in these expressions into products of second order moments invoking the Gaussianity of the initial distribution. We shall only keep pairs of the form $\langle A_{\mathbf{p}_1} A_{\mathbf{p}_2}^* \rangle = n_{\mathbf{p}_1} \delta(\mathbf{p}_1 - \mathbf{p}_2)$ since pairs of the form $\langle A_{\mathbf{p}_1} A_{\mathbf{p}_2} \rangle$ or $\langle A_{\mathbf{p}_1}^* A_{\mathbf{p}_2}^* \rangle$ are zero because of averaging over fast oscillations. Bearing this in mind we write

$$M_4^{(0)}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4) = n_{\mathbf{p}_1} n_{\mathbf{p}_2} (\delta_{p_1, p_3} \delta_{p_2, p_4} + \delta_{p_1, p_4} \delta_{p_2, p_3}), \quad (33)$$

$$M_6(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3; \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6) = n_{\mathbf{p}_1} n_{\mathbf{p}_2} n_{\mathbf{p}_3} (\delta_{p_1, p_4} \delta_{p_2, p_5} \delta_{p_3, p_6} + \delta_{p_1, p_4} \delta_{p_2, p_6} \delta_{p_3, p_5} + \delta_{p_1, p_5} \delta_{p_2, p_4} \delta_{p_3, p_6} \\ + \delta_{p_1, p_5} \delta_{p_2, p_6} \delta_{p_3, p_4} + \delta_{p_1, p_6} \delta_{p_2, p_4} \delta_{p_3, p_5} + \delta_{p_1, p_6} \delta_{p_2, p_5} \delta_{p_3, p_4}). \quad (34)$$

Upon substitution into (30) and integrating out the pair delta functions arising from these decompositions we find

$$M_2^{(2)}(\mathbf{p}_1; \mathbf{p}_2)(t) = 0, \quad (35)$$

$$M_2^{(4)}(\mathbf{p}_1; \mathbf{p}_2)(t) = \frac{2g^2}{(2\pi)^{2d}} \int [n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{p}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{p}_1} n_{\mathbf{k}_1} n_{\mathbf{k}_3} - n_{\mathbf{p}_1} n_{\mathbf{k}_1} n_{\mathbf{k}_2}] \\ \times (E[0, \Omega_{p_1 k_1, k_2 k_3}](t) + E[0, -\Omega_{p_1 k_1, k_2 k_3}](t)) \delta(\mathbf{p}_1 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_{123} \delta_{p_1, p_2}. \quad (36)$$

Since the order ε^2 term in the expansion (20) for $M_2(\mathbf{p}_1, \mathbf{p}_2)(t)$ is zero there is no secular behaviour and the multiple scale consistency condition arising at this order is the trivial one:

$$\frac{\partial M_2^{(2)}(\mathbf{p}_1; \mathbf{p}_2)}{\partial \mathcal{T}_2} = 0. \quad (37)$$

However the order ε^4 term does exhibit secular behaviour. It arises when we take the long time limit:

$$E[0, x](t) \sim \left(\pi \delta(x) + iPV\left(\frac{1}{x}\right) \right) \left(t - i \frac{\partial}{\partial x} \right) \quad \text{as } t \rightarrow \infty, \quad (38)$$

where $PV(\cdot)$ denotes the Cauchy Principal Value. Thus at this order, the multiple scale procedure yields a non-trivial consistency condition:

$$\frac{\partial M_2^{(0)}(\mathbf{p}_1; \mathbf{p}_2)}{\partial \mathcal{T}_4} = \frac{4\pi g^2}{(2\pi)^{2d}} \int [n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{p}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{p}_1} n_{\mathbf{k}_1} n_{\mathbf{k}_3} - n_{\mathbf{p}_1} n_{\mathbf{k}_1} n_{\mathbf{k}_2}] \delta(\Omega_{p_1 k_1, k_2 k_3}) \delta_{p_1 k_1, k_2 k_3} d\mathbf{k}_{123} \delta_{p_1, p_2} \\ = \frac{\partial n_{\mathbf{p}_1}}{\partial \mathcal{T}_4} \delta_{p_1, p_2}. \quad (39)$$

Note that the delta function expressing spatial homogeneity factors out. Recalling that $\mathcal{T}_4 = \varepsilon^4 t$ we note that this equation is actually the well-known kinetic equation governing the relaxation of the spectral particle density in an ensemble of weakly interacting waves:

$$\frac{\partial n_{\mathbf{p}_1}}{\partial t} = \varepsilon^4 \frac{4\pi g^2}{(2\pi)^{2d}} \int (n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{p}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{p}_1} n_{\mathbf{k}_1} n_{\mathbf{k}_3} - n_{\mathbf{p}_1} n_{\mathbf{k}_1} n_{\mathbf{k}_2}) \\ \times \delta(\omega_{\mathbf{p}_1} + \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3}) \delta(\mathbf{p}_1 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_{123}. \quad (40)$$

Note that the structure of this equation is very reminiscent of the B–N equation (1). We shall be primarily interested in the idealised situation where the momentum distribution, $n_{\mathbf{p}_1}$, is statistically isotropic. In this case, $n_{\mathbf{p}_1}$ is a function of $p_1 = |\mathbf{p}_1|$.

