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C. R. Physique 5 (2004) 1081–1090
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String theory and fundamental forces/Théorie des cordes et forces fondamentales

Semiclassical integrability in AdS/CFT

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Available online 11 November 2004

Presented by Guy Laval

Abstract

Integrable structures on both sides of the AdS/CFT correspondence are reviewed, with emphasis on the Bethe ansatz. ***To cite this article: K. Zarembo, C. R. Physique 5 (2004).***

Résumé

Intégrabilité semi-classique dans AdS/CFT. Des structures intégrables de chaque coté de la correspondance AdS/CFT sont rappelées, avec un accent sur la conjecture de Bethe. ***Pour citer cet article : K. Zarembo, C. R. Physique 5 (2004).***

Keywords: AdS/CFT; Bethe ansatz

Mots-clés : AdS/CFT ; Conjecture de Bethe

1. Introduction

The large-N limit of $\mathcal{N} = 4$ super-Yang–Mills (SYM) is a string theory that has a geometric description at strong coupling in terms of the $AdS_5 \times S^5$ background with RR flux [1–3]. A remarkable property of the large-N SYM is its complete integrability. It arises in CFT as a quantum symmetry of the operator mixing [4–8] and in AdS as a classical symmetry on the string world-sheet [9–11]. The quantum nature of integrability in CFT suggests that the world-sheet integrability should also survive quantization. Unfortunately not much is know about quantum string theory in $AdS_5 \times S^5$ and it is not possible to compare integrable structures on both sides of AdS/CFT directly, but it is possible to do such comparison in the semiclassical limit which is accurate for states with large quantum numbers [12,13]. I will shortly review the semiclassical integrability in AdS/CFT following [14].

There are two mechanisms by which SYM operators can acquire large quantum numbers. First, the majority of the SYM operators get large scaling dimensions at strong coupling [2], in the stringy regime of AdS/CFT. Or one can consider operators that contain large number of constituent fields [12]. Such operators have huge quantum numbers already at the tree level. It is qualitatively clear why both types of operators behave stringy. At strong coupling, diagrams with large number of vertices and propagators that resemble continuous string world-sheets obviously dominate. Planar diagrams for long operators always contain a large number of propagators and are stringy even at lowest orders of perturbation theory.

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1631-0705/$ – see front matter
doi:10.1016/j.crhy.2004.09.006

2. Integrability in CFT

The field content of $\mathcal{N}=4$ SYM theory consists of gauge fields A_μ, six scalars Φ_i and four Majorana fermions Ψ_α^A, all in the adjoint representation of $\mathrm{U}(N)$. The action is

$$S=\frac{1}{g^2}\int \mathrm{d}^4x\ \mathrm{tr}\left\{-\frac{1}{2}F_{\mu\nu}^2+(D_\mu\Phi_i)^2+[\Phi_i,\Phi_j]^2+\text{fermions}\right\}. \tag{1}$$

I will consider local gauge-invariant operators composed of two types of complex scalars $Z=\Phi_1+\mathrm{i}\Phi_2$ and $W=\Phi_3+\mathrm{i}\Phi_4$:

$$\mathcal{O}=\mathrm{tr}(Z^{L-M}W^M+\text{permutations}). \tag{2}$$

These operators transform non-trivially under a $\mathrm{SU}(2)\times\mathrm{U}(1)$ subgroup of the SO(6) R-symmetry of $\mathcal{N}=4$ SYM. (Z,W) transform as a doublet under U(2), so that the length of the operator L is its U(1) charge and $L-M$ is its SU(2) spin. The operators of the same length L obviously have the same bare dimension: $\Delta_0=L$. The number of operators with the same L and M grows exponentially and if the length is sufficiently large perturbation theory becomes highly degenerate.[2] This makes computation of anomalous dimensions for long operators highly non-trivial problem even at one loop.

