Strings, gravity, and the quest for unification/Cordes, gravitation, et la quête d'unification

Anomalies, Gauss laws, and Page charges in M-theory

Gregory W. Moore

Department of Physics, Rutgers University, Piscataway, NJ 08854-8019, USA

Presented by Guy Laval

Abstract

We review the E_8 model of the M-theory 3-form, and its applications to anomaly cancellation, Gauss laws, quantization of Page charge, and the 5-brane partition function. We discuss the potentially problematic behavior of the model under parity. ***To cite this article: G.W. Moore, C. R. Physique 6 (2005).***
© 2005 Published by Elsevier SAS on behalf of Académie des sciences.

Résumé

Anomalies, lois de Gauss, et charges de Page dans M-théorie. Nous rappelons le modèle E_8 de la 3-forme de M-théorie, et ses applications pour l'annulation des anomalies, lois de Gauss, quantification de la charge de Page, ainsi que pour la fonction de partition de la 5-brane. Nous envisageons le comportement potentiellement problématique du modèle sous la parité. ***Pour citer cet article : G.W. Moore, C. R. Physique 6 (2005).***
© 2005 Published by Elsevier SAS on behalf of Académie des sciences.

1. Introduction

In 1978 Cremmer, Julia, and Scherk found the action for 11-dimensional supergravity [1]. Twenty-six years later the theory has come to be regarded as a low energy limit of some hypothetical more fundamental 'M-theory'. A satisfactory formulation of M-theory is still unknown. One set of clues to finding such a formulation lies in the issues one encounters in formulating 11-dimensional supergravity in topologically nontrivial situations. While the action principle in [1] is simple, it contains a very subtle Chern–Simons term. In this article we review some recent work aimed at clarifying the mathematical nature of that term [2–6]. We will also describe briefly some related new results [7,8]. Another motivation for this recent work is the clarification of anomaly cancellation issues in M-theory. This is discussed in section three below. A further motivation is the possibility that there are new topological terms in the action. (Such terms were found for type IIA supergravity in [5] in exactly this way. For a general discussion see Section 5.5 of [7].) As discussed in Section 5, the Chern–Simons term also leads to a noncommutative structure in the theory leading to some important subtleties in flux quantization. Finally, the considerations touched on here are of importance in understanding aspects of the M-theory 5-brane in topologically interesting situations. As we remark below, they are not without applications to currently fashionable topics.

E-mail address: gmoore@physics.rutgers.edu (G.W. Moore).

While the E_8 formalism is very useful for studying some of the topological complexities of 11-dimensional supergravity it appears to have two important drawbacks. The first is that the action of parity is subtle (and possibly impossible) to formulate in a satisfactory way. We describe some of the salient points in Section 7 below. A second important challenge is the incorporation of elementary 5-branes in the formalism. In the spacetime external to the 5-brane one finds a nontrivial E_8 instanton linking the 5-brane worldvolume. However, to describe 5-branes we wish to include their worldvolume. Unfortunately, the inclusion of nonzero magnetic current in the E_8 formalism presents an unsolved difficulty.

2. Defining the Chern–Simons term

Let Y be an 11-dimensional, oriented, spin manifold. In topologically trivial situations M-theory has an abelian gauge field, a globally defined 3-form $C \in \Omega^3(Y)$ with fieldstrength $G = dC \in \Omega^4(Y)$.[1] The exponentiated Euclidean action for the theory is (schematically):

$$\exp\left[-2\pi \int_Y \frac{1}{\ell^9} \operatorname{vol}(g) R(g) + \frac{1}{2\ell^3} G \wedge *G + \bar\psi \slashed{D} \psi \right] \Phi(C), \tag{1}$$

$$\Phi(C) = \exp\left(2\pi i \int_Y \frac{1}{6} C G^2 - C I_8(g) \right) \tag{2}$$

where g is the metric, ψ is the gravitino and ℓ is the 11-dimensional Planck length. This form of the action cannot apply in topologically interesting situations in which the cohomology class $[G] \neq 0$. If $\partial Y = \emptyset$ the usual definition of a Chern–Simons term involves an extension to a bounding 12-manifold Z:

$$\Phi(C) \overset{?}{=} \exp\left(2\pi i \int_Z \frac{1}{6} G^3 - G I_8(g) \right). \tag{3}$$

As it stands, this definition appears to depend on the extension. The existence of M2-branes implies $[G] = \bar a - \frac{1}{2}\bar\lambda$ where $\bar a \in \overline{H}^4(Y; \mathbb{Z})$ [2]. (The bar denotes reduction modulo torsion and λ is the characteristic class of the spin-bundle on Y.) Thus the factor of 1/6 looks problematic. In fact, since $[I_8(g)] = (p_2 - \lambda^2)/48$, the definition (3) appears to be ambiguous by a 96th root of unity. It was pointed out by Witten in [2] that E_8 index theory shows the situation is actually not that bad. Isomorphism classes of principal E_8 bundles on manifolds M of dimension $\leqslant 15$ are in 1–1 correspondence with integral classes $a \in H^4(M, \mathbb{Z})$. Let $P(a)$ denote an E_8 bundle with characteristic class $a \in H^4(Y, \mathbb{Z})$. If we identify $[G] = [\operatorname{tr} F^2 - \frac{1}{2} \operatorname{tr} R^2]$, where F is the fieldstrength of a connection A on the bundle $P(a)$, then there is a remarkable identity

$$\frac{1}{6} G^3 - G I_8 = \left[\frac{1}{2} i(\slashed{D}_A) + \frac{1}{4} i(\slashed{D}_{RS}) \right]^{(12)} \tag{4}$$

where $i(\slashed{D})$ denotes the standard index density. The first term is for the Dirac operator coupled to A in the adjoint representation while the second is for the Dirac operator coupled to $T^*Y - 4$. We extract the 12-form piece of the right-hand side. (The simple formula (4) summarizes all the nontrivial group-theoretic identities used in Green–Schwarz anomaly cancellation for the $E_8 \times E_8$ theory, as well as the identities used by Horava and Witten [9].) Since the index is even in 12-dimensions $\Phi(C)$ is in fact well-defined up to a sign. The sign ambiguity cannot be removed without introducing other fields. See Section 3 below.

