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String theory and fundamental forces/Théorie des cordes et forces fondamentales

Supersymmetric backgrounds from generalized Calabi–Yau manifolds

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Abstract

We show that the supersymmetry transformations for type II string theories on six-manifolds can be written as differential conditions on a pair of pure spinors, the exponentiated Kähler form e^{iJ} and the holomorphic form Ω. The equations are explicitly symmetric under exchange of the two pure spinors and a choice of even or odd-rank RR field. This is mirror symmetry for manifolds with torsion. Moreover, RR fluxes affect only one of the two equations: e^{iJ} is closed under the action of the twisted exterior derivative in IIA theory, and similarly Ω is closed in IIB. This means that supersymmetric SU(3)-structure manifolds are always complex in IIB while they are twisted symplectic in IIA. Modulo a different action of the B-field, these are all generalized Calabi–Yau manifolds, as defined by Hitchin. ***To cite this article: M. Graña et al., C. R. Physique 5 (2004).***

Résumé

Fonds supersymetriques à partir de variétés de Calabi–Yau généralisées. On montre que les transformations de supersymétrie pour les théories des cordes de type II peuvent être traduites dans des équations différentielles pour une paire de spineurs purs, l'exponentiel de la forme de Kähler e^{iJ} et la forme holomorphe Ω. Ces équations sont symétriques sous l'échange des deux spineurs purs et des formes de RR de rang pair ou impair. Cette propriété est la symétrie miroir pour les variétés avec torsion. On voit aussi que les fluxes de RR entrent seulement dans une des deux équations : e^{iJ} est fermé sous l'action de la dérivée extérieure « twisted » dans la corde de type IIA, et de la même manière Ω est fermé en type IIB. Cela implique que les variétés supersymétriques de structure SU(3) sont toujours complexes en type IIB ou bien symplectiques « twisted » en IIA. Ces variétés sont donc des variétés des Calabi–Yau généralisées selon la définition de Hitchin, mais avec une action du champ B différente. ***Pour citer cet article : M. Graña et al., C. R. Physique 5 (2004).***

Keywords: Supersymmetric backgrounds; Generalized Calabi–Yau manifolds

Mots-clés : Fonds supersymetriques ; Variétés de Calabi–Yau généralisées

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1. Introduction

Compactifications with fluxes have received much attention recently due to a number of interesting features. In many ways these can be seen as extensions of the more conventional compactifications on Ricci-flat manifolds. On the other hand, many aspects of the latter, most notably in the case of Calabi–Yau manifolds, still have to find their generalized counterparts. Mirror symmetry has been one of the most prominent and useful features of Calabi–Yau compactifications, and the question of its extension to compactifications with fluxes is both of conceptual and of practical interest.

The issue of extending mirror symmetry to compactifications with fluxes has been studied recently in [1–5]. A first question is, of course, within which class of manifolds this symmetry should be defined. A natural proposal comes from the formalism of G-structures, recently used in many contexts of compactifications with fluxes. As shown in [3,4], mirror symmetry can be defined on manifolds of SU(3) structure, thus generalizing the usual Calabi–Yau case. One of the points which makes this symmetry non-trivial is that, as expected, geometry and NS flux mix in the transformation. On the other hand, RR fluxes are mapped by mirror symmetry into RR fluxes and their transformation is well-understood. However, for many reasons it would be better to have a formalism that would incorporate geometrical data and fluxes in a natural way. As a step forward in that direction, we propose to use pure spinors of Clifford$(6,6)$ as a formalism to describe SU(3)-structure compactifications.

As far as we are concerned in this introduction, Clifford$(6,6)$ spinors are simply formal sums of forms, in analogy with usual spinors, which are often realized as formal sums of $(p,0)$ forms. A spinor is called pure if it is annihilated by half of the gamma matrices. A pure spinor defines an $\mathrm{SU}(3,3)$ structure on the bundle $T+T^*$ on the manifold. If the spinor is also closed, the manifold is called by Hitchin [6] a generalized Calabi–Yau.

For a SU(3) structure on T, there are two pure spinors, φ_1 and φ_2 which are orthogonal and of unit norm. An SU(3) structure is defined by a two-form J and a three-form Ω obeying $J\wedge\Omega=0$ and $\mathrm{i}\Omega\wedge\overline{\Omega}=(2J)^3/3!$. Then, the two pure spinors are $\mathrm{e}^{\mathrm{i}J}$ and Ω. We will show that supersymmetry equations imply differential equations for the pure spinors, which are, schematically

$$\begin{aligned}&\mathrm{e}^{-f_1}\mathrm{d}(\mathrm{e}^{f_1}\varphi_1)=H\bullet\varphi_1,\\&\mathrm{e}^{-f_2}\mathrm{d}(\mathrm{e}^{f_2}\varphi_2)=H\bullet\varphi_2+(F,\varphi).\end{aligned}\tag{1}$$

