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Nonlinear mean-field Fokker–Planck equations and their applications in physics, astrophysics and biology

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Abstract

We discuss a general class of nonlinear mean-field Fokker–Planck equations [P.-H. Chavanis, Phys. Rev. E 68 (2003) 036108] and show their applications in different domains of physics, astrophysics and biology. These equations are associated with generalized entropic functionals and non-Boltzmannian distributions (Fermi–Dirac, Bose–Einstein, Tsallis, ...). They furthermore involve an arbitrary binary potential of interaction. We emphasize analogies between different topics (two-dimensional turbulence, self-gravitating systems, Debye–Hückel theory of electrolytes, porous media, chemotaxis of bacterial populations, Bose–Einstein condensation, BMF model, Cahn–Hilliard equations, ...) which were previously disconnected. All these examples (and probably many others) are particular cases of this general class of nonlinear mean-field Fokker–Planck equations. ***To cite this article: P.-H. Chavanis, C. R. Physique 7 (2006).***

Résumé

Équations de Fokker–Planck non-linéaires en champ moyen et leurs applications en physique, astrophysique et biologie. Je présente une classe générale d'équations de Fokker–Planck non-linéaires en champ moyen [P.-H. Chavanis, Phys. Rev. E 68 (2003) 036108] et montre leurs applications dans différents domaines de la physique, de l'astrophysique et de la biologie. Ces équations sont associées à des fonctionnelles entropiques généralisées et à des distributions non-Boltzmanniennes (Fermi–Dirac, Bose–Einstein, Tsallis, ...). De plus, elles incluent un potentiel d'interaction binaire arbitraire. Je souligne des analogies entre différents domaines (turbulence bidimensionnelle, systèmes auto-gravitants, théorie des electrolytes de Debye–Hückel, milieux poreux, chimiotactie des populations bactériennes, condensation de Bose–Einstein, modèle BMF, équations de Cahn–Hilliard, ...) qui étaient auparavant déconnectés. Tous ces exemples (et probablement beaucoup d'autres) sont des cas particuliers de cette classe générale d'équations de Fokker–Planck non-linéaires en champ moyen. ***Pour citer cet article : P.-H. Chavanis, C. R. Physique 7 (2006).***

Keywords: Generalized Fokker–Planck equations; Vlasov equation; Long-range interactions

Mots-clés : Classe générale d'équations de Fokker–Planck ; Équation de Vlasov ; Interaction à longue portée

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1. Introduction

Recently, several researchers have questioned the 'universality' of the Boltzmann distribution in physics. This problem goes back to Einstein himself who did not accept Boltzmann's principle $S = k \ln W$ on a general scope because he argued that the statistics of a system (W) should follow from its dynamics and cannot have a universal expression [1]. In 1988, Tsallis introduced a more general form of entropic functional in an attempt to describe complex systems [2]. This was the starting point for several generalizations of thermodynamics, statistical mechanics and kinetic theories. A lot of experimental and numerical studies have then shown that complex media exhibit non-standard (non-Boltzmannian) distributions and that Tsallis q-distributions can fit a wide diversity of results when the Boltzmann distribution fails. However, there also exists situations that are described neither by the Tsallis nor by the Boltzmann distribution. The important point is to explain *why* a given system exhibits non-Boltzmannian distributions and find the reason for that. We cannot simply invoke the Tsallis entropy each time the system is 'complex'.

In a recent series of papers [3], we have argued that the Tsallis distributions and the associated entropic functionals are just *particular* (yet important) types of non-Boltzmannian distributions and entropies, and that there is no fundamental reason why they should play a prominent, or universal, role in physics. Other distributions can emerge as well in complex systems for different reasons. On general grounds, we have suggested that non-Boltzmannian distributions arise when 'hidden constraints' are in action and prevent the a priori accessible microstates to be equiprobable. These constraints can be of different nature: (i) There can be exclusion or inclusion constraints on the probabilities of transition of the system from one state to the other. A fundamental example is the Pauli exclusion principle in quantum mechanics which prevents two fermions to occupy the same site in phase space. This leads to the Fermi–Dirac statistics, instead of the Boltzmann statistics, placing a bound $f(\mathbf{r}, \mathbf{v}, t) \leqslant \eta_0$ on the distribution function. Similarly, one can account for excluded volume constraints in classical systems of particles. In simplest models, this leads to the Fermi–Dirac statistics in physical space placing a bound $\rho(\mathbf{r}, t) \leqslant \sigma_0$ on the spatial density. Alternatively, we can account for inclusion constraints (like for bosons) by favoring the transition to a site that is already occupied. This leads to the Bose–Einstein statistics. Nonlinear Fokker–Planck equations taking into account these inclusion or exclusion constraints have been discussed by Kaniadakis [4]. (ii) We can imagine situations where some microscopic constraints prevent the system from sampling the energetically accessible phase space uniformly. These microscopic constraints will modify the equilibrium distribution with respect to the expected Boltzmann distribution. One example is the case of porous media where the system has a complicated phase space structure (fractal or multi-fractal) leading to anomalous diffusion. The Tsallis entropy [2] is adapted to this situation. The dynamical evolution of the system can be described by a nonlinear Fokker–Planck equation introduced by Plastino and Plastino [5] that reproduces the Tsallis statistics at equilibrium. (iii) In biology, the diffusion coefficient and the mobility of the particles are often assumed to depend on the local concentration so as to account for microscopic constraints that cannot be easily modeled. This is the case for the Keller–Segel model of chemotaxis [6] which can lead to non-standard distributions at equilibrium. (iv) Finally, collisionless stellar systems and inviscid 2D turbulent flows described by the Vlasov–Poisson system and by the 2D Euler–Poisson system spontaneously form meta-equilibrium states (galaxies and vortices) that are usually described by non-Boltzmannian distributions [7]. This is due, on the one hand, to the existence of an infinite family of Casimir invariants for the fine-grained dynamics which lead to non-standard distributions on the coarse-grained scale. These Casimirs play the role of 'hidden constraints' because they are not directly accessible from the coarse-grained flow where the observations are made. Generalized Fokker–Planck equations have been introduced by Robert and Sommeria [8] and Chavanis et al. [9] to take into account these constraints. On the other hand, collisionless mixing is incomplete in general and other forms of distributions can emerge that are different from those predicted by the statistical mechanics of violent relaxation. These quasi-stationary states (QSS) are nonlinearly dynamically stable stationary solutions of the Vlasov or the 2D Euler equations selected by the dynamics. In this context, the Tsallis distributions are particular stationary solutions of the Vlasov equation known as stellar polytropes in astrophysics. They can sometimes arise as a result of incomplete violent relaxation, like in 2D turbulence, but this is not generic [10]. For collisionless stellar systems and 2D vortices described by the Vlasov and the 2D Euler equations, non-standard distributions are explained by microscopic constraints (the Casimirs) and incomplete violent relaxation (non-ergodicity) [11].