The set of operators (2) is closed under renormalization and does not mix with operators that contain other fields or derivatives [15]. Including such operators is certainly possible [6,8] but will not be discussed here for the sake of simplicity. Indeed operators (2) admit a very simple parametrization. If we associate the Z field with spin up and the W with spin down, an operator of the form (2) can be associated with a distribution of spins on a periodic one-dimensional lattice of length L:

$$\mathrm{tr}\,ZZZWWZZZWWWZWZZZZ\ldots \quad\longleftrightarrow\quad |\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\ldots\rangle.$$

The map between operators and states of a spin chain is one-to-one if we impose an additional constraint that the states should be translationally invariant. Translational invariance is equivalent to the cyclicity of trace. This condition can be implemented as a constraint on admissible states. The mixing matrix acts linearly on operators and can thus be interpreted as a Hamiltonian of a lattice spin system.

The mixing matrix is defined as $\Gamma=\mathbf{Z}^{-1}\,\mathrm{d}\mathbf{Z}/\mathrm{d}\ln\Lambda$, where Λ is a UV cutoff and $(\mathbf{Z}-1)\mathcal{O}$ is the counterterm that makes correlation functions of $\mathbf{Z}\mathcal{O}$ finite. The counterterm need not be proportional to the bare operator, so $\mathbf{Z}_\mathbf{A}^\mathbf{B}$ and hence $\Gamma_\mathbf{A}^\mathbf{B}$ are matrices with multi-indices that parameterize all operators of the same length. The eigenvectors of the mixing matrix are conformal operators and eigenvalues are their anomalous dimensions: $\Gamma_\mathbf{A}^\mathbf{B}\mathcal{O}_B^{(n)}=\gamma_n\mathcal{O}_B^{(n)}$, so that $\Delta_n=L+\gamma_n$.

The mixing matrix can be easily computed at one loop. There are three diagrams that contribute (Fig. 1): the scalar self-interaction, the gluon exchange and the self-energy graphs. While the gluon exchange and the self-energy produce the same renormalized operator, the scalar vertex can lead to the interchange of Z and W fields. The really simplification occurs at large N, when the interchange can only occur between nearest neighbors. Indeed, an insertion of a vertex between a pair of non-nearest-neighbor propagators produces a non-planar diagram which is suppressed by $1/N$. As a result, the planar mixing matrix is a Hamiltonian of a spin chain with nearest-neighbor interactions [4]:

$$\Gamma=\frac{\lambda}{16\pi^2}\sum_{l=1}^{L}(1-P_{l,l+1}), \tag{3}$$

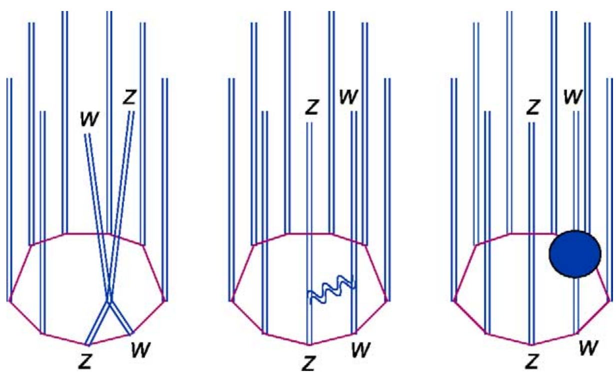

Fig. 1. The one-loop diagrams.

[2] It is assumed here that $N\to\infty$. At finite N the number of degenerate operators grows linearly with L.

Fig. 2. The spectrum of the Heisenberg spin chain.

where $\lambda = g^2 N$ is the 't Hooft coupling and P is the permutation operator: $P\, a \otimes b = b \otimes a$. The use of the identity $P = (1 + \sigma \otimes \sigma)/2$ brings the mixing matrix to the form

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} (1 - \sigma_l \cdot \sigma_{l+1}). \tag{4}$$

This is the Hamiltonian of the Heisenberg spin chain. The remarkable property of this model is its completely integrability.

Though the Heisenberg Hamiltonian contains no adjustable parameters except for the length of the chain, it is possible to identify several energy scales in its spectrum (Fig. 2). The ground state is the ferromagnetic vacuum (all spin up). The ground state energy is zero because the corresponding operator $\operatorname{tr} Z^L$ belongs to a short multiplet and has zero anomalous dimension to all orders in perturbation theory. The excited states are created by flipping one or more spins. The approximate creation operators are

$$a_n^\dagger = \frac{1}{\sqrt{L}} \sum_{l=1}^{L} \mathrm{e}^{2\pi \mathrm{i} nl/L} \sigma_l^-. \tag{5}$$

The operator $a_n^\dagger$ creates magnon with mode number n and momentum $p = 2\pi n/L$. If L is large, Fock states $a_{n_1}^\dagger \cdots a_{n_M}^\dagger |0\rangle$ (with $M \ll L$) approximate the eigenstates of the Heisenberg Hamiltonian sufficiently well. These states correspond to the BMN operators [12], which are dual to string states in the pp-wave limit of the $AdS_5 \times S^5$ geometry. Their anomalous dimensions

$$\gamma = \frac{\lambda}{2L^2} \sum_{k=1}^{M} n_k^2 \tag{6}$$

match with the energies computed on the string side.