We can simplify our discussion by writing the kinetic equation in energy space rather than momentum space. We introduce the variables ϵ_i as follows:

$$\epsilon_1 = \frac{1}{2} p_1^2, \quad \epsilon_2 = \frac{1}{2} k_1^2, \quad \epsilon_3 = \frac{1}{2} k_2^2, \quad \epsilon_4 = \frac{1}{2} k_3^2.$$

Some algebra then allows the kinetic equation to be re-written as

$$\frac{\partial n_{\epsilon_1}}{\partial t} = \frac{1}{\sqrt{\epsilon_1}} T_4[n_{\epsilon_1}], \quad (41)$$

where

$$T_4[n_{\epsilon_1}] = \int S_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4} (n_{\epsilon_2} n_{\epsilon_3} n_{\epsilon_4} + n_{\epsilon_1} n_{\epsilon_3} n_{\epsilon_4} - n_{\epsilon_1} n_{\epsilon_2} n_{\epsilon_4} - n_{\epsilon_1} n_{\epsilon_2} n_{\epsilon_3}) \delta(\epsilon_1 + \epsilon_1 - \epsilon_2 - \epsilon_2) d\epsilon_2 d\epsilon_3 d\epsilon_4, \quad (42)$$

and

$$S_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4} = \varepsilon^4 \frac{4\pi g^2}{(2\pi)^{2d}} \sqrt{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4} \int \delta(\mathbf{p}_1 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\Omega_2 d\Omega_3 d\Omega_4. \quad (43)$$

In this formula, $d\Omega_2 d\Omega_3 d\Omega_4$ represents the integration over polar angles in $\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3$ space. Note that $S_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4}$ is a homogeneous function of degree $1/2$. The angular integration of the delta function can be done explicitly in two and three dimensions. We are interested in the three-dimensional case where $d = 3$, it is found [13]

$$S_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4} = \varepsilon^4 \frac{4\pi g^2}{(2\pi)^6} \min\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\}. \quad (44)$$

The two-dimensional case, along with many other aspects of the kinetic theory for the nonlinear Schrödinger equation, is explained in detail in [10].

Eq. (41) is the wave kinetic equation well known in the theory of wave turbulence. In this article we shall refer to (41) as the Gross–Pitaevskii kinetic equation to distinguish it from the Boltzmann–Nordheim kinetic equation, (1). Having shown how the G–P kinetic equation can be derived from averaging solutions of the nonlinear Schrödinger equation for the condensate wave-function, we now turn our attention to its solutions. We shall first examine the time-independent solutions which describe stationary equilibrium and non-equilibrium momentum distributions. Then we shall study time-dependent solutions.

3. Equilibrium and non-equilibrium stationary states

It is a trivial matter to verify that the kinetic equation, (40), formally conserves the total energy and total particle number given by

$$E = \int \omega_{\mathbf{k}} n_{\mathbf{k}} d\mathbf{k} \quad (45)$$

and

$$N = \int n_{\mathbf{k}} d\mathbf{k} \quad (46)$$

respectively. The stationary solutions are intimately linked with these conservation laws. There are equilibrium stationary states, which describe equipartition of conserved quantities, and non-equilibrium stationary states, which describe constant fluxes of conserved quantities in momentum space. The two thermodynamic equilibrium solutions can be seen from (41) almost by inspection. They are

$$n_{\epsilon} = c_1, \quad (47)$$

$$n_{\epsilon} = c_2 \epsilon^{-1}, \quad (48)$$

where c_1 and c_2 are constants. The first of these corresponds to an equidistribution of particle number, the second to an equidistribution of energy.

To study the non-equilibrium solutions it is convenient to write the conservation laws as continuity equations in energy space:

$$\frac{\partial N_{\epsilon}}{\partial t} = \frac{\partial Q}{\partial \epsilon}, \quad (49)$$

$$\frac{\partial(\epsilon N_{\epsilon})}{\partial t} = -\frac{\partial P}{\partial \epsilon}, \quad (50)$$

where N_ϵ is the integrated particle number obtained from

$$\int n_{\mathbf{p}} \, d\mathbf{p} = \int N_\epsilon \, d\epsilon. \tag{51}$$

The fluxes, P and Q , are easily obtained from (41):

$$Q_\epsilon = - \int_\epsilon^\infty T_4[n_{\epsilon'}] \, d\epsilon', \tag{52}$$

$$P_\epsilon = \int_0^\epsilon \epsilon' T_4[n_{\epsilon'}] \, d\epsilon'. \tag{53}$$

Note that these fluxes are defined such that Q is positive for particles flowing to the left corresponding to an inverse cascade and P is positive for energy flowing to the right corresponding to a direct cascade in the energy scale. If we consider power law spectra of the form $n = c \epsilon^{-x}$ then dimensional analysis suggests that the fluxes, Q and P are independent of ϵ for $x = 7/6$ and $x = 4/3$ respectively. These power laws are the Kolmogorov–Zakharov spectra corresponding to constant fluxes of particles and energy respectively. For BEC it is the inverse cascade which is of particular interest since it is responsible for the transfer of particles to low momentum states which characterises the final state of the condensate.

The Kolmogorov–Zakharov spectra discussed above are more than dimensionally consistent however. They are exact stationary solutions of the kinetic equation [14,10]. This fact can be demonstrated and the value of the Kolmogorov constant computed using an ingenious transformation due to Zakharov [15,16]. The details of Zakharov’s method are well explained in the references so here we shall only give a brief outline.

