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C. R. Physique 6 (2005) 626–639

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Interaction of electromagnetic fields with the environment/Interaction du champ électromagnétique avec l'environnement

State of the art in computational methods for the prediction of radar cross-sections and antenna–platform interactions

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Available online 30 August 2005

Abstract

In this paper, a review is presented on computational methods for the prediction of Radar Cross-Sections (RCS) and antenna–platform interactions. In a first part the techniques for RCS computations are considered. A list of frequency and time domain solvers for the Maxwell's equations are given with their performances in memory requirements and run-time. Boundary Elements Methods, Finite Difference Time Domain Methods, Finite Elements—Finite Volume Methods, Hybridization and Factorization Techniques, are reviewed. The exceptional performances of the Fast Multipole Method compared to those of the classical Moment Method are especially highlighted. We have also made mention of some recent research work on numerical techniques conducted in France. Asymptotic methods are mainly discussed in the second part of this article devoted to antenna–platform interactions. After a brief description of the historical evolution of Geometrical Theory of Diffraction tools for antenna analysis and design, the advantages and drawbacks of different techniques for generating the geometry and searching the rays are discussed. Then a list of unsolved problems and lines of future research on asymptotic techniques are presented together with an example of a computer code founded on the Uniform Theory of Diffraction. In the conclusion some new research topics such as higher order finite elements defined on surfaces represented by B-Splines and macro-basis functions containing information on the phase or derived from analytical or asymptotic solutions are briefly introduced. ***To cite this article: F. Molinet et al., C. R. Physique 6 (2005).***

Résumé

État de l'art sur les méthodes de calcul pour la prédiction des surfaces équivalentes radar et des interactions antennes–structures. Dans cet article une synthèse sur les méthodes de calcul pour la prédiction des Surfaces Equivalentes Radar (SER) et des interactions antennes–structures est présentée. Dans une première partie les techniques de calcul des SER sont considérées. Une liste de solveurs en régime harmonique et stationnaire est donnée avec leurs performances en place mémoire et temps de calcul. Les méthodes d'éléments finis de frontière, de différences finies en régime temporel, d'éléments finis volumiques ainsi que les techniques d'hybridation et de factorisation sont passées en revue. En particulier, les performances exceptionnelles de la méthode des multipôles rapides comparées à celles de la méthode classique des moments sont mises en évidence. Nous avons aussi mentionné les travaux de recherche récents sur les techniques numériques conduits en France. Les méthodes asymptotiques sont surtout traitées dans la seconde partie de cet article consacrée à l'interaction antennes–structures. Après une courte description de l'évolution historique des outils de calcul fondés sur la Théorie Géométrique de la Diffraction, les avantages et inconvénients des différentes méthodes de représentation de la géométrie et de recherche des rayons sont discutés. Puis

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doi:10.1016/j.crhy.2005.06.013

une liste des problèmes non résolus et les axes de recherche futurs sur les méthodes asymptotiques sont présentés ainsi qu'un exemple de code de calcul fondé sur la Théorie Uniforme de la Diffraction. Dans la conclusion quelques nouveaux sujets de recherche tels que les éléments finis d'ordre supérieur définis sur des surfaces décrites par des B-Splines et les macro-fonctions de base contenant une information sur la phase ou dérivées de solutions analytiques ou asymptotiques sont brièvement introduits.
Pour citer cet article : F. Molinet et al., C. R. Physique 6 (2005).

Keywords: Radar Cross-section; Antenna–platform interaction; Numerical methods; Asymptotic methods; Uniform Theory of Diffraction

Mots-clés : Surface Equivalente Radar ; Interaction antenne–structure ; Méthodes numériques ; Méthodes asymptotiques ; Théorie Uniforme de la Diffraction

1. Introduction

Computational electromagnetics constitutes nowadays a wide domain of research and development (R & D) in a world in which the chief actors are

(i) the Universities and the research centres, mostly involved in the conception of new algorithms,
(ii) industry, active in the development of software adapted to its needs, assisted by software development companies and
(iii) some SME highly specialized in a particular technique.

From this R & D activity throughout the world, emerges every year, an enormous number of publications on new algorithms, most of which have only a short lifetime, while a few of them constitute a major advance which will profit by long term development.

In this article we limit our state of the art review to this last category. However, some new lines of research which have not yet passed through the whole chain of investigation running from the research state to industrial use, will also be given.

Computational methods can be classified in two categories: numerical methods for the resolution of the Maxwell equations, called also exact methods, and asymptotic high frequency methods, which are only valid when the characteristic dimensions of the scatterer are large compared to the wavelength.

Numerical methods are limited by the electrical size of the body measured in wavelengths. Thus, as the frequency is increased, the computer storage or the CPU time required to set up and solve Maxwell equations becomes prohibitive at some frequency.

In the past, for targets in and just above resonance, numerical techniques were used. For radar targets in the microwave band, asymptotic high frequency methods formed the basis for computation. However, owing to the rapid progress in numerical techniques, especially during the last fifteen years, their limit of applicability has been shifted to higher frequencies and now includes a large part of the diffraction problems encountered at radar frequencies, such as radar cross-section (RCS) computation, electromagnetic compatibility applications, and antenna design.

However, at the same time, the size and complexity of objects have also risen, especially in the field of radiocommunications, but also in the field of radars, where objects which are both complex in shape and partially composed of lossy dielectric or anisotropic material are potential radar targets. While most of these objects can now be treated by a combination of numerical techniques, especially for the prediction and analysis of their radar cross-sections, the antenna–platform interaction, especially on large platforms like airplanes, spacecrafts and ships, still fall outside their domain of applicability. For this reason, we have divided our review in two parts: the prediction of radar cross-sections where numerical methods predominate, and the prediction of antenna–platform interactions where asymptotic methods remain absolutely necessary. These two subjects are treated respectively in Sections 2 and 3. Of course, numerical methods are also of great importance in antenna analysis and design, and some first applications of these methods to antennas mounted on a spacecraft have been reported recently [1]. New, promising algorithms are also described in this special issue [2]. However, since most of the numerical techniques presented in Section 2 concerning RCS predictions are also applied in antenna analysis, these methods will not be presented again in the Section 3 devoted to antenna–platform interactions. On the other hand, high frequency methods [3] are still of importance for RCS computation especially at higher frequencies. Moreover, they give a physical insight in the diffraction processes which can be useful in the design of low RCS targets. Again, since the corresponding software tools employ the same techniques for the geometrical modelling of surfaces and for ray searching as those discussed in the antenna–platform interaction part, we will limit our analysis in the RCS part to some fundamental discussions on asymptotic methods, and especially on the role of the caustics which gives greater importance to current-based methods to the detriment of field based methods in RCS computation.