The situation changes when the number of magnons becomes macroscopically large: $M \sim L$. The interaction between magnons then cannot be neglected and the Fock space generated by simple operators (5) is no longer a good approximation for the spectrum. Fortunately, the spectrum of the Heisenberg model can be computed exactly [16,17]. The energy eigenvalues can be found by solving a set of algebraic Bethe equations:

$$\left(\frac{u_j + \mathrm{i}/2}{u_j - \mathrm{i}/2}\right)^L = \prod_{k \neq j} \frac{u_j - u_k + \mathrm{i}}{u_j - u_k - \mathrm{i}}. \tag{7}$$

The Bethe roots u_j, $j = 1, \ldots, M$, are rapidities of individual magnons. The rapidity is related to the momentum as

$$\mathrm{e}^{\mathrm{i}p} = \frac{u + \mathrm{i}/2}{u - \mathrm{i}/2},$$

so the momentum constraint takes the form

$$\prod_j \frac{u_j + \mathrm{i}/2}{u_j - \mathrm{i}/2} = 1. \tag{8}$$

The anomalous dimension is given by

$$\gamma = \frac{\lambda}{8\pi^2} \sum_j \frac{1}{u_j^2 + 1/4}. \tag{9}$$

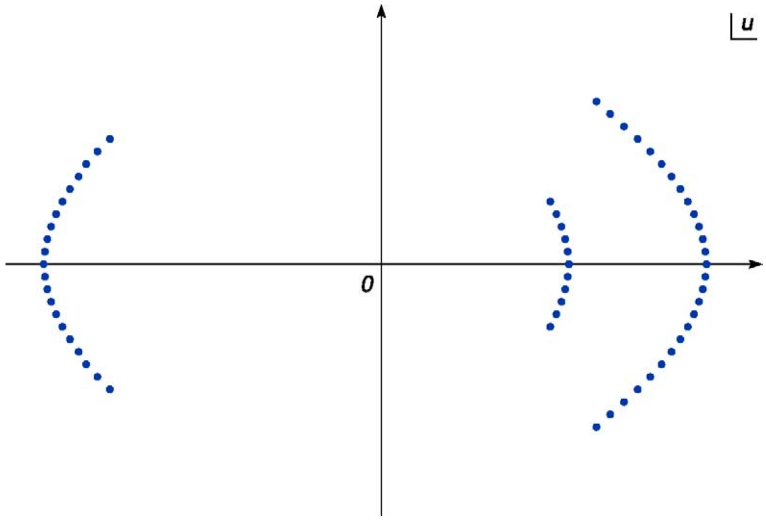

Fig. 3. Macroscopically large Bethe strings.

The simplest solution of Bethe equations that satisfies the momentum constraint contains two roots:

$$u_1 = -u_2 = \cot \frac{2\pi n}{L-1}, \quad n = 1, 2, \ldots, \tag{10}$$

with

$$\gamma = \frac{\lambda}{\pi^2} \sin^2 \frac{\pi n}{L-1}. \tag{11}$$

When L is large, $u_{1,2} \approx \pm L/2\pi n$ and γ is well approximated by the BMN formula (6). Since $u_i \sim L$, the right-hand side of Bethe equations can be replaced by 1, magnons do not interact and the spectrum is the Fock space of free magnons. In fact, the scattering can be neglected only if all mode numbers are different. The Bethe ansatz does not allow different Bethe roots to coincide completely. What happens when we try to put two magnons in the same momentum state is that they form a bound state and their rapidities become complex. Because $(u + \mathrm{i}/2)/(u - \mathrm{i}/2)$ is no longer a pure phase when u is complex, the left-hand side of Bethe equations turns either to zero or to infinity as $L \to \infty$. Hence, the right-hand side should develop either zero or pole. This can only happen if two of the rapidities are separated by $\pm\mathrm{i}$. Bethe roots with the same mode number form arrays parallel to the imaginary axis with separation between adjacent roots equal to i. Such arrays are usually called strings. The number of Bethe roots in a string can be arbitrary, even macroscopically large [18] (Fig. 3). In the latter case strings bend along some contours in the complex plane. Corresponding Bethe states describe macroscopic spin waves and are dual to semiclassical strings in $AdS_5 \times S^5$ [19].