Bearing in mind that $\epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1 \geq 0$, the integration region for the collision integral, (42), in the (ϵ_3, ϵ_4) plane is the shaded region shown in Fig. 1. The idea of the Zakharov transformation is to assume a power law distribution, $n_\epsilon = c \epsilon^{-x}$, and then to map the the three regions R2, R3 and R4 into the region R1 in a scale-invariant way. It will then be possible to spot the criterion whereby the integrand vanishes. We define new variables $(\epsilon'_2, \epsilon'_3, \epsilon'_4)$ in each of the regions R2, R3, R4 as follows

$$Z2: R2 \mapsto R1 : (\epsilon_2, \epsilon_3, \epsilon_4) \rightarrow \left(\frac{\epsilon_1^2}{\epsilon'_2}, \frac{\epsilon_1 \epsilon'_3}{\epsilon'_2}, \frac{\epsilon_1 \epsilon'_4}{\epsilon'_2} \right), \tag{54}$$

$$Z3: R3 \mapsto R1 : (\epsilon_2, \epsilon_3, \epsilon_4) \rightarrow \left(\frac{\epsilon_1 \epsilon'_3}{\epsilon'_4}, \frac{\epsilon_1 \epsilon'_2}{\epsilon'_4}, \frac{\epsilon_1^2}{\epsilon'_4} \right), \tag{55}$$

$$Z4: R4 \mapsto R1 : (\epsilon_2, \epsilon_3, \epsilon_4) \rightarrow \left(\frac{\epsilon_1 \epsilon_4}{\epsilon'_3}, \frac{\epsilon_1^2}{\epsilon'_3}, \frac{\epsilon_1 \epsilon'_2}{\epsilon'_3} \right). \tag{56}$$

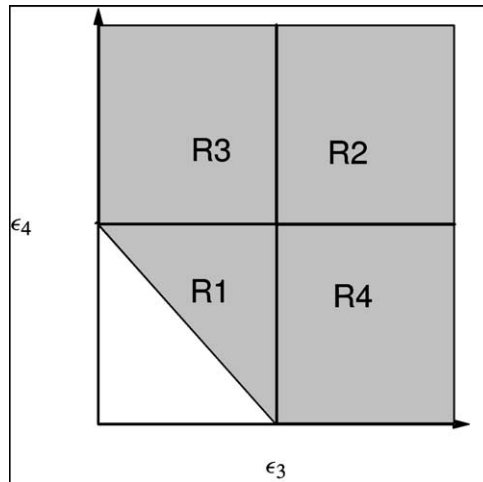


Fig. 1. Region of integration for the RHS of Eq. (42).

Adding the resulting four contributions together we obtain

$$c^3 \int_{\text{R1}} S_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4} (\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4)^{-x} (\epsilon_1^x + \epsilon_2^x - \epsilon_3^x - \epsilon_4^x) \epsilon_1^y (\epsilon_1^{-y} + \epsilon_2^{-y} - \epsilon_3^{-y} - \epsilon_4^{-y}) \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) d\epsilon_{234},$$

where $y = 3x - 7/2$. It is immediately clear that the collision integral, $T_n[n]$, vanishes for

$$\left. \begin{array}{l} x = 0 \\ x = 1 \end{array} \right\} \text{thermodynamic equilibrium solutions, (47) and (48),} \quad (57)$$

$$\left. \begin{array}{l} y = 1 \Rightarrow x = 3/2 \\ y = 0 \Rightarrow x = 7/6 \end{array} \right\} \text{K-Z solutions.} \quad (58)$$

Of course, the K–Z solutions cannot exist in isolation. The fluxes Q or P must be sustained by coupling the system to external sources. It is to be understood that these sources are localised at sufficiently large or small scales to allow for the existence of ‘inertial ranges’ where the dynamics is entirely dominated by the wave-wave interactions. In such situations the K–Z solution is the relevant one within the inertial range of energies. However this solution must then match to the particular forcing and dissipation profiles which provide the fluxes. The question of when the K–Z spectrum is a universal object, independent of the details of sources and sinks at large and small scales is a nontrivial one from the point of view of analysis. On the other hand, experimental observations of various wave turbulence systems suggest that in many cases the K–Z spectrum is more robust than the mathematical simplifications underlying its derivation might suggest.

The Kolmogorov constant, c , can be computed in terms of the flux carried by the K–Z spectrum; for details see [12,14].

4. Nonstationary solutions and condensate formation

Suppose the initial particle distribution, $n(\epsilon, t = 0)$, is far from the equilibrium configuration for the system which would like to see particles drop into the zero momentum state in order to generate the uniform condensate solution (10). The time dependent solution of (41) describes the relaxation process which then occurs. However, since we know from our discussion of the previous section, the inverse cascade only has finite capacity to absorb particles so the solution of Eq. (41) will eventually break particle number conservation. We denote the time at which this failure occurs by t^* . The failure of the conservation law can be interpreted as the generation of a non-zero flux of particles to $\epsilon = 0$. Beyond this time, Eq. (41) no longer provides an appropriate description of the physics, since it fails to account for the singular nature of the zero mode. Lacaze et al. [17]¹ have proposed a system of kinetic equations for the post- t^* evolution based on an ansatz which splits the particle distribution into smooth and singular parts. We consider these equations in the next section.