The operator $H\bullet$ is a certain action of the three-form H, involving contractions and wedges but different from $H\wedge$. So, both in IIA and IIB there is a 'preferred' pure spinor (of the same parity as F – namely $\mathrm{e}^{\mathrm{i}J}$ in IIA and Ω in IIB) which does not receive any back reaction from the RR fluxes, i.e. which is 'twisted' closed. Then supersymmetry implies that 6-dimensional manifolds are all 'twisted' generalized Calabi–Yau [6]. The twisting refers to the presence of the H field. In the mathematical literature (and in some physical applications [7]) this twisting is actually always appearing in the form $(\mathrm{d}+H\wedge)$. It is interesting to see that, in general, the inclusion of RR fluxes requires a different form of twisting than the one usually assumed. Understanding the origin of this twisting from first principles remains an important open problem.

2. SU(3) structure and torsion versus fluxes

We start by briefly introducing the notions of SU(3)-structure and intrinsic torsion with the help of which we will describe the non-Ricci-flat geometries under consideration. For a more extensive review, see for example [3] and references therein. A manifold with SU(3)-structure has all the group-theoretical features of a Calabi–Yau, namely invariant two- and three forms, J and Ω, respectively. On a manifold of SU(3) holonomy, not only J and Ω are well defined (nowhere vanishing, SU(3) invariant), but they are also closed: $\mathrm{d}J=0=\mathrm{d}\Omega$. If they are not closed, $\mathrm{d}J$ and $\mathrm{d}\Omega$ give a good measure of how far the manifold is from having SU(3) holonomy. Decomposing $\mathrm{d}J$ and $\mathrm{d}\Omega$ in the different $\mathrm{SU}(3)$ representations, we can write

$$\begin{aligned}&\mathrm{d}J=-\tfrac{3}{2}\,\mathrm{Im}(W_1\overline{\Omega})+W_4\wedge J+W_3,\\&\mathrm{d}\Omega=W_1J^2+W_2\wedge J+\overline{W}_5\wedge\Omega.\end{aligned}\tag{2}$$

The Ws are the $(3\oplus\bar{3}\oplus 1)\otimes(3\oplus\bar{3})$ components of the intrinsic torsion: W_1 is a complex zero-form in $1\oplus 1$, W_2 is a complex primitive two-form, so it lies in $8\oplus 8$, W_3 is a real primitive $(2,1)\oplus(1,2)$ form and it lies in $6\oplus\bar{6}$, W_4 is a real one-form in $3\oplus\bar{3}$, and finally W_5 is a complex $(1,0)$-form (notice that in (2) the $(0,1)$ part drops out), so its degrees of freedom are again $3\oplus\bar{3}$. These W_i allow to classify the differential type of any SU(3) structure.

An SU(3) structure can be defined also by a spinor η, which is nowhere vanishing, SU(3) invariant, but not covariantly constant, unless the manifold has SU(3) holonomy. In terms of this, J and Ω above are defined as bilinears:

$$\begin{aligned}&\eta^\dagger\gamma_{mn}\gamma\eta=\mathrm{i}J_{mn},\\&-\mathrm{i}\eta^\dagger\gamma_{mnp}(1+\gamma)\eta=\Omega_{mnp}.\end{aligned}\tag{3}$$

Table 1
Decomposition of torsion and fluxes into SU(3) representations

	$1 \oplus \bar{1}$	$3 \oplus \bar{3}$	$6 \oplus \bar{6}$	$8 \oplus 8$
Torsion	1 (W_1)	2 (W_4, W_5)	1 (W_3)	1 (W_2)
H_3	1	1	1	0
IIA: F_{2n}	2 (F_0, F_2, F_4)	2 (F_2, F_4)	0	1 (F_2, F_4)
IIB: F_{2n+1}	1 (F_3)	3 (F_1, F_3, F_5)	1 (F_3)	0

Torsion is induced by flux, so in any solution to the equations of motion any nonvanishing torsion has to be compensated by a nonvanishing flux in the same representation. We can gain a lot of insight just by decomposing the fluxes into the different SU(3) representations, and searching for any missing one. Table 1 shows the number of times each representation appears for torsion, NS and RR fluxes.