2.1. Boundaries

The extension to the case with boundary, $\partial Y = X$ is nontrivial. It is best described by making a choice of a 'model' for the C-field. We will now explain what we mean by a 'model'. The membrane coupling provides us with the *gauge equivalence class* of a C-field. Thus, an isomorphism class $[C]$ may be identified with a map

$$[C] : \Sigma \to \exp\left(2\pi i \int_\Sigma C \right) \tag{5}$$

---

[1] In general we follow the notation of [6].

from the space of closed 3-cycles to $U(1)$, such that, if $\Sigma = \partial B$ then $[C] \colon \Sigma \to \exp(2\pi i \int_B G)$. Mathematically, the membrane coupling is (a torsor for) the Cheeger–Simons group $\check{H}^4(Y)$.[2]

While the mathematical formulation of the gauge equivalence class of a C-field is clear, there are different ways of expressing C in terms of redundant variables. This issue does not arise in Yang–Mills theory, where there is a natural way: one uses the space of connections $\mathrm{Conn}(P)$ on a principal bundle P. In the case of the C-field, the language of categories turns out to be useful. This language applies to all gauge theories. Abstractly the space of C-fields should be viewed as a groupoid, i.e., a category all of whose morphisms are invertible. The gauge potentials are the objects, while the gauge transformations are the morphisms. The group of global gauge transformations is the automorphism group of the object. Different models for the C-field correspond to equivalent categories. In this note we use a particular model, the 'E_8 model for the C-field'. Another model, based on the differential cohomology theory of Hopkins and Singer [10] is described in [6] and is developed further in [11].

In the E_8 model, a 'C-field' on Y with characteristic class a is a pair (A, c) in $\mathcal{C}(Y) := \mathrm{Conn}(P(a)) \times \Omega^3(Y)$. The gauge invariant fieldstrength is $G = \mathrm{tr}\, F^2 - \frac{1}{2}\mathrm{tr}\, R^2 + dc$ so that, morally speaking,

$$C = CS(A) - \frac{1}{2}CS(g) + c, \tag{6}$$

can be written in terms of Chern–Simons forms. The objects of the groupoid are points in $\mathcal{C}(Y)$. The morphisms are defined by a gauge group $\mathcal{G}$, described in Section 4 below.[3] Note the metric dependence in C: the space of bosonic fields in M-theory is fibered over the space of metrics, the fiber is the space of C-fields.

In the E_8 model we can write the Chern–Simons term of M-theory as

$$\Phi(C) = \exp\left[2\pi i\left\{\frac{1}{4}\eta(\not{D}_A) + \frac{1}{8}\eta(\not{D}_{RS})\right\} + 2\pi i I_{\mathrm{local}}\right] \tag{7}$$

where η is the Atiyah–Patodi–Singer invariant and

$$I_{\mathrm{local}} = \int\limits_Y \left(c\left(\frac{1}{2}G^2 - I_8\right) - \frac{1}{2}cdcG + \frac{1}{6}c(dc)^2\right). \tag{8}$$

The importance of Eq. (7) is that the formula is intrinsically formulated in 11-dimensions and moreover the same formula holds on a manifold with boundary. However, we must now pay a price. Φ cannot be viewed as a $U(1)$ valued function but rather must be considered as a section of a line bundle $\mathcal{L} \to \mathcal{C}(Y) \times Met(Y)$. This line bundle has a connection. When Y is closed the connection is only nontrivial in the metric directions, is flat, and has ± 1 holonomy on $Met(Y)/Diff^+(Y)$. When $\partial Y = X$ is nonempty the connection has nontrivial components in the C-field directions. Heuristically $\mathcal{A} = 2\pi \int_Y (\frac{1}{2}G^2 - I_8)\delta C$. An important point is that the curvature of $\mathcal{L}$ is nonzero:

$$\mathcal{F} = \pi \int\limits_X G\delta C\delta C. \tag{9}$$

3. Anomaly cancellation and setting the quantum integrand

This section covers some new work done with D. Freed [7].

Quite generally the quantum integrand of a path integral is a section of a line bundle $\mathcal{L}_{qi}$ over the space of bosonic fields. This bundle is equipped with a connection ∇. In theories with fermions and/or Chern–Simons terms this line bundle with connection might well be nontrivial. If this is the case the path integral does not make sense—even formally—since one cannot add vectors in different lines. This is the geometrical interpretation of anomalies. In order to define a sensible path integral one must introduce a trivialization, i.e., a globally nonvanishing section $\mathbf{1}$ of $\mathcal{L}_{qi}$ so that, if e^{-S} is the quantum integrand then $e^{-S}/\mathbf{1}$ is a globally well-defined function, which can be integrated. Note that this requires that $\mathcal{L}_{qi}$ be topologically trivial. Moreover, the connection ∇ must be flat: This is the cancellation of local anomalies. Furthermore, the flat connection ∇ must have no holonomy: This is the cancellation of global anomalies. In other words, in a well-defined theory $(\mathcal{L}_{qi}, \nabla)$ must be

---

[2] It is a torsor because of a shift in C needed to cancel worldvolume anomalies on the membrane [2]. This may also be understood as being due to background magnetic current induced by w_4.