We consider for now the dynamics pre- t^* . Let us look for a self-similar solution of the form

$$n_\epsilon = \frac{1}{\tau^a} \varphi(\eta), \quad (59)$$

where the similarity variables, η and τ , are defined as

$$\eta = \frac{\epsilon}{\tau^b}, \quad \tau = t^* - t. \quad (60)$$

The exponents a and b are to be determined. Note that if there is a quasi-stationary power law distribution at large momenta, $n_\epsilon \sim \epsilon^{-x}$, then the ansatz (59) implies that $x = -a/b$. Thus the exponents of the similarity transformation are related to the scaling of the high momentum part of the solution. Upon substitution of (59) and (60) into the equation we find that we can remove the time dependence from the problem by choosing

$$4b + 1 = 2a. \quad (61)$$

We are left with an integro-differential equation for the function $\varphi(\eta)$:

$$a\varphi_{\eta_1} - b\eta_1 \frac{d\varphi_{\eta_1}}{d\eta} = \eta_1^{-1/2} \int S_{\eta_1 \eta_2 \eta_3 \eta_4} (\varphi_{\eta_2} \varphi_{\eta_3} \varphi_{\eta_4} + \varphi_{\eta_1} \varphi_{\eta_3} \varphi_{\eta_4} - \varphi_{\eta_1} \varphi_{\eta_2} \varphi_{\eta_4} - \varphi_{\eta_1} \varphi_{\eta_2} \varphi_{\eta_3}) \delta(\eta_1 + \eta_1 - \eta_2 - \eta_2) d\eta_1 d\eta_2 d\eta_3, \quad (62)$$

¹ This reference shows in particular how a condensate can be created from a continuous momentum distribution, via a finite time singular solution of the Boltzmann–Nordheim kinetic equation.

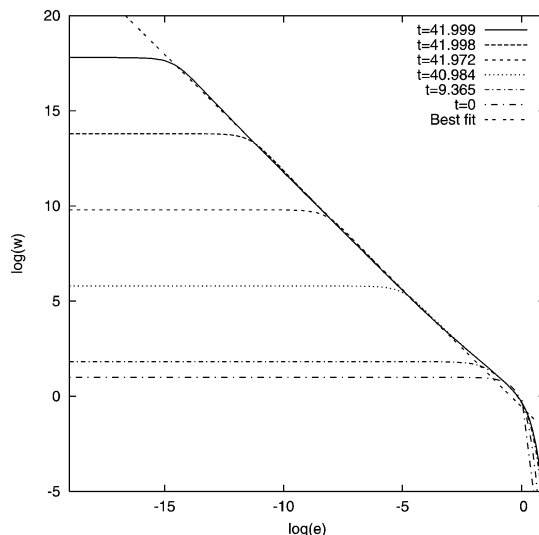


Fig. 2. Computation of the inverse cascade in the unforced kinetic equation for parameter values describing Bose–Einstein condensation in 3 dimensions (NLS). This is a finite capacity system. The traces shown are snapshots of the spectrum at a series of consecutive times. The final trace is close to the finite time singularity at zero momentum. The fitted power law at large momenta has exponent 1.2345.

with $S_{\eta_1 \eta_2 \eta_3 \eta_4}$ given by Eq. (43). We must supplement this equation with a second relation between a and b in order to solve the problem. In some cases, it makes sense to assume that the total amount of some conserved quantity grows linearly in time due to a constant input flux. This can sometimes yield a second relation between a and b , which as shown by Falkovich and Shafarenko [18], implies that the self-similar solution exhibits the Kolmogorov–Zakharov scaling at large momenta.

In our case, the total number of particles contained in the spectrum cannot increase linearly in time since we know that the final spectrum contains a finite amount of particles. We do not know of any other rational constraint which the system must satisfy in order to furnish a second relation although we have made some conjectures for the analogous problem for the direct cascade in wave turbulence [19].

Since we cannot analytically determine the values of a and b and hence the scaling at large momenta, a priori, we solved the original equation numerically to measure what scaling is selected. A numerical computation of the solution of equation (41) is shown in Fig. 2 with an initial condition given by

$$n(\epsilon, t = 0) = 10e^{-3\epsilon}. \quad (63)$$

In line with recent investigations of finite capacity systems in wave turbulence [19] and elsewhere [20] we find that the pre- t^* scaling is not the Kolmogorov–Zakharov scaling. We measure a quasi-stationary power law distribution at high momenta with exponent 1.234 providing independent verification of the exponent measured by Lacaze et al. [17] in their simulations.