It was impossible to present in detail the different numerical methods mentioned in our review. Instead we have given the references of some of the main articles where the information can be found. We have also quoted the work on numerical techniques of French researchers in applied mathematics which is not so well known by the electromagnetic community.

2. Predictions of Radar Cross-Section: synthesis on numerical methods

Numerical methods in electromagnetism have increased in efficiency in the last decades. These techniques are nowadays the most frequently used for RCS, CEM applications and antenna design. The choice of the methods to be used depends on:

- Time discretization: time harmonic or time domain formulations;
- Spatial discretization: finite differences / volumes or elements methods;
- Solver: explicit, direct or iterative;
- Sensors (emitters/receivers): monostatic, bistatic, ...;
- Media: in/homogeneous, an/isotropic, dispersive;
- Computer resources: distributed / shared memory, ...;
- The complexity of the problem to be solved: multiscale, large body, ...

In this section, we first consider three main families of numerical methods: the boundary elements methods (BEM), finite difference time domain methods (FDTD), and finite elements/volumes methods (FEM/FVM). To solve complex problems (large body and many apertures), we present some hybridization and factorization techniques. Asymptotic methods are also briefly discussed.

2.1. Boundary Elements Methods (BEM)

These methods are derived from the discretization of the Maxwell integral equations. Thirty years ago, the Method of Moments (MoM) appeared (Harrington [4]): from harmonic domain integral equations we obtain an algebraic system constituted from a complex dense matrix, right-hand sides for each incident wave and current density as unknowns. The most common formulation is the well-known Electric Field Integral Equation (EFIE) using the Hdiv 'edge' boundary elements on triangular meshes (Rao Wilton Glisson basis functions [5]). A good accuracy of the solutions could be obtained by taking the length of each edge of elements of the order $\lambda/7$. Some extensions to Magnetic Field Integral Equation (MFIE) and Combined Field Integral Equation (CFIE) could be used to solve problems constituted by homogeneous domains of media, but these formulations are less accurate than EFIE and require at least a $\lambda/10$ mesh size. However, the CFIE is more stable and is a remedy to the problem of spurious frequencies.

2.1.1. Direct solvers

Most of the industrial BEM codes use direct solvers (LU factorization) on parallel machines and are now very efficient for problem sizes less than $N = 200\,000$ unknowns. The limitation is due to memory requirements of $\mathrm{O}(N^2)$ to store the matrix (in core or out of core) and CPU of $\mathrm{O}(N^3 + sN^2)$ to solve s linear systems (s the number of sources).

2.1.2. Iterative solvers

Iterative solvers (conjugate gradients, GMRES, QMR) can reduce the CPU overheads to $\mathrm{O}(spN^2)$ where s is the number of sources and p is the number of iterations, but matrix products are costly. To decrease the cost of matrix products, recently, during the years 1994–1995, Multi-Level Fast Multipol Algorithm (MLFMA) using the CFIE formulation (Chew [6,7]) leads to CPU use of $\mathrm{O}(spN\log^2 N)$ and memory storage of $\mathrm{O}(N\log N)$. Problem sizes of more than $N = 10$ million unknowns could be now available for RCS problems (up to S and X bands).

The Table 1 shows a comparison between MoM and FMM in case of a bistatic perfectly conducting sphere (report made by EMSS from ESA [1]).

The use of preconditioning (SPAI for example) often decreases run-time and gives a convergence of the solutions. In the case of monostatic or multi-sources problems (when the number of sources s is large), efficiency could be increased by using block multi right-hand side iterative solvers as Block GMRES, MGCR (Sylvand [8], Simon [9]) or interpolation techniques (Carayol [10]). Mer [11] uses FMM methods in Desprès Integral Equations to obtain accurate solutions. Meanwhile, to improve FMM methods, more evaluations have to be made in terms of accuracy, for cavities and bodies including absorbing material.

2.1.3. Time domain

Time domain boundary element methods (BEMTD) has been developed during the 1990s [12,13]. This method uses mixed finite elements to compute potentials and an implicit, stable, temporal scheme. At each time step, it needs to solve a sparse

Table 1
Comparison between FMM (CFIE without preconditioning, mesh in $\lambda/10$) and MoM. CPU time is obtained with an AMD Opteron 248, 2.2 GHz

Sphere diameter	Number of unknowns	MLFMA memory	MLFMA run-time	MoM memory	MoM run-time
2.566λ	6 372	41.4 Mbytes	83.0 s	620 Mbytes	655 s
5.132λ	25 050	160.5 Mbytes	355.9 s	9.35 Gbytes	4.74 h
10.264λ	100 005	636.3 Mbytes	0.46 h	149 Gbytes	not solved
20.528λ	398 304	2.475 Gbytes	2.61 h	2.31 Tbytes	not solved

linear system (conjugate gradient or direct solver) and a matrix convolution. BEMTD are well suited for large band problems and, to increase the efficiency, FMM techniques could be used during the matrix convolution (see Terrasse [14]).

2.2. Finite Difference Time Domain methods (FDTD)

These methods are derived from the discretization of the time domain Maxwell equations on structured meshes. At each time, we obtain a fully explicit scheme (Yee scheme [15]) allowing us to compute quite easily several millions of unknowns (electromagnetic fields). We find many applications in CEM and in computation of fields in heterogeneous media. The main drawback of FDTD is the accuracy of the solution: firstly, absorbing boundary condition for the outer boundary in free space gives spurious solutions if the distance between this boundary and the body is too small, and secondly, the stair case approximation of the body produces dispersive solutions. Recently Béranger in 1994 [16], introduced Perfectly Matched Layer (PML) which removes the drawbacks of absorbing boundary conditions. To remove the stair case approximation, a coupling between FDTD (in free space) and FEM or FVM (a region around the body discretized with tetrahedrons, see next subsection) could be used.