[3] In this definition we have fixed a bundle $P(a)$ for each a, at the cost of some naturality. Section 3.5 of [6] describes an equivalent category where no such choice is made.

geometrically trivial. Note that in an anomaly free theory there might still be a nontrivial choice of trivializing section **1**. In [7] this choice is called a 'setting of the quantum integrand'.

In the case of M-theory both the C-field and the gravitino theories are quantum-mechanically inconsistent. That is, both $\Phi(C)$ and the gravitino partition function are sections of nontrivial line bundles with non-flat connections. However, it is shown in [7] that the tensor product is geometrically trivial. This is the Green–Schwarz anomaly cancellation. Moreover, it is shown in [7] that there is a canonical trivialization, thus leading to a canonical setting of the quantum integrand.

There are already anomalies in the case when Y is closed. This is the sign ambiguity mentioned below Eq. (4). The gravitino partition function $\mathrm{pf}(\not{D}_{RS})$ is a section of the Pfaffian line bundle

$$L := \mathrm{PF}(\not{D}_{RS}) \to Met(Y), \tag{10}$$

L is a *complex line bundle* with real structure. L has a connection compatible with the real structure so the holonomies on $Met(Y)/Diff^+(Y)$ are ± 1. In fact, the gravitino has a global anomaly. In [7, Section 2], one finds a natural geometric isomorphism $L \cong \mathcal{L}$ leading to global anomaly cancellation. That is,

$$\mathrm{Pf}(\not{D}_{RS}) \cdot \Phi \tag{11}$$

is a well-defined function on $\mathcal{C}(Y) \times Met(Y)/\mathcal{G} \times Diff^+(Y)$. This Green–Schwarz mechanism was already indicated in [2] and [7] establishes it rigorously.

When we consider anomaly cancellation on manifolds with boundary we need to distinguish temporal boundaries from spatial boundaries because of the boundary conditions which we will impose. In the case of temporal boundary conditions, we put global, or APS boundary conditions on the fermion fields. In this case, a similar but rather more subtle story applies to establish the cancellation of Hamiltonian anomalies. This is described in detail in [7, Section 4.2].

We now consider spatial boundaries. With local (i.e., chiral) boundary conditions on fermions one can still define elliptic operators and study geometric invariants. (D. Freed's student M. Scholl is studying a general class of local boundary conditions for Dirac operators on manifolds with boundary [12].) Using these results one can produce a geometric isomorphism between the line bundle of the Chern–Simons term and that of the fermion partition function. In this way one can give a rigorous proof of anomaly cancellation in the Horava–Witten model. The advantage of this proof is that it covers simultaneously both local and global anomalies, and moreover it becomes crystal clear that the anomaly cancellation is completely *local*. (It has been pointed out that this issue is nontrivial [13].) We are not being very precise here about the meaning of locality, but we note that the anomaly cancels boundary component by boundary component. In particular, there is no topological obstruction to putting M-theory on an 11-manifold with any number of boundary components. On each component we choose, arbitrarily, a sign $\varepsilon_i = \pm$ determining the chirality projection. Each component carries an independent E_8 super-Yang–Mills multiplet and we choose boundary conditions such that $G|_{X_i} = \varepsilon_i(\mathrm{tr}\, F^2(A_i) - \frac{1}{2}\,\mathrm{tr}\, R^2(g_i))$. There are a number of subtle details one encounters in checking this cancellation. Perhaps the most surprising is that, in some circumstances, the *Pfaffian* line bundle admits, globally, a well-defined square root. Again, for the many details we refer to [7, Section 4.3].

The existence of these topological sectors of M-theory raises the interesting question of whether there are *solutions* of the equations of motion on manifolds of this type. This curious question remains open.

4. The Gauss law

Our next goal is to write the Gauss law for C-field gauge invariance. In the E_8 model $C = (A, c)$. Small gauge transformations act by $c \to c + d\Lambda$, $\Lambda \in \Omega^2(Y)$. It is usually said that gauge transformations are $c \to c + \omega$, $\omega \in \Omega^3_{\mathbb{Z}}(Y)$.[4] However, this does not properly account for global gauge transformations '$\Lambda \sim$ constant'. The correct choice is to replace $\Omega^3_{\mathbb{Z}}(Y)$ by the Cheeger–Simons group $\check{H}^3(Y)$, and interpret ω as the fieldstrength of the differential character. What we stress here is that the Cheeger–Simons group $\check{H}^3(Y)$ is an extension:

$$0 \to H^2\big(Y, U(1)\big) \to \check{H}^3(Y) \to \Omega^3_{\mathbb{Z}}(Y) \to 0. \tag{12}$$

We interpret $H^2(Y, U(1))$ as the group of global gauge transformations. In the categorical language, these are the automorphisms of the object: If $\alpha \in H^2(Y, U(1))$ then $\gamma_\alpha : (A, c) \to (A, c)$. But the automorphism still has nontrivial physical effects. Firstly, it has a nontrivial effect on *open* membrane amplitudes. A second nontrivial effect emerges in the formulation of the Gauss law for the gauge group $\mathcal{G}$.[5]

---

[4] $\Omega^p_{\mathbb{Z}}(M)$ denotes the space of p forms on M with integral periods. Such forms are necessarily closed.

[5] Actually, $\mathcal{G} = \Omega^1(\mathrm{ad}P) \ltimes \check{H}^3(Y)$, where the first factor shifts $A \to A + \alpha$, see [6].