5. Post-singularity dynamics

Lacaze et al. [17] have proposed a system of kinetic equations for the post- t^* evolution based on an ansatz which splits the particle distribution into smooth and singular parts as follows:

$$n(\mathbf{p}, t) = n_0(t)\delta(\mathbf{p}) + \psi(\mathbf{p}, t), \quad (64)$$

where ψ is a regular function at $\mathbf{p} = 0$ and $n_0(t)$ is the amplitude of the singular part of the particle distribution which we loosely think of as the condensate (see Section 6 below). These equations couple the smooth part of the particle distribution to the singular part allowing for an exchange of mass between finite momentum particles and the condensate. If we assume an isotropic particle distribution we can average the equations of Lacaze et al. over angles and write them in terms of energy variables, $\epsilon = p^2$. The resulting system of equations can then be written:

$$\frac{\partial \psi_1}{\partial t} = T_4[\psi] + n_0 T_3[\psi], \quad (65)$$

$$\frac{dn_0}{dt} = Q_0 + n_0 U_3[\psi], \quad (66)$$

where Q_0 is the Kolmogorov–Zakharov flux at zero momentum coming from the inverse cascade² in the continuous part of the spectrum and

$$T_4[\psi] = \frac{1}{\sqrt{\epsilon_1}} \int \min(\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}) [\psi_2 \psi_3 \psi_4 + \psi_1 \psi_3 \psi_4 - \psi_1 \psi_2 \psi_4 - \psi_1 \psi_2 \psi_3] \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) d\epsilon_2 d\epsilon_3 d\epsilon_4, \quad (67)$$

$$\begin{aligned} T_3[\psi] &= \frac{1}{\sqrt{\epsilon_1}} \int [\psi_2 \psi_3 - \psi_1 \psi_2 - \psi_1 \psi_3] \delta(\epsilon_1 - \epsilon_2 - \epsilon_3) d\epsilon_2 d\epsilon_3 \\ &+ \frac{1}{\sqrt{\epsilon_1}} \int [\psi_2 \psi_3 - \psi_1 \psi_2 + \psi_1 \psi_3] \delta(\epsilon_2 - \epsilon_3 - \epsilon_1) d\epsilon_2 d\epsilon_3 \\ &+ \frac{1}{\sqrt{\epsilon_1}} \int [\psi_2 \psi_3 + \psi_1 \psi_2 - \psi_1 \psi_3] \delta(\epsilon_3 - \epsilon_1 - \epsilon_2) d\epsilon_2 d\epsilon_3, \end{aligned} \quad (68)$$

$$U_3[\psi] = 2\pi \int [\psi_3 \psi_4 - \psi_2 \psi_3 - \psi_2 \psi_4] \delta(\epsilon_2 - \epsilon_3 - \epsilon_4) d\epsilon_2 d\epsilon_3 d\epsilon_4. \quad (69)$$

The first term written symbolically as $T_4[\psi]$ is the regular collision term of the G–P kinetic theory, the second term written as $T_3[\psi]$ is the collision term arising from 3-wave interactions [14] and describes the nonlocal coupling of the smooth part of the spectrum to the condensate. The term $U_3[\psi]$ couples the condensate back to the smooth part of the spectrum. We are developing a numerical code to integrate these equations in time with the aim of exploring the behaviour of the system after the singularity time, t^* , described in the previous section. In this section we report the preliminary findings of our investigations.

We performed the following numerical experiment. We took an initial distribution for $\psi(\epsilon)$ which behaves like $\epsilon^{-1.234}$ at low energies and decays exponentially for high energies. This profile mimics the essential features of the pre-singularity distribution shown in Fig. 2 which we have discussed in the previous section. For n_0 we took an initial ‘seed’ value which was of the order of 1×10^{-5} of the *total* mass of the system. This arbitrary assignment is necessary because by Eq. (66) an initially zero amplitude condensate remains of zero amplitude for all time if the flux, Q_0 is also zero for $t > t^*$ as our simulations

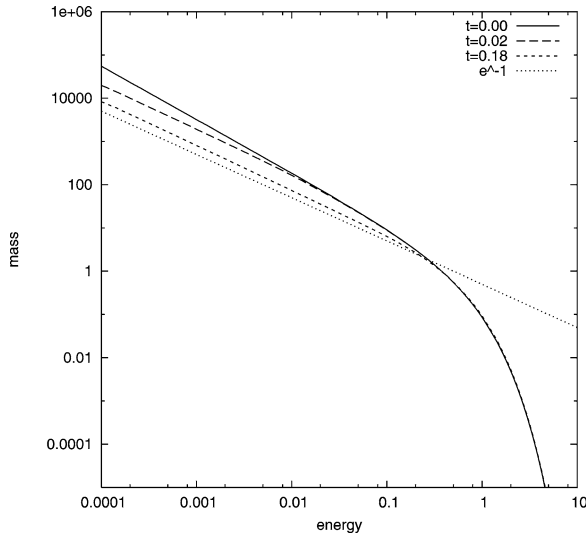


Fig. 3. Post- t^* relaxation of an initial profile exhibiting the transient scaling, $\epsilon^{-1.234}$, characteristic of the pre-singularity regime. The low momentum behaviour very quickly changes over to a spectrum close to ϵ^{-1} characteristic of thermodynamic equilibrium.

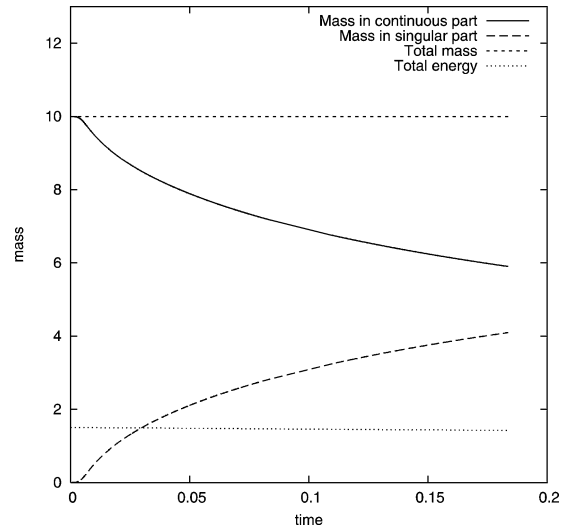


Fig. 4. Monitoring conservation laws for the post- t^* evolution. We see how the mass of the singular part of the spectrum grows at the expense of the smooth part. The total mass and total energy remain constant throughout this exchange. Presumably the system comes to equilibrium if one waits a sufficiently long time.