2.3. Finite Elements/Volume Methods (FEM/FVM)

These methods are derived from the discretization of the Maxwell equations on unstructured meshes (tetrahedrons, prisms, hexahedrons).

2.3.1. Harmonic time domain

FEM are based on Hcurl ‘edge’ elements (Nédélec basis functions [17]). These elements are free of spurious modes and lead to the correct field continuity properties at the interfaces between different media particularly well suited for the modelling of inhomogeneous anisotropic scatterers. To model homogeneous regions and particularly the unbounded free-space, integral equations (BEM) have been combined to FEM and the compatibility between Hcurl end Hdiv elements has been established. The combined equations lead to solve a quite-sparse linear system and direct and/or iterative solvers could be used (see [18] for some applications).

2.3.2. Time domain

FVTD was first introduced in electromagnetic by Shankar in 1989 [19] and was derived from codes used in CFD (Euler equations), but solutions were too dissipative. Later, more accurate explicit high order schemes have appeared [20].

Concerning FETD (mixed finite elements) or DGTD (Discontinuous Galerkin elements) [21,22] we introduce a high order non dissipative explicit scheme, with very accurate solutions for long time simulation. To reduce the free-space region, PML techniques could also be combined with this time domain method.

2.4. Hybridization/factorization

Coupling together different methods is a good way to reduce computer requirements in terms of memory and CPU time for solving multiscale and complex structures for electromagnetic problems: large bodies with cavities, protuberances, different dielectric materials, ... It consists in first separating the global problem in smaller local problems, then computing partial solutions with an appropriate method for each sub domain, and, at the end, in summarizing the solutions.

Collaborative simulations are also a good application for coupling techniques: the electromagnetic simulations of Antenna or RCS contributors must combine models under the responsibility of various partners: i.e., antennas developed by an electronic systems manufacturer, missiles by a weapon manufacturer, engine by an engine manufacturer. The aircraft manufacturer must be able to gather these models to get the integrated behaviour of all sub-systems.

We find a discussion and a full description of some hybridization and factorization techniques in [23,24]. These techniques have been first introduced in [25] and further reported in [26].

2.4.1. Factorization

This consists of a matched domain decomposition, separated by interfaces. Each domain is reduced to operators on the interfaces by using a suitable numerical method. Then a gathering of operators leads to the final solutions. If the factorized operators process on the same basis functions than the solver, factorization is only an algebraic manipulation and does not provide any loss of accuracy.

Factorization is an efficient method and we obtain a reduction of cost from n to n^2 where n is the number of sub domains. Limitations of this method concern the decomposition of unbounded regions and difficulties to model interfaces in the free-space, leading to spurious solutions.

2.4.2. Hybridization

We decompose the global problem in two sub problems: in the case of apertures in a body the first problem, called the short circuited problem, contains the body without apertures (for an air duct the aperture or interface will be the cross section of the air intake); we obtain the radar cross section RCS1, the source currents J_{s} and receiver currents J_{r} at the interface. The second problem, called the local problem, contains the apertures and near regions around apertures; we use $-J_{\mathrm{s}}$ currents as sources at the interface and solve the M_{s} currents at the interface. Then we compute an integral reaction at interfaces between M_{s} currents inducted by emitters from the local problem, and J_{r} currents inducted by receivers. Finally we summarize RCS1 and integral reaction to obtain an approximate RCS of the global problem.

Hybridization is well suited for the computation at high frequency when the aperture size is > 5 wavelength. The short circuit problem could be solved by a high frequency method: asymptotic or FMM for example and the local problems could be solved by numerical methods adapted to the apertures: BEM for example.

For an application to the channel mock-up at 7 GHz, see Fig. 1. The measurements have been made at ONERA. The problem is decomposed in 9 domains: an outer region (the cylinder) is solved by the method of asymptotic currents (MoASC in Dassault Aviation Spectre code) and the inner region (6 inlet sections and 2 engine wheels) is solved by factorization (using BEM methods in Dassault Aviation Spectre code).

Fig. 2 shows a comparison of monostatic near field RCS in polarization $\theta\theta$ function of θ (x, z angle, where z is along the duct, $\theta = 90°$ for an incident wave entering in the duct and $\theta = 180°$ for an incident wave perpendicular to the outer cylinder): red crosses for measurements, blue line for reference factorization method (9 BEM methods, involving 500 000 unknowns), cyan line for hybrid method (1 MoASC + 8 BEM methods involving 200 000 unknowns) and green line for factorization method (9 BEM methods with an truncated outer cylinder involving 200 000 unknowns). We observe a very good agreement between all solutions, except for the factorization with a truncated cylinder when $\theta > 160°$.

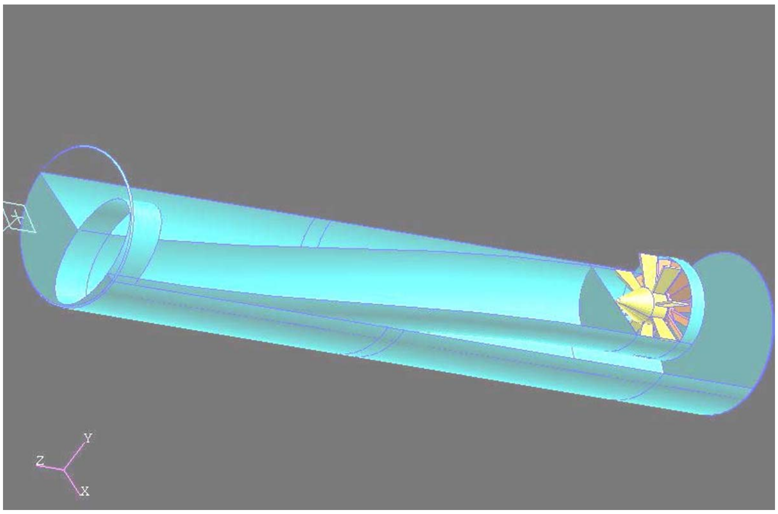

Fig. 1. Perspective of the channel mock-up.