In this paper, we discuss a general class of nonlinear mean-field Fokker–Planck equations associated with generalized entropic functionals (leading to non-standard distributions) and an arbitrary binary potential of interaction [3]. These equations are consistent with an effective generalized thermodynamical formalism (E.T.F.). The study of these equations is interesting in its own right due to their rich mathematical properties but we provide here explicit ex-

amples (2D turbulence, self-gravitating systems, porous media, bacterial colonies, Bose–Einstein condensation, ...) where these nonlinear mean-field Fokker–Planck equations can have physical applications. In Section 2, we present the general formalism and the connection between nonlinear Fokker–Planck equations in phase space (Kramers) and in physical space (Smoluchowski) in a strong friction limit. In Sections 3 and 4, we describe the statistical mechanics of violent relaxation for 2D vortices and stellar systems described by the Euler and Vlasov equations and show how generalized Fokker–Planck equations can describe the dynamics on the coarse-grained scale. In Sections 5 and 6 we consider the case of self-gravitating Brownian particles and show the analogy with the Bose–Einstein condensation and with the chemotaxis of bacterial populations.

2. Nonlinear mean-field Fokker–Planck equations

We consider a generalized class of nonlinear mean-field Kramers equations of the form [3]:

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}-\nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}}=\frac{\partial}{\partial \mathbf{v}}\cdot\left\{D\left[fC''(f)\frac{\partial f}{\partial \mathbf{v}}+\beta f\mathbf{v}\right]\right\} \tag{1}$$

where $f=f(\mathbf{r},\mathbf{v},t)$ is the distribution function, C is a convex function (i.e., $C''\geqslant 0$), $\beta=1/T$ is the inverse temperature and $\Phi(\mathbf{r},t)$ is a mean-field potential. The friction parameter satisfies an Einstein relation $\xi=D\beta$. Since the temperature T is fixed, the above equation describes a canonical situation. We assume that the potential is related to the density by a general relation of the form

$$\Phi(\mathbf{r},t)=\int u\bigl(|\mathbf{r}-\mathbf{r}'|\bigr)\rho(\mathbf{r}',t)\,\mathrm{d}\mathbf{r}' \tag{2}$$

where u is a binary potential of interaction and $\rho=\int f\,\mathrm{d}\mathbf{v}$ is the spatial density. We introduce the functional

$$F[f]=\frac{1}{2}\int fv^2\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}+\frac{1}{2}\int\rho\Phi\,\mathrm{d}\mathbf{r}+T\int C(f)\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v} \tag{3}$$

This functional can be interpreted as a free energy $F=E-TS$, where S is a 'generalized entropy' and $E=K+W$ is the energy including a kinetic term and a potential term. When $S[f]=-\int f\ln f\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}$ is the Boltzmann entropy, Eq. (1) reduces to the ordinary Kramers equation. More generally, it is straightforward to check that $F[f]$ plays the role of a Lyapunov functional satisfying $\dot{F}\leqslant 0$. This is the equivalent, in the canonical ensemble, of the H-theorem. The stationary solutions $f_{\mathrm{eq}}(\mathbf{r},\mathbf{v})$ are determined by the *integro-differential* equation

$$C'(f_{\mathrm{eq}})=-\beta\left[\frac{v^2}{2}+\Phi(\mathbf{r})\right]-\alpha \tag{4}$$

where Φ depends on f_{eq} by Eq. (2). This equation results from the individual cancelation of the advective term (L.H.S.) and collision term (R.H.S.) in Eq. (1) [3]. Since C is convex, Eq. (4) can be inversed so that $f_{\mathrm{eq}}=f_{\mathrm{eq}}(\varepsilon)$ is a decreasing function of the individual energy $\varepsilon=v^2/2+\Phi(\mathbf{r})$. The linearly dynamically *stable* stationary solutions of Eq. (1) are *minima* of the free energy $F[f]$ at fixed mass [3].

In the strong friction limit $\xi\to+\infty$, or for large times $t\gg\xi^{-1}$, the distribution function $f(\mathbf{r},\mathbf{v},t)$ is close to the *isotropic* distribution function determined by the relation

$$C'(f)=-\beta\left[\frac{v^2}{2}+\lambda(\mathbf{r},t)\right]+\mathrm{O}\bigl(\xi^{-1}\bigr) \tag{5}$$

The pressure $p\equiv\frac{1}{d}\int fv^2\,\mathrm{d}\mathbf{v}=p[\lambda(\mathbf{r},t)]$ and the density $\rho\equiv\int f\,\mathrm{d}\mathbf{v}=\rho[\lambda(\mathbf{r},t)]$ are related to each other by a barotropic equation of state $p=p(\rho)$ which is entirely specified by the function $C(f)$. This equation of state is obtained by eliminating $\lambda(\mathbf{r},t)$ between the foregoing relations. Taking the hydrodynamic moments of the generalized Kramers equation (1) and closing the hierarchy by considering the limit $\xi\to+\infty$, it is shown in [3] that the time evolution of the density $\rho(\mathbf{r},t)$ is governed by the generalized Smoluchowski equation

$$\frac{\partial\rho}{\partial t}=\nabla\cdot\left[\frac{1}{\xi}(\nabla p+\rho\nabla\Phi)\right] \tag{6}$$

A formal derivation of the generalized Smoluchowski equation (6) from the generalized Kramers equation (1) can be realized by performing a Chapman–Enskog expansion in powers of ξ^{-1} [12]. The generalized Smoluchowski equation monotonically decreases the Lyapunov functional

$$F[\rho]=\int \rho \int^{\rho} \frac{p(\rho')}{\rho'^2}\,\mathrm{d}\rho'\,\mathrm{d}\mathbf{r}+\frac{1}{2}\int \rho\Phi\,\mathrm{d}\mathbf{r} \tag{7}$$

which is the simplified form of free energy (3) obtained by using the isotropic distribution function (5) to express $F[f]$ as a functional of ρ [3,12]. A stationary solution satisfies the condition of hydrostatic equilibrium

$$\nabla p+\rho\nabla\Phi=\mathbf{0} \tag{8}$$

Taking the hydrodynamic moments of the nonlinear mean-field Kramers equation (1) and closing the hierarchy by a condition of local thermodynamic equilibrium (in the canonical ensemble) one gets a system of hydrodynamical equations

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0 \tag{9}$$

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\frac{1}{\rho}\nabla p-\nabla\Phi-\xi\mathbf{u} \tag{10}$$

that we called the damped barotropic Euler equations [3]. In the strong friction limit $\xi\to+\infty$, one can neglect the inertial term in Eq. (10) to obtain $\rho\mathbf{u}=-\frac{1}{\xi}(\nabla p+\rho\nabla\Phi)+\mathrm{O}(\xi^{-2})$. Substituting this relation in the continuity equation (9), we recover the generalized Smoluchowski equation (6). The generalized Smoluchowski equation can also be written

$$\frac{\partial\rho}{\partial t}=\nabla\cdot\left\{D\left[\rho C''(\rho)\nabla\rho+\beta\rho\nabla\Phi\right]\right\} \tag{11}$$

In the general formulation, D can depend on $\mathbf{r}$ and t, so on $\rho(\mathbf{r},t)$ [3]. The generalized Smoluchowski equation (11) monotonically decreases the generalized free energy

$$F[\rho]=E-TS=\frac{1}{2}\int\rho\Phi\,\mathrm{d}\mathbf{r}+T\int C(\rho)\,\mathrm{d}\mathbf{r} \tag{12}$$

The stationary solutions of Eq. (11) are solutions of the integro-differential equation

$$C'(\rho_{\mathrm{eq}})=-\beta\Phi(\mathbf{r})-\alpha \tag{13}$$

where Φ depends on ρ_{eq} by Eq. (2). Since C is convex, Eq. (13) can be inversed so that $\rho_{\mathrm{eq}}=\rho_{\mathrm{eq}}(\Phi)$ is a decreasing function of the potential (assuming $\beta>0$). The linearly dynamically *stable* stationary solutions of Eq. (11) are *minima* of the free energy $F[\rho]$ at fixed mass [3].