Since Bethe roots scale linearly with L, it is natural to define $x_i = u_i/L$, which then stays finite as $L \to \infty$. Taking logarithm of the Bethe equations (7) and expanding in $1/L$, we get

$$\frac{1}{x_j} + 2\pi n_j = \frac{1}{L} \sum_{k \neq j} \frac{2}{x_j - x_k}, \tag{12}$$

where the phases $2\pi n_j$ parameterize different branches of the logarithm. The rapidities of magnons with the same mode number n_j form a string with the center of mass at $x = 2\pi n_j$. The distance between adjacent roots $x_k - x_{k+1} \sim 1/L$, so the distribution of roots can be characterized by a continuous density in the scaling limit:

$$\rho(x) = \frac{1}{L} \sum_j \delta(x - x_j). \tag{13}$$

The density is non-zero on a collection of contours $C = C_1 \cup \cdots \cup C_K$ in the complex plane and is normalized by

$$\int_C \mathrm{d}x\, \rho(x) = \frac{M}{L}, \tag{14}$$

where M is the total number of magnons or the number of W fields in the operator (2). Equivalently, the distribution of Bethe roots can be characterized by the resolvent:

$$G(x)=\frac{1}{L}\sum_j\frac{1}{x-x_j}=\int_C \mathrm{d}y\,\frac{\rho(y)}{x-y}. \tag{15}$$

The Taylor expansion of $G(x)$ at zero generates local conserved charges of the Heisenberg model [20,21]. In particular, the total momentum is $-G(0)$. Translational invariance of the physical states requires

$$-G(0)=\int_C \mathrm{d}x\,\frac{\rho(x)}{x}=2\pi m. \tag{16}$$

The Taylor coefficient before x determines the anomalous dimension:

$$\gamma=\frac{\lambda}{8\pi^2L}\int_C \mathrm{d}x\,\frac{\rho(x)}{x^2}. \tag{17}$$

The Bethe equations reduce to a singular integral equation for the density:

$$\fint_{\mathbf{C}}\frac{\mathrm{d}y\,\rho(y)}{x-y}=\frac{1}{x}+2\pi n_k,\quad x\in C_k. \tag{18}$$

This equation can be solved in hyperelliptic integrals. The details can be found in [14]. The Riemann surface associated with the hyperelliptic integrals is obtained by gluing together two copies of the complex plane with cuts along the contours C_k.

The simplest one- and two-cut solutions have been worked out in detail [19,22,14] and were compared to Frolov–Tseytlin string solitons [23]. Even $1/L$ corrections can be calculated in the simplest cases [24]. The scaling dimensions of operators agree with the energies of the string solitons up to two loops. At three loops the agreement generally breaks down [25,14]. The two-loop agreement can be established quite generally at the level of equations of motion or effective actions [26]. Even the higher charges of the integrable hierarchies, that do not have geometric interpretation in AdS/CFT, agree for simplest solutions [20]. In fact, it is possible to match the whole integrable hierarchies, including Bethe ansatz equations [14]. Usually, Bethe ansatz is associated with quantum systems, but it turns out that classical solutions of the sigma-model can also be parameterized by an integral equation which resembles the scaling limit of Bethe equations in the Heisenberg model.