Just by looking at Table 1, we realize that in IIB there is no flux capable of compensating the torsion class W_2. Thus, we can conclude that in any IIB solution, W_2, which is an obstruction for getting complex geometry, has to vanish. In IIA there is no RR flux capable of compensating W_3 so, if this last torsion is not zero, it must be compensated by NS flux. This means that in IIA there should be a relation $W_3 \sim H^{(6)}$ (the 6 denotes the representation). W_3 appears in the derivative of J, so it is an obstruction to have symplectic geometry. Another torsion class, W_1, appears in both $\mathrm{d}J$ and $\mathrm{d}\Omega$, and represents an obstruction to have either complex or symplectic geometries. If additionally $W_1 = 0$, which is true in any IIA and IIB supersymmetric solution with SU(3) structure,[1] then supersymmetric 6-manifolds with SU(3) structure are always complex in IIB while they are 'twisted symplectic' in IIA (twisting refers to H-flux in the relation $W_3 \sim \mathrm{d}J \sim H^6$, we will expand on this later).

Since IIA is related to symplectic geometries while IIB is associated to complex ones, one immediately wonders if there is a mathematical construction that contains, or even more, extends, both. That mathematical construction is generalized complex geometry. It has been introduced by Hitchin [6] (see [9] for details and further developments), and recently used in string theory related context by [10–13]. It is clear that this formalism must be useful for mirror symmetry: although for the physical string mirror symmetry is an exchange of Calabi–Yau's, for the topological string it can be formulated as sending symplectic manifolds into complex ones, and vice versa.

3. Generalized complex geometry

Usual complex geometry deals with the tangent bundle of a manifold T, whose sections are vectors X, and separately, with the cotangent bundle T^*, whose sections are 1-forms ζ. In generalized complex geometry one deals with the direct sum of the tangent and cotangent bundle, $T \oplus T^*$ rather than the tangent (or cotangent) bundle itself, whose sections are the sum of a vector field plus a one-form $X + \zeta$. The standard machinery of complex geometry can be generalized to this bundle.

To start with, let us consider the almost complex structure. If ordinary almost complex structures J are bundle maps from T to itself that square to $-\mathbb{I}_d$ (d is the real dimension of the manifold), generalized almost complex structures $\mathcal{J}$ are maps of $T \oplus T^*$ to itself that square to $-\mathbb{I}_{2d}$. As for an almost complex structure, $\mathcal{J}$ must also satisfy the hermiticity condition $\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$, with the respect to the natural metric on $T \oplus T^*$, $\mathcal{I} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Such generalized almost complex structures have the form

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix}, \tag{4}$$

where $J : TM \to TM$, $P : T^*M \to TM$, $L : TM \to T^*M$ and $K : T^*M \to T^*M$.

The condition $\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$ leads to $K = -J^t$, $P = -P^t$ and $L = -L^t$, so the matrix (4) can be expressed:

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & -J^t \end{pmatrix}, \tag{5}$$

with P and L antisymmetric matrices. The condition $\mathcal{J}^2 = -\mathbb{I}_{2d}$ imposes further constrains for J, P and L, in particular $J^2 + PL = -\mathbb{I}_d$. From this, it is easy to see that usual complex structures are naturally embedded into $\mathcal{J}$: they correspond to the choice

$$\mathcal{J}_1 \equiv \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}, \tag{6}$$

[1] This statement cannot be concluded just by looking at representations, since both in IIA and IIB there are enough scalars in the flux to compensate W_1. It is derived by looking at all supersymmetry equations, as done in [8].

with $J_m{}^n$ an almost complex structure (i.e. $J^2 = -\mathbb{I}_d$). Another example of generalized almost complex structure can be built using a non degenerate two-form ω,

$$\mathcal{J}_2 \equiv \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{7}$$

Given an almost complex structure one can build holomorphic and antiholomorphic projectors $\pi_\pm = \frac{1}{2}(1_d \pm \mathrm{i}J)$. Correspondingly, projectors can be build out of a generalized almost complex structure, $\Pi_\pm = \frac{1}{2}(1_{2d} \pm \mathrm{i}\mathcal{J})$. There is an integrability condition for generalized almost complex structures, analogous to the integrability condition for usual almost complex structures. For the usual complex structures integrability, namely the vanishing of the Nijenhuis tensor, can be written as the condition $\pi_\mp[\pi_\pm X, \pi_\pm Y] = 0$, i.e. the Lie bracket of two holomorphic vectors should again be holomorphic. For generalized almost complex structures, integrability condition reads exactly the same, with π and X replaced by Π and $X + \zeta$, and the Lie bracket replaced by certain bracket on $T \oplus T^*$, called Courant bracket.[2] This bracket does not satisfy Jacobi identity in general, but it does on the i-eigenspaces of $\mathcal{J}$. In case these new conditions are fulfilled, we can drop the 'almost' and speak of generalized complex structures.

For the two examples of generalized almost complex structure given above, $\mathcal{J}_1$ and $\mathcal{J}_2$, integrability condition turns into a condition on the building blocks J and ω. For $\mathcal{J}_1$, integrability of the generalized almost complex structure turns into the condition of J being integrable as an almost complex structure in T, thus making it a complex structure. For $\mathcal{J}_2$, which was built from a two-form ω, the condition becomes $\mathrm{d}\omega = 0$, thus making ω into a symplectic form.