These generalized Kramers and Smoluchowski equations can be obtained in different ways:

(i) from the linear thermodynamics of Onsager [13,3];
(ii) from a variational principle called Maximum Entropy Production Principle (MEPP) adapted to the canonical description [3];
(iii) from stochastic Langevin equations where the diffusion coefficient (noise) and the friction or mobility coefficients explicitly depend on the density of particles [14,3]; and
(iv) from the Master equation, by allowing the transition probabilities to depend on the population in the initial and arrival states [4].

These generalized kinetic equations can be justified as effective equations modeling heuristically microscopic constraints that are not directly accessible to the observer. It is clear that the mathematical study of these equations is of considerable interest. A systematic study of the generalized mean-field Smoluchowski equation (6) has been undertaken in [15–20] for different equations of state $p(\rho)$, or generalized entropies $S[\rho]$, and different potentials of interaction $u(\mathbf{r}-\mathbf{r}')$ in different dimensions of space d. Note that when the potential of interaction is short-range, the mean-field potential is given by $\Phi\simeq-\rho-R^2\Delta\rho$ and, when substituted in Eq. (11), we get a generalized form of the Cahn–Hilliard equation (see [3,12]). We now give explicit examples where these nonlinear mean-field Fokker–Planck equations have physical applications.

3. Two-dimensional turbulence

Two-dimensional flows with high Reynolds numbers are described by the 2D Euler equations

$$\frac{\partial\omega}{\partial t}+\mathbf{u}\cdot\nabla\omega=0,\quad \mathbf{u}=-\hat{\mathbf{z}}\times\nabla\psi,\quad \omega=-\Delta\psi \tag{14}$$

In geophysical fluid dynamics, one also considers the Quasi-Geostrophic (QG) equations where the vorticity ω is replaced by the potential vorticity q related to the streamfunction ψ by $q=-\Delta\psi+\frac{1}{R^2}\psi$ where R is the Rossby radius. Starting from a generically unstable initial condition $\mathbf{u}_0(\mathbf{r})$, the 2D Euler equations are known to develop a complicated mixing process which ultimately leads to the emergence of a large-scale coherent structure, typically a jet or a vortex. Jovian atmosphere shows a wide diversity of structures (Jupiter's great red spot, white ovals, brown barges, ...). One question of fundamental interest is to predict the structure and the stability of these 'equilibrium' states depending on the initial conditions. To that purpose, Robert and Sommeria [21] have proposed a statistical mechanics of the 2D Euler equation. The idea is to replace the deterministic evolution of the flow $\omega(\mathbf{r},t)$ by a probabilistic description where $\rho(\mathbf{r},\sigma,t)$ gives the density probability of finding the vorticity level $\omega=\sigma$ in $\mathbf{r}$ at time t. The observed (coarse-grained) vorticity field is then expressed as $\overline{\omega}(\mathbf{r},t)=\int\rho\sigma\,\mathrm{d}\sigma$.

Consider first the case where the initial vorticity field consists of an ensemble of patches with vorticity $\omega=\sigma_0$ surrounded by irrotational flow $\omega=0$. These patches will mix in a complicated way but their vorticity and their total area γ_0 will remain constant. In this two-levels approximation, the conservation of the area is equivalent to the conservation of the circulation $\Gamma=\int\overline{\omega}\,\mathrm{d}\mathbf{r}=\gamma_0\sigma_0$. The energy $E=\frac{1}{2}\int\overline{\omega}\psi\,\mathrm{d}\mathbf{r}$ is also conserved. Using a result of large deviations, it can be shown [21] that the optimal probability $p(\mathbf{r})$ of finding the level σ_0 in $\mathbf{r}$ at equilibrium is obtained by maximizing the mixing entropy

$$S[p]=-\int\big\{p\ln p+(1-p)\ln(1-p)\big\}\,\mathrm{d}\mathbf{r} \tag{15}$$

at fixed $\gamma_0=\int p\,\mathrm{d}\mathbf{r}$ (or Γ) and E. This expression of entropy can be obtained from a combinatorial analysis taking into account the 'incompressibility' of the vorticity field. Using the method of Lagrange multipliers, the optimal probability $p_*(\mathbf{r})$, and consequently the coarse-grained vorticity $\overline{\omega}=p_*\sigma_0$, is given by

$$\overline{\omega}(\mathbf{r})=\frac{\sigma_0}{1+\lambda\mathrm{e}^{\beta\sigma_0\psi(\mathbf{r})}}\quad(\lambda>0) \tag{16}$$

Note that $\overline{\omega}\leqslant\sigma_0$, as the coarse-grained vorticity locally averages over the levels $\omega=0$ and $\omega=\sigma_0$. This constraint is similar to the Pauli exclusion principle in quantum mechanics and this is why (15) and (16) look similar to the Fermi–Dirac entropy and the Fermi–Dirac distribution. Non-standard (non-Boltzmannian) distributions arise here because the patches (levels) cannot overlap (or cannot be compressed).

Robert and Sommeria [8] have proposed a dynamical model to describe the relaxation of the system towards statistical equilibrium. By using a Maximum Entropy Production Principle (MEPP), they obtain in the two-levels case an equation of the form

$$\frac{\partial\overline{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\overline{\omega}=\nabla\cdot\big[D\big(\nabla\overline{\omega}+\beta(t)\overline{\omega}(\sigma_0-\overline{\omega})\nabla\psi\big)\big],\quad \overline{\omega}=-\Delta\psi \tag{17}$$

$$\beta(t)=-\frac{\int D\nabla\overline{\omega}\cdot\nabla\psi\,\mathrm{d}\mathbf{r}}{\int D\overline{\omega}(\sigma_0-\overline{\omega})(\nabla\psi)^2\,\mathrm{d}\mathbf{r}},\quad D(\mathbf{r},t)\propto\omega_2=\overline{\omega}(\sigma_0-\overline{\omega}) \tag{18}$$

This relaxation equation enters in the class of generalized mean-field Fokker–Planck equations of Section 2. It is associated with the Fermi–Dirac entropy in physical space $S[\overline{\omega}]=-\int\{(\overline{\omega}/\sigma_0)\ln(\overline{\omega}/\sigma_0)+(1-\overline{\omega}/\sigma_0)\ln(1-\overline{\omega}/\sigma_0)\}\,\mathrm{d}\mathbf{r}$ and with a potential $u(|\mathbf{r}-\mathbf{r}'|)=-\frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'|$. Note, however, that in the present case the inverse temperature $\beta(t)$ evolves with time according to Eq. (18) in order to conserve the energy. Therefore, Eqs. (17) and (18) monotonically increase the entropy S at fixed E and Γ. This corresponds to a microcanonical description while the equations presented in Section 2 correspond to a canonical description [3].

