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Bose–Einstein condensates: recent advances in collective effects/Avancées récentes sur les effets collectifs dans les condensats de Bose–Einstein

Normal fluid/condensate interaction

Yves Pomeau

Laboratoire de physique statistique de l'École normale supérieure, associé au CNRS, 24, rue Lhomond, 75231 Paris cedex 05, France

Presented by Guy Laval

Abstract

In a dilute Bose gas the interaction between the normal gas and the condensate can be studied both at equilibrium and for its relevance to superfluid mechanics. In this article, one reviews first the thermodynamics of the interacting Bose gas in the low density limit and in the low temperature–finite density case. This is extended to thermodynamics with a velocity difference between the two components, normal and superfluid. It is argued that, if the initial value of the momentum related to this velocity difference is larger than the one given by thermodynamic stability, a new condensate should form in a finite time with a velocity close to the that of the normal gas. In the dilute case this occurrence of a new condensate is reminiscent of the condensation in the isotropic case at high enough density. ***To cite this article: Y. Pomeau, C. R. Physique 5 (2004).***

Résumé

Sur l'interaction du fluide normal/condensat. Dans un gaz de Bose dilué l'interaction entre gaz normal et condensat est étudiée à la fois pour l'équilibre et dans le cas d'un écoulement superfluide. On revoit d'abord la thermodynamique d'un gaz de Bose dilué et aussi le cas d'un fluide à base température. Les résultats sont étendus à la thermodynamique avec différence de vitesse. Si la vitesse relative entre les deux composantes excède la valeur limite de stabilité thermodynamique, un nouveau condensat doit se former avec une vitesse proche de celle du gaz normal. Dans le cas d'un gaz dilué cette apparition d'un nouveau condensat rappelle la condensation dans le cas isotrope à densité suffisante. ***Pour citer cet article : Y. Pomeau, C. R. Physique 5 (2004).***

Keywords: Bose gas; Low density limit; Low temperature–finite density limit

Mots-clés : Gaz de Bose ; Gaz de Bose dilué ; Thermodynamique d'un fluide à base température

1. Introduction

The theory of superfluidity remains dominated by the ideas of Landau [1]. They explain both the equilibrium and non-equilibrium properties of superfluid liquid Helium 4 by relying on the concept of quasiparticles. In this framework, Landau builds up on an analogy with the Debye theory of phonons in solids: the relevant degrees of freedom of a crystal lattice at low temperature are long wave-low energy excitations described by the mechanics of continuous media, elasticity in solids. Landau assumes that in liquid Helium and at low enough temperatures the long wave excitations are somehow described by hydrodynamic theory. This lead him to the concept of the two fluid theory, where the fluid is dynamically divided into a quantum coherent part, and the normal fluid made of the thermal excitations. However brilliant is this theory, it is difficult to assess or to derive it from first principles without guesses that are sometimes not easy to check in a rigorous way, basically because there is no expansion parameter in a dense liquid such as superfluid Helium. However, there are two possibilities for an expansion

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E-mail address: yves.pomeau@lps.ens.fr (Y. Pomeau).

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with respect to a small parameter. First of all, the Bose–Einstein condensation of *noninteracting* Bosonic systems provides a starting point for an expansion in powers of the density: the infinitely dilute limit is represented by the calculation of Bose and Einstein, and the next order should account for the various interaction effects. This program was started in two seemingly independent ways by Bogoliubov and by Lee and Yang in the late 1940s and early 1950s. Some of the results obtained then will be discussed later on. Bogoliubov managed to show that the interaction changes the free particles of the Bose–Einstein theory into quasiparticles with an energy spectrum of the type assumed by Landau, at least for long waves. Later, Lee and Yang expanded the thermodynamic quantities near the free Bose gas. The main lesson from their work is that Landau theory does *not* give the correct result when applied to a dilute Bose gas at nonzero temperature. A central assumption of Landau is that the quasiparticles have a vanishing chemical potential. By rereading him carefully on this point, it is hard to find a clear justification, besides a formal analogy with Debye theory. A systematic expansion near the infinitely dilute limit shows that such a chemical potential exists and is not zero. Actually this chemical potential is there because the quasiparticles do carry some mass (meaning that the quasiparticles operators do not commute with the particle operators), so that a chemical potential is needed to impose that at equilibrium the normal gas and the condensate exchange mass reversibly. The two components need to have the same chemical potential to make this exchange compatible with equilibrium, as usual in thermodynamics. Without it there should be a new constant of the motion (the mass of the condensate for instance) that should be brought explicitly into the thermodynamics. This is obviously wrong and was not assumed by Einstein, who maximized the entropy under the exchange of mass between the two components. Einstein's calculation only shows that this chemical potential is zero because of the peculiarities of the free Bose gas. Outside of this situation it is generally not zero.

However, as shown in Section 4 below, this chemical potential should tend to zero as the temperature tends to zero, independently of the fact that the system is dilute or not. Somehow this is precisely the limit that was taken by Landau, that remains therefore correct.