These two examples are not exhaustive, and the most general generalized complex structure interpolates between complex and symplectic manifolds. A generalized complex manifold is locally equivalent to the product $\mathbb{C}^k \times (\mathbb{R}^{d-2k}, \omega)$, where $\omega = \mathrm{d}x^{2k+1} \wedge \mathrm{d}x^{2k+2} + \cdots + \mathrm{d}x^{d-1} \wedge \mathrm{d}x^d$ is the standard symplectic structure and $k \leqslant d/2$ is called rank, and can be constant or vary over the manifold.

3.1. Pure spinors in generalized complex geometry

There is an algebraic correspondence between generalized almost complex structures and pure spinors of Clifford(6, 6). In string theory, the picture of generalized almost complex structures emerges naturally from the worldsheet point of view [12], while that of pure spinors arises from the space-time side. Since it this last approach that we deal with, let us first review the formalism of Clifford(6, 6) spinors, and then show how to use pure spinors in the context of generalized complex geometry.

Spinors on T transform under Clifford(6), whose algebra is $\{\gamma^m, \gamma^n\} = 2g^{mn}$. There is a representation of this algebra in terms of forms. Using holomorphic and anitholomorphic indices, we can take $\gamma^i = \mathrm{d}z^i \wedge$, $\gamma^{\bar{\imath}} g^{\bar{\imath}j} \iota_j$.[3] The (3, 0)-form Ω can be used as a Clifford vacuum to construct a basis of spinors. Ω is a pure spinor of Clifford(6), which means that it is annihilated by half of the gamma matrices ($\gamma^i \Omega = 0$). Acting with the rest of the gamma matrices $\gamma^{\bar{\imath}}$, $\gamma^{\bar{\imath}\bar{\jmath}}$ and $\gamma^{\bar{\imath}\bar{\jmath}\bar{k}}$, we can construct a basis of 'spinors' made out of $(p, 0)$-forms. So Clifford(6) spinors are equivalent to $(p, 0)$-forms.

A similar story can be done with Clifford(6, 6). To start with, there are twice the number of generators as in Clifford(6), i.e. twelve. These are given by matrices λ^m, ρ_n obeying

$$\{\lambda^m, \lambda^n\} = 0, \qquad \{\lambda^m, \rho_n\} = \delta^m_n, \qquad \{\rho_m, \rho_n\} = 0.$$

We have chosen two different symbols, λ and ρ, instead of the more commonly used γ^m and γ_m, to emphasize that these matrices are independent, they cannot be obtained from each other by raising and lowering indices with the metric. The representation of this algebra in terms of forms which is usually taken, and to which we will stick, is $\lambda^m = \mathrm{d}x^m \wedge$, and $\rho_n = \iota_n$. Ω is still a good vacuum of Clifford(6, 6), as it is annihilated by λ^i and $\rho_{\bar{\imath}}$, which are half of the gamma matrices, thus making it a pure spinor. Acting with the other half, $\lambda^{\bar{\imath}}$ and ρ_i we get forms of all possible degrees. So Clifford(6, 6) spinors are equivalent to (p, q)-forms.

On a space with SU(3) structure on T, there two invariant forms, namely Ω and J. Ω is a pure spinors, but J is not. What is a pure spinor instead is $\mathrm{e}^{\mathrm{i}J} \equiv 1 + \mathrm{i}J - \frac{1}{2}J \wedge J - \frac{\mathrm{i}}{6}J \wedge J \wedge J$. It is annihilated by $\rho_m + \mathrm{i}J_{mn}\lambda^n$, as it is easy to check using $J^n_m J^p_n = -\delta^p_m$. Thus on a space of SU(3) structure there are always two pure spinors, Ω and $\mathrm{e}^{\mathrm{i}J}$. It is shown in [4] that the action of mirror symmetry for manifolds with SU(3) structure that are T^3 fibrations over a 3-dimensional base is

$$\mathrm{e}^{\mathrm{i}J} \leftrightarrow \Omega. \tag{8}$$

Furthermore, [4] conjectured that this is the action of mirror symmetry for any manifold with SU(3) structure. By this proposal, mirror symmetry is the exchange of two pure spinors.

[2] The Courant bracket is defined as follows: $[X + \zeta, Y + \eta]_C = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\zeta - \frac{1}{2}\mathrm{d}(\iota_X\eta - \iota_Y\zeta)$.

[3] $\iota_n : \Lambda^p T^* \to \Lambda^{p-1}T^*$, $\iota_n\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} = p\delta^{[i_1}_n \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_p]}$.