² We are not excluding the possibility that $Q_0 = 0$ and all the transfer of mass to the condensate is carried by the nonlocal interactions described by $T_3[\psi]$ and $U_3[\psi]$. Indeed our initial numerical results suggest that Q_0 is probably zero or at least very small. This is in agreement with the arguments in Section 6.

suggest. At present we are trying to determine a consistent way of determining the value for this ‘seed’. However it is our opinion that the detailed mechanism whereby the physical system passes through the singularity at $t = t^*$ in order to generate this seed is outside of the scope of the kinetic theory for reasons which we shall discuss in the next section.

Some early results from our computations are shown in Figs. 3 and 4. Fig. 3 illustrates our first main observation: the spectrum very quickly begins to relax back to a spectrum which is very close to the thermodynamic equilibrium spectrum, ϵ^{-1} . With an ϵ^{-1} spectrum we know $Q_0 = 0$. There is some evidence to suggest that the spectrum is slightly steeper than ϵ^{-1} although this might be an effect coming from the relatively small scaling range which we have been able to resolve so far. Our second major observation, illustrated in Fig. 4, is the exchange of mass between the continuous part of the spectrum and the singular part. This exchange conserves the total mass and energy of the system as can be easily checked from Eqs. (65) and (66). At present the maintenance of these conservation laws is problematic in our code and requires some tweaking and tuning (as was done to produce Fig. 4) with mass tending to leak from the system. We are currently trying to pin down the reason for this sensitivity in order to perform some more robust simulations over larger ranges of scales and longer times. Despite the difficulties which still require resolution within our numerical scheme, we believe that the main features of our computations as described above are quite robust.

6. Connection between Boltzmann–Nordheim and Gross–Pitaevskii kinetic theories

One notices a striking similarity between the B–N equation and the kinetic equations for the amplitude of waves interacting via the nonlinear term in the Gross–Pitaevskii equation for a statistically homogeneous condensate. Therefore one may legitimately ask if there is some connection between the two descriptions: B–N versus G–P, the point we shall consider below.

The connection comes from the fact that the occupation numbers associated to the long-wave excitation are large. At equilibrium, the Bose distribution predicts that this number diverges like $n_p \sim 2mk_B T/p^2$ at small momentum p , with m mass of the identical particles, k_B Boltzmann constant and T absolute temperature (less than the temperature of transition). For energies notably less than the thermal energy $k_B T$, n_p is much bigger than 1. Therefore quantum fluctuations in those states are relatively small, by the familiar Bogoliubov argument extended to non-zero momenta: the quantum amplitude associated to $\langle n_p \rangle = \langle \hat{\psi}_p^\dagger \hat{\psi}_p \rangle$ is of order of magnitude $\hat{\psi}_p \sim \sqrt{2mk_B T/p^2}$, and is large. Therefore the quantum commutation condition $[\hat{\psi}_p^\dagger, \hat{\psi}_p] = 1$ introduces negligible correction, to the large order of magnitude estimate $\hat{\psi}_p \sim \hat{\psi}_p^\dagger \sim \sqrt{2mk_B T/p^2}$. This means that in the range p small, one can neglect the quantum fluctuations and consider that $\hat{\psi}_p^\dagger$ and $\hat{\psi}_p$ are the Fourier transform of a classical complex-valued field $\Psi(x)$. This function has to be the solution of a nonlinear field equation, that is the Gross–Pitaevskii equation.

This G–P equation, when the nonlinear term is considered as a perturbation, yields a wave kinetic equation that has exactly the same form as the B–N kinetic equation, when restricted to its cubic terms as discussed in Section 2. This is not surprising since, in the limit of large occupation numbers, one expects that a classical field equation, like G–P, approximates well the full dynamical problem. However, this is not the complete solution of the problem of extending the B–N kinetic equation to the full range of possible wavenumbers. This only shows that the wave kinetic equation and the B–N kinetic equation overlap in some range of wavenumbers, the range where the occupation numbers are very large compared to one, or— for a thermal gas—the range of energies far smaller than the thermal energy.

This leaves, however, another domain of very small wavenumbers, where the occupation numbers are still very large, but where the interaction energy and the kinetic energy are of the same order of magnitude. For a number density n and a scattering length f , the cross-over occurs whenever the wavenumber is of order of or smaller than $p_{\text{cross}} = \hbar(fn)^{1/2}$. In this range of very small wavenumbers, one cannot no longer neglect the interaction energy. However, on the other hand, the quantum fluctuations are totally negligible in the same domain. Therefore the relevant equation is G–P. However this equation should be completed to include some information coming from the short wave part of the spectrum, the one described by the B–N kinetic theory. This information comes in in two different ways:

- (i) There is a coupling between the short wave part of the spectrum and the long wave part that amounts to change the cubic interaction term in G–P from the usual $2\pi \frac{\hbar^2 f}{m} |\Psi_0|^2 \Psi_0$ to $2\pi \frac{\hbar^2 f}{m} (|\Psi_0|^2 + 2n_n)$, where n_n is the density of the normal gas, mostly accounted for by the short wave part of the spectrum.
- (ii) There is a mass exchange between the condensate described by the term proportional to n_0 derived from B–N by assuming a component of momentum distribution with a Dirac delta part at zero momentum as described in Section 5.