The Gauss law is the statement that physical wavefunctions of the C-field must be gauge invariant:

$$\gamma \cdot \Psi(C) = \Psi(\gamma \cdot C) \quad \forall \gamma \in \mathcal{G},\ C \in \mathcal{C}(X). \tag{13}$$

Now the wavefunction is a section of the line $\mathcal{L}$ in which Φ is valued. Thus, to formulate the Gauss law we must define a lift:

$$
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{\ \mathcal{G}\ } & \mathcal{L} \\
\downarrow & & \downarrow \\
\mathcal{C}(X) & \xrightarrow{\ \mathcal{G}\ } & \mathcal{C}(X)
\end{array}
\tag{14}
$$

To define the lift we combine the parallel transport using the connection on $\mathcal{L}$ with a cocycle for the group action:

$$\gamma \cdot \Psi(C) = \varphi(C, \gamma)^* \cdot \exp\left(\int_C^{\gamma \cdot C} \mathcal{A}\right) \cdot \Psi \tag{15}$$

where $\varphi(C, \gamma)$ is a cocycle, that is

$$\varphi(C, \gamma_1)\varphi(\gamma_1 \cdot C, \gamma_2) = e^{-i\pi \int_X G\omega_1\omega_2} \varphi(C, \gamma_1\gamma_2) \tag{16}$$

where $\gamma_1, \gamma_2 \in \mathcal{G}$ are C-field gauge transformations with fieldstrength $\omega_1, \omega_2 \in \Omega_{\mathbb{Z}}^3(X)$. We will refer to $\varphi(C, \gamma)$ as the 'lifting phase'. Following a construction in [3] (described more fully in [6]), given $C \in \mathcal{C}(X)$ and $\gamma \in \mathcal{G}$ we construct a twisted C-field C_γ on $Y = X \times S^1$: $C_\gamma(x, 1) = \gamma \cdot C_\gamma(x, 0) = \gamma \cdot C$. Then we define $\varphi(C, \gamma) := \Phi(C_\gamma)$.

Since (12) is an extension the Gauss law consists of two statements. For $\gamma = \gamma_\alpha$, $\alpha \in H^2(X, U(1))$, we obtain the electric charge tadpole condition. Once this law is satisfied we can study the Gauss law for $\gamma \in \Omega_{\mathbb{Z}}^3(X)$. This leads to the quantization of 'Page charge'.

The tadpole condition has been described in detail in [6]. Assume X is compact. A global gauge transformation γ_α, $\alpha \in H^2(X, \mathbb{R}/\mathbb{Z})$, acts nontrivially on quantum wavefunctions. If $\Psi \in \mathcal{L}_{A,c}$ then $\gamma_\alpha \cdot \Psi = \exp[2\pi i\langle Q, \alpha\rangle]\Psi$, where $Q \in H^8(X, \mathbb{Z})$ is the C-field electric charge. Thus, if $Q \neq 0$ then $\Psi = 0$. From the definition of the group lift we get a formula for Q. It only depends on the characteristic class a, so we may write $Q(a)$. It is easy to show that $\bar{Q} = [\frac{1}{2}G^2 - I_8]_{DR}$ thus recovering the usual condition of [14]. Nevertheless, $Q(a)$ is an *integral refinement* of $[\frac{1}{2}G^2 - I_8] = \frac{1}{2}\bar{a}(\bar{a} - \bar{\lambda}) + 30\hat{A}_8$, and hence $Q(a) = 0$ carries further information related to torsion. Not much is known about $Q(a)$. It is a quadratic refinement of the cup product. This and some other pertinent facts can be found in [6].

When $Q = 0$ we can have nonzero gauge invariant wavefunctions $\Psi(C) \in \Gamma(\mathcal{L})$. There is still further information in the statement of gauge invariance. In order to demonstrate the physical interpretation it is convenient to trivialize $\mathcal{L}$. This entails choosing a basepoint so $C = C_\bullet + c$, and replacing the wavesection $\Psi(C)$ by a wavefunction $\psi(c)$. The result of a careful analysis [8] is that the Gauss law may be written:

$$\psi(c + \omega) = e_\omega(c)\psi(c) \quad \forall \omega \in \Omega_{\mathbb{Z}}^3(X), \tag{17}$$

where

$$e_\omega(c) := \varphi(C_\bullet, \omega)^* e^{2\pi i \int_X (\frac{1}{2}G_\bullet + \frac{1}{6}dc)c\omega}. \tag{18}$$

5. Page charges

Eq. (17) can be interpreted physically by rewriting it in the form

$$\exp\left(2\pi i \int_X \omega P\right)\psi = f_\bullet(\omega)\psi \quad \forall \omega \in \Omega_{\mathbb{Z}}^3(X), \tag{19}$$

where P is an operator-valued 7-form. In order to prove this one notes that on spin 10-manifolds the cocycle in (16) is in fact $\mathbb{Z}_2$-valued (this is nontrivial since $[G]$ has half-integer periods). Then it follows that $\varphi(C_\bullet, \omega)$ is *linear* on $\Omega_{2\mathbb{Z}}^3$ and hence of the form $\varphi(C_\bullet, \omega) = \exp[2\pi i \int \omega T_\bullet]$. The 7-form $T_\bullet \in \Omega^7(X)$ is a trivialization $dT_\bullet = \frac{1}{2}G_\bullet^2 - I_8$. It is only defined modulo a form with half-integer periods. We make a definite choice and define $f_\bullet(\omega) := \varphi(C_\bullet, \omega)^* e^{2\pi i \int \omega T_\bullet}$ for all $\omega \in \Omega_{\mathbb{Z}}^3(X)$. This is a $\mathbb{Z}_2$-valued cocycle satisfying (16). It is then elementary to show that (17) is equivalent to (19) provided

$$P = \frac{1}{2\pi}\Pi + \left(\frac{1}{2}G_\bullet c + \frac{1}{6}cdc\right) + T_\bullet \tag{20}$$

where Π is the canonical momentum of c. The expression (20) is nothing other than the 'Page charge' of supergravity, formulated in the canonical formalism. This 7-form flux should be considered as the electro-magnetic dual of the flux G. Morally speaking, $P = dC_6$ where C_6 is the 6-form potential that couples to the 5-brane.