In the general case, the optimal vorticity distribution $\rho(\mathbf{r},\sigma)$ maximizes the mixing entropy

$$S[\rho]=-\int\rho\ln\rho\,\mathrm{d}\mathbf{r}\,\mathrm{d}\sigma \tag{19}$$

at fixed E, Γ and $\Gamma_{n>1} = \int \overline{\omega^n}\,\mathrm{d}\mathbf{r} = \int \rho\sigma^n\,\mathrm{d}\sigma\,\mathrm{d}\mathbf{r}$. The last constraints are equivalent to the conservation of the Casimirs or of the area $\gamma(\sigma) = \int \rho\,\mathrm{d}\mathbf{r}$ of each vorticity level. We must also account for the local normalization condition $\int \rho\,\mathrm{d}\sigma = 1$. This yields the Gibbs state

$$\rho(\mathbf{r},\sigma) = \frac{1}{Z(\mathbf{r})}\chi(\sigma)\mathrm{e}^{-(\beta\psi+\alpha)\sigma}, \quad Z = \int_{-\infty}^{+\infty} \chi(\sigma)\mathrm{e}^{-(\beta\psi+\alpha)\sigma}\,\mathrm{d}\sigma \tag{20}$$

where $\chi(\sigma) = \exp(-\sum_{n>1}\alpha_n\sigma^n)$ encapsulates the Lagrange multipliers $\alpha_{n>1}$ accounting for the conservation of the fragile constraints $\Gamma_{n>1}$ while β and α account for the conservation of the robust constraints E and Γ (Z is the normalization constant determined by $\int \rho\,\mathrm{d}\sigma = 1$).[1] The coarse-grained vorticity $\overline{\omega} = \int \rho\sigma\,\mathrm{d}\sigma$ is now given by

$$\overline{\omega} = \frac{\int \chi(\sigma)\sigma\mathrm{e}^{\sigma(\beta\psi+\alpha)}\,\mathrm{d}\sigma}{\int \chi(\sigma)\mathrm{e}^{\sigma(\beta\psi+\alpha)}\,\mathrm{d}\sigma} = F(\beta\psi+\alpha) = f(\psi) \tag{21}$$

where $F(\Phi) = -(\ln\hat{\chi})'(\Phi)$ and $\hat{\chi}(\Phi) = \int_{-\infty}^{+\infty}\chi(\sigma)\mathrm{e}^{-\sigma\Phi}\,\mathrm{d}\sigma$. In this approach, $\chi(\sigma)$ is determined *a posteriori* by the initial conditions through the conservation of the Casimirs. Indeed, the α_n are Lagrange multipliers which must be related to the constraints Γ_n. The general relaxation equations have the form

$$\frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho = \nabla\cdot\big[D\big(\nabla\rho + \beta(t)\rho(\sigma-\overline{\omega})\nabla\psi\big)\big], \quad D \propto \omega_2 = \int \rho(\sigma-\overline{\omega})^2\,\mathrm{d}\mathbf{r} \tag{22}$$

where the time evolution of $\beta(t)$ is determined by the energy constraint [8]. These equations for $\rho(\mathbf{r},\sigma,t)$ conserve all the Casimirs (in addition to the circulation and the energy) and monotonically increase the mixing entropy (19) until the maximum entropy state (20) is reached. Note also that the diffusion coefficient is not constant but depends on the local fluctuations of vorticity. This can block the relaxation in a sub-region of the flow (maximum entropy bubble) and account for incomplete relaxation [8,9].

In the case of flows that are forced at small-scales, Ellis et al. [22] and Chavanis [23] have proposed to treat the constraints associated with the fragile moments $\Gamma_{n>1}$ canonically and fix the Lagrange multipliers $\alpha_{n>1}$, or the *prior* distribution $\chi(\sigma)$, instead of $\Gamma_{n>1}$ (by contrast, the robust constraints Γ and E are still treated microcanonically). If we view the vorticity levels as species of particles, this amounts to fixing the chemical potentials instead of the total number of particles in each species. It is argued that the prior $\chi(\sigma)$ is imposed by the small-scale forcing so it must be regarded as given a priori (it is external to the system under consideration). In this point of view, one must work with the relative entropy

$$S_\chi[\rho] = -\int \rho\ln\left[\frac{\rho}{\chi(\sigma)}\right]\mathrm{d}\mathbf{r}\,\mathrm{d}\sigma \tag{23}$$

where $\chi(\sigma)$ is assumed given. The relative entropy (23) can be viewed as the Legendre transform $S_\chi = S - \sum_{n>1}\alpha_n\Gamma_n$ of the mixing entropy (19) with respect to the fragile moments $\Gamma_{n>1} = \int\overline{\omega^n}\,\mathrm{d}\mathbf{r}$ [23]. The equilibrium state is obtained by maximizing (23) while conserving only the robust constraints E and Γ. This again yields the Gibbs state (20) except that now $\chi(\sigma)$ is fixed a priori by the small-scale forcing. The optimal coarse-grained vorticity is given by (21) where $F(\Phi) = -(\ln\hat{\chi})'(\Phi)$ is now completely specified by the prior $\chi(\sigma)$. Note that $\overline{\omega}$ satisfies a result of large deviations [22]. It maximizes a generalized entropy [23]:

$$S[\overline{\omega}] = -\int C(\overline{\omega})\,\mathrm{d}\mathbf{r} \tag{24}$$

at fixed circulation and energy, where C is a convex function directly determined by the prior according to the relation [24]:

$$C(\overline{\omega}) = -\int^{\overline{\omega}} F^{-1}(x)\,\mathrm{d}x = -\int^{\overline{\omega}}\big[(\ln\hat{\chi})'\big]^{-1}(-x)\,\mathrm{d}x \tag{25}$$

[1] E and Γ are called *robust constraints* because they can be expressed in terms of the coarse-grained field. By contrast, the $\Gamma_{n>1}$ are called *fragile constraints* because they are not conserved by the coarse-grained vorticity as $\overline{\omega^n} \neq \overline{\omega}^n$ and they must be expressed in terms of the fine-grained vorticity distribution $\rho(\mathbf{r},\sigma)$. This implies that E and Γ can be computed at any time from the coarse-grained flow while the $\Gamma_{n>1}$ are only accessible from the initial conditions or from the fine-grained flow. Since they are not accessible from the macroscopic dynamics, they often behave as 'hidden constraints'.