That the condensate and the normal gas exchange mass poses at once the question of stability of superflows: how can mass be exchanged without exchange of momentum? If momentum can be brought to (or taken out of) an immobile condensate by the normal gas, there is no superfluidity at all, since the normal gas is dissipative. However, the microscopic equations of motion of the full system are such that the interaction between the two components takes place without an exchange of momentum (this is valid in the frame of reference of an immobile condensate), which justifies the existence of superfluidity. As emphasized by Landau, the relative velocity between the two components (condensate and normal gas) is a thermodynamical parameter. It is associated to a constant of the motion representing (for instance) the arbitrary momentum of the normal gas in the rest frame of the condensate. However, things are not so simple, because this concerns situations that are homogeneous in space. Said otherwise, a uniform velocity difference is possible at equilibrium, but thermodynamics says nothing whenever the velocity difference is not uniform in space. This is an important issue in the theory of superfluidity: suppose that we have a steady flow of superfluid around an obstacle (a sphere for instance), although the normal gas/fluid is at rest. There the velocity difference is obviously not uniform in space. Therefore, equilibrium thermodynamics is unable to show that such a system is at equilibrium and remains so forever.

All this was based upon the analysis of the low density limit of a Bosonic system, clearly not applicable to liquid Helium that is dense. It shall be indicated, briefly however, in Section 4, how the low temperature limit there can be handled in a way inspired by the calculation made in the low density limit. At this low temperature limit the thermal excitations are rare enough to be noninteracting in a first approximation.

2. Thermodynamics of the dilute Bose gas – a summary

In the dilute gas limit the two body interaction can be considered as a relatively small effect, compared to the kinetic energy. This holds true for most particles only: particles with a momentum small enough may have an interaction energy and a kinetic energy of the same order. Because of that, a regular perturbation method fails to account for the interaction as a relatively small effect (practically of first order in the small positive scattering length f). Bogoliubov [2] retains part of these first order terms. However, if the density of the thermal gas and of the condensate are comparable (not considered by Bogoliubov, who studied the zero temperature limit), the interaction between two particles in an excited state and between one particle in an excited state and the condensate are of the same order of magnitude. The starting point of [3] is the energy operator $\mathcal{H}$ in second quantization formalism, keeping all relevant contributions at order f, the positive scattering length of the two body potential (other contributions are considered thereafter):

$$\mathcal{H} = \sum_p \varepsilon_H(p) a_p^\dagger a_p + \frac{2\pi\hbar^2 f}{m\Omega}\left[2(N-n_0)^2 + n_0^2\right] + \frac{2\pi\hbar^2 f}{m}\sum_{p\neq 0}\left(\Psi_0^2 a_p^\dagger a_{-p}^\dagger + \Psi_0^{*2} a_p a_{-p} + 2|\Psi_0|^2 a_p^\dagger a_p^\dagger\right). \qquad (1)$$

In (1), $a_p^\dagger(/\ a_p)$ is the creation(/annihilation) operator in the state of momentum p, $n_p = a_p^\dagger a_p$ the number of particles in this state and $\varepsilon_H(p) = p^2/(2m)$ is their bare kinetic energy. All particles are identical of mass m; N, fixed, is their total (and large) number, Ω is the enclosing volume and n_0 is the number of particles in the ground state $p = 0$ (assumed here to be at rest). In the low density limit, the dimensionless combination, f^3N/Ω is small. This is the formal expansion parameter, but it shall not be used explicitly; it is only assumed that f is small.

The energy operator written in (1) is not the exact energy. It keeps the same interaction terms as Bogoliubov (like $\Psi_0^2 a_p^\dagger a_{-p}^\dagger$) and the self interaction of the normal gas (proportional to $(2\pi\hbar^2 f/(m\Omega))[2(N-n_0)^2]$). This term was shown by Lee and Yang [4] to be accurate to first order. A full proof that it yields a valid approximation would require one to go to the next order and to compute the difference between the 'exact' theory and the one presented here, something that requires [5] rather complex algebraic manipulations.

The canonical partition function is the trace (denoted as $\mathrm{Tr}(\cdots)$):

$$Q_N = \mathrm{Tr}\left(\exp\left[-\frac{\mathcal{H}}{k_B T}\right]\right). \tag{2}$$

Because $\mathcal{H}$ is a known function of n_0 and N and an operator quadratic in the operators $a_p^\dagger$ and a_p, Q_N can be computed explicitly. We shall not reproduce this derivation that can be found in [3]. It follows the principles laid down by Lee and Yang. Recently there has been some interest in this question, and exact mathematical results [6] have been obtained for the partition function of models similar to ours but not exactly the same because, compared to us, various terms are absent in the interaction operator. The final result is a set of relations. The first one relates the number density of the normal gas to a chemical potential μ:

$$\rho(\mu) = -\frac{1}{(2\pi\hbar)^3}\int \mathrm{d}^3p\,\frac{1}{\mathrm{e}^{\varepsilon_b(p,\mu,\rho_s)/(k_BT)}-1}\,\frac{\partial\varepsilon_b(p,\mu,\rho_s)}{\partial\mu}, \tag{3}$$

where

$$\varepsilon_b(p,\mu,\rho_s) = \sqrt{\left(\frac{p^2}{2m}-\mu\right)^2 + 2\eta\rho_s\left(\frac{p^2}{2m}-\mu\right)},$$

$\rho_s = n_0/\Omega$ number density of the condensate and $\eta = 4\pi f\hbar^2/m$. This 'spectrum' (that is a relation between the 'energy' of a quasiparticle and its momentum) is also found in [6]. It may be called a 'gap' spectrum because the energy $\varepsilon_b(p,\mu,\rho_s)$ does not vanish at $p = 0$.[1] Another relation follows from the condition that the full partition function is stationary under the exchange of mass between condensate and normal gas:

$$\mu + \eta(\rho - \rho_s) = \frac{1}{(2\pi\hbar)^3}\int \mathrm{d}^3p\,\frac{1}{\mathrm{e}^{\varepsilon_b(p,\mu,\rho_s)/(k_BT)}-1}\,\frac{\partial\varepsilon_b(p,\mu,\rho_s)}{\partial\rho_s}, \tag{4}$$

$\rho = N/\Omega = \rho_s + \rho(\mu)$ being the total number density.