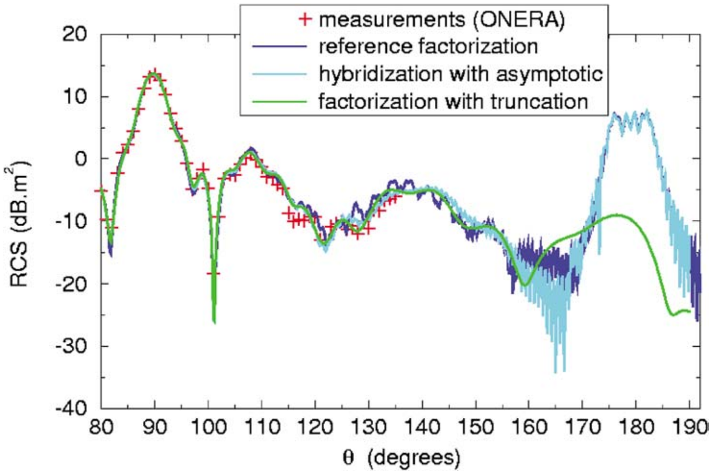

Fig. 2. RCS of the channel mock-up as a function of θ for 7 GHz.

2.5. Asymptotic methods

Geometrical Optics (GO) predicts an infinite value for the RCS of a flat plate of finite dimensions. This wrong result could be explained by the fact that the rays reflected by the surface of the plate are all parallel and therefore pass through a point caustic at infinity.

A more refined analysis consists in observing that in the vicinity of the shadow boundaries of the reflected GO field, the rays diffracted by the rim of the plate lie in the transition regions of the reflected field. At large distance from the plate, the beam of rays reflected by the plate is entirely situated in the transition zones and GO is no longer valid and must be replaced by the Uniform Theory of Diffraction (UTD). It can be shown that by adding to the reflected field at an observation point located at finite distance, the field diffracted by the rim given by UTD and by letting the observation point tend to infinity in the direction of reflection, the terms which do not satisfy the Sommerfeld radiation condition cancel, and the rest of the formula gives the correct result. This procedure, which involves a passage to a limit, is not easy to apply to a general polygonal plate. Moreover, the same problem arises when the plate is slightly bent, in which case the derivation of the correct result is much more complicated. As a consequence, field-based methods are not suitable for RCS computations of flat or quasi-flat plates. An easier approach consists in determining first the currents on the plate and then in calculating the fields radiated by these currents. This technique, which is called a current-based method, is widely used in RCS computation of complex targets, composed of curved and quasi-flat surfaces, such as airplanes, ships and tanks. In the past, most of the codes used the Physical Optics (PO) approximation for the determination of the currents associated with the Physical Theory of Diffraction (PTD) giving a correction to PO when sharp edges are present. Since PO considers only the currents on the illuminated part of an object, this method is not valid for large bistatic angles. It gives also inaccurate results for the monostatic RCS for nose on illumination of an airplane or a missile. The asymptotic current method, which consists in completing the GO contribution to the currents on the surface by transition regions currents close to the light shadow boundary, and by the creeping wave current in the shadow region, permits us to overcome the shortcomings of PO. The most recent versions of asymptotic methods codes for RCS computation integrate the asymptotic current method, the rays being determined by a shooting of rays procedure described in the antenna–platform interaction section of this article.

3. Antenna–platform interactions

3.1. Introduction

The interactions of an antenna with its platform or with other surrounding objects is a fundamental problem which arises in various domains, such as radars mounted on an airplane or a ship, terrestrial or satellite radiocommunications and electromagnetic compatibility between equipments.

This problem differs from the computation of the Radar Cross-Section (RCS) by the fact that the source, or, reciprocally, the observation point, are at a finite distance from the platform supporting the antenna, whereas, for the RCS, the object is illuminated by a plane wave and the scattered field is observed at infinity. In addition, practically all algorithms which have been developed in the past have been limited to monostatic configurations, the emitter and the receiver being superposed, whereas the antenna–platform interaction is essentially bistatic. These differences are at the origin of the development of algorithms which take into account the near field interactions and which are specifically tuned to antenna analysis and design.

Generally, the modelisation of an antenna in its environment implies the use of several methods depending on the size and the complexity of the structure of the antenna itself, and on the natural or artificial obstacles which intercept the radiated field.

In this review, we limit our investigations to artificial man-made obstacles constituted by the platform supporting the antenna: mast, tower, building, terrestrial vehicle, ship, spacecraft and aircraft. In addition to the deformation of the radiation pattern of the antenna (amplitude, phase, polarisation, directivity), the surrounding obstacles may also enhance or reduce the coupling between antennas located on the same platform. It is well known that the electromagnetic characteristics of an antenna can be strongly modified and its performance reduced by the platform on which it is mounted. Since most antennas are not designed for a particular platform and a specific location on it, it was important to develop appropriate software to compute the interaction of the antenna with the platform as well as with other neighbouring antennas.

The computational methods which have been developed for the prediction of antenna–platform interactions can be classified in two main categories: numerical methods and asymptotic high frequency methods.

Numerical methods are mainly used for the modelisation of the antenna itself. The different techniques are the same as those described in Section 2 for the computation of the RCS. Despite the rapid augmentation of their performance during the last fifteen years, due mainly to the Fast Multipole Method, the treatment of the interactions between an antenna and its platform has only started very recently [1] for medium sized platforms and antennas defined by their free space radiation diagram. For complex antennas mounted on a platform which is very large compared to the wavelength, or for an array of elements conformed to the surface of an aircraft, the size of the problem is still too big for strictly numerical methods.