In this context, Chavanis [3,23] has proposed a thermodynamical small-scale parameterization of 2D forced geophysical flows where the coarse-grained vorticity evolves according to

$$\frac{\partial\overline{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\overline{\omega}=\nabla\cdot\left\{D\left[\nabla\overline{\omega}+\frac{\beta(t)}{C''(\overline{\omega})}\nabla\psi\right]\right\},\quad \overline{\omega}=-\Delta\psi \tag{26}$$

$$\beta(t)=-\frac{\int D\nabla\overline{\omega}\cdot\nabla\psi\,\mathrm{d}\mathbf{r}}{\int D\frac{(\nabla\psi)^2}{C''(\overline{\omega})}\,\mathrm{d}\mathbf{r}},\quad D\propto\omega_2=\frac{1}{C''(\overline{\omega})} \tag{27}$$

This relaxation equation for $\overline{\omega}(\mathbf{r},t)$ conserves only the robust constraints (circulation and energy) and monotonically increases the generalized entropy (24) fixed by the prior vorticity distribution $\chi(\sigma)$. It differs from the relaxation equations (22) of Robert and Sommeria in the sense that the specification of the prior $\chi(\sigma)$ (determined by the small-scale forcing) replaces the specification of the Casimirs (determined by the initial conditions). However, in both models, the robust constraints E and Γ are treated microcanonically (i.e., they are rigorously conserved). We thus have to solve *one* differential equation (26) instead of N coupled equations (22) for each discretized level σ_k. The relaxation equation (26) enters in the class of generalized mean-field Fokker–Planck equations presented in Section 2. It is however associated with a microcanonical description since $\beta(t)$ varies in time so as to conserve the energy E [3].

Because of incomplete relaxation (non-ergodicity), the coarse-grained vorticity can converge towards a meta-equilibrium state which is different from that predicted by the statistical theory [11]. This coarse-grained vorticity field is a stable stationary solution of the 2D Euler equation which is incompletely mixed. The 2D Euler equation admits an infinite number of stationary solutions specified by an arbitrary $\overline{\omega}-\psi$ relationship. A solution of the form $\overline{\omega}=f(\psi)$, where f is monotonic, which maximizes an H-function $H[\overline{\omega}]=-\int C(\overline{\omega})\,\mathrm{d}\mathbf{r}$ (where C is convex) at fixed energy E and circulation Γ is nonlinearly dynamically stable with respect to the 2D Euler equation [22,23]. For example, the Tsallis functional $H_q=-\frac{1}{q-1}\int(\overline{\omega}^q-\overline{\omega})\,\mathrm{d}\mathbf{r}$ is a particular H-function. Its maximization at fixed E and Γ determines a particular class of stationary solutions of the 2D Euler equation (polytropic vortices) corresponding to $\overline{\omega}=[\lambda-\beta(q-1)\psi/q]^{1/(q-1)}$. Minus the enstrophy $\Gamma_2=\int\overline{\omega}^2\,\mathrm{d}\mathbf{r}$ can also be viewed as an H-function leading to a linear $\overline{\omega}-\psi$ relationship. This is a particular case of the Tsallis functional corresponding to $q=2$. Note that the relaxation equations (26) and (27) can serve as a numerical algorithm to construct nonlinearly dynamically stable stationary solutions of the 2D Euler equation since the steady state of these equations maximizes, by construction, an H-function specified by $C(\overline{\omega})$ at fixed circulation and energy.

4. Collisionless stellar systems

For most stellar systems, the encounters between stars are negligible [25] so that the galactic dynamics is described by the Vlasov–Poisson system

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial\mathbf{r}}-\nabla\Phi\cdot\frac{\partial f}{\partial\mathbf{v}}=0,\quad \Delta\Phi=4\pi G\int f\,\mathrm{d}\mathbf{v} \tag{28}$$

The Vlasov–Poisson and the 2D Euler–Poisson systems present deep similarities as shown in [7]. In particular, the Vlasov–Poisson system develops a complicated mixing process in phase space associated with the damped oscillations of a proto-galaxy initially out-of-equilibrium and in search of a steady (virialized) state. Due to 'phase mixing', a collisionless stellar system can reach an 'equilibrium' state (meta-equilibrium) on a very short timescale. This collisionless relaxation has been called violent relaxation. Lynden-Bell [26] proposed a statistical mechanics of this process which is similar to that exposed previously in 2D turbulence. However, this process takes place in a six-dimensional phase space instead of the plane [9].

In the two-levels approximation where $f=\eta_0$ or $f=0$, the optimal probability $p(\mathbf{r},\mathbf{v})$ of finding the level η_0 in $\mathbf{r},\mathbf{v}$ at equilibrium is obtained by maximizing the mixing entropy

$$S[p]=-\int\left\{p\ln p+(1-p)\ln(1-p)\right\}\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v} \tag{29}$$

at fixed $\gamma_0 = \int p\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}$ (or total mass M) and energy $E = \frac{1}{2}\int \overline{f} v^2\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v} + \frac{1}{2}\int \rho\Phi\,\mathrm{d}\mathbf{r}$. Using the method of Lagrange multipliers, the optimal probability $p_*(\mathbf{r},\mathbf{v})$, and consequently the coarse-grained distribution function $\overline{f} = p_*\eta_0$, is given by

$$\overline{f}(\mathbf{r},\mathbf{v}) = \frac{\eta_0}{1+\lambda \mathrm{e}^{\beta\eta_0(\frac{v^2}{2}+\Phi)}} \quad (\lambda > 0) \tag{30}$$

Apart from a reinterpretation of the constants, this is similar to the Fermi–Dirac distribution [26]. Using the analogy with 2D turbulence, Chavanis et al. [9] have proposed a dynamical model to describe the relaxation of the system towards equilibrium. By using the MEPP, they obtain in the two-levels case an equation of the form

$$\frac{\partial \overline{f}}{\partial t} + \mathbf{v}\cdot\frac{\partial \overline{f}}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial \overline{f}}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\cdot\left\{D\left[\frac{\partial \overline{f}}{\partial \mathbf{v}} + \beta(t)\overline{f}(\eta_0-\overline{f})\mathbf{v}\right]\right\} \tag{31}$$

$$\beta(t) = -\frac{\int D\frac{\partial \overline{f}}{\partial \mathbf{v}}\cdot\mathbf{v}\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}}{\int D\overline{f}(\eta_0-\overline{f})v^2\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}}, \qquad \Delta\Phi = 4\pi G\int \overline{f}\,\mathrm{d}\mathbf{v} \tag{32}$$

This relaxation equation enters in the class of generalized mean-field Fokker–Planck equations of Section 2 associated with the Fermi–Dirac entropy in phase space $S[\overline{f}] = -\int\{(\overline{f}/\eta_0)\ln(\overline{f}/\eta_0) + (1-\overline{f}/\eta_0)\ln(1-\overline{f}/\eta_0)\}\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}$ and with a Newtonian potential $u = -G/|\mathbf{r}-\mathbf{r}'|$. Note, however, that in the present case the inverse temperature evolves with time according to Eq. (32) so as to conserve the energy. Therefore, Eq. (31) monotonically increases the entropy S at fixed E and M. This describes a microcanonical situation [3].