3. Integrability in AdS

The R-charges of the operator $\mathrm{tr}(Z^{L-M}W^M+\cdots)$, L and M, translate into two angular momenta of the string on S^5. A string moving in the $S^3\subset S^5$ and sitting in the middle of AdS_5 has the right quantum numbers, and I will concentrate on this particular subsector by all setting transverse coordinates to zero. Then the world sheet is parameterized by the global AdS time X^0 and by four Cartesian coordinates X^i constrained by $X^iX^i=1$. A point on the three-sphere defines a group element of SU(2):

$$g=\begin{pmatrix}X_1+\mathrm{i}X_2 & X_3+\mathrm{i}X_4\\ -X_3-\mathrm{i}X_4 & X_1-\mathrm{i}X_2\end{pmatrix}\equiv\begin{pmatrix}Z_1 & Z_2\\ -\overline{Z}_2 & \overline{Z}_1\end{pmatrix}\in \mathrm{SU}(2). \tag{19}$$

The string action and the equations of motion can be formulated in terms of currents

$$j_a=g^{-1}\partial_a g=\frac{\sigma^A}{2\mathrm{i}}\,j_a^A. \tag{20}$$

In the conformal gauge,[3]

$$S_{\sigma m}=-\frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi}\mathrm{d}\sigma\int \mathrm{d}\tau\left[\frac{1}{2}\,\mathrm{Tr}\,j_a^2+(\partial_a X_0)^2\right], \tag{21}$$

which give the following equations of motion:

[3] The world-sheet metric is $(+-)$.

$$\partial_+\partial_- X_0 = 0, \tag{22}$$

$$\partial_+ j_- + \partial_- j_+ = 0, \tag{23}$$

$$\partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0, \tag{24}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$ and $j_\pm = j_\tau \pm j_\sigma$. The equation of motion for X^0 always has a trivial solution

$$X^0 = \kappa\tau.$$

The equation of motion should be supplemented by Virasoro constraints

$$\frac{1}{2}\,\mathrm{tr}\, j_\pm^2 = -(\partial_\pm X^0)^2 = -\kappa^2. \tag{25}$$

The global symmetry of the sigma-model (21) is $\mathrm{SU}_L(2) \times \mathrm{SU}_R(2) \times \mathbb{R}$. The first two factors are associated with the left and right group multiplication: $g \to hg$ and $g \to gh$. The Noether current of these symmetries are l_a and j_a, where

$$l_a = g j_a g^{-1} = \partial_a g\, g^{-1} = \frac{\sigma^A}{2\mathrm{i}} l_a^A. \tag{26}$$

Therefore

$$Q_L^A = \frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} \mathrm{d}\sigma\, l_\tau^A, \qquad Q_R^A = \frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} \mathrm{d}\sigma\, j_\tau^A \tag{27}$$

generate left and right group shifts. The dual R-charges in the SYM theory can be easily identified. The six scalars of the SYM transform under SO(6) in the same way as the six Cartesian coordinates that parameterize the sphere: $\Phi_i \sim X^i$. Since $Z = \Phi_1 + \mathrm{i}\Phi_2$ and $W = \Phi_3 + \mathrm{i}\Phi_4$, these fields transform as Z_1 and Z_2 in (19). Thus (Z_1, Z_2) and (Z, W) are doublets of $\mathrm{SU}_R(2)$, so that Z has $Q_R^3 = 1$ and W and $Q_R^3 = -1$. For the operator (2),

$$Q_R^3 = L - 2M. \tag{28}$$

Under the left shifts, $(Z_1, -\overline{Z}_2)$ and $(Z_2, -\overline{Z}_1)$ transform as doublets. Therefore, $(Z, -\overline{W})$ and $(W, -\overline{Z})$ are doublets of $\mathrm{SU}_L(2)$ and both fields Z and W have $Q_L^3 = 1$. The left charge of the operator (2) is just the length of the spin chain:

$$Q_L^3 = L. \tag{29}$$

The time translations $X^0 \to X^0 + t$ generate scale transformations on the boundary of AdS_5. The energy of the string thus should be identified with the scaling dimension of the dual operator:

$$\Delta = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma\, \partial_\tau X_0 = \sqrt{\lambda}\,\kappa. \tag{30}$$

The equation of motion for the chiral field (22), (23) are completely integrable [27,28] and can be effectively linearized with the help of the inverse scattering transformation [29]. This method is based on the zero-curvature representation of the equations of motion [30]. The rescaled currents [28]

$$J_\pm(x) = \frac{j_\pm}{1 \mp x} \tag{31}$$

are flat for any value of x:

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0, \tag{32}$$

as a consequence of the equations of motion. The converse is also true: If (32) is satisfied for any x, then $j_\pm$ are solutions of the equations of motion. The monodromy of the flat connection (31) defines the quasi-momentum $p(x)$:

$$\Omega(x) = P\exp\left(-\int_0^{2\pi} \mathrm{d}\sigma\, J_\sigma\right) = P\exp\int_0^{2\pi} \mathrm{d}\sigma\, \frac{1}{2}\left(\frac{j_+}{x-1} + \frac{j_-}{x+1}\right) \tag{33}$$

and

$$\mathrm{tr}\,\Omega(x) = 2\cos p(x), \tag{34}$$

where the integral is taken along a fixed-time section of the world-sheet. Because the connection is flat on-shell, the result of integration does not depend on which particular section is taken and therefore the quasi-momentum is conserved as soon as the equations of motion are satisfied. The quasi-momentum can thus be regarded as a generating function for an infinite set of integrals of motion.

The $\mathrm{SU}_{L,R}(2)$ charges appear in the expansion of $p(x)$ in $1/x$ and in x at large and small x, respectively. Since $J_\sigma(x) = j_\tau/x + \cdots$ at infinity,

$$\operatorname{tr}\Omega = 2 + \frac{1}{2x^2}\int_0^{2\pi} \mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\,\operatorname{tr} j_0(\sigma_1) j_0(\sigma_2) + \cdots = 2 - \frac{4\pi^2 Q_R^2}{\lambda x^2} + \cdots = 2 - \frac{4\pi^2 (L-2M)^2}{\lambda x^2} + \cdots. \tag{35}$$

Hence,

$$p(x) = -\frac{2\pi(L-2J)}{\sqrt{\lambda}\,x} + \cdots \quad (x \to \infty). \tag{36}$$

The expansion of the monodromy matrix $\Omega(x)$ at infinity before taking the trace generates Yangian charges, which are potentially important in the AdS/CFT correspondence in general [31].

At $x \to 0$, $\partial_\sigma + J_\sigma(x) = \partial_\sigma + j_\sigma - x j_\tau + \cdots = g^{-1}(\partial_\sigma - x l_\tau + \cdots) g$, so

$$\operatorname{tr}\Omega = 2 + \frac{x^2}{2}\int_0^{2\pi} \mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\,\operatorname{tr} l_0(\sigma_1) l_0(\sigma_2) + \cdots = 2 - \frac{4\pi^2 Q_L^2}{\lambda} x^2 + \cdots = 2 - \frac{4\pi^2 L^2}{\lambda} x^2 + \cdots, \tag{37}$$

which yields

$$p(x) = 2\pi m + \frac{2\pi L}{\sqrt{\lambda}} x + \cdots \quad (x \to 0), \tag{38}$$

where m is an arbitrary integer.

The local charges are obtained by expanding the quasi-momentum in the Lorant series at $x = \pm 1$. I will only compute the leading order here, but it is possible to develop a systematic procedure that recursively determines all local integrals of motion [30]. A slightly different but equivalent definition of the quasi-momentum is more appropriate for that purpose. Let us consider the auxiliary linear problem

$$\left[\partial_\sigma - \frac{1}{2}\left(\frac{j_+}{x-1} + \frac{j_-}{x+1}\right)\right]\psi = 0, \tag{39}$$

where $\psi(\sigma; x)$ is a two-component vector ($j_\pm$ are anti-Hermitian 2×2 matrices). This equation can be regarded as an eigenvalue problem for a one-dimensional Dirac operator with a periodic potential, where x plays the role of the spectral parameter. Two linearly independent solutions can be chosen quasiperiodic: $\psi(\sigma + 2\pi; x) = \mathrm{e}^{\pm \mathrm{i} p(x)}\psi(\sigma; x)$. This is the standard definition of the quasi-momentum. It is equivalent to the previous one, because $\psi(\sigma + 2\pi; x) = \Omega(x)\psi(\sigma; x)$, which is true for any solutions, not only quasi-periodic. Quasi-periodicity corresponds to diagonalization of the monodromy matrix, whose eigenvalues are precisely $\mathrm{e}^{\pm \mathrm{i} p(x)}$.