There is a one-to-one correspondence between a generalized almost complex structure $\mathcal{J}$ and a pure spinor φ. The six-dimensional space that annihilates the pure spinor should be equal to the $+\mathrm{i}$ eigenspace of the generalized almost complex structure that it is mapped to. Integrability condition for the generalized complex structure corresponds on the pure spinor side to the condition

$$\mathcal{J}\ \text{is integrable} \quad \Leftrightarrow \quad \exists \text{ vector } v \text{ and 1-form } \zeta \text{ such that } \mathrm{d}\varphi = (v\llcorner + \zeta\wedge)\varphi.$$

A generalized Calabi–Yau, as defined by Hitchin [6], is a manifold on which a closed pure spinor exists.

There is also a possibility of adding a three-form H to the story. Using a three-form, the Courant bracket can be modified,[4] and with it the integrability condition. Not surprisingly, this corresponds also to a modification of the condition on the pure spinor, which now becomes

$$(\mathrm{d} + H\wedge)\varphi = (v\llcorner + \xi)\varphi. \tag{9}$$

If we decompose φ in forms, $\sum \varphi_{(k)}$, the condition means that $\mathrm{d}\varphi_{(k)} + H \wedge \varphi_{(k-2)} = v\llcorner\varphi_{(k+2)} + \zeta \wedge \varphi_{(k)}$ for every k.

3.2. Supersymmetry equations for pure spinors

In this section we will use the supersymmetry equations in type IIA and type IIB supergravity to derive equations on the two pure spinors. The equations we derive do not encode all the information coming from the supersymmetry conditions. They are rather the counterpart of the internal gravitino, in that they encode derivatives of J and Ω, which come from covariant derivatives of the spinor in the original internal gravitino equation. They capture the information about the intrinsic torsion of the manifold; but in general from supersymmetry there are more conditions arising, equaling components of fluxes (and derivatives of the dilaton and warping) among each other. These conditions are explicitly given in [8].

To get equations for the pure spinors one starts with the internal gravitino equation which, in the democratic formulation of [14] can be expressed:

$$\delta\psi_m = D_m\epsilon + \frac{1}{4}H_m\mathcal{P}\epsilon + \frac{1}{16}\,\mathrm{e}^{\phi}\sum_n \not{F}_{2n}\Gamma_m\mathcal{P}_n\epsilon, \tag{10}$$

where $F_{2n} = \mathrm{d}C_{2n-1} - H \wedge C_{2n-3}$ are the modified RR field strengths with non standard Bianchi identities, that we will call from now on simply RR field strengths; $n = 0, \ldots, 5$ for IIA and $n = 1/2, \ldots, 9/2$ for IIB and $H_M \equiv \frac{1}{2}H_{MNP}\Gamma^{NP}$ and $\mathcal{P} = \Gamma_{11}$, $\mathcal{P}_n = \Gamma_{11}^n\sigma^1$ for IIA, while $\mathcal{P} = -\sigma^3$, $\mathcal{P}_n = \sigma^1$ for $n + 1/2$ even and $\mathcal{P}_n = \mathrm{i}\sigma^2$ for $n + 1/2$ odd for IIB. The two Majorana–Weyl supersymmetry parameters of type II supergravity are arranged in the doublet $\epsilon = (\epsilon_1, \epsilon_2)$.

The 'total' RR field involves both the field strengths and their duals, and a self-duality relation is still to be imposed

$$F_{2n} = (-1)^{\mathrm{Int}[n]} \star_{10} F_{10-2n}. \tag{11}$$

In order to preserve 4d Poincaré invariance, RR fluxes should be of the form

$$F_{2n} = \widehat{F}_{2n} + \mathrm{Vol}_4 \wedge \widetilde{F}_{2n-4}. \tag{12}$$

Here $\widehat{F}_{2n}$ stands for purely internal fluxes. The self-duality of F_{2n}, Eq. (11) becomes $\widetilde{F}_{2n-4} = (-1)^{\mathrm{Int}[n]} \star_6 \widehat{F}_{10-2n}$, and allows to write the RR part of (10) in terms of the internal fluxes only. From now on we will work only with internal fluxes, and drop the hats in F.