We are now in a position to study the quantization of Page charge. Here we encounter a surprise. If $[G] = 0$, the quantum Gauss law for large C-field gauge transformations implies $[P] \in \check{H}^7(X; \mathbb{Z})$. This is the naive electro-magnetic dual to the naive quantization of magnetic flux: $[G] \in \check{H}^4(X; \mathbb{Z})$. However, when $[G] \neq 0$, things are quite different. For $\phi \in H^3_{DR}(X)$ define $P(\phi) := \int_X \phi \wedge P$. An easy computation shows that

$$\left[P(\phi_1), P(\phi_2) \right] = \frac{\mathrm{i}}{2\pi} \int \phi_1 \wedge \phi_2 \wedge G. \tag{21}$$

Eq. (21) is important. It means, first of all, that not all $P(\phi)$ can be simultaneously diagonalized. Moreover, $[P]$ is not even gauge invariant. If $U(\omega) := \exp[2\pi \mathrm{i} \int \omega P]$ implements large gauge transformations then (as was noted in a special case in [15])

$$U(\omega) P(\phi) U(\omega)^{-1} = P(\phi) - \int \omega \phi G. \tag{22}$$

In general, the conserved gauge invariant 'Page charges' or electric fluxes should be regarded as characters of a certain group which we will call the *magnetic translation group*. When $[G] = 0$ this group is simply $H^3(X, U(1))$, and hence we recover the lattice of fluxes, $H^7(X, \mathbb{Z})$. In general, with $[G] \neq 0$, the group is generated by the gauge invariant operators $W(\phi) := \mathrm{e}^{2\pi \mathrm{i} P(\phi)}$ where ϕ is such that: $\int \phi \omega G \in \mathbb{Z}$ for all $\omega \in H^3(X, \mathbb{Z})$. Note that the group is in general nonabelian:

$$W(\phi_1) W(\phi_2) = \mathrm{e}^{-\mathrm{i}\pi \int \phi_1 \phi_2 G} W(\phi_1 + \phi_2) = \mathrm{e}^{-2\pi \mathrm{i} \int \phi_1 \phi_2 G} W(\phi_2) W(\phi_1). \tag{23}$$

In summary, the naive lattice of (magnetic,electric) fluxes $H^4(X, \mathbb{Z}) \oplus H^7(X, \mathbb{Z})$ is modified in two ways. The first factor is constrained by the tadpole constraint $Q(a) = 0$. The second factor is replaced by the character group of the magnetic translation group.

A comparison with ordinary gauge theory might help in understanding better what is going on here. Consider $U(1)$ gauge theory on spacetimes of the form $X \times \mathbb{R}$, where X is an n-dimensional Riemannian manifold. If we take the action $S = \int_{X \times \mathbb{R}} -\frac{1}{2\mathrm{e}^2} F * F$ then the Hilbert space of the theory is graded by $H^2(X, \mathbb{Z}) \oplus H^{n-1}(X, \mathbb{Z})$. The first component is c_1 of the line bundle on which A is a connection, while the second component is the quantized electric flux. This grading can be understood elegantly as follows.[6] The space of gauge equivalence classes of line bundles with connection on X is the Cheeger–Simons group $\check{H}^2(X)$, and therefore the Hilbert space is—formally—$L^2(\check{H}^2(X))$. Now, note that $\check{H}^2(X)$ is an abelian group. Quite generally, if A is an abelian group then a Heisenberg extension of $A \times \hat{A}$ acts on $L^2(A)$ where $\hat{A}$ is the group of characters of A. If X is oriented the Poincaré dual group to $\check{H}^2(X)$ is $\check{H}^{n-1}(X)$. The subgroup $H^1(X, U(1)) \times H^{n-2}(X, U(1))$ of $A \times \hat{A}$ acts on Hilbert space with trivial extension. The characters of this subgroup are simply $H^2(X, \mathbb{Z}) \oplus H^{n-1}(X, \mathbb{Z})$. Now, let us consider 3d massive abelian gauge theory with action

$$S = \int_{\Sigma \times \mathbb{R}} -\frac{1}{2\mathrm{e}^2} F * F + 2\pi \int_{\Sigma \times \mathbb{R}} k A \, dA \tag{24}$$

where Σ is a Riemann surface. The exponentiated Chern–Simons term must be considered as a section of a line bundle $\mathcal{L}_k \to \check{H}^2(\Sigma)$. We now identify the Hilbert space as a space of L^2 sections $\Gamma(\check{H}^2(\Sigma); \mathcal{L}_k)$. The wavefunction is only nonzero on the component with $c_1 = 0$ (this is the analog of the tadpole condition $Q(a) = 0$ above). Moreover, because $\mathcal{L}_k$ carries a nontrivial connection the translation symmetry is broken and replaced by a Heisenberg group extension of $H^1(\Sigma, \mathbb{Z}/k\mathbb{Z})$. In the analogy with Chern–Simons theory k corresponds to $\frac{1}{2}[G]$ and the (noncommuting) Wilson line operators correspond to the operators $W(\phi)$.