In the dilute gas limit we are concerned with, the number density of the normal gas and of the condensate are close to their value in the perfect gas. Therefore it is legitimate to see μ as a byproduct of the properties of this perfect gas, that is (and outside the vicinity of the transition temperature) one can compute it by putting in Eq. (4) the equilibrium value of ρ_s. Moreover, μ can be neglected [3] in the integrand on the right-hand side of (4) at the dominant order. However μ must be negative or zero: a positive μ would give a negative ε_b^2 in a range of values of p.

Before considering the equilibrium motion of the normal gas with respect to the condensate we shall need the value of μ in the small f limit as derived from (4). Reference [3] looks at the neighborhood of the transition temperature, where ρ_s is also small. In the temperature-density range where ρ_s is of the same order as ρ, by combining Eqs. (3) and (4), and taking into account that $\rho(\mu) = \rho - \rho_s$, one obtains:

$$\mu = -\frac{\eta^2\rho_s}{(2\pi\hbar)^3}\int \mathrm{d}^3p\,\frac{1}{(\mathrm{e}^{\varepsilon_b(p,\mu,\rho_s)/(k_BT)}-1)\varepsilon_b(p,\mu,\rho_s)}. \tag{5}$$

The integral on the right-hand side of (5) is dominated by contributions such that $p^2/(2m)$ is small compared to k_BT. As found a posteriori, μ is of order $f^{3/2}$ for f small. Therefore $\varepsilon_b^2(p,\mu,\rho_s) \approx (p^2/2m)(p^2/(2m) + 2\eta\rho_s)$ and:

$$\int \frac{\mathrm{d}^3p}{(\mathrm{e}^{\varepsilon_b(p,\mu,\rho_s)/(k_BT)}-1)\varepsilon_b(p,\mu,\rho_s)} \approx 4\pi m k_B T\int_0^\infty \frac{\mathrm{d}p}{p^2/(2m)+2\eta\rho_s}.$$

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[1] That there is a gap at nonzero temperature has been discussed in the literature quite early [7]; see also [8]. There the gap was excluded based on uncontrolled assumptions on various correlation functions. The mathematical rigour of the work of Zagrebnov [6] clearly shows that such a gap is present at finite temperature. It is of interest too to notice that a nonzero gap yields normal fluctuations [9].

Thus μ is of order $f^{3/2}$:

$$\mu = -\frac{(\eta m)^{3/2} \rho_s^{1/2} k_B T}{(2\pi\hbar)^3}. \tag{6}$$

In the next section, the thermodynamics when there is a velocity difference between the normal gas and the condensate will be derived. This requires the understanding of why this velocity difference should be seen as a thermodynamical parameter.

3. Thermodynamics with a velocity difference between the condensate and the normal gas

Thermodynamics with a velocity difference between the condensate (assumed at rest first) and the normal gas relies on the following remarks. In the energy operator (1), not every term has been included. Other terms exist, that represent either interactions between the thermal particles or between the thermal particles and the condensate. The general form of the interaction energy is [4]:

$$V_{\mathrm{int}} = \frac{2\pi f \hbar^2}{m\Omega} \sum_{\alpha\ldots\omega} a_\alpha^\dagger a_\beta^\dagger a_\nu a_\omega \delta(p_\alpha + p_\beta - p_\nu - p_\omega),$$