The ten-dimensional Majorana–Weyl spinors ϵ_1, ϵ_2, which have opposite chirality in IIA and the same chirality in IIB, can be decomposed

$$\begin{aligned} \epsilon_1 &= \zeta_+ \otimes \eta_+^1 + \zeta_- \otimes \eta_-^1, \\ \epsilon_2 &= \zeta_+ \otimes \eta_-^2 + \zeta_- \otimes \eta_+^2, \end{aligned} \tag{13}$$

in IIA, where ζ and η^i are chiral spinors in 4 and 6 dimensions, respectively. The Majorana condition implies also $(\zeta_+)^* = \zeta_-$, $(\eta_+^i)^* = \eta_-^i$. For IIB, the two spinors can be decomposed

$$\epsilon_i = \zeta_+ \otimes \eta_+^i + \zeta_- \otimes \eta_-^i. \tag{14}$$

On a manifold of SU(3) structure there is only one nowhere vanishing invariant spinor, η. So η_1 and η_2 should be related to η, which also means that ϵ_1 and ϵ_2 are related, as should be the case for $\mathcal{N} = 1$ supersymmetry. We write the relation as

$$\eta_+^1 = a\eta_+, \qquad \eta_+^2 = b\eta_+. \tag{15}$$

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4 $[X + \zeta, Y + \eta]_H = [X + \zeta, Y + \eta]_C + \iota_X\iota_Y H$.

In supersymmetry equations, we will use the combinations

$$\alpha \equiv a + \mathrm{i}b, \qquad \beta \equiv a - \mathrm{i}b. \tag{16}$$

Coming back to the pure spinors, the strategy to get equations for them is to use the fact that we can map a form (or a formal sum of them) to an element of the usual Clifford algebra, Clifford(6):

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1\dots i_k} \mathrm{d}x^{i_i} \wedge \cdots \wedge \mathrm{d}x^{i_k} \quad \longleftrightarrow \quad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1\dots i_k} \gamma^{i_i\dots i_k}_{\alpha\beta}. \tag{17}$$

An object in Clifford(6) can also be seen as a bispinor, since it has two free spinor indices. So we have realized Clifford(6, 6) spinors as bispinors, which are more useful in string theory. Another useful technical fact is that one can realize λ and ρ also as combinations of the more familiar γ's acting on the left and on the right of a bispinor. For example, $\lambda^m C^{(k)} \leftrightarrow \frac{1}{2}(\gamma^m \not{C}^{(k)} \pm \not{C}^{(k)}\gamma^m)$ when the plus (minus) sign corresponds to k even (odd).

A crucial fact is that $\mathrm{e}^{\mathrm{i}J}$ and $\not{\Omega}$ can be re-expressed in terms of tensor products of η. Using Fierz rearrangement, one can show

$$\eta_\pm \otimes \eta_+^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_+^\dagger \gamma_{i_1\dots i_k} \eta_\pm \gamma^{i_k\dots i_1}. \tag{18}$$

Using the expression for J and Ω in terms of η, Eq. (3), it is possible to express the pure spinors as tensor products of the standard spinor defining the SU(3) structure

$$\begin{aligned} \eta_\pm \otimes \eta_\pm^\dagger &= \frac{1}{8} \mathrm{e}^{\mp iJ}, \\ \eta_+ \otimes \eta_-^\dagger &= -\frac{\mathrm{i}}{8} \not{\Omega}, \\ \eta_- \otimes \eta_+^\dagger &= -\frac{\mathrm{i}}{8} \overline{\not{\Omega}}, \end{aligned} \tag{19}$$

where the extra factor of 1/2 with respect to (19) comes from the normalization chosen for the spinors, $\eta_\pm^\dagger \eta_\pm = \frac{1}{2}$. Then, the exterior derivative $\mathrm{d}(\mathrm{e}^{-\mathrm{i}J})$ can be re-expressed in the bispinor picture as the anticommutator

$$\{\gamma^m, D_m(\eta_+ \otimes \eta_+^\dagger)\}.$$

The covariant derivative here is meant to be a bispinor covariant derivative, which corresponds to the ordinary covariant derivative of forms under the Clifford map, and which anyway reduces to exterior derivative when we fully antisymmetrize, as usual. To compute this object, one can use Leibniz rule for the covariant derivative of the bispinor, reducing it to $\{\gamma^m, D_m(\eta_+) \otimes \eta_+\}$ plus its complex conjugate. Using the internal gravitino equation (10) for the covariant derivative of the spinor, gives

$$\begin{aligned} &\text{IIA:} \quad -\left[\not{\partial}(2A - \phi + \log\alpha) + \frac{\beta}{4\alpha}\not{H}\right]\eta_+ \otimes \eta_+^\dagger - \left(\partial_m\alpha + \frac{\beta}{4\alpha}H_m\right)\eta_+ \otimes \eta_+^\dagger \gamma^m, \\ &\text{IIB:} \quad -\left[\not{\partial}(2A - \phi + \log\alpha) - \frac{\beta}{4\alpha}\not{H}\right]\eta_+ \otimes \eta_+^\dagger - \left(\partial_m\alpha - \frac{\beta}{4\alpha}H_m - \frac{\mathrm{i}}{4\alpha}\mathrm{e}^{\phi}\not{F}_B\gamma_m\right)\eta_+ \otimes \eta_+^\dagger \gamma^m, \end{aligned}$$

where α and β are defined in (16), A is the warp factor, i.e. the metric has the form