In the general case, we get a system of relaxation equations for $\rho(\mathbf{r},\mathbf{v},\eta,t)$ of the form [9]:

$$\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\frac{\partial \rho}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial \rho}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\cdot\left\{D\left[\frac{\partial \rho}{\partial \mathbf{v}} + \beta(t)\rho(\eta-\overline{f})\mathbf{v}\right]\right\} \tag{33}$$

They converge at equilibrium towards the Gibbs state

$$\rho(\mathbf{r},\mathbf{v},\eta) = \frac{1}{Z(\mathbf{r},\mathbf{v})}\chi(\eta)\mathrm{e}^{-(\beta\varepsilon+\alpha)\eta} \tag{34}$$

which maximizes the mixing entropy

$$S[\rho] = -\int \rho\ln\rho\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\eta \tag{35}$$

at fixed E, M and $\gamma(\eta) = \int \rho\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}$. At statistical equilibrium, the coarse-grained distribution function $\overline{f} = \int \rho\eta\,\mathrm{d}\eta$ is given by

$$\overline{f} = \frac{\int \chi(\eta)\eta\mathrm{e}^{\eta(\beta\varepsilon+\alpha)}\,\mathrm{d}\eta}{\int \chi(\eta)\mathrm{e}^{\eta(\beta\varepsilon+\alpha)}\,\mathrm{d}\eta} = F(\beta\varepsilon+\alpha) = \overline{f}(\varepsilon) \tag{36}$$

We note that the statistical theory of violent relaxation leads to non-standard (non-Boltzmannian) coarse-grained distribution functions. This is due to the existence of an infinite family of conserved quantities, the Casimirs, which play the role of 'hidden constraints' in our general interpretation of non-standard distributions. The coarse-grained distribution functions arising in theories of violent relaxation can be viewed as sorts of *superstatistics* [24] since they are expressed as a superposition of Boltzmann distributions weighted by a non-universal factor depending on the initial conditions.

As in 2D turbulence, the violent relaxation of collisionless stellar systems is usually incomplete [26]. The Vlasov equation admits an infinite number of stationary solutions specified by the Jeans theorem. A solution of the form $\overline{f} = f(\varepsilon)$ with $f'(\varepsilon) < 0$ which maximizes an H-function $H[\overline{f}] = -\int C(\overline{f})\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}$ (where C is convex) at fixed energy E and mass M is nonlinearly dynamically stable with respect to the Vlasov–Poisson system [27]. This determines a subclass of spherical stellar systems. For example, the Tsallis functional $H_q = -\frac{1}{q-1}\int(\overline{f}^q - \overline{f})\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}$ is a particular H-function. Its maximization at fixed E and M determines a particular class of stationary solutions of the Vlasov–Poisson system with DF $\overline{f} = [\lambda - \beta(q-1)\varepsilon/q]^{1/(q-1)}$ called stellar polytropes in astrophysics (the standard polytropic index is related to the q parameter by $n = 3/2 + 1/(q-1)$). Like in 2D turbulence, we can use generalized Fokker–Planck equations as numerical algorithms to construct stable stationary solutions of the Vlasov equation (see [3]).

5. Self-gravitating Brownian particles and related systems

In a series of papers, Chavanis and Sire [15] have introduced and studied a model of self-gravitating Brownian particles. In this model, the particles interact via self-gravity but, in addition, they experience a friction force (against an inert medium) and a stochastic force. This Brownian model is described by the canonical ensemble in contrast to the usual Hamiltonian model of stellar systems which is described by the microcanonical ensemble. In the mean-field approximation, the evolution of the distribution function of the Brownian gas is governed by the Kramers–Poisson system. In the strong friction limit, we get the Smoluchowski–Poisson (SP) system

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\left[\frac{1}{\xi}(T\nabla\rho + \rho\nabla\Phi)\right] \tag{37}$$

$$\Delta\Phi = S_d G\rho \tag{38}$$

Comparing with Eq. (6), this corresponds to an isothermal equation of state $p = \rho T$ associated with the Boltzmann entropy $S = -\int \rho\ln\rho\,\mathrm{d}\mathbf{r}$. In $d \geqslant 2$, there exists a critical temperature T_c for box-confined systems. For $T > T_c$, the system reaches a stable equilibrium state corresponding to the Boltzmann statistics

$$\rho_{\mathrm{eq}} = A\mathrm{e}^{-\beta\Phi}. \tag{39}$$

For $T < T_c$, there is no equilibrium state anymore. The density blows up and finally forms a Dirac peak (condensate) [15]. Chavanis and Sire [16] have also studied more general models where the diffusion of particles is anomalous so that the evolution of the particles is described by the nonlinear Smoluchowski–Poisson (NSP) system

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\left[\frac{1}{\xi}(K\nabla\rho^\gamma + \rho\nabla\Phi)\right] \tag{40}$$

$$\Delta\Phi = S_d G\rho \tag{41}$$

Comparing with Eq. (6), this corresponds to a polytropic equation of state $p = K\rho^\gamma$ associated with the Tsallis entropy $S = -\frac{1}{\gamma-1}\int(\rho^\gamma - \rho)\,\mathrm{d}\mathbf{r}$. The steady states (polytropes) reproduce the statistics introduced by Tsallis

$$\rho_{\mathrm{eq}} = \left[\lambda - \frac{\beta(q-1)}{q}\Phi\right]^{1/(q-1)} \tag{42}$$

The nonlinear Smoluchowski equation (40) also arises in the physics of porous media where anomalous diffusion is due to the complicated (fractal, multi-fractal) phase-space structure of the medium and Φ is an external potential [5]. More generally, we can consider the evolution of particles described by the generalized Smoluchowski–Poisson (GSP) system

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\left[\frac{1}{\xi}(\nabla p + \rho\nabla\Phi)\right] \tag{43}$$

$$\Delta\Phi = S_d G\rho \tag{44}$$

The case of self-gravitating Brownian fermions where $p = p(\rho)$ is the Fermi–Dirac equation of state has been studied in [17]. Some general properties of the GSP system, including the Virial theorem and a stability analysis of the steady states, have been obtained in [18].

We can also consider other forms of potential of interaction than the gravitational one. For example, the Smoluchowski–Poisson system with the repulsive Coulombian potential describes electrolytes in the theory of Debye and Hückel [28]. On the other hand, when the particles evolve on a ring and the potential of interaction is truncated to one Fourier mode, we get the Brownian Mean Field (BMF) model introduced in [19]:

$$\frac{\partial\rho}{\partial t} = T\frac{\partial^2\rho}{\partial\theta^2} + \frac{k}{2\pi}\frac{\partial}{\partial\theta}\left\{\rho\int_0^{2\pi}\sin(\theta-\theta')\rho(\theta',t)\,\mathrm{d}\theta'\right\} \tag{45}$$

which is the canonical version of the Hamiltonian Mean Field (HMF) model.