From Eq. (65) various important consequences can be drawn. First there does not seem to be a possibility of Kolmogorov–Zakharov solution post t^* , except perhaps in a very transient state. In principle such K–Z solutions are interesting because they give a constant flux toward zero momentum. The inverse cascade has a power law behaviour like $\psi = C\epsilon^{-7/3}$ as described

in Section 3, this cancels the first collision term, $T_4[\psi]$ in Eq. (65). However, such a constant flux solution cannot be a steady solution near zero energies with a smooth time dependent coefficient C . This is because, by plugging this type of behaviour in the collision term written explicitly, one obtains a contribution to the time derivative $\partial\psi(\epsilon_1)/\partial t$ that is of order $\epsilon^{-17/6}$, far bigger near ϵ small than the expected $\epsilon^{-7/6}$ power law.

This leads one to look for other type of behaviour of $\psi(\epsilon)$ for ϵ small. Actually, we expect that the solution will evolve in one way or the other to equilibrium. This equilibrium solution behaves like $\psi(\epsilon) \sim C_0/\epsilon$ near ϵ small. Therefore we try an expansion of the solution of the kinetic equation (65) as

$$\psi(\epsilon) = \frac{C_0}{\epsilon} + \delta\psi(\epsilon), \quad (70)$$

where $\delta\psi(\epsilon)$ is a function of ϵ that is much smaller than the dominant contribution C_0/ϵ to ψ in the limit ϵ small. This perturbation series is necessary to get an equation of motion for C_0 , because by putting the one over ϵ term in the collision terms, one gets zero, because it is the behaviour of the equilibrium solution at small energies. Simple scaling arguments show that by inserting into the collision term explicitly written in Eq. (65), one gets a function of ϵ that scales like $C_0\delta\psi/\sqrt{\epsilon}$, this being restricted to terms linear in the relatively small $\delta\psi$. Balancing this with the dominant term on the left side, namely $(1/\epsilon)(dC_0/dt)$ one gets that $\delta\psi(\epsilon)$ must behave like $C_1/\sqrt{\epsilon}$ near ϵ small. From this we derive the equation of motion for C_0 , like $dC_0/dt = KnC_0C_1 + K'C_0^2C_1$. The constant K is computed by rather complicated multidimensional integrals. The same is true for the constant K' that represents another contribution to the time derivative of C_0 coming from the term $T_4[\psi]$ in the kinetic equation.

One could continue at higher order in ϵ to get coupled equations for the time derivative of the coefficients of the Laurent expansion of $\psi(\epsilon)$ near $\epsilon = 0$. We are presently working on an explicit way of using all this information to get practically a numerically solvable coupled set of kinetic equations for the condensate density and the smooth momentum/energy distribution.

However, this does not end the story. We still have to deal with the question of the condensate by itself. We have in mind a rather obvious question: given initial conditions for the density and energy such that the equilibrium state has a condensate at zero momentum, how does this initial condition evolve toward the final equilibrium state? We shall limit ourselves to rather sketchy remarks on this question, that would require a rather long discussion.

At the moment, from numerical and analytical studies [17], one understands well that, if the initial density is too large for a given energy, a smooth initial condition of the B–N kinetic equation becomes singular in a finite time. The solution blows up because its value at zero energy becomes infinite at a well defined finite time, t^* , depending on the initial conditions. This does not mean however that something like a condensate per se is formed at the singularity time. This is for two reasons. First at the time of the singularity the mass inside the singularity is zero, because the momentum distribution, although singular at zero energy, remains integrable. Furthermore, the condensate, as usually understood, implies infinite range correlations in space: all particles there are in the same homogeneous ground state. This infinite range (phase) correlation cannot build-up instantaneously at the collapse time. It cannot even build-up at infinite distances in any finite time interval [21]. Therefore, one expects that after the initial blow-up, there will be something like a continuous process of growth of the range of the correlations associated to a modulus of the wavefunction growing itself in the course of time until it reaches its equilibrium value.

This process cannot be described by the B–N kinetic theory because it assumes that the wavelength is less than the one for which the interaction energy becomes significant compared to the kinetic energy of the excitations. This leads one quite naturally to assume that the long wave part of the spectrum is described by the G–P equation, that has to be matched somehow with the B–N equation at short scales. This is done in two different ways, depending on the time at which it is done. The wave kinetic equation is the same as the B–N equation restricted to its cubic terms, the one relevant to describe the collapse. Therefore it is a good approximation to the collapse process to take as initial condition of the G–P equation a function that represents accurately the self-similar solution of the kinetic equation. This means that it must have the spectral distribution known numerically [17] with random phases for each spectral mode. The time evolution afterwards should be without singularity (for the G–P equation) and should describe a smooth transition from a regime with finite range spatial correlations to a regime of phase correlations extending at larger and larger distances as time goes on. Once the phase correlations extend at distances far longer than the typical length scale where the kinetic energy and the interaction are of the same order of magnitude, one can take the density associated to these long wave fluctuations as the density of the singular part of the momentum distribution, n_0 in the kinetic equation. This gives a way to get the initial condition for this n_0 in the B–N kinetic equations coupling the condensate amplitude and the smooth part of the distribution. At the same time, the initial value for the smooth part of this distribution can be taken as the spectrum found in the solution of the G–P equation with the self-similar initial condition. This gives a well defined (although rather complicated) procedure for describing the transition across the singularity of the B–N kinetic equation.