The asymptotic high frequency methods used in antenna analysis and design comprise principally the Geometrical Optics (GO) associated to the Geometrical Theory of Diffraction (GTD) which give directly the scattered field along rays and the asymptotic current method which give the currents on the surface of the scattering object. In connection to these methods, other techniques have been developed mainly to remedy locally some of their insufficiencies; for instance, the Uniform Theory of Diffraction giving correctly the field in the transition regions close to the shadow boundaries, the Spectral Theory of Diffraction allowing one to extend the theory to non local plane waves, the Incremental Theory of Diffraction valid in the vicinity of an edge and verifying the boundary conditions, the Equivalent Edge Currents giving the field on a caustic of the edge diffracted rays. On account of the bistatic behaviour of the interaction between an antenna and the surrounding structures, techniques like the Physical Optics (PO) approximation, which consists in calculating the currents on the illuminated region of an object using the GO field, and the Physical Theory of Diffraction (PTD), giving a correction to the field radiated by these currents due to the existence of fringe currents close to the edge of a wedge, which are both very important in RCS computations, are of less importance here. An exception is the computation of the radiation pattern of reflector antennas where GO and GTD are not valid, owing to the presence of a caustic of the reflected field and of the field diffracted by the rim, at infinity.

On the other hand, the asymptotic current method which gives the currents on both the illuminated region and the shadowed region by taking into account the effect of creeping waves, plays an important role, especially in hybrid methods combining a numerical technique with asymptotic solutions.

In Section 3.2, after a brief description of the historical evolution of GTD tools for antenna analysis and design, some advantages and drawbacks of different techniques for generating the geometry and searching the rays, are discussed.

In Section 3.3, we present a list of problems which remain to be solved. These problems will be the basis from which different lines of future research and development will be defined.

In Section 3.4, a typical UTD code for antenna analysis is presented with some comments and illustrations on the geometrical modelling of the platform, the ray searching technique and the types of outputs provided.

3.2. Historical development and state of the art of GTD tools for the computation of antenna–platform interactions

At high frequencies, or, more precisely, when $kD \gg 1$ where k is the wave number ($k = 2\pi/\lambda$) and D is a characteristic dimension of the scatterer, the reflected GO field constitutes the dominant contribution to the scattered field. It is the first term of an asymptotic expansion in entire or fractional powers of $1/k$ and is of order zero with respect to this parameter. The next term, of order $k^{-1/2}$ corresponds to the field diffracted by a sharp wedge. Creeping waves, which are of order $k^{-1/3}$, are generally a weaker contribution, owing to the exponential decay along their propagation path. However, this argument is only valid for RCS computation in a monostatic diffraction process since there, the creeping waves travel a long distance because they have to circumvent the object in order to shed energy in the direction opposite to the direction of propagation of the incident wave, unless it is diffracted by an edge and consequently of lower order. In the case of an antenna interacting with a platform having curved surfaces, creeping waves may exist which travel on a very short distance. In this case, their contribution can no longer be neglected. Since the beginning of the development of codes for antenna analysis and design, the effort has been put on these three contributions.

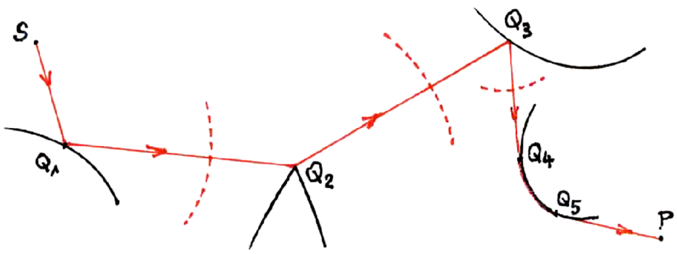

Fig. 3. Chain of interactions comprising reflections, edge diffractions and creeping waves.

For computing the GO field reflected by curved surfaces, it is necessary to know the principal radii of curvature and the principal directions of the surface at the point of reflection. Since the curvatures depend on the second derivatives of the surface we see that the geometrical modelisation of the latter is constrained by completely different limitations, compared to numerical methods. In the first codes which have been constructed (code SARGASSES [27] from THALES, France, code NEC-BSE [28] from University of Ohio, USA), the structures were represented with the aid of elementary analytical surfaces (cylinder, cone, ellipsoid, flat plates, etc. ...). The rays were searched by solving the equations verified by the co-ordinates of the interaction points, obtained by applying the Fermat principle. The knowledge of the rays and of the surface characteristics at the interaction points allows one to calculate the reflected and diffracted fields. In order to make this operation easier, MOTHESIM has developed the library PROMETHEE [29] which is composed of modules, each of which treats a specific interaction (reflection, edge diffraction, creeping waves) and has general inputs and outputs, allowing them to be used at any place in the interaction chain along a ray trajectory. For instance the same module corresponding to a reflection is used at Q_1 and Q_3 in the chain of interactions of Fig. 3. These modules which take also into account the necessity to use specific asymptotic solutions in the vicinity of the shadow boundaries given by the Uniform Theory of Diffraction, have been integrated in the software SARGASSES at the end of the 1980s.

The geometrical modelisation of a complex object (an airplane for instance), with a collection of analytical surfaces, is an expensive operation (3 weeks for an engineer) and its accuracy is difficult to control. For this reason, this technique has been progressively replaced by CAD procedures, which have, very rapidly, been installed on workstations.

The second version of SARGASSES makes use of curved surfaces represented by NURBS (Non Uniform Rational B-Spline). The rays are still searched directly by applying Fermat's principle and solving the corresponding equations on Bésier squares. This is, however, a very heavy procedure, especially when applied to double interactions (double reflections, reflection-diffraction) on a general complex object, without a first trial of the rays. For a systematic search of ray trajectories with three interactions, the computer time needed nowadays remains still too long and unfeasible on a workstation.

In order to overcome this difficulty, researchers turned towards a new technique called the 'shooting of rays', which consists in emitting a ray or a thin pencil of rays in a given direction and in following its path by applying at each interaction point with a surface, the laws of reflection. By emitting rays in all directions and selecting those which reach a small volume around the observation point, it is possible to determine all simple and multiple reflected rays. This procedure applies also to the diffraction by an edge. In this case, new elementary pencils of rays are emitted from the interaction point on an edge, in the direction of the generatrixes of the Keller cone. This procedure is used in the software SPECTRE [30] from Dassault Aviation for complex objects (air-planes), modelled geometrically with the CAD tool CATIA.