We expect that the above remarks will have some important implications for the classification of RR fluxes in type II string theory. It is commonly believed that the topological sectors are classified by twisted K-theory. (See [16,17] for recent reviews.) Naively one might expect the classification of RR fluxes in the background of a nontorsion H field to be given in terms of the image of the Chern-character of twisted K-theory [18], analogous to the quantization condition proposed in [19,20]. A discussion of this proposal (and other relevant matters) can be found in [21]. Dimensional reduction of the above formulae indicate that the situation is more complex and needs further investigation.

The phenomenon we have described is probably closely related to the Hanany–Witten effect [22] and to the noncommuting brane charges of [23]. Similar noncommutative structures have appeared in compactifications of M-theory on tori [24] and in formulations of M-theory using the C-field together with its electromagnetic dual [25].

---

[6] Thanks to G. Segal for some illuminating remarks.

6. Application: the 5-brane partition function

In the 3D Chern–Simons theory of Eq. (24) the dynamics of the topological (flat) modes of A is that of an electron on a torus $H^1(\Sigma; U(1))$ in a constant magnetic field. In a long distance approximation of M-theory, '$\ell \to 0$', where ℓ is the 11-dimensional Planck length one only keeps the harmonic modes of the C-field and an analogous story holds. If we introduce a basis ω^a of the space $\mathcal{H}^3(X)$ of harmonic 3-forms on X then we may expand $c = \sum_a c_a \omega^a$, and the effective Hamiltonian for these modes may be shown to be

$$H_{\text{eff}} = h_{ab}\left(-\mathrm{i}\frac{\partial}{\partial c_a} - \pi \mathcal{B}^{aa'} c_{a'}\right)\left(-\mathrm{i}\frac{\partial}{\partial c_b} - \pi \mathcal{B}^{bb'} c_{b'}\right) \tag{25}$$

where $h^{ab} = \int_X \omega^a * \omega^b$ and the 'magnetic field' is $\mathcal{B}^{ab} = \int_X G\omega^a \omega^b$. We effectively have a Landau-level problem on the torus $H^3(X, \mathbb{R})/\bar{H}^3(X, \mathbb{Z})$. The Page charge operator corresponds to the magnetic translation operator.

As an application, we can use the above formalism to derive Witten's prescription for the 5-brane partition function [3]. In the process of doing so we will underscore a point which is almost always misunderstood in the literature. Our approach will be via the AdS/CFT correspondence. We consider $X = D \times S^4$, where D is a compact 6-fold, so X is a conformal boundary at infinity for an asymptotically AdS space Y:

$$ds^2 \to \left(k^{2/3}\ell^2\right)\left[dr^2 + \mathrm{e}^{2r}\,ds_D^2 + \frac{1}{4}\,ds_{S^4}^2\right], \tag{26}$$

and $G \to G_\infty = k\omega_{S^4} + \widetilde{G}$, where $\widetilde{G} \in \Omega^4(D)$. According to AdS/CFT for $k \gg 1$ the partition function of M-theory on Y is the partition function of the $U(k)$ $(2, 0)$ theory on D. Now $U(k) = \frac{SU(k) \times U(1)}{\mathbb{Z}_k}$ where the $U(1)$ couples to the center of mass degree of freedom of the 5-branes. This couples to the harmonic modes of c at infinity (for simplicity we denote these as c) and, contrary to what is usually stated, does not completely decouple. In fact, the partition function of the $(2, 0)$ theory may be written as

$$Z\left[U(k) \quad (2,0)\text{-theory}\right] = \sum_{\beta \in \Lambda_1/k\Lambda_1} \zeta^\beta \Psi_\beta(c) \tag{27}$$

where $H^3(D, \mathbb{Z}) = \Lambda_1 \oplus \Lambda_2$ is a Lagrangian decomposition of $H^3(D, \mathbb{Z})$ with its canonical symplectic structure. (For a discussion of similar decompositions in AdS_3 and AdS_5 see [26–28].) In Eq. (27) ζ^β is the contribution of the $SU(k)/\mathbb{Z}_k$ $(0, 2)$ theory. As pointed out in [26], β should be considered as a label for the 't Hooft sectors of the $SU(k)/\mathbb{Z}_k$ $(0, 2)$ theory. (Note that for $D = D' \times S^1$, the theory reduces to $SU(k)/\mathbb{Z}_k$ gauge theory on D' and we have a natural symplectic splitting with $\Lambda_1 = H^2(D', \mathbb{Z})$, but this is precisely the group classifying 't Hooft sectors.) On the other hand, the magnetic translation group is a Heisenberg group extending $H^3(D, \mathbb{Z}_k)$ and the formula for Ψ_β given below makes it clear that

$$W(\phi_1)\Psi_\beta = \Psi_{\beta+\phi_1}, \quad \phi_1 \in \Lambda_1/k\Lambda_1, \tag{28}$$

$$W(\phi_2)\Psi_\beta = \mathrm{e}^{2\pi \mathrm{i}k\langle\phi_2,\beta\rangle}\Psi_\beta, \quad \phi_2 \in \Lambda_2/k\Lambda_2, \tag{29}$$

giving the standard representation of the Heisenberg group. Thus, the 't Hooft sector label is AdS/CFT dual to the Page charge.