Finally, a dynamical model of the Bose–Einstein condensation has been studied recently by Sopik et al. [20]. The dynamical evolution of a gas of non-interacting bosons in contact with a thermal bath is governed by the bosonic Kramers equation

$$\frac{\partial\rho}{\partial t}=\frac{1}{\xi}\nabla_{\mathbf{k}}\cdot\bigl[T\nabla_{\mathbf{k}}\rho+\rho(1+\rho)\mathbf{k}\bigr] \tag{46}$$

which takes into account an inclusion principle [4]. This equation enters in the generalized class of Fokker–Planck equations of Section 2. It is associated with the Bose–Einstein entropy $S=-\int\{\rho\ln\rho-(1+\rho)\ln(1+\rho)\}\,\mathrm{d}\mathbf{k}$. We have found that the Bose–Einstein condensation (in phase space) shares deep similarities with the collapse of the self-gravitating Brownian gas (in position space). In particular, for $d\geqslant 3$, there exists a critical temperature T_c. The density blows up for $T<T_c$ leading ultimately to a Dirac peak (Bose condensate) in $\mathbf{k}$-space [20].

6. Chemotactic aggregation of bacterial populations

The name chemotaxis refers to the motion of organisms (amoeba) induced by chemical signals (acrasin). In some cases, the biological organisms secrete a substance that has an attractive effect on the organisms themselves. Therefore, in addition to their diffusive motion, they move systematically along the gradient of concentration of the chemical they secrete (chemotactic flux). When attraction prevails over diffusion, the chemotaxis can trigger a self-accelerating process until a point at which aggregation takes place. This is the case for the slime mold *Dictyostelium Discoideum* and for the bacteria *Escherichia coli*.

A model of slime mold aggregation has been introduced by Keller and Segel [6] in the form of two PDEs:

$$\frac{\partial\rho}{\partial t}=\nabla\cdot(D_2\nabla\rho)-\nabla\cdot(D_1\nabla c) \tag{47}$$

$$\frac{\partial c}{\partial t}=-k(c)c+f(c)\rho+D_c\Delta c \tag{48}$$

In these equations $\rho(\mathbf{r},t)$ is the concentration of amoebae and $c(\mathbf{r},t)$ is the concentration of acrasin. Acrasin is produced by the amoebae at a rate $f(c)$. It can also be degraded at a rate $k(c)$. Acrasin diffuse according to Fick's law with a diffusion coefficient D_c. Amoebae concentration changes as a result of an oriented chemotactic motion in the direction of a positive gradient of acrasin and a random motion analogous to diffusion. In Eq. (47), $D_2(\rho,c)$ is the diffusion coefficient of the amoebae and $D_1(\rho,c)$ is a measure of the strength of the influence of the acrasin gradient on the flow of amoebae. This chemotactic drift is the fundamental process in the problem.

A first simplification of the Keller–Segel model is provided by the system of equations

$$\frac{\partial\rho}{\partial t}=D\Delta\rho-\chi\nabla\cdot(\rho\nabla c) \tag{49}$$

$$\frac{\partial c}{\partial t}=D'\Delta c+a\rho-bc \tag{50}$$

where the parameters are positive constants. An additional simplification, introduced by Jäger and Luckhaus [29], consists in ignoring the temporal derivative in Eq. (50). This is valid in the case where the diffusion coefficient D' is large. Taking also $b=0$, we obtain

$$\frac{\partial\rho}{\partial t}=D\Delta\rho-\chi\nabla\cdot(\rho\nabla c) \tag{51}$$

$$\Delta c=-\lambda\rho \tag{52}$$

where $\lambda=a/D'$. These equations are isomorphic to the Smoluchowski–Poisson system (37), (38) which describes self-gravitating Brownian particles in a high friction limit. The analogy between self-gravitating systems and the chemotaxis of bacterial colonies is developed in [17].

The Keller–Segel model ignores clumping and sticking effects. However, at the late stages of the blow-up, when the density of amoebae has reached high values, finite size effects and stickiness must clearly be taken into account. As a first step, we have proposed in [17] to replace the classical equation (51) by an equation of the form

$$\frac{\partial\rho}{\partial t}=D\Delta\rho-\chi\nabla\cdot\bigl(\rho(1-\rho/\sigma_0)\nabla c\bigr) \tag{53}$$

which enforces a limitation $\rho \leqslant \sigma_0$ on the maximum concentration of bacteria in physical space. This equation monotonically decreases the Lyapunov functional

$$F[\rho] = -\frac{1}{2}\int \rho c\,\mathrm{d}\mathbf{r} + \frac{D\sigma_0}{\chi}\int\left[\frac{\rho}{\sigma_0}\ln\frac{\rho}{\sigma_0} + \left(1-\frac{\rho}{\sigma_0}\right)\ln\left(1-\frac{\rho}{\sigma_0}\right)\right]\mathrm{d}\mathbf{r} \tag{54}$$

This functional can be interpreted as a free energy $F = E - T_{\rm eff}S$ associated with a Fermi–Dirac entropy in physical space [3,17] where $T_{\rm eff} = D/\chi$ is an effective temperature. This form of entropy can be obtained by introducing a lattice model preventing two particles to be on the same site. The lattice creates an exclusion principle in physical space similar to the Pauli exclusion principle in phase space for fermions. Then, $S[\rho]$ can be obtained by a standard combinatorial analysis respecting this exclusion principle. The equilibrium states of Eq. (53) are given by a Fermi-type distribution in physical space

$$\rho = \frac{\sigma_0}{1+\lambda \mathrm{e}^{-\frac{\chi}{D}c}} \quad (\lambda > 0) \tag{55}$$

which minimizes the effective free energy (54) at fixed mass. On the other hand, the relation between the concentration of amoebae and acrasin may be more complex that simply given by the Poisson equation (52). For example, taking $b \neq 0$ in the original Keller–Segel model, we obtain a relation of the form

$$\Delta c - k_S^2 c = -\lambda\rho \tag{56}$$

where $k_S^2 = b/D'$. The second term is similar to the Debye shielding in plasma physics or to the Rossby shielding in Quasi-Geostrophic (QG) flows. We note that Eq. (53) enters in the class of nonlinear mean-field Fokker–Planck equations of Section 2 associated with a Fermi–Dirac entropy in physical space. We also note the resemblance with the relaxation equations (17) of 2D turbulence except that here $\beta \equiv \chi/D$ is fixed (which is similar to a 'canonical' situation) while in 2D turbulence E is fixed (microcanonical situation).