where $\delta(\cdot)$ is for a Kronecker discrete function, equal to one if its argument is zero and to 0 otherwise. Furthermore the momenta p_α, p_β, etc. are the discrete 3D momenta arising from the quantization of plane wave modes in a rectangular box for instance. The sum $\sum_{\alpha\ldots\omega}$ may be decomposed in various terms, depending on the way the condensate wavefunction enters into those terms. If there is condensation in the state of zero momentum, the operators a_p and $a_p^\dagger$ with $p = 0$ become c-numbers [2]: $a_0 = \Psi_0 \Omega^{1/2}$, $a_0^\dagger = \Psi_0^*$, Ψ_0 complex number and Ψ_0^* its complex conjugate. The terms including this ground state wavefunction in V_{int} bear either one, two or four zero indices. The ones with four and with two indices are kept in (1). It remains for us to look at the interaction with only one zero wavenumber. This would bring a contribution to the thermodynamic functions of an order in f higher than the one retained in [3]. But this kind of interaction (contrary to the one taken into account in the Bogoliubov renormalization) yields irreversible collision effects accounted for in the kinetic theory [10]. As shown in [11],[2] the Boltzmann–Nordheim kinetic theory for Bosons with a condensate splits into an equation for the amplitude of the condensate and another one for the momentum distribution of the normal gas. This describes possible exchanges between the two components that are ruled by the square of the matrix element in V_{int} with one momentum equal to zero, say $p_\alpha = 0$ (the particle in the condensate) and the three others nonzero (particles in an excited state-actually quasiparticles but this does not matter here). The conservation of momentum implies $p_\beta = p_\nu + p_\omega$. This has the important consequence that this process does conserve the total momentum in the normal gas (although, for instance the mass of the condensate is not conserved, because in such an process, a particle goes from the condensate to the normal gas, which explains that the thermodynamic sum must be carried in such a way that the free energy is stationary under that exchange of mass). One could think that only the forward process $(p_\alpha, p_\beta) \to (p_\nu, p_\omega)$ is relevant with $p_\alpha = 0$, since it could appear very unlikely that the reverse interaction of two thermal particles of (arbitrary) momenta p_ω and p_ν yields exactly a particle of zero momentum in the condensate. This is not so for Bosons, because of the enhancement by quantum interferences of the process giving back a particle in the condensate. This shows up clearly in the structure of the Boltzmann–Nordheim kinetic theory with condensate [12],[3] [13]: the kinetic equation for the amplitude of the condensate through the microscopic process just described shows gain and loss terms of the same order. There is also a condition of energy conservation in the collision process. At large momenta the energy of the quasiparticles becomes the bare particle energy $p^2/(2m)$, so that the energy conservation in the elementary process reads $p_\alpha^2 = p_\nu^2 + p_\omega^2$. Together with the conservation of momentum, this yields $p_\nu \cdot p_\omega = 0$ (scalar product). At smaller momenta, one has to take into account the Bogoliubov renormalization. The condition of energy conservation becomes more complex but remains compatible with momentum conservation. The equilibrium state (supposing again that the condensate is at rest) will depend on the conserved quantities, as usual. Besides the total mass and energy, the momentum of the normal component is determined not by some equilibrium condition, but by its initial value, since it is dynamically conserved. According to the principles of equilibrium statistical mechanics, the value of this momentum is imposed by adding to the energy operator a Lagrange term proportional to this momentum. At the end the average momentum is taken as equal to the prescribed value of

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[2] It is fair to say that at the present time, no kinetic theory has been written fully consistent with the rather complex equilibrium properties of the dilute Bose gas at finite temperature, beyond the Boltzmann–Nordheim kinetic theory (L.W. Nordheim, Proc. Roy. Soc. London Ser. A 119 (1928) 689).

[3] This reference and the paper in this journal by Connaughton and Pomeau show how a condensate grows out of a continuous momentum distribution, via a finite time singular solution of the Boltzmann–Nordheim kinetic equation.

the invariant. This amounts to computing the thermodynamic sums with $\mathcal{H}' = \mathcal{H} - u \cdot \sum_p p a_p^\dagger a_p$ instead of $\mathcal{H}$, u velocity of the normal gas with respect to the condensate. The calculations are very similar to the ones summarized before: one replaces $\varepsilon_H(p) = p^2/(2m)$ by $\varepsilon_H(p) - u \cdot p$ everywhere. The new relation between the density of the normal gas and μ is as in (3), except that $\varepsilon_b(p, \mu, \rho_s)$ is to be replaced by

$$\varepsilon'_b(p, \mu, \rho_s) = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + 2\eta\rho_s\left(\frac{p^2}{2m} - \mu\right)} - p \cdot u.$$

This follows from the same algebra as done in Section 4 below that is therefore not reproduced here. The calculation is as that in [3] until (5) which becomes:

$$\mu = -\frac{\eta^2\rho_s}{(2\pi\hbar)^3} \int \mathrm{d}^3 p \frac{1}{(\mathrm{e}^{\varepsilon'_b(p,\mu,\rho_s)/(k_B T)} - 1)\varepsilon'_b(p, \mu, \rho_s)}. \tag{7}$$

As shown in the Section 5 below, there is a limit speed, that is a limit value of u such that if u is larger, the quantity denoted as $\varepsilon'_b(p, \mu, \rho_s)$ may become negative for some values of p, which makes the system thermodynamically unstable.

4. Thermodynamics at low temperature and finite density

This section presents a derivation of the thermodynamical functions of a dense quantum fluid at low temperature. Because of the finite density assumption, the various parameters involved cannot be derived exactly from molecular calculations. However, the fundamental quantity of this theory is well defined in principle; it is the energy of the quantum ground state as a function of the number density ρ. This energy is proportional to the volume and it is written as $\Omega e(\rho)$. In the dilute model just considered, $e(\rho) = 2\pi\hbar^2 f \rho^2/m$. The other idea in this derivation is borrowed from the end section of Landau's paper. There, Landau remarks that a long wave excitation corresponds to a wavefunction of the total system that is the ground state wave function $\Psi_0(r_1, r_2, \ldots)\,\mathrm{e}^{\mathrm{i}t E_0/\hbar}$ times a product of single argument functions $\prod_{i=1}^N \chi(r_i, t)$. Since the ground state wavefunction satisfies the permutation symmetries of the many body system (Fermionic or Bosonic), the ground state wavefunction times a product of single particle wave functions satisfies obviously the same permutation symmetries. More generally, a product of the ground state wavefunction times a symmetric function of the argument $(r_1, r_2, \ldots)$ will also satisfy the required permutation symmetries. Therefore the long wave excitations can be described by Bosonic operators acting on the ground state, independently of the fact that this ground state is for Bosons or Fermions.