Let us now come to the explicit formulae for the conformal blocks of the 5-brane theory. To derive the 5-brane partition function, in the $\ell \to 0$ approximation, we solve for the eigenstates of Eq. (25). The ground state on $\mathcal{H}^3(X)$ is the lowest Landau level. We may take $c \in \mathcal{H}^3(D)$, and then an overcomplete basis of wavefunctions has the form $\Psi_v(c) = \mathrm{e}^{-\frac{\pi k}{2}\int_D c*c + \int_D v(1+i*)c}$. Here $v \in \mathcal{H}^3(D)$, and the Landau level is infinitely degenerate. However, we must project these wavefunctions onto gauge invariant states, so we average over large gauge transformations:

$$\overline{\Psi}_v = \sum_{\omega \in \mathcal{H}^3_{\mathbb{Z}}(D)} \left(e_\omega(c)\right)^* \Psi_v(c+\omega) \tag{30}$$

where $e_\omega(c)$ was defined in Eq. (18). Written out explicitly this becomes

$$\overline{\Psi}_v = \sum_{\omega \in \mathcal{H}^3_{\mathbb{Z}}(D)} \varphi(\check{C}_\bullet, \omega)\exp\left\{-\frac{\pi k}{2}\int_D (c+\omega)*(c+\omega) - \mathrm{i}\pi k\int_D c \wedge \omega\right\}\exp\left\{\int_D v \wedge (1+i*)(c+\omega)\right\}. \tag{31}$$

The span of these wavefunctions is finite-dimensional, as is most easily seen by performing a Poisson resummation with respect to Λ_2. One then obtains

$$\overline{\Psi}_v = \sum_{\beta \in \Lambda_1/k\Lambda_1} \Psi_\beta(c)\Psi_\beta(v)^* \tag{32}$$

where $\Psi_\beta(c) = e^Q \Theta_{\beta,k/2}$ with Q a quadratic (nonholomorphic) form in c and $\Theta_{\beta,k/2}$ a holomorphic level $k/2$ theta function. Holomorphy refers to the complex structure on $\mathcal{H}^3(D)$ defined by Hodge $*$ [3]. The argument of the theta function is shifted by characteristics, which can be deduced from $\varphi(C_\bullet, \omega)$. In this way one derives explicit formulae for the conformal blocks.

We recognize in the sum and the first exponential in Eq. (31) the 5-brane partition function of Witten [3]. The sum over ω is therefore interpreted as a sum over instantons for the chiral 2-form on the 5-brane. Of course, our derivation is only valid for $k \gg 1$, but we expect that the formulae hold for all values of k. In particular, for $k = 1$ (32) is a holomorphic square. Note the inclusion of the lifting phase $\varphi(\check{C}_\bullet, \omega)$. Without this phase, Poisson resummation will not produce theta functions of the correct level, or with the correct characteristics. In particular, without the phase one finds a sum over level $2k$ theta functions. Moreover, the lifting phase shows that the characteristics of the theta function depend on the metric. Indeed, one can show that if we change the metric, holding (A, c) fixed then

$$\frac{\varphi(C_{\bullet,1}, \omega)}{\varphi(C_{\bullet,2}, \omega)} = \exp\left[2\pi i k \int_D \omega CS(g_1, g_2) \right] \tag{33}$$

where $CS(g_1, g_2)$ is the relative Chern–Simons form for the two metrics. There are also potential contributions to the characteristics from quantum corrections to the Born–Oppenheimer approximation one uses when separating harmonic from nonharmonic modes of the c-field.

The issue of characteristics can be important in applications, such as 5-brane instantons. A theta function with characteristics has an expansion schematically of the form $\Theta \sim q^{\theta^2/2} + \cdots$. Thus, if q is small (e.g., because some coupling is weak) and θ is nonzero, there can be suppression of 5-brane instanton amplitudes. Such suppressions can have consequences. For example, using these considerations it might be possible to derive an interesting lower bound on the values of the string coupling for which the constructions of [29] are self-consistent.

7. The problem with parity

M-theory is parity invariant, and should in principle be formulated in a way which makes sense on unoriented, and possibly nonorientable, manifolds. The formalism described above makes heavy use of an orientation on Y. Extending the E_8 formalism to a parity invariant formalism is subtle and potentially problematic.[7] There is no difficulty at all describing the action of parity on isomorphism classes of the C-field. We take $[C]^P = -[C]$, that is, any parity transform $C \to C^P$ must satisfy

$$\exp 2\pi i \int_\Sigma C^P = \left(\exp 2\pi i \int_\Sigma C \right)^*. \tag{34}$$

Note that $G^P = -G$ and $a^P = \lambda - a$. In the E_8 model we understand C in this equation as in Eq. (6). However, there is no natural way to map $A \in \text{Conn}(P(a))$ to $A^P \in \text{Conn}(P(\lambda - a))$. By contrast, in the rival model [6,11] based on Hopkins–Singer cocycles the action of parity is simple and natural. In the latter model a C-field is represented by a triple $(a, h, G) \in \check{C}(Y) := C^4(Y, \mathbb{Z}) \times C^3(Y, \mathbb{R}) \times \Omega^4(Y)$ and parity is simply the transformation $(a, h, G) \to (\lambda(g) - a, -h, -G)$ (there is a functorial choice of a representative $\lambda(g)$ of the class $\lambda \in H^4(Y, \mathbb{Z})$). This presents a serious problem for the E_8 model. It can be traced to the fact that there is a natural group structure on $\check{C}(Y)$, but there is no natural group structure on $\coprod_a \text{Conn}(P(a))$.