More generally, in the primitive Keller–Segel model (47), (48) the coefficients D_1 and D_2 can both depend on the concentration of particles. This takes into account 'hidden constraints' that act on the microscopic dynamics. Eq. (53) is an example of nonlinear mean-field Fokker–Planck equations where the mobility depends on the concentration while the diffusion coefficient is constant. Alternatively, if the diffusion coefficient depends on the concentration while the mobility is constant, we can write Eq. (47) in the form

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\left[\chi(\nabla p - \rho\nabla c)\right] \tag{57}$$

where $p = p(\rho)$ is an effective pressure. Therefore, the Keller–Segel model is an example of nonlinear mean-field Fokker–Planck equations presented in Section 2. We can thus develop an Effective Thermodynamical Formalism (ETF) to investigate the chemotactic problem [3]. In [17,18], we have also introduced an inertial model of chemotaxis based on the hydrodynamic equations

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{58}$$

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nabla c - \xi\mathbf{u} \tag{59}$$

$$\frac{\partial c}{\partial t} = -k(c)c + f(c)\rho + D_c\Delta c \tag{60}$$

These hydrodynamical equations can be derived from generalized Fokker–Planck equations of the form (1) [18]. They are similar to the damped Euler equations (9), (10) for self-gravitating Brownian particles [3] except that the Poisson equation is replaced by a more general field equation taking into account the specificities of the chemotactic model. For $\xi = 0$, we recover the hyperbolic model introduced by Gamba et al. [30] and for $\xi \to +\infty$, we can neglect the inertial term in Eq. (59), leading to $\rho\mathbf{u} = -\frac{1}{\xi}(\nabla p + \rho\nabla\Phi) + \mathrm{O}(\xi^{-2})$. When substituted in Eq. (58), we recover the parabolic Keller–Segel model (57). It has been shown that parabolic models of chemotaxis lead to point-wise blow-up while hyperbolic models show the formation of a network that is interpreted as the beginning of a vasculature. This phenomenon is responsible of angiogenesis, a major factor for the growth of tumors.

7. Conclusion

In this article, we have given several physical examples where the evolution of the system is governed by nonlinear mean-field Fokker–Planck equations introduced in [3]. These equations are consistent with an Effective Thermodynamical Formalism (ETF) and lead to non-standard distributions at equilibrium. We stress that: (i) The Tsallis distributions and the Tsallis generalized entropies just represent a particular case of this general formalism; (ii) the physical reason why non-standard distributions arise in a problem is different from case to case and must be sought for each particular system. In general, this reflects the existence of 'hidden constraints' [3] that can be of various forms as discussed in the Introduction. If the system does not 'mix well' (for some reason) there is no universal form of entropy to account for non-ergodicity. (iii) In many cases, the resemblance with a 'generalized thermodynamical formalism' is essentially effective or formal. In some cases, there is no relation with thermodynamics at all. Yet, it may be of interest to develop a *thermodynamical analogy* and use the same vocabulary as in thermodynamics. This allows us to unify the formalisms and transpose the results obtained in one context to another context. Indeed, we have found many *analogies* between systems of a very different nature: stellar systems, 2D vortices, bacteria, Bose–Einstein condensation, ... [3,7,17,20].

References

[1] A. Pais, Subtle is the Lord, Oxford Univ. Press, New York, 1982.
[2] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[3] P.-H. Chavanis, Phys. Rev. E 68 (2003) 036108;
P.-H. Chavanis, Physica A 332 (2004) 89;
P.-H. Chavanis, Banach Center Publ. 66 (2004) 79;
P.-H. Chavanis, Physica A 340 (2004) 57.
[4] G. Kaniadakis, Physica A 296 (2001) 405.
[5] A.R. Plastino, A. Plastino, Physica A 222 (1995) 347.
[6] E. Keller, L.A. Segel, J. Theor. Biol. 26 (1970) 399.
[7] P.-H. Chavanis, Statistical mechanics of two-dimensional vortices and stellar systems, in: T. Dauxois, S. Ruffo, E. Arimondo, M. Wilkens (Eds.), Dynamics and Thermodynamics of Systems with Long Range Interactions, in: Lecture Notes in Phys., Springer-Verlag, Berlin/New York, 2002, cond-mat/0212223.
[8] R. Robert, J. Sommeria, Phys. Rev. Lett. 69 (1992) 2776;
R. Robert, C. Rosier, J. Stat. Phys. 86 (1997) 481;
P.-H. Chavanis, Phys. Rev. Lett. 84 (2000) 5512.
[9] P.-H. Chavanis, J. Sommeria, R. Robert, Astrophys. J. 471 (1996) 385;
P.-H. Chavanis, Mon. Not. R. Astron. Soc. 300 (1998) 981.
[10] H. Brands, P.-H. Chavanis, J. Sommeria, R. Pasmanter, Phys. Fluids 11 (1999) 3465.
[11] P.-H. Chavanis, cond-mat/0509726.
[12] P.-H. Chavanis, P. Laurençot, M. Lemou, Physica A 341 (2004) 145.
[13] T.D. Frank, Phys. Lett. A 290 (2001) 93.
[14] L. Borland, Phys. Rev. E 57 (1998) 6634.
[15] P.-H. Chavanis, C. Rosier, C. Sire, Phys. Rev. E 66 (2002) 036105;
C. Sire, P.-H. Chavanis, Phys. Rev. E 66 (2002) 046133;
C. Sire, P.-H. Chavanis, Phys. Rev. E 69 (2004) 066109;
P.-H. Chavanis, C. Sire, Phys. Rev. E 70 (2004) 026115;
C. Sire, P.-H. Chavanis, Banach Center Publ. 66 (2004) 287;
J. Sopik, C. Sire, P.-H. Chavanis, Phys. Rev. E 72 (2005) 026105.
[16] P.-H. Chavanis, C. Sire, Phys. Rev. E 69 (2004) 016116.
[17] P.-H. Chavanis, M. Ribot, C. Rosier, C. Sire, Banach Center Publ. 66 (2004) 103.
[18] P.-H. Chavanis, C. Sire, cond-mat/0504718.
[19] P.-H. Chavanis, J. Vatteville, F. Bouchet, Eur. Phys. J. B 46 (2005) 61.
[20] J. Sopik, C. Sire, P.-H. Chavanis, cond-mat/0511347.
[21] R. Robert, C. R. Acad. Sci. Paris (I) 311 (1990) 575;
J. Miller, Phys. Rev. Lett. 65 (1990) 2137;
R. Robert, J. Sommeria, J. Fluid Mech. 229 (1991) 291;
P.-H. Chavanis, J. Sommeria, J. Fluid Mech. 314 (1996) 267.
[22] R. Ellis, K. Haven, B. Turkington, Nonlinearity 15 (2002) 239.
[23] P.-H. Chavanis, Physica D 200 (2005) 257;
P.-H. Chavanis, in: Proceedings of the Workshop on "Interdisciplinary Aspects of Turbulence" at Ringberg Castle, Tegernsee, Germany, Max-Planck Institut für Astrophysik, 2005, physics/0601087.

[24] P.-H. Chavanis, Physica A 359 (2006) 177.
[25] J. Binney, S. Tremaine, Galactic Dynamics, Princeton Series in Astrophysics, 1987.
[26] D. Lynden-Bell, Mon. Not. R. Astron. Soc. 136 (1967) 101;
P.-H. Chavanis, J. Sommeria, Mon. Not. R. Astron. Soc. 296 (1998) 569.
[27] S. Tremaine, M. Hénon, D. Lynden-Bell, Mon. Not. R. Astron. Soc. 219 (1986) 285;
P.-H. Chavanis, C. Sire, Physica A 356 (2005) 419;
P.-H. Chavanis, astro-ph/0602429, Astron. Astrophys., in press.
[28] P. Debye, E. Hückel, Phys. Z. 24 (1925) 305.
[29] W. Jäger, S. Luckhaus, Trans. Amer. Math. Soc. 329 (1992) 819.
[30] A. Gamba, et al., Phys. Rev. Lett. 90 (2003) 118101.