The shooting of rays is particularly rapid for searching multiply reflected rays on an object, the surface of which is modelled by plane facets, since, in this case, the divergence per unit length of a pencil of rays remains constant, so that the global divergence can be easily controlled. In the case of curved surfaces however, the divergence of a pencil of rays can change very rapidly, especially close to shadow boundaries as illustrated on Fig. 4.

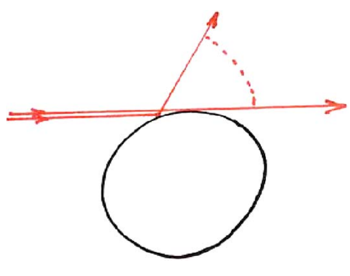

Fig. 4. Divergence of the reflected rays near the shadow boundary.

In this case, it is necessary to reduce drastically the width of the pencil of rays in some directions of space and start again the shooting of rays procedure until a pencil sufficiently thin reaches the observation point. This difficulty, which leads to much longer CPU time, is at the origin of the preference accorded in the ray search tools to facetted surfaces. An example is the code AAPG 2000 (Aircraft inter-Antenna Propagation with Graphics) de Matis Inc. in collaboration with IIT Research Institute, USA [31] which operates on a three-dimensional platform–surface representation, consisting of a collection of plane triangular facets. The ray searching procedure starts with a set of initial trial paths which can be obtained by the shooting of rays technique. In a second step each of the initial paths is optimized by an iterative algorithm which searches for an extremum of the curve length. The geometrical parameters of the true surface which enter in the UTD formulas are evaluated via a double spline interpolation, one for smoothing the field of tangent vectors and the other for smoothing the field of normals in the vicinity of the interaction point for reflection or edge diffraction and along the facetized surface path for creeping rays.

The computation of the geometrical data of the true non facetted surface at the interaction points of the ray path with the surface is essential for applying the UTD formulas. Otherwise the information concerning the crossing of a caustic of reflected or edge diffracted rays would be lost and the phase shift of $\pi/2$ (in the time convention $\exp(i\omega t)$) affecting a wave crossing a caustic would not be taken into account. There are still some GTD codes in France and elsewhere which do not take care of this problem.

It is also important to mention that the number of facets needed for the geometrical modelling of a surface is not independent of the frequency and augment with the latter. The criterium which is usually adopted is that the maximum distance between the surface and the facet is less than $\lambda/16$, where λ is the wavelength. When a fixed sampling is used at higher frequencies, a ‘facet noise’ appears in the radiation diagram.

Another GTD code which operates on surfaces represented by plane triangular facets coupled with a shooting of rays procedure has recently been developed in France by ONERA (code FERMAT [32]). This code is well adapted to very large scenes composed of buildings, trees and vegetation. Some more information on the coupling between the shooting of rays technique and the asymptotic methods used in the code FERMAT may be found in [33,34]. For aerodynamic forms like an aircraft or a missile which are mainly composed of curved surfaces, geometrical modelling by a parametric representation using NURBS is now in strong competition and takes advantage of the rapid augmentation of the possibilities of the computer. FASANT [35] is a well-known code using such a representation. It has been developed by the University of Alcalà (Spain). The rays are obtained by solving the equations resulting from the direct application of Fermat’s principle. Since this problem is computationally complex, acceleration techniques based on visibility tests (z-Buffer) are used. A similar representation is used in the code IDRA [36] of IEEA (France). More details on this code and some numerical results are presented in Section 3.4.

3.3. Unsolved problems and lines of future research

In all the computer codes described so far for GTD applications, the antenna is represented either by a phase centre and by its radiation pattern at infinity transposed by similarity to a finite distance R corresponding to the distance of the phase centre to an interaction point on the platform, or by numerical data of electric and magnetic equivalent currents on the surface of the antenna or on a surface close to the antenna and surrounding it. Since some of the current elements may be located close to the platform–surface, appropriate asymptotic solutions are needed for computing the interaction of the field radiated by these elements with the platform. Now the asymptotic solutions which are available for smooth convex surfaces are limited to the following situations:

(1) Source and observation point are both located at far distance from the obstacle [37],
(2) The source is located on the surface or very close to it ($kh \ll 1$, $h =$ height), the observation point being at large distance from it [38,39].

When the height of the source above a convex smooth surface is of the order of a few wavelengths, or less, or when both the source and the observation point are close to the surface, the uniform solution (1) becomes inaccurate, especially in the transition regions close to shadow boundaries through which it is no longer continuous. The last situation is encountered when two neighbouring antennas are located close to the platform–surface and when the coupling between these antennas has to be evaluated.

Other interactions like the diffraction of a creeping wave by the edge of a wedge in a curved convex surface into a space wave or another creeping wave [40] are also important for computing the deformation of the radiation pattern and, particularly, for evaluating the coupling between antennas. Here again, existing asymptotic solutions are not valid when the source or the observation point or both of them are close to the edge of the wedge.

In the future, we will see platforms with antennas structurally integrated in them. The region in which these antennas will reside will be complex, both geometrically and materially. A hybrid numerical-asymptotic code (finite elements for instance, coupled to the outside through GTD) is a possibility for treating such configurations. If the platform is a strongly elongated

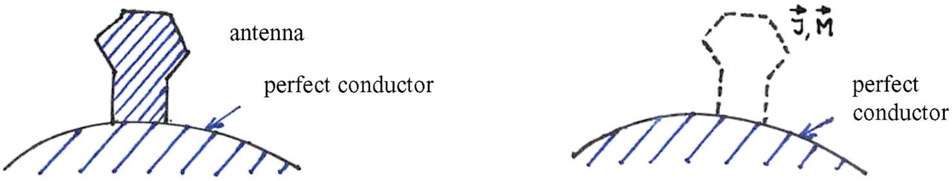

Fig. 5. Equivalence theorem for the hybridization of an integral equation method with asymptotic solutions.

object (like the fuselage of an aircraft or a missile) new asymptotic solutions are needed in the paraxial direction. Some work on generalized creeping waves propagating with very weak attenuation along on elongated object has been reported in [41]. Further research work on this subject, the results of which could also be applied to the coupling between elements of an array conformed to the surface of an elongated platform, will also be of interest for the future.