$$\mathrm{d}s^2 = \mathrm{e}^{2A}(\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu) + \mathrm{d}s_6^2$$

and $F_B = \alpha F_1 - \beta F_3 + \alpha F_5$ is a sum of IIB RR fluxes. Notice that in IIA F has disappeared. This is because it would have multiplied $\gamma_m \eta_- \otimes \eta_+^\dagger \gamma^m$. This expression is zero because $\eta_- \otimes \eta_+^\dagger = -\frac{\mathrm{i}}{8}\overline{\not{\Omega}}$, and $\gamma_m \gamma^{npq} \gamma^m = 0$ in six dimensions. This technical circumstance is what allows us to make F disappear in one of the pure spinor equations for both IIA and IIB. It is now only required to go from the bispinor picture back to the form picture, inverting the Clifford map (17). The equations we obtain are the following. For type IIA we have

$$\mathrm{e}^{-f}\,\mathrm{d}(\mathrm{e}^f \mathrm{e}^{\mathrm{i}J}) = -\frac{1}{2} \frac{\mathrm{Re}(\alpha\bar{\beta})}{|\alpha|^2 + |\beta|^2} H \bullet \mathrm{e}^{\mathrm{i}J}, \tag{20}$$

$$\mathrm{e}^{-g}\,\mathrm{d}(\mathrm{e}^g \Omega) = -\frac{1}{4} \frac{\beta^2 + \alpha^2}{2\alpha\beta} H \bullet \Omega - \frac{\mathrm{e}^\phi}{16} \frac{1}{2\alpha\beta} \left(F \cdot \left(-\frac{1}{4}\mathrm{e}^{-\mathrm{i}J} + 1 + \mathrm{i}\,\mathrm{vol}\right) - \left(-\frac{1}{4}\mathrm{e}^{\mathrm{i}J} + 1 - \mathrm{i}\,\mathrm{vol}\right) \cdot F^*\right), \tag{21}$$

and in type IIB

$$\mathrm{e}^{-f}\,\mathrm{d}(\mathrm{e}^{f}\,\mathrm{e}^{\mathrm{i}J}) = \frac{1}{2}\frac{\mathrm{Re}(\alpha\bar{\beta})}{|\alpha|^2+|\beta|^2} H \bullet \mathrm{e}^{\mathrm{i}J} - \mathrm{i}\frac{\mathrm{e}^{\phi}}{16}\frac{1}{|\alpha|^2+|\beta|^2}\left(F\cdot\left(-\frac{1}{4}\mathrm{e}^{-\mathrm{i}J}+1+\mathrm{i}\,\mathrm{vol}\right)-\left(-2\,\mathrm{e}^{-\mathrm{i}J}+1+\mathrm{i}\,\mathrm{vol}\right)\cdot F\right), \tag{22}$$

$$\mathrm{e}^{-g}\,\mathrm{d}(\mathrm{e}^{g}\Omega) = \frac{1}{4}\frac{\beta^2+\alpha^2}{2\alpha\beta} H \bullet \Omega. \tag{23}$$

In both cases $f = 2A - \phi + \log(|\alpha|^2 + |\beta|^2)$ and $g = 2A - \phi + \log(\alpha\beta)$, and $F \equiv (|\alpha|^2 - |\beta|^2)F_+ + (\alpha\bar{\beta} - \bar{\alpha}\beta)F_-$, where F_+ is the Hermitian piece of the RR total form ($F_+ = F_0 + F_4$ in IIA, $F_+ = F_1 + F_5$ in IIB) and F_- is the antihermitian piece ($F_- = F_2 + F_6$ in IIA and $F_- = F_5$ in IIB). A dot $\cdot$ indicates the Clifford product between forms[5] and vol is the volume form. The operator $H\bullet$ is the same for all equations and is defined by

$$H\bullet \equiv H_{mnp}\left(\mathrm{d}x^m\,\mathrm{d}x^n\,\iota^p - \frac{1}{3}\iota^m\iota^n\iota^p\right). \tag{24}$$

Although the RR piece is not very nice, it has a similar form in both cases too. Most importantly, the action of the NS sector is always the same.

Given the mathematical discussion, it is natural to wonder if the operator $H\bullet$ we found has a realization in terms of a twisting of the Courant bracket. This remains as an open problem. Note however that the combination $\mathrm{d} + H\bullet$ does not square to zero, unlike $\mathrm{d} + H\wedge$.

With this caveat (or technical clarification) in mind, we will call any action of H-flux a twisting. The main outcome of Eqs. (20), (21) for IIA and (22), (23) for IIB is that in each case there is one pure spinor equation that contains an exterior derivative and H-twist. Thus, having a twisted closed pure spinor, or in other words twisted generalized Calabi–Yau, is a necessary condition for having an $\mathcal{N} = 1$ vacuum. All the backgrounds with SU(3) structure constructed so far satisfy this condition.