One way to address the parity problem was discussed in [6]. Let Y_d be the orientation double cover of Y and let σ be the Deck transformation so that $Y_d/\langle\sigma\rangle = Y$. We then define a 'parity invariant C-field on Y' to be a C-field on Y_d such that $\sigma^*[C] = [C]^P$. If Y is orientable this definition amounts to defining a parity invariant C-field on Y as a pair of ordinary C-fields on Y, namely, $((A, c), (A', c'))$ such that

$$\exp 2\pi i \int_\Sigma C = \left(\exp 2\pi i \int_\Sigma C' \right)^*. \tag{35}$$

The morphisms of the groupoid are simply $\mathcal{G} \times \mathcal{G}$. The space of isomorphism classes is the same as before. However, at this point we encounter a new problem: The automorphism group of an object in our new groupoid is $H^2(Y, U(1)) \times H^2(Y, U(1))$ and hence the groupoid is *inequivalent* to the previous one, even when Y is orientable! A potential solution to this difficulty is that one must require (35) hold for open membrane worldvolumes Σ. Such a constraint reduces the automorphism group to a single copy of $H^2(Y, U(1))$, as desired, but introduces yet another difficulty. For open membranes, the left- and right-hand

---

[7] This section is based on discussions with D. Freed.

sides of (35) are sections of line bundles (over the space of 2-cycles in Y). These line bundles are isomorphic, but not naturally so. The set of isomorphisms is a torsor for $H^2(Y, U(1))$, which accounts for the 'second' copy in the automorphism group of an object in our parity-invariant groupoid. Fortunately, this extra factor of $H^2(Y, U(1))$ appears to have no physical effect, and hence we effectively have an equivalent groupoid. Thus, in the author's current opinion, the parity invariant C-field model is physically viable. However, this issue clearly deserves further scrutiny.

Note that the above formulation of the E_8 model has the elegant consequence that the underlying topological gauge group is $E_8 \times E_8$ when Y is orientable, while it is simply a single copy of E_8 when Y is nonorientable.

Acknowledgements

References

[1] E. Cremmer, B. Julia, J. Scherk, Supergravity theory in 11 dimensions, Phys. Lett. B 76 (1978) 409.
[2] E. Witten, On flux quantization in M-theory and the effective action, J. Geom. Phys. 22 (1997) 1, hep-th/9609122.
[3] E. Witten, Five-brane effective action in M-theory, J. Geom. Phys. 22 (1997) 103, hep-th/9610234.
[4] D.E. Diaconescu, G.W. Moore, E. Witten, A derivation of K-theory from M-theory, hep-th/0005091.
[5] D.E. Diaconescu, G.W. Moore, E. Witten, E(8) gauge theory, and a derivation of K-theory from M-theory, Adv. Theor. Math. Phys. 6 (2003) 1031, hep-th/0005090.
[6] E. Diaconescu, D.S. Freed, G. Moore, The M-theory 3-form and E(8) gauge theory, hep-th/0312069.
[7] D. Freed, G. Moore, Setting the quantum integrand of M-theory, hep-th/0409135.
[8] G. Moore, Quantization of Page charges in M-theory, in preparation.
[9] P. Horava, E. Witten, Eleven-dimensional supergravity on a manifold with boundary, Nucl. Phys. B 475 (1996) 94, hep-th/9603142.
[10] M.J. Hopkins, I.M. Singer, Quadratic functions in geometry, topology, and M-theory, math.at/0211216.
[11] D. Freed, M. Hopkins, Differential functions, cubic forms, and string theory, in preparation.
[12] M. Scholl, PhD thesis, in preparation.
[13] A. Bilal, S. Metzger, Anomaly cancellation in M-theory: A critical review, Nucl. Phys. B 675 (2003) 416, hep-th/0307152.
[14] S. Sethi, C. Vafa, E. Witten, Constraints on low-dimensional string compactifications, Nucl. Phys. B 480 (1996) 213, hep-th/9606122.
[15] C. Beasley, E. Witten, A note on fluxes and superpotentials in M-theory compactifications on manifolds of G(2) holonomy, JHEP 0207 (2002) 046, hep-th/0203061.
[16] E. Witten, Overview of K-theory applied to strings, Internat. J. Mod. Phys. A 16 (2001) 693, hep-th/0007175.
[17] G. Moore, K-theory from a physical perspective, hep-th/0304018.
[18] V. Mathai, D. Stevenson, Chern character in twisted K-theory: Equivariant and holomorphic cases, Commun. Math. Phys. 236 (2003) 161, hep-th/0201010.
[19] G.W. Moore, E. Witten, Self-duality, Ramond–Ramond fields, and K-theory, JHEP 0005 (2000) 032, hep-th/9912279.
[20] D.S. Freed, M.J. Hopkins, On Ramond–Ramond fields and K-theory, JHEP 0005 (2000) 044, hep-th/0002027.
[21] V. Mathai, H. Sati, Some relations between twisted K-theory and E(8) gauge theory, JHEP 0403 (2004) 016, hep-th/0312033.
[22] A. Hanany, E. Witten, Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics, Nucl. Phys. B 492 (1997) 152, hep-th/9611230.
[23] S. Gukov, M. Rangamani, E. Witten, Dibaryons, strings, and branes in AdS orbifold models, JHEP 9812 (1998) 025, hep-th/9811048.
[24] N.A. Obers, B. Pioline, E. Rabinovici, M-theory and U-duality on T**d with gauge backgrounds, Nucl. Phys. B 525 (1998) 163, hep-th/9712084.
[25] E. Cremmer, B. Julia, H. Lu, C.N. Pope, Dualisation of dualities. II: Twisted self-duality of doubled fields and superdualities, Nucl. Phys. B 535 (1998) 242, hep-th/9806106.
[26] E. Witten, AdS/CFT correspondence and topological field theory, JHEP 9812 (1998) 012, hep-th/9812012.
[27] J.M. Maldacena, G.W. Moore, N. Seiberg, D-brane charges in five-brane backgrounds, JHEP 0110 (2001) 005, hep-th/0108152.
[28] S. Gukov, E. Martinec, G. Moore, A. Strominger, Chern–Simons gauge theory and the AdS(3)/CFT(2) correspondence, hep-th/0403225.
[29] F. Denef, M.R. Douglas, B. Florea, Building a better racetrack, JHEP 0406 (2004) 034, hep-th/0404257.