Now we are ready to write the energy operator for a general long wave perturbation to the ground state. We start from the classical (= nonquantum) Lagrangian of the slightly perturbed ground state:

$$\mathcal{L} = \int \mathrm{d}^3 r\, \mathrm{d}t \left[\frac{\hbar^2}{2m}|\nabla\chi|^2 - e(\rho) - \mathrm{i}\frac{\hbar}{2}\left(\bar{\chi}\frac{\partial\chi}{\partial t} - \chi\frac{\partial\bar{\chi}}{\partial t}\right)\right]. \tag{8}$$

The first term in the integral represents the kinetic energy contribution, and the second one the potential energy. Both terms depend on the amplitude $\chi(r)$, that we shall consider for a moment as a classical field, although it is a Bosonic field. The difference does not matter in the calculation to be made. In the ground state, χ is just a complex constant, called χ_0, and by suitable normalization one can manage to have for the ground state number density $\rho_0 = \chi_0\bar{\chi}_0$. In the low temperature limit, one expects that the deviation of the density brought by the thermal fluctuations is small compared to the ground state density. This perturbation will change ρ_0, a constant, into a r-dependant quantity written as:

$$\rho(r) = \big(\chi_0 + \delta\chi_0 + \varphi(r)\big)\big(\bar{\chi}_0 + \delta\bar{\chi}_0 + \bar{\varphi}(r)\big). \tag{9}$$

We make in this expression a distinction between $\delta\chi_0$ that is the change of amplitude (and ultimately in the number of particles) of the ground state and the pertubation brought by the thermal excitations, namely the field $\varphi(r)$. The two are linked by the condition that the total number of particles is fixed. This shows up as a condition on the space average of $\rho(r)$. Denoting the space average by brackets, this condition of conservation of the number of particles can be expressed:

$$0 = \big\langle\big(\rho(r) - |\chi_0|^2\big)\big\rangle = \chi_0\delta\bar{\chi}_0 + \delta\chi_0\bar{\chi}_0 + \delta\chi_0\delta\bar{\chi}_0 + \big\langle\varphi(r)\bar{\varphi}(r)\big\rangle. \tag{10}$$

Note that in Eq. (10), no space average is written in front of products like $\chi_0\delta\bar{\chi}_0$ or $\delta\chi_0\bar{\chi}_0$ since the quantities inside are just constants, not functions of r. Moreover the average of any quantity linear in $\varphi(r)$ and $\bar{\varphi}(r)$ cancel. We want to compute the change in the energy of the system due to the thermal excitations. As already said, we want to get an expression for the various thermodynamical quantities in the low temperature limit. This means that the amplitude of the thermal fluctuations is small, which permits us to neglect a number of terms in the calculation to be made now. Besides the contribution of the square gradient term to the energy, the other part of the energy of the system is the volume integral of $e(\rho(r))$, $e(\rho)$ being the same function as in the ground state. Because of the fluctuations are relatively small, we try to estimate $\langle e(\rho)\rangle$ by assuming $\rho(r)$ close to its ground state value, ρ_0, that is that $\rho(r) = \rho_0 + \delta\rho(r)$ where $\delta\rho(r)$ is somehow small, and where $\rho_0 = N/\Omega$ is well defined, independent on the actual value of the density $|\chi_0|^2$. Therefore we expand $e(\rho)$ in powers of $\delta\rho(r)$. Because of the constraint (10), no term linear in $\delta\rho$ will show up in $\langle e(\rho)\rangle$. At the next order:

$$\big\langle e(\rho) - e(\rho_0)\big\rangle \approx \frac{1}{2}\frac{\mathrm{d}^2 e}{\mathrm{d}\rho^2}\big\langle \delta\rho(r)^2\big\rangle. \tag{11}$$

Many terms disappear when computing this quantity from Eq. (9) at the dominant order (that is for small fluctuations), either because they are linear in $\delta\varphi(r)$, or because they involve fluctuations to the fourth power, negligible compared to the dominant ones that are second order:

$$\big\langle e(\rho) - e(\rho_0)\big\rangle \approx \frac{1}{2}\frac{\mathrm{d}^2 e}{\mathrm{d}\rho^2}\big[2\rho_0\varphi(r)\bar{\varphi}(r) + \chi_0^2\bar{\varphi}(r)\bar{\varphi}(r) + \bar{\chi}_0^2\varphi(r)\varphi(r)\big]. \tag{12}$$

Putting all this in the Lagrange function, Eq. (8), doing the usual derivation of the energy operator, and finally substituting for the c-number amplitudes $\varphi(r)$ Hermitian operators of creation and annihilation, one gets the following expression for the energy operator in Fourier space:

$$\mathcal{H} = \sum_p \frac{p^2}{2m} a_p^\dagger a_p + \frac{1}{2}\frac{\mathrm{d}^2 e(\rho)}{\mathrm{d}\rho^2}\sum_p \big[2\rho_0 a_p^\dagger a_p + \chi_0^2 a_{-p}^\dagger a_p^\dagger + \bar{\chi}_0^2 a_p a_{-p}\big]. \tag{13}$$