When x is close to 1 or -1, we can keep only the singular term in the potential and rewrite (39) as

$$(\hbar\partial_\sigma - j_\pm)\psi = 0, \quad \hbar \equiv 2(x \mp 1). \tag{40}$$

The limit $x \to \pm 1$ is equivalent to $\hbar \to 0$. As usual in the semiclassical limit, we look for the solution in the form $\psi = \chi\,\mathrm{e}^{\mathrm{i}S/\hbar}$. Plugging this into (40), we find that χ satisfies an algebraic equation

$$(\mathrm{i}\partial_\sigma S - j_\pm)\chi = 0,$$

which has solutions only if $\det(j_\pm - \mathrm{i}\partial_\sigma S) = 0$. It follows from the Virasoro constraints (25) that the two eigenvalues of $j_\pm$ are $\pm \mathrm{i}\kappa$. Choosing the upper sign we find that $S(\sigma) = \kappa\sigma$ and

$$\psi(\sigma + 2\pi; x) = \exp\left(\frac{\mathrm{i}\pi\kappa}{x \mp 1}\right)\psi(\sigma; x).$$

Therefore[4]

$$p(x) = -\frac{\pi\kappa}{x \mp 1} + \cdots \quad (x \to \pm 1). \tag{41}$$

[4] The local analysis determines $p(x)$ only up to a sign. Here the sign ambiguity is fixed. As a result, some solutions, like for instance pulsating strings [32,21], can be lost. See Section 5.3 of [14] for more details.

The linear problem (39) has a band spectrum, but since the Dirac operator in (39) does not possess any particular Hermiticity properties allowed bands do not lie on the real axis. The quasi-momentum is real on the allowed zones, so $\operatorname{tr}\Omega$ is real there and $|\operatorname{tr}\Omega| < 2$. Forbidden zones can be defined as contours on which the quasi-momentum is pure imaginary ($\operatorname{tr}\Omega$ real, $|\operatorname{tr}\Omega| > 2$). At zone boundaries $\operatorname{tr}\Omega = 2$ and the monodromy matrix Ω degenerates into a Jordan cell.[5] The two quasi-periodic solutions of the Dirac equation, let us denote them $\psi_\pm(\sigma; x)$, become degenerate at zone boundaries. Going around a zone boundary interchanges the two solutions, so $\psi_\pm(x)$ have branch points there. By cutting the complex plane along the forbidden zones and gluing two copies of it together we get a Riemann surface on which $\psi_\pm$ are globally defined as two branches of a single meromorphic function. The same is true for the eigenvalues of the monodromy matrix $e^{\pm ip(x)}$. This is easy to understand. The trace of the monodromy matrix $\operatorname{tr}\Omega(x)$ is non-analytic at $x = \pm 1$, where the potential in the Dirac operator has a pole, but has no other singularities. However, solving $e^{ip} + e^{-ip} = \operatorname{tr}\Omega$ for e^{ip} produces square root branch points precisely at the zone boundaries.

Let us now pick up a particular branch of the quasi-momentum $p(x)$ which is an analytic function of x on the complex plane with cuts and has single poles at $x = \pm 1$. Subtracting the poles we get a function

$$G(x) = p(x) + \frac{\pi\kappa}{x-1} + \frac{\pi\kappa}{x+1}, \tag{42}$$

which has only branch cut singularities and therefore is completely determined by its discontinuity: $G(x + i0) - G(x - i0) \equiv 2i\pi\rho(x)$. It is straightforward to prove that $G(x)$ admits the dispersion representation

$$G(x) = \int_C dy\, \frac{\rho(y)}{x-y}, \tag{43}$$

which is an analog of (15).

The density $\rho(x)$ satisfies an integral equation which reflects the unimodularity of the monodromy matrix. To derive this equation, let us consider the behavior of the quasi-momentum near a forbidden zone where it experiences a jump, so that $e^{ip(x\pm i0)}$ are two branches of the double-valued analytic function. The two branches are the two eigenvalues of the monodromy matrix $\Omega(x)$ and thus satisfy $e^{ip(x+i0)}\, e^{ip(x-i0)} = 1$, or

$$p(x + i0) + p(x - i0) = 2\pi n_k, \quad x \in C_k, \tag{44}$$

where C_k is one of the forbidden zones. Taking into account (42) and the spectral representation (43), we get

$$G(x + i0) + G(x - i0) = 2\,⨍ dy\, \frac{\rho(y)}{x-y} = \frac{2\pi\kappa}{x-1} + \frac{2\pi\kappa}{x+1} + 2\pi n_k, \quad x \in C_k. \tag{45}$$