In all the techniques using asymptotic methods described so far, it has been supposed that the electric characteristics of the antenna, and particularly the currents on its surface, have not been influenced by the platform. This is an approximation which is all the more coarse if the antenna is located close to the platform surface. Again, it is possible to take into account this coupling by using a hybrid numerical–asymptotic method based on the equivalence theorem illustrated on Fig. 5.

As shown in Fig. 5, the exterior surface of a complex antenna is replaced by surface densities of electric and magnetic currents $\vec{J}$ and $\vec{M}$, radiating in the presence of the platform which is supposed to be a perfect conductor limited by a smooth convex surface. An integral equation method, restricted to the outer surface of the antenna can therefore be applied in which the coupling with the platform is described by the Green's function of the space limited by the exterior surface of the platform and verifying the boundary conditions on this surface. For very large platforms asymptotic solutions allowing one to calculate this Green's function have to be developed for convex, but also for concave and, more general smooth surfaces, like convex-concave surfaces having an inflection line with a source and an observation point, both located close to the surface.

3.4. *UTD asymptotic code used for antenna implementation on electrically large structures*

The implementation of antennas in a complex environment still remains a problem when high frequencies are considered. The Uniform geometrical Theory of Diffraction (UTD) is one of the most convenient techniques to solve this problem. This method is applied in the software IDRA developed at IEEA. Compared with other methods, the UTD has some interesting advantages. It is an efficient tool to understand the phenomenology because the global field results from localised contributors. In addition, the computational time is reduced. It is frequency independent and enables the software to handle electrically large structures.

3.4.1. *Structure geometry*

In IDRA, the structure geometry is based on NURBS curves and surfaces, which are imported from common CAD formats, such as, for example, IGES or CATIA. NURBS is a parametric representation of a 3D curve or surface. It allows an accurate description of any arbitrary shape. The surface curvature is easily derived. It is an important parameter for UTD coefficients computation. Fig. 6 presents some examples of structures described with NURBS. In these examples, very few NURBS surfaces are needed to describe complex geometries.

3.4.2. *Ray tracing*

Once the environment is geometrically described, the software performs a two steps calculation:

- ray tracing;
- once the interaction point is found, information about angles and curvatures are gathered to compute the UTD coefficients. The details of the UTD coefficients will not be explained here.

The ray tracing method used on arbitrary shaped NURBS will be explained for the case of reflection. The geometry is presented in Fig. 7(a). The total length of the ray path from source S to observation O (incident ray + reflected ray) depends on the position of a point R on the NURBS surface. This point follows the NURBS parametric equation. That is why the length is a function of two parameters (u, v). According to Fermat's Principle, the reflection point is found when the length reaches an extremum. A conjugate gradient routine is used to compute the parameters u and v minimizing or maximizing the ray length. It is not difficult to extend the method to all interactions, except one: creeping rays.

A creeping wave propagates on a surface along a geodesic path. The ray tracer has to find a whole curve and not only a finite number of points. The geodesic path is described by Eq. (1) on a parametric surface in which the coefficients Γ are

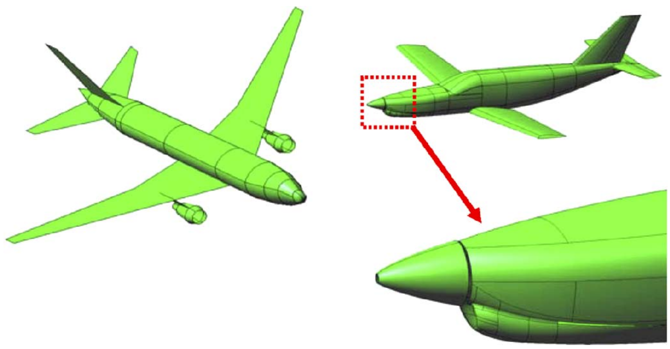

Fig. 6. Two examples of aircrafts described with NURBS surfaces and curves. A detail of the nose is shown to see the complex form of the fuselage.

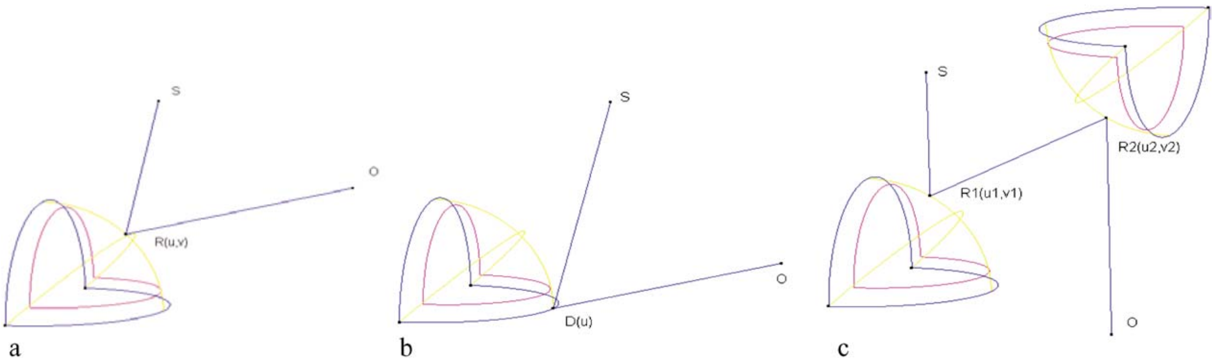

Fig. 7. Ray tracing on arbitrary NURBS curves or surfaces: (a) simple reflection, (b) simple diffraction, (c) double reflection.