The operators a_p and $a_p^\dagger$ are the Bosonic creation and annihilation operators of long wave excitations of the density in the state of momentum p. The coefficient $\rho_0 \mathrm{d}^2 e(\rho)/\mathrm{d}\rho^2$ is the mass m times the speed of sound squared:

$$\rho_0 \frac{\mathrm{d}^2 e(\rho)}{\mathrm{d}\rho^2} = m c_s^2.$$

The expression of the energy operator in (13) is obviously very reminiscent of the one of the same operator for a dilute gas (Eq. (1)). The algebra giving the partition function in the canonical ensemble is very similar to the one exposed for the dilute case with the important difference, however, that the small parameter here, somewhat hidden, is the temperature. Moreover, the expression (13) of the energy operator and the one valid in the low density limit are different, because no energy change is associated formally to the variation in the number density of the ground state. This is because the variation of energy due to density variations of the ground state and of the excited states cancel at the dominant order. Therefore the number of quasiparticles in the excited state can be freely chosen and the chemical potential associated to the excited particles is zero. Let us emphasize again that this happens only because of the low temperature limit, not because of a general principle.

The derivation of the thermodynamics parameters associated to the energy operator (13) is quite straightforward and follows directly from a Bogoliubov transformation. Let us sketch the main steps. The Bogoliubov transformation amounts to diagonalize the part of the energy quadratic in $a_p^\dagger$, a_p, $a_{-p}^\dagger$ and a_{-p}. This is done by computing the energy eigenvalue associated to the contributions to the full energy operator associated to the wavenumbers p and $-p$:

$$\mathcal{H}_p = \left(\frac{p^2}{2m} + m c_s^2\right)\big(a_p^\dagger a_p + a_{-p}^\dagger a_{-p}\big) + \frac{\mathrm{d}^2 e(\rho)}{\mathrm{d}\rho^2}\big[\chi_0^2 a_{-p}^\dagger a_p^\dagger + \bar{\chi}_0^2 a_p a_{-p}\big]. \tag{14}$$

Note that the factor $\frac{1}{2}$ in front of the quantities like $a_{-p}^\dagger a_p^\dagger$ has disappeared in $\mathcal{H}_p$ because it adds the contribution to $\mathcal{H}$ with wavenumber p and $-p$. In Heisenberg representation, the equation of motion for the operators $a_p^\dagger$, a_p, $a_{-p}^\dagger$ and a_{-p} can be written, in general

$$\mathrm{i}\hbar\frac{\mathrm{d}A}{\mathrm{d}t} = [A, \mathcal{H}_p], \tag{15}$$

where A is anyone of the operators $a_p^\dagger, a_p, \ldots$, although $[\cdot\,,\cdot]$ is the usual commutator. This yields for example:

$$\mathrm{i}\hbar\frac{\mathrm{d}a_p^\dagger}{\mathrm{d}t} = [a_p^\dagger, \mathcal{H}_p] = \left(\frac{p^2}{2m} + m c_s^2\right)a_p^\dagger + \frac{\mathrm{d}^2 e(\rho)}{\mathrm{d}\rho^2}\chi_0^2 a_{-p}. \tag{16}$$

Diagonalizing $\mathcal{H}_p$ amounts to find the real eigenvalues λ such that $\mathrm{i}\hbar\,\mathrm{d}a_p^\dagger/\mathrm{d}t = \lambda a_p^\dagger$. Inserting this into the set of linear equations of motion yields a four by four determinant and a fourth order algebraic equation for λ:

$$\left(-\lambda + \left(\frac{p^2}{2m} + m c_s^2\right)\right)\left(\lambda + \left(\frac{p^2}{2m} + m c_s^2\right)\right) = m^2 c_s^4. \tag{17}$$

The roots of Eq. (17) are $\lambda = \pm\sqrt{(p^2/(2m) + 2mc_s^2)p^2/(2m)}$, the familiar Bogoliubov spectrum (although derived in a different approximation – the low temperature–finite density limit). This last result can be extended to the case where $p \cdot u$ (scalar product) is added to $p^2/(2m) + mc_s^2$. Then Eq. (17) becomes

$$\left((\lambda - p \cdot u)^2 - \left(\frac{p^2}{2m} + 2mc_s^2\right)\frac{p^2}{2m}\right)\left((\lambda + p \cdot u)^2 - \left(\frac{p^2}{2m} + 2mc_s^2\right)\frac{p^2}{2}\right) = 0. \tag{18}$$

Its roots are

$$\lambda = \pm\left(\sqrt{\left(\frac{p^2}{2m} + 2mc_s^2\right)\frac{p^2}{2m}} \pm u \cdot p\right).$$

This is the case relevant when there is an uniform velocity difference, u, between the condensate and the normal fluid. This expression shows that, as assumed by Landau, the quantity $p \cdot u$ has to be added to the energy of the quasiparticles, so that the thermodynamic functions are derived from the following expression of the momentum distribution of the quasiparticles:

$$\phi(p) = \frac{1}{\exp((pc_s - p \cdot u)/(k_B T)) - 1}. \tag{19}$$

In the above equation p in the product pc_s is just the modulus of the momentum, although $p \cdot u$ is the scalar product of two vectors. This expression of the Bose factor is written with the low energy approximation to the Bogoliubov spectrum because we are concerned with the low temperature limit.