The density also satisfies several normalization conditions which follow from (36), (38) and (41):

$$\int dx\, \rho(x) = \frac{2\pi}{\sqrt{\lambda}}(\Delta + 2J - L), \tag{46}$$

$$\int dx\, \frac{\rho(x)}{x} = 2\pi m, \tag{47}$$

$$\int dx\, \frac{\rho(x)}{x^2} = \frac{2\pi}{\sqrt{\lambda}}(\Delta - L). \tag{48}$$

Eqs. (45)–(48) are direct analogs of the classical Bethe equations in the spin chain. The change of variables $x \to 4\pi Lx/\sqrt{\lambda}$ makes this analogy explicit:

$$2\,⨍ dy\, \frac{\rho(y)}{x-y} = \frac{x}{x^2 - \lambda/(16\pi^2 L^2)}\, \frac{\Delta}{L} + 2\pi n_k, \quad x \in C_k, \tag{49}$$

$$\int dx\, \rho(x) = \frac{M}{L} + \frac{\Delta - L}{2L}, \tag{50}$$

$$\int dx\, \frac{\rho(x)}{x} = 2\pi m, \tag{51}$$

$$\Delta - L = \frac{\lambda}{8\pi^2 L} \int dx\, \frac{\rho(x)}{x^2}. \tag{52}$$

In the limit $\lambda/L^2 \to 0$ these equations coincide with (18), (14), (16) and (17).

[5] Not all points with $\operatorname{tr}\Omega = 2$ are zone boundaries! The eigenvalues of Ω can become degenerate while Ω still has two independent eigenvectors.

4. Discussion

The main result of the long derivation in Section 3 is an integral equation that parameterizes classical solution of the string sigma-model [14]:

$$2\fint \mathrm{d}y \, \frac{\rho(y)}{x-y} = \frac{x}{x^2 - \lambda/(16\pi^2 L^2)} \, \frac{\Delta}{L} + 2\pi n_k. \tag{53}$$

At $\lambda/L^2 \to 0$ this equation reduces to the classical Bethe equation for the spin chain:

$$2\fint \mathrm{d}y \, \frac{\rho(y)}{x-y} = \frac{1}{x} + 2\pi n_k. \tag{54}$$

That equation in its turn is an approximation to the exact quantum Bethe equations

$$\left(\frac{u_j + \mathrm{i}/2}{u_j - \mathrm{i}/2}\right)^L = \prod_{k \neq j} \frac{u_j - u_k + \mathrm{i}}{u_j - u_k - \mathrm{i}}. \tag{55}$$

In view of the analogy between (53) and (54), it is natural to call (53) classical Bethe equation. Is there quantum Bethe equation for which (53) is a scaling limit? Such equation, if exists, will describe non-perturbative spectrum in the SU(2) subsector of the AdS string and hence of the large-N SYM. It is not clear at present how quantum Bethe equations in string theory could be derived from first principles, but some hints come from the known solution of the SU(2) chiral field (sigma-model on S^3) [33]. According to [33] the sigma-model is equivalent to a four-fermion model which is solvable by Bethe ansatz. Of course, restriction to the SU(2) sector only makes sense at the classical level, but Bethe ansatz equations usually have rather rigid group structure [34] and in many cases Bethe equations for a larger group can be guessed by looking at the Bethe ansatz for its subgroup. It is not inconceivable that the classical limit uniquely fixes quantum Bethe equations. In any case, a putative quantum Bethe ansatz for the AdS string should reproduce (53) in the classical limit. In other words quantum Bethe equations should be a discretization of (53). A particular discretization was proposed in [35] and passed several non-trivial checks. Though the equations of [35] are valid at strong coupling, they do have a spin-chain interpretation [36]. The Hamiltonian of this spin chain starts to deviate from the mixing matrix of $\mathcal{N}=4$ SYM at three loops.

Acknowledgements

I would like to thank V. Kazakov, A. Marshakov and J. Minahan for fruitful collaboration.

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