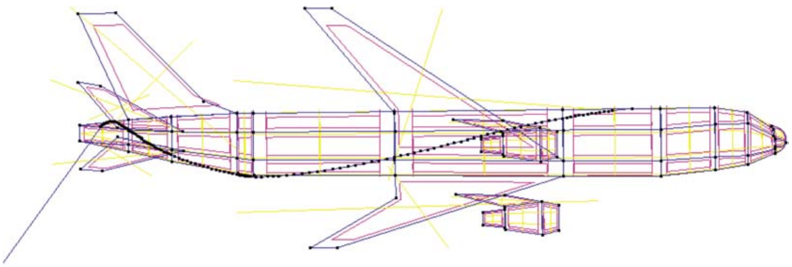

Fig. 8. Geodesic path computed from a source located on a curved surface.

the Christoffel coefficients of the surface, the indices 1 and 2 corresponding respectively to u and v. For an arbitrary shaped geometry, Eq. (1) must be solved numerically. In IDRA, the ray tracer uses a Runge–Kutta solver. Fig. 8 presents a solution of Eq. (1).

$$\frac{\mathrm{d}^2 v}{\mathrm{d}u^2} = \Gamma_{22}^{1}\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^3 + \left(2\Gamma_{12}^{1} - \Gamma_{22}^{2}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^2 + \left(\Gamma_{11}^{1} - 2\Gamma_{12}^{2}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right) - \Gamma_{11}^{2} \tag{1}$$

3.4.3. An application: antenna implementation

Once the rays are traced, the UTD coefficients are applied to compute the electric field. Fig. 9 presents an example of output: coupling parameter. Other outputs can also be provided like near field maps or radiation patterns. These values are important parameters for antenna design and may be highly dependent on the antenna environment.

As the computation speed is very high, many iterations may be made in limited time. This feature makes the software very suitable for optimisation routines. The input of the problem is the position of the antenna. The cost function is the difference between the parameter to reach and the computed value of this parameter. For example, the cost function may be the difference

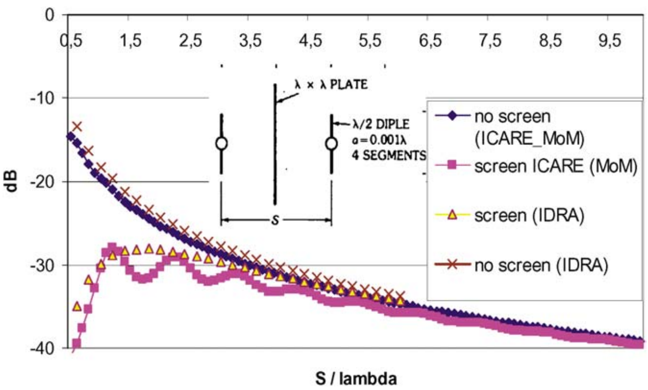

Fig. 9. Coupling between two dipole antennas separated by a square plate.

between the free space radiation pattern and the computed radiation pattern. In that case, the aim is minimizing the influence of the environment. In other cases, the aim may be using the environment to reduce the coupling between two antennas.

An interesting class of optimisation methods is the genetic algorithms (or other related stochastic methods). There is usually very little information on the cost function. In addition, this function may have several local extrema. The genetic algorithms are able to manage this situation.

As a conclusion, the software IDRA based on UTD provides an efficient solution for fast evaluation of the radiation pattern of an antenna mounted on an electrically large carrier, or of the coupling between two antennas in a complex environment. Coupled with a set of optimisation utilities, it is a convenient tool for antenna implementation on structures.

The example of Fig. 9 is taken from Burnside and Marhefka [42, Fig. 48, chap. 20]. It applies to the calculation of the S-matrix parameters at the dipole feeding points: S11 (blue lozenge and crosses) and S12 (squares and triangles).Two cases are considered, one with the square screen and another without the square screen.

In the last case each dipole is in the shadow of the other dipole. The agreement is quite satisfactory. The small differences are explained by the fact that in the asymptotic technique the currents are imposed and are not modified by the coupling.

4. Conclusion and future trends

Frequency domain numerical solvers of the Maxwell equations have made a very rapid progress in the last decade due especially to major breakthroughs in iterative solvers. The fast multipole method is now able to perform computations with more than ten million unknowns. Some improvements and extensions of this method are still matter for research, especially its application in time domain solvers which is very promising. However it seems that we have now reached a stair-head in the number of unknowns which can be handled. Rather than continuing to augment the number of unknowns, the new lines of research go towards a reduction of the numbers of unknowns of a given problem either by using higher order finite elements, or by employing macro-basis functions. The convergence of numerical algorithms founded on finite elements of order 1 needs a number of elements per square wavelength which augments with the desired accuracy. For an accuracy of 0.1 dB, about 60 elements per square wavelength are needed whereas the same accuracy may be obtained with 30 elements of order 2. This result is only valid if the geometrical modelisation of the surface by planar triangles is satisfactory. Curved triangular facets may be necessary when the order of the finite elements for the expansion of the currents augments. Some research work is performed at the moment on the application of finite elements of higher order defined on surfaces described by B-splines [43,44]. The use of higher order finite elements in the multipole method is also a topic of future research.

Another way to limit the number of unknowns consists in including in the representation of the currents some information on the phase. The usual shape functions in the finite elements are polynomials in the local co-ordinates of the element and can therefore not properly follow the oscillations of the solution. A natural idea consists in incorporating an exponential with a linear phase variation corresponding to a plane wave. Since the direction of propagation along the elements is not known, a superposition of waves with uniformly distributed propagation directions has been chosen [45,46]. The introduction of the phase in the shape function permits one to choose large elements covering about six wavelengths [46]. Since, in the method of asymptotic currents, the propagation directions of the waves on the surface are known, this information could also be used for reducing the number of unknowns. Some work has been reported recently on analytically or asymptotically derived characteristic basis functions [47,48].

Hybrid numerical-asymptotic methods in the sense of building macro-basis functions, but also in the classical sense of coupling two methods, which already constitute an important domain of research will remain a topic of further research together with the development of industrial codes in the form of a toolbox which will couple together codes that are based on different methods and which are needed for performing a specific task. An example is the software toolbox ADF (Antenna Design Framework) developed by ESA for the computation of antenna-spacecraft interactions.

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