The pure spinor that is twisted close in each case has the same parity as the RR-flux: even for IIA ($\mathrm{e}^{\mathrm{i}J}$) and odd for IIB (Ω). This respects the mirror symmetry exchange (8).

The condition $\mathrm{e}^{\mathrm{i}J}$ being twisted closed in IIA means that in IIA supersymmetric manifolds are twisted symplectic. In IIB, on the contrary, Ω is twisted closed. Decomposing (1) order by order, one gets $H \llcorner \Omega = 0$ (so H does not contribute to W_1), and $\mathrm{d}\Omega$ is $(3,1)$ (H does not – and cannot – contribute to W_2). So supersymmetric manifolds with SU(3) structure in IIB are always complex.

4. Discussion

To summarize, we obtained that supersymmetry implies that the 6-dimensional compactification manifolds of type II with SU(3) structure are always twisted generalized Calabi–Yau's. This means that they have one twisted closed pure spinor, $\mathrm{e}^{\mathrm{i}J}$ for IIA and Ω for IIB, which has the same parity as the RR-flux. Twisting refers to the action of the 3-form H, $\mathrm{d}_H = \mathrm{d} + H\bullet$ (see (24)), which works differently from the way considered by [6], $\mathrm{d}_H = \mathrm{d} + H\wedge$. Understanding the supergravity twisting from first principles remains an open problem.

There are quite a few other open problems related to generalized Calabi–Yau 'compactifications'. One is the issue about global tadpoles: what kind of compact manifolds are suitable, i.e. evade no-go theorems? In the case type IIB on warped-Calabi–Yau's, which are a particular case of generalized Calabi–Yau, O3 planes give the appropriate negative tension and RR-charge source to cancel tadpoles. For other kind of generalized Calabi–Yau's, which are supersymmetric given a set of fluxes the orientifold planes needed to cancel global tadpoles break supersymmetry (for example, in IIB solutions corresponding to bound states of D3- and D5-branes, there is no known combination of O3- and O5-planes that preserves supersymmetry).

Another key open question is the deformation problem for twisted operators (while the discussion of $H\wedge$ twisting started as early as in [7] and is still far from being complete, as mentioned the very origin of $H\bullet$ is yet to be understood). It seems very likely that the generalized complex geometry provides the right framework for these problems, and the understanding of the moduli spaces of the generalized Calabi–Yau's, and consequently the string spectra in flux compactifications will hopefully be achieved soon.

[5] $F \cdot G$ is obtained by first building the bispinor $\not{F}\not{G}$ and then using the map (17) to get back the corresponding form.

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References

[1] C. Vafa, Superstrings and topological strings at large N, J. Math. Phys. 42 (2001) 2798, hep-th/0008142.
[2] S. Kachru, M.B. Schulz, P.K. Tripathy, S.P. Trivedi, New supersymmetric string compactifications, JHEP 03 (2003) 061, hep-th/0211182.
[3] S. Gurrieri, J. Louis, A. Micu, D. Waldram, Mirror symmetry in generalized Calabi–Yau compactifications, Nucl. Phys. B 654 (2003) 61, hep-th/0211102.
[4] S. Fidanza, R. Minasian, A. Tomasiello, Mirror symmetric SU(3)-structure manifolds with NS fluxes, hep-th/0311122.
[5] O. Ben-Bassat, Mirror symmetry and generalized complex manifolds, math.aG/0405303.
[6] N. Hitchin, Generalized Calabi–Yau manifolds, math.dg/0209099.
[7] R. Rohm, E. Witten, The antisymmetric tensor field in superstring theory, Ann. Phys. 170 (1986) 454.
[8] M. Graña, R. Minasian, M. Petrini, A. Tomasiello, Supersymmetric backgrounds from generalized Calabi–Yau manifolds, hep-th/0406137.
[9] M. Gualtieri, Generalized complex geometry, Ph.D. thesis, Oxford University, math.DG/0401221.
[10] D. Huybrechts, Generalized Calabi–Yau structures, K3 surfaces, and B-fields, math.ag/0306162.
[11] A. Kapustin, Topological strings on noncommutative manifolds, hep-th/0310057.
[12] U. Lindstrom, R. Minasian, A. Tomasiello, M. Zabzine, Generalized complex manifolds and supersymmetry, hep-th/0405085.
[13] A. Kapustin, Y. Li, Topological sigma-models with H-flux and twisted generalized complex manifolds, hep-th/0407249.
[14] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, A. Van Proeyen, New formulations of $D = 10$ supersymmetry and D8–O8 domain walls, Classical Quant. Grav. 18 (2001) 3359, hep-th/0103233.