Finally it should be noticed that the leftover terms in the energy of the ground state formally diverge there, as in the case of the Bogoliubov transformation in the dilute limit [14]. In the latter case, this divergence disappears because it is cancelled by an opposite divergence in the rest of the energy (specifically in the next order term in the Born approximation to two-body scattering needed to write the interaction energy). In the present situation of a dense fluid, no such explicit calculation can be made, but this divergence is obviously outside of the range of validity of the long wave approximation, since it occurs at large wavenumbers.

5. Fluid mechanics of the normal fluid–superfluid interaction: a few remarks

Below some remarks concerning the normal fluid–superfluid interaction are presented. This requires us to understand first the status of the pure superfluid problem. Recently some progress has been made there, relying on detailed studies of solutions of the Gross–Pitaevskii equations [15]. The first had to do with the stability of a superflow, without any normal fluid.

This problem of stability of a superflow remains a difficult issue: Landau [1] predicted that a superflow becomes unstable against the excitation of quasiparticles of momentum p whenever its velocity v is larger than the minimum of $\varepsilon(p)/|p|$, $\varepsilon(p)$ energy of the quasiparticle. The Landau criterion has been spectacularly verified in theoretical studies of superflows without normal gas around large obstacles [13]: beyond Landau's critical speed, dissipation sets in either through the generation of vortices if $\varepsilon(p)/|p|$ is minimum at $p = 0$ or generation of a Cerenkov wake [16] if it is at finite p.

This stability criterion remains almost the same in the framework of the equilibrium statistics: the condition of stability is that a pair of eigenvalues found in the Bogoliubov transformation remain positive. This requires that the eigenvalue

$$\varepsilon_b'(p, \mu, \rho_s) = \sqrt{\left(\frac{p^2}{2m} + 2mc_s^2 - \mu\right)\left(\frac{p^2}{2m} - \mu\right)} - p \cdot u$$

is positive for any p (recall that $2mc_s^2 = 2\eta\rho_s$). A little algebra shows that this happens if and only if u^2 is smaller than the root of

$$m\left(c_s^2 - u^2\right)^2 + 2\mu u^2 = 0.$$

Because μ is small compared to mc_s^2 (μ is of order $f^{3/2}$ although c_s^2 is of order f), this condition of thermodynamic stability is almost the same as the condition of dynamical stability in the sense of Landau.

Nevertheless, this is only part of the story. As already said, even though the extension of the results of [3] to the case of a nonzero velocity difference u is not very hard, it works if the velocity difference between the normal gas and the condensate is a thermodynamic parameter in the usual sense, that is, if the state of the system is uniform in space. There is no reason why a non-uniform velocity difference should be compatible with thermodynamical equilibrium. The situation is somehow reminiscent of that in a normal fluid: a uniform velocity is well compatible with equilibrium, just by uniform Galilean boost on the rest state.

Khalatnikov [17] derived a version of the two fluid equations, including viscous effects. This clearly requires careful handling of the way velocities, normal fluid and superfluid velocity, enter into the momentum distribution of the quasi particles. His derivation seems incorrect because of his choice of the local equilibrium momentum distribution. In general, if $\phi(p)$ is the momentum distribution of the quasiparticles in a certain frame of reference, this distribution becomes $\phi(p - mv)$ in a frame moving with velocity v with respect to the first one. This is a condition of general consistency for any form of momentum distribution. Actually, Khalatnikov, on top of page 117 in [17], gives the following expression for $\phi(p)$ ($=$ his $n_0(p)$):

$$\phi(p) = \left[\exp\left(\frac{\varepsilon(p) + p \cdot v_s - p \cdot v_n}{k_B T}\right) - 1\right]^{-1}. \tag{20}$$

It is inconsistent with the requirement that with no velocity difference between the two components (normal and condensate) $\phi(p)$ should be just the equilibrium distribution with its argument shifted by $-mV$. This is $\phi_{\text{eq}}(p - mV)$ with $V = v_n = v_s$, and

$$\phi_{\text{eq}} = \left[\exp\left(\frac{\varepsilon(p)}{k_B T}\right) - 1\right]^{-1}.$$

This is likely because most studies made since Landau tried to derive by a Galilean transformation the expression of the momentum distribution with $v_s = 0$ and an arbitrary v_n. When the superfluid component is at rest, Landau gives the (correct) expression of the momentum distribution

$$\phi(p) = \left[\exp\left(\frac{\varepsilon(p) - p \cdot v_n}{k_B T}\right) - 1\right]^{-1}. \tag{21}$$

Let us derive from this the general form of the momentum distribution at equilibrium with both an arbitrary superfluid velocity and an arbitrary normal velocity as well, both velocities being uniform. For that purpose one needs first to write the momentum distribution in the frame where the coherent component is at rest. This changes the distribution from $\phi_{\text{eq}}(\varepsilon(p))$ to $\phi_{\text{eq}}(\varepsilon(p) - p \cdot v_n)$. Now we can perform the Galilean boost from the rest frame to a frame moving with velocity v_s with respect to the lab frame. This Galilean boost changes momenta from p to $p - mv_s$ and add $-v_s$ to the speed v_n in the expression of $\phi(p)$ given in (21). The final result is

$$\phi(p) = \left[\exp\left(\frac{\varepsilon(p - mv_s) - (p - mv_s) \cdot (v_n - v_s)}{k_B T}\right) - 1\right]^{-1}, \tag{22}$$

a formula in agreement with the effect of the Galilean boost if $v_n = v_s$. This expession of the local equlibrium distribution, written in the frame of reference of the laboratory, is the correct one to begin a Hilbert–Enskog derivation of the hydrodynamic equation.