The B–N kinetic equation is self-contained, and it describes the final evolution toward equilibrium, but for one piece of information, the ever going process of phase synchronization of the condensate throughout all space. This is described by the G–P equation, as already explained. However this is not sufficient, because G–P, when left to itself, yields a never ending cascade of energy toward the small scales [22]. This is clearly incompatible with the final relaxation toward equilibrium, since

the ultimate result of this cascade should be all the mass concentrated in the zero wavenumber part of the spectrum, plus a noise at infinitely small wavelength. We expect instead finite occupation numbers diverging like $1/\epsilon$ for small energies, not something tending to zero. To represent this one has to add to the G–P equation a noise and a damping term ensuring that at any given time the spectrum of the solution of this equation matches the one given by the solution of the kinetic equation near zero energies: this is the usual strategy of inner-outer expansion, the ‘large’ energies or momenta of a solution of the G–P equation is actually a small energy (or momentum) when seen from the point of view of the B–N kinetic equation. Therefore it makes sense to use the small energy behaviour of a solution of B–N to determine the properties of a noise term at small scale for the G–P equation.

This is done by adding to the G–P equation a Gaussian noise $\eta(t, x)$ and a damping term. They intend to represent different physical processes. The damping term (added to the right-hand side of G–P) must be of the form $-i\hbar\frac{\zeta}{2}\psi(x, t)$, with ζ real positive. It represents the loss of particles in the long wave domain by collisions with particles in the thermal bath. If this damping term were alone, the density of the condensate would decay at the constant rate ζ . In the kinetic picture this takes into account all loss terms of the equation for n_0 . Therefore ζ has a well defined expression given by the B–N equation. Once this is done, the other piece of the puzzle, namely the noise term, is easy to find. It is a Gaussian noise in time because it represents the addition of random waves to the condensate. Furthermore we expect this noise term to have no correlation in time, since it concerns waves with frequencies much larger than what is described by G–P. Moreover it should have space correlations fitted to yield back the $C_0/\epsilon = 2mC_0/p^2$ spectrum in the wavenumber space. This concerns the short wave part of the G–P equation for which the nonlinear term is negligible. Therefore it is only a matter of *linear* algebra to find the explicit form of the correlations of the noise term that yield back the prescribed $C_0/\epsilon = 2mC_0/p^2$ in the wave number space for solution of the G–P equation. We refer the interested reader to a coming publication on this topic.

7. Concluding remarks

We conclude with a brief discussion of some of the many questions which remain to be answered before a full understanding of the connection between kinetic theory and BEC is obtained. Firstly, both the B–N equation (1) and the G–P kinetic equation (41) are in some sense mean field results. Both are obtained by truncating an expansion at the lowest nontrivial order and neglecting the higher order terms which describe fluctuations arising from higher order nonlinear interactions. It is not clear a priori that these fluctuations can always be neglected. Indeed it is highly likely (see, for example, the arguments in [12]) that the higher order terms do become relevant at very low momenta and are expected to modify the K–Z spectrum in some way. To further understand these issues is a very challenging problem which is not unique to the particular physical situation under consideration here. It arises in many areas of equilibrium and non-equilibrium statistical mechanics where mean field a picture requires modification to account for fluctuations.

A second issue relates to the scaling behaviour of the pre- t^* solution of the kinetic theory. Since the scaling observed is non-trivial in the sense that it corresponds neither to the K–Z scaling nor to the equilibrium one, it makes sense to wonder whether there is some physical basis for the scaling exponent, 1.234? It is possible that this exponent is nothing more than the solution of the nonlinear eigenvalue problem (62) and contains nothing deeper. From a physical perspective however, it would be nice to have a more rational explanation for the value of the exponent.

We would also like to understand more clearly what happens to this system after the singularity time, t^* . On the one hand, the derivation of a modified kinetic theory which takes into account the possibility of a uniform condensate, $\langle A_{\mathbf{p}} \rangle \neq 0$, might shed some light on this. The inclusion of a non-zero mean should be possible in the approach of Section 2 although it will result in a significant jump in algebraic complexity of a problem which already contains a lot of terms. The resulting equations presumably would resemble those suggested in [17] and studied numerically in the present article. On the other hand such a theory would be getting rather far away from the real physical system we began with and even further from the experimental situation. It is clear that in a real physical system we cannot grow infinite correlations within a finite time as the singular solution of Section 4 suggests. Furthermore real BEC experiments are very spatially inhomogeneous. Probably more interesting insights could be found by trying to understand how spatial correlations grow and how the inhomogeneity of the trap influences the physics of the inverse cascade phenomenon described here. Both these issues are somewhat outside of the remit of standard kinetic theories.

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