Another related issue concerns the possible interaction between normal component and coherent state. We have shown that, independent on any kind of expansion, if the condensate has zero speed, the interactions with the normal gas do not yield any exchange of momentum between the two components. Of course this does not exclude exchange of momentum in general: if the velocity of the condensate is not zero, any change of its mass will be also a change of momentum, even if its velocity does not change. This gives very tight constraints on the form of the two fluid equations including viscous effects. For instance this is not compatible with the two dissipative terms written by Khalatnikov [17], Eq. (9.15), p. 66. He writes the equation of motion for the superfluid velocity as

$$\frac{\partial v_s}{\partial t} + \nabla\left[\mu + \frac{v_s^2}{2}\right] = \nabla\left[\zeta_3 \operatorname{div}\bigl(\rho_s(v_n - v_s)\bigr) + \zeta_4 \operatorname{div}(v_n)\right]. \tag{23}$$

The terms on the left-hand side are the ones given by Landau on the basis of thermodynamic arguments, although the ones on the right-hand side are dissipative effects introduced by Khalatnikov ($\zeta_{3,4}$ are phenomenological viscosity coefficients). Neither should be there, because none satisfies the constraint that, if $v_s = 0$, there is no exchange of momentum and so no change of velocity of the condensate. Indeed the terms written by Khalatnikov are expected to be very small, because they depend on velocity divergences that are small at small Mach number. However, even so, they should not be there.

6. Final remarks and comments

(i) Consider the following (‘gedanken’ in atomic vapors) experiment: let us start to rotate a bucket full of superfluid at finite temperature from rest, the final peripheral velocity being bigger than the critical thermodynamic speed. Supposing the condensate remains at rest, the normal gas will not be able to move at equilibrium at the speed of the external boundary, because this would exceed the thermodynamic speed. So it is not clear at first sight what would happen. Most likely a new condensate is formed, with a nonzero speed close enough to the speed of the normal gas, by the same mechanism of creation of condensate in a too dense gas via the formation of a singular piece of momentum distribution [12]. In a rotating bucket, superfluid vortices should be created as well.

(ii) A rather natural question is how to include this thermodynamic limit speed in Landau's two fluid equations. When it concerns the small p limit, Landau's criterion is built in the two fluid equations, if the superfluid is treated as compressible, although the study of the generation of vortices requires a microscopic description. The thermodynamic limit speed would require probably the introduction of the velocity difference between the two components as a true thermodynamic parameter, and some free boundary problem would likely show up to represent the constraint on the limit of the velocity difference.

(iii) Estimates like the one of μ in Eq. (6) assume that the temperature is not too low, specifically that $k_BT \gg \eta\rho_s$, to allow the expansion of the exponential in (5) near its zero argument. At very low temperatures, one has $\mu \sim (\eta/\rho_s)^{(1/2)m^{3/2}(k_BT)^2/\hbar^3}$.

References

[1] L.D. Landau, J. Phys. USSR 5 (1941) 71.
[2] N.N. Bogoliubov, J. Phys. 11 (1947) 23.
[3] Y. Pomeau, S. Rica, J. Phys. A 33 (2000) 691, Europhys. Lett. 51 (2000) 20.
[4] T.D. Lee, C.N. Yang, Phys. Rev. 112 (1958) 1419.
[5] Y. Pomeau, in preparation.
[6] J.P. Bru, V. Zagrebnov, J. Phys. A 31 (1998) 9377.
[7] P.C. Hohenberg, P.C. Martin, Ann. Phys. (NY) 34 (1965) 291.
[8] A. Griffin, Phys. Rev. B 53 (1996) 9341.
[9] Y. Pomeau, Int. J. Bifurcation and Chaos 12 (2002) 2349.
[10] A good presentation of quantum kinetic theory is in R. Balescu, Equilibrium and Nonequilibrium Statistical Mechanics, Wiley, New York, 1975.
[11] Y. Pomeau, C. R. Acad. Sci., Sér. IV 1 (2000) 91;
Y. Pomeau, Chaos, Solitons and Fractals 12 (2001) 2675;
Y. Pomeau, M.E. Brachet, S. Métens, S. Rica, C. R. Acad. Sci., Sér. IIb 327 (1) (1999) 791.
[12] P. Lacaze, P. Lallemand, Y. Pomeau, S. Rica, Physica D 152–153 (2001) 779.
[13] C. Josserand, Y. Pomeau, Nonlinearity 14 (2001) R25–R62.
[14] K. Huang, Statistical Mechanics, Wiley, New York, 1963, Chapters 10 and 12.
[15] L.P. Pitaevskii, Sov. Phys. JETP 7 (1958) 808;
E.P. Gross, J. Math. Phys. 4 (1963) 195.
[16] Y. Pomeau, S. Rica, Phys. Rev. Lett. 71 (1993) 247.
[17] I.M. Khalatnikov, Introduction to the Theory of Superfluidity, Benjamin, New